Online Appendix to " Farsighted Stability in Service Platforms Competition"

Hongqiao Chen, Pengfei Guo, Qingying Li, Yulan Wang

OA.1. Notations and Auxiliary Functions

The notations are defined as follows:

$$N_0 = hv\mu/(v\mu - r)^2, \quad \underline{N} = h/(\sqrt{v\mu} - \sqrt{r})^2,$$
$$\bar{N} = 2h(v\mu + r)/(v\mu - r)^2, \quad \Lambda_0 = 4hr\mu/(v\mu - r)^2.$$

The auxiliary functions are defined as follows:

$$\bar{\Lambda}_m(N) = \Big(1 - \sqrt{\frac{h}{vN\mu}}\Big)N\mu, \quad \underline{\Lambda}_m(N) = \frac{1}{4}\Big(\sqrt{\frac{h\mu}{r} + 4N\mu} - \sqrt{\frac{h\mu}{r}}\Big)^2,$$

$$\underline{\Lambda}_m^{-1}(\Lambda) = \Big(1 + \sqrt{\frac{h\mu}{r\Lambda}}\Big)\frac{\Lambda}{\mu}, \quad \hat{\Lambda}_m(N) = \frac{1}{2v}\big(rN + vN\mu - h - \sqrt{(rN + vN\mu - h)^2 - 4rvN^2\mu}\big),$$

$$\underline{\Lambda}(N) = \frac{1}{4}\Big(\sqrt{\frac{2h\mu}{r} + 4N\mu} - \sqrt{\frac{2h\mu}{r}}\Big)^2, \quad \bar{\Lambda}(N) = \Big(1 - \sqrt{\frac{2h}{vN\mu}}\Big)N\mu,$$

$$\hat{\Lambda}(N) = \frac{1}{2v}\big(rN + vN\mu - 2h - \sqrt{(rN + vN\mu - 2h)^2 - 4rvN^2\mu}\big),$$

$$\underline{\Lambda}'(N) = \frac{1}{4}\Big(\sqrt{\frac{h\mu}{r} + 4\Big(N\mu - \frac{2h\mu(v\mu + r)}{(v\mu - r)^2}\Big)} - \sqrt{\frac{h\mu}{r}}\Big)^2 + \frac{4hr\mu}{(v\mu - r)^2},$$

$$\bar{\Lambda}'(N) = N\mu - \frac{h\sqrt{\mu}}{(\sqrt{v\mu} - \sqrt{r})\sqrt{v}} - \sqrt{\frac{h\mu[(\sqrt{v\mu} - \sqrt{r})^2N - h]}{(\sqrt{v\mu} - \sqrt{r})^2 v}}, \quad {\underline{\Lambda}'}^{-1}(\Lambda) = \frac{\Lambda}{\mu} + \sqrt{\frac{h\Lambda_0}{r\mu}} + \sqrt{\frac{h(\Lambda - \Lambda_0)}{r\mu}}.$$

$$\pi(\lambda, n) = \big(v - \frac{h}{n\mu - \lambda}\big)\lambda - rn. \tag{OA.1}$$

For any $t \geq 0$, we define an auxiliary problem P^t as follows.

$$\begin{aligned} P^t: \qquad & \max \qquad \pi_2(\lambda_2, n_2) = \Big(v - \frac{h}{n_2\mu - \lambda_2}\Big)\lambda_2 - rn_2 \\ & s.t. \qquad \pi_1(\lambda_1, n_1) = t \ \text{ and } (\lambda_1, n_1; \lambda_2, n_2) \in S', \end{aligned}$$

where $S' = \{(\lambda_1, n_1; \lambda_2, n_2) | n_i \geq \underline{N}, n_1 + n_2 = N, \max\{\hat{\Lambda}_m(n_i), \underline{\Lambda}_m(n_i)\} \leq \lambda_i \leq \bar{\Lambda}_m(n_i), \lambda_1 + \lambda_2 = \Lambda\}$. Denote P_0 be the optimal objective value to Problem P^0.

OA.2. Auxiliary Results in Section 4.2

To identify Pareto-efficient strategy profiles, we construct the auxiliary problem $P_i'|_{(p_j^*, w_j^*)}$:

$$\begin{aligned} P_i'|_{(p_j, w_j)}: \quad & \max \quad \pi_i = (p_i - w_i)\lambda_i \\ & s.t. \quad (p_i, w_i) \in S_i|_{(p_j, w_j)}, \end{aligned}$$

where $S_1|_{(p_2, w_2)} = \{(p, w) | (p, w; p_2, w_2) \in S\}$ and $S_2|_{(p_1, w_1)} = \{(p, w) | (p_1, w_1; p, w) \in S\}$. We first give a necessary condition of Pareto-efficient strategy profile in the following lemma.

Lemma OA.1. *(i) For $i, j = 1, 2$ and $i \neq j$, if a feasible strategy profile $(p_1^*, w_1^*; p_2^*, w_2^*)$ is Pareto efficient, then (p_i^*, w_i^*) must be the optimal solution to problem $P_i'|_{(p_j^*, w_j^*)}$. (ii) If (p_i^*, w_i^*) is the optimal solution of problem $P_i'|_{(p_j^*, w_j^*)}$, then (1) and (2) hold.*

By substituting (1) and (2) into problem $P_i'|_{(p_j, w_j)}$, it can be rewritten as problem $P_i|_{(\lambda_j, n_j)}$:

$$P_i|_{(\lambda_j,n_j)}: \quad \max_{\lambda_i>0, n_i>0} \quad \pi_i = \Big(v - \frac{h}{n_i\mu - \lambda_i}\Big)\lambda_i - rn_i$$
$$s.t. \quad N - n_j \geq n_i, \quad \Lambda - \lambda_j \geq \lambda_i, \quad v - \frac{h}{n_i\mu - \lambda_i} - \frac{rn_i}{\lambda_i} \geq 0.$$

Then, we can identify Pareto-efficient strategy profiles from the jointly optimal solution to problems $P_1|_{(\lambda_2^*, n_2^*)}$ and $P_2|_{(\lambda_1^*, n_1^*)}$. Lemma OA.2 first establishes a key property, while Lemma OA.3 provides the sufficient and necessary conditions for the optimal solution.

Lemma OA.2. *For $i, j = 1, 2$ and $i \neq j$, if (λ_i^*, n_i^*) is the optimal solution to problem $P_i|_{(\lambda_j^*, n_j^*)}$, then $n_1^* + n_2^* = N$, or $\lambda_1^* + \lambda_2^* = \Lambda$, or both hold.*

Lemma OA.3. *For $i, j = 1, 2$ and $i \neq j$, the sufficient and necessary conditions of (λ_i^*, n_i^*) is the optimal solution to problem $P_i|_{(\lambda_j^*, n_j^*)}$ are given as follows:*
(i) If $N \geq 2\underline{N}$ and $\Lambda > \bar{\Lambda}(N)$, $n_i^* \geq \underline{N}$, $n_1^* + n_2^* = N$ and $\lambda_i^* = \bar{\Lambda}_m(n_i^*)$.
(ii) If (1) $2\underline{N} \leq N \leq 2\overline{N}$ and $\hat{\Lambda}(N) \leq \Lambda \leq \bar{\Lambda}'(N)$ or (2) $N > 2\overline{N}$ and $\underline{\Lambda}(N) \leq \Lambda \leq \bar{\Lambda}'(N)$, $n_i^* \geq \underline{N}$, $n_1^* + n_2^* = N$, $\max\{\hat{\Lambda}_m(n_i^*), \underline{\Lambda}_m(n_i^*)\} \leq \lambda_i^* \leq \bar{\Lambda}_m(n_i^*)$ and $\lambda_1^* + \lambda_2^* = \Lambda$.
(iii) If $N > 2\bar{N}$ and $2\Lambda_0 \leq \Lambda < \underline{\Lambda}'(N)$, $n_i^* = \underline{\Lambda}_m^{-1}(\lambda_i^*)$, $\lambda_i^* \geq \Lambda_0$ and $\lambda_1^* + \lambda_2^* = \Lambda$.

OA.3. Auxiliary Results in Section 5.2

Lemma OA.4. *The optimal solution to problem P_m' satisfies*

$$p_m^* = v - \frac{h}{n_m^*\mu - \lambda_m^*}, \tag{OA.2}$$
$$w_m^* = \frac{rn_m^*}{\lambda_m^*}. \tag{OA.3}$$

Lemma OA.5. *The optimal solution to problem P_m satisfies $n_m^* = N$ or $\lambda_m^* = \Lambda$, where problem P_m is defined as follows:*

$$P_m: \quad \max_{\lambda_m>0, n_m>0} \quad \pi_m = \left(v - \frac{h}{n_m\mu - \lambda_m}\right)\lambda_m - rn_m$$
$$s.t. \quad N - n_m \geq 0, \quad \Lambda - \lambda_m \geq 0, \quad v - \frac{h}{n_m\mu - \lambda_m} - \frac{rn_m}{\lambda_m} \geq 0.$$

OA.4. Auxiliary Results in Section 5.3

We fine some notations and auxiliary functions.

$$\underline{\Lambda}^o = 2\mu + \frac{h}{v} - \sqrt{\frac{8h\mu}{v} + \frac{h^2}{v^2}}, \quad \bar{\Lambda}^o = 2\big(\mu - \sqrt{\frac{h\mu}{v}}\big),$$

$$g_1(x,\Lambda) = \big(v - \frac{h}{\mu - x}\big)x - \frac{4h\Lambda}{(2\mu-\Lambda)^2}y, \quad g_2(x,\Lambda) = \frac{hx}{(\mu + x - \Lambda)^2} + \frac{h\mu}{(\mu - x)^2},$$

where y is uniquely given by $v - \frac{4h\Lambda}{(2\mu-\Lambda)^2} - \frac{h}{\mu-y} + \frac{h}{\mu-(\Lambda-y)} = \frac{h}{\mu-(\Lambda-x)}$.

Lemma OA.6. *If the potential arrival rate $\Lambda < \bar{\Lambda}^o$, the strategy profile $(p_1^o; p_2^o)$ is Pareto efficient if and only if $p_i^o = v - \frac{h}{\mu - \lambda_i^o}$, $\lambda_i^o \le \mu - \sqrt{\frac{h\mu}{v}}$, and $\lambda_1^o + \lambda_2^o = \Lambda$, $i = 1, 2$. Otherwise, when $\Lambda \ge \bar{\Lambda}^o$, the strategy profile $(p_1^o; p_2^o)$ with $p_i^o = v - \frac{h}{\mu - \lambda_i^o}$ and $\lambda_i^o = \mu - \sqrt{\frac{h\mu}{v}}$, $i = 1, 2$, is Pareto efficient.*

To facilitate the comparison between our farsightedly stable outcomes and the myopically stable outcomes in Chen and Wan (2003), below we list both our results and the results of Chen and Wan (2003) in Table OA.1, where we use $\tilde{\cdot}$ to denote the myopically stable outcomes.

Table OA.1 Myopically versus Farsightedly Stable Outcome for One-sided Competition

	Myopic service platforms $(i=1,2)$		Farsighted service platforms $(i=1,2)$	
	Myopically Stable Outcome	Customer Utility	Farsightedly Stable Outcome	Customer Utility
Scarce Demand $\Lambda < \underline{\Lambda}^o$	$\tilde{p}_i^o = \frac{4h\Lambda}{(2\mu-\Lambda)^2}, \tilde{\lambda}_i^o = \Lambda/2$	>0	$p_i^o = v - \frac{h}{\mu-\lambda_i^o}$, $g_1(\lambda_i^o, \Lambda) \ge 0$, $\lambda_1^o + \lambda_2^o = \Lambda$	0
Moderate Demand $\underline{\Lambda}^o \le \Lambda < \bar{\Lambda}^o$	$\tilde{p}_i^o = v - \frac{h}{\mu-\tilde{\lambda}_i^o}$, $\tilde{\lambda}_i^o \le \mu - \sqrt{\frac{h\mu}{v}}$, $g_2(\tilde{\lambda}_i^o, \Lambda) \ge v$ $\tilde{\lambda}_1^o + \tilde{\lambda}_2^o = \Lambda$	0	$p_i^o = v - \frac{h}{\mu-\lambda_i^o}$, $\lambda_i^o \le \mu - \sqrt{\frac{h\mu}{v}}$, $g_2(\lambda_i^o, \Lambda) \ge v$ $\lambda_1^o + \lambda_2^o = \Lambda$	0
Ample Demand $\Lambda \ge \bar{\Lambda}^o$	$\tilde{p}_i^o = v - \frac{h}{\mu-\tilde{\lambda}_i^o}$, $\tilde{\lambda}_i^o = \mu - \sqrt{\frac{h\mu}{v}}$	0	$p_i^o = v - \frac{h}{\mu-\lambda_i^o}$, $\lambda_i^o = \mu - \sqrt{\frac{h\mu}{v}}$	0

OA.5. Main Proofs

Proof of Proposition 1. We first show $\{s\}$ is a singleton vNM FSS if and only if s is Pareto effcient.

"*Necessity*": We prove this by contradiction. Suppose $\{s\}$ forms a singleton vNM FSS but s is not Pareto effcient. Then there exists a feasible strategy profile s' that satisfying $\pi_1(s) \le \pi_1(s')$ and $\pi_2(s) \le \pi_2(s')$. However, by external stability in Definition 1, s' is indirectly dominated by s, which implies $\pi_1(s) > \pi_1(s')$ or $\pi_2(s) > \pi_2(s')$. This is a contradiction.

"*Sufficiency*": Let $(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2)$ be a Pareto efficient outcome. It is sufficiently to show any feasible strategy profile $(p_1, w_1; p_2, w_2) \ne (\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2)$ can be indirectly dominated by $(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2)$ due to theorem 4.8 in Bloch and van den Nouweland (2021). The strategy profile $(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2)$ is Pareto efficient implies that at lest one of the following two inequalities holds: $\pi_1(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2) > \pi_1(p_1, w_1; p_2, w_2)$ and $\pi_2(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2) > \pi_2(p_1, w_1; p_2, w_2)$. Without of loss generality, we suppose $\pi_1(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2) > \pi_1(p_1, w_1; p_2, w_2)$. Consider the path $(p_1, w_1; p_2, w_2) \to_1 (\bar{p}_1, \bar{w}_1; p_2, w_2) \to_2 (\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2)$. Given that platform 1's strategy $(\bar{p}_1, \bar{w}_1)$, (p_2, w_2) is a feasible solution to problem $P_2'|_{(\bar{p}_1, \bar{w}_1)}$, and that $(\bar{p}_2, \bar{w}_2)$ is the optimal solution by Lemma OA.1(i), we have $\pi_2(\bar{p}_1, \bar{w}_1; p_2, w_2) \le \pi_2(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2)$ (with strict inequality when $(p_2, w_2) \ne (\bar{p}_2, \bar{w}_2)$). Therefore, we show that $(p_1, w_1; p_2, w_2)$ is indirectly dominated by $(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2)$.

We next verify that any vNM FSS is a singleton by contradiction. Suppose there exists a vNM FSS, denoted by A, which contains more than one element. We first show there is no Pareto efficient outcome in A. Recall that we have proved any feasible strategy profile can be indirectly dominated by a Pareto-efficient strategy profile in last paragraph. We know there is no Pareto efficient outcome in A. Otherwise, other elements in A can be indirectly dominated by this Pareto efficient outcome, which violates the internal stability in Definition 1. For $s =$

$\{p_1, w_1; p_2, w_2\} \in A$, there exists a Pareto-efficient strategy profile $\bar{S} = \{\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2\}$ dominate s, i.e., $\pi_i(s) \le \pi_i(\bar{s})$ and strict inequality holds for some $i = 1, 2$. Then, s can't indirectly dominate $\bar{s}$. By external stability in Definition 1, there exists $s' = \{p_1', w_1'; p_2', w_2'\} \in A$ indirectly dominates $\bar{s}$. We next show s' indirectly dominates s, which violates internal stability in Definition 1. Because s' indirectly dominates $\bar{s}$, without loss of generality, we assume $\pi_1(\bar{s}) < \pi_1(s')$. Then there is an indirectly dominate path $(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2) \to_1 (p_1', w_1'; \bar{p}_2, \bar{w}_2) \to_2 (p_1', w_1'; p_2', w_2')$, which implies $\pi_2(p_1', w_1'; \bar{p}_2, \bar{w}_2) \le \pi_2(p_1', w_1'; p_2', w_2')$. Hence, there exists an indirectly dominate path from s to s': $s = (p_1, w_1; p_2, w_2) \to_1 (p_1'', w_1''; p_2, w_2) \to_2 (p_1'', w_1''; \bar{p}_2, \bar{w}_2) \to_1 (p_1', w_1'; \bar{p}_2, \bar{w}_2) \to_2 (p_1', w_1'; p_2', w_2') = s'$, where $\pi_2(p_1'', w_1''; p_2, w_2) = 0$. We will check the path is indeed an indirectly dominate path. That is, $\pi_1(s) \le \pi_1(\bar{s}) < \pi_1(s')$, $\pi_2(p_1'', w_1''; p_2, w_2) = 0 < \pi_2(s')$, $\pi_1(p_1'', w_1''; \bar{p}_2, \bar{w}_2) \le \pi_1(\bar{p}_1, \bar{w}_1; \bar{p}_2, \bar{w}_2) = \pi_1(\bar{s}) < \pi_1(s')$ (given that platform 2's strategy $(\bar{p}_2, \bar{w}_2)$, (p_1'', w_1'') is a feasible solution to problem $P_1'|_{(\bar{p}_2, \bar{w}_2)}$, and that $(\bar{p}_1, \bar{w}_1)$ is the optimal solution by Lemma OA.1(i)), $\pi_2(p_1', w_1'; \bar{p}_2, \bar{w}_2) \le \pi_2(p_1', w_1'; p_2', w_2')$.

Proof of Lemma 1. Lemma 1 comes from Lemmas OA.1 and OA.2 directly. □

Proof of Proposition 2. By the analysis in Online Appendix OA.2, any Pareto-efficient strategy profile should satisfy the conditions in Lemma OA.3. We show Proposition 2(i) by considering the three cases in Table 1. Thus, we get Proposition 2(ii) by applying Proposition 1.

Case I: Ample demand and limited supply. We show the strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ that satisfying $n_i^* \ge \underline{N}$, $n_1^* + n_2^* = N$ and $\lambda_i^* = \bar{\Lambda}_m(n_i^*)$ is Pareto efficient by contradiction. In other words, any strategy profile that satisfying the conditions in Lemma OA.3(i) is Pareto efficient. Let (p_1^*, w_1^*) be the optimal solution of problem $P_i'|_{(p_j, w_j)}$ for $i, j = 1, 2$ and $i \neq j$, and $(\pi_1^*; \pi_2^*)$ is the platforms' profit with the strategy profile $(p_1^*, w_1^*; p_2^*, w_2^*)$. Suppose that there exists a strategy profile $(p_1', w_1'; p_2', w_2')$ with profit $(\pi_1'; \pi_2')$, and $\pi_1^* < \pi_1'$ and $\pi_2^* < \pi_2'$. Let $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ and $(\lambda_1', n_1'; \lambda_2', n_2')$ be the corresponding equilibrium choices for the customers and workers. Without loss of generality, we consider the strategy profiles $(p_1', w_1'; p_2', w_2')$ and $(\lambda_1', n_1'; \lambda_2', n_2')$ satisfying Lemma 1. Otherwise, we can easily construct a new strategy profile with better utilized workers or customers, and thus induce a higher profit. From any strategy profile $(p_1, w_1; p_2, w_2)$ such that (1) and (2) are satisfied, including both $(p_1^*, w_1^*; p_2^*, w_2^*)$ and $(p_1', w_1'; p_2', w_2')$, it is easy to check that $\pi_i^* = \pi(\lambda_i^*, n_i^*)$ and $\pi_i' = \pi(\lambda_i', n_i')$, where $i = 1, 2$.

We consider the strategy profile $(\lambda_1', n_1'; \lambda_2', n_2')$. The relation $n_1^* + n_2^* = N$ together with $n_1' + n_2' \le N$ implies that either $n_1^* \ge n_1'$ or $n_2^* \ge n_2'$. Without of loss generality, we let $n_1^* \ge n_1'$. Recall that we restrict our attention to the strategy profile $(\lambda_1', n_1'; \lambda_2', n_2')$ satisfying Lemma 1. Thus, from (OA.1), platform 1's profit satisfies $\pi_1' = \pi_1(\lambda_1', n_1') = \pi(\lambda_1', n_1') \le \pi(\bar{\Lambda}_m(n_1^*), n_1^*) = \pi_1(\bar{\Lambda}_m(n_1^*), n_1^*) = \pi_1^*$, where the first inequality holds because given n, $\pi(\lambda, n)$ is maximized at $\lambda = \bar{\Lambda}_m(n)$ from Lemma OS.6, and the second inequality holds because $\pi(\bar{\Lambda}_m(n), n)$ increases in n from Lemma OS.6, and we assume $n_1^* > n_1'$. This contradicts to $\pi_1' > \pi_1^*$.

Case II: Balanced demand and supply. We show a strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ that satisfying the conditions in Lemma OA.3(ii) is Pareto efficient if and only if $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \cup_{0 \le t \le P_0} \mathcal{F}(t)$ with $\mathcal{F}(t) = \{(\lambda_1, n_1; \lambda_2, n_2) | \pi_1(\lambda_1, n_1) = t \text{ and } (\lambda_2, n_2) \text{ optimizes problem } P^t\}$.

First, we show the "*necessity*" by contradiction. Suppose there is a strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ that satisfying the conditions in Lemma OA.3(ii) is Pareto efficient, but $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \notin \cup_{0 \le t \le P_0} \mathcal{F}(t)$. We assume $\pi_1(\lambda_1^*, n_1^*) = t$. For a strategy profile $(\bar{\lambda}_1^*, \bar{n}_1^*; \bar{\lambda}_2^*, \bar{n}_2^*) \in \mathcal{F}(t)$, we have $\pi_1(\bar{\lambda}_1^*, \bar{n}_1^*) = t$ and $\pi_2(\bar{\lambda}_2^*, \bar{n}_2^*) > \pi_2(\lambda_2^*, n_2^*)$. This contradicts to $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ is Pareto efficient.

Second, we show the "*sufficiency*", i.e., if $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \cup_{0\le t\le P_0}\mathcal{F}(t)$, it is Pareto efficient. Note any Pareto-efficient strategy profile should satisfy the conditions in Lemma OA.3. By Lemma OS.8, to show $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ is Pareto efficient is equivalent to show: for any strategy profile $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ that satisfying the conditions in Lemma OA.3, we have $\pi_1(\lambda_1^*, n_1^*) > \pi_1(\bar\lambda_1^*, \bar n_1^*)$ or $\pi_2(\lambda_2^*, n_2^*) > \pi_2(\bar\lambda_2^*, \bar n_2^*)$, or $\pi_1(\lambda_1^*, n_1^*) = \pi_1(\bar\lambda_1^*, \bar n_1^*)$ and $\pi_2(\lambda_2^*, n_2^*) = \pi_2(\bar\lambda_2^*, \bar n_2^*)$. Assume $\pi_1(\lambda_1^*, n_1^*) = t$, then $\pi_2(\lambda_2^*, n_2^*) = P_t$. We consider the following three subcases.

Subcase II.1: $N \ge 2\underline{N}$ and $\bar\Lambda(N) < \Lambda \le \bar\Lambda'(N)$. By Lemma OS.9(i), $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ satisfies the conditions in Lemma OA.3(i) or (ii). We assume $n_1^* \le N/2$, and let $\pi_1(\bar\Lambda_m(\bar n), \bar n) = \bar t$.

If $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ satisfies the conditions in Lemma OA.3(i), then $\bar\lambda_i^* = \bar\Lambda_m(\bar n_i^*)$. We will show $\pi_2(\lambda_2^*, n_2^*) = P_t > \pi_2(\bar\lambda_2^*, \bar n_2^*)$. Thus, $t = \pi_1(\lambda_1^*, n_1^*) \le \pi_1(\bar\Lambda_m(n_1^*), n_1^*) \le \pi_1(\bar\Lambda_m(\bar n), \bar n) = \bar t$, where the first inequality is due to $\pi_1(\lambda, n) = \pi(\lambda, n)$ is maximized at $\lambda = \bar\Lambda_m(n)$ from Lemma OS.6, the second inequality is due to $\pi_1(\bar\Lambda_m(n), n) = \pi(\bar\Lambda_m(n), n)$ is increasing in n (Lemma OS.6) and $n_1^* \le \bar n$ (Lemma OS.9(i)). We thus have $\pi_2(\bar\Lambda_m(N-\bar n), N-\bar n) \le P_{\bar t} \le P_t$, where the second inequality is due to $t \le \bar t$ and P_t is decreasing in t from Lemma OS.10. If $\bar n_1^* > N/2$, $\bar n_1^* > N/2 \ge \bar n$, if $\bar n_1^* \le N/2$, $\bar n_1^* > \bar n$ by Lemma OS.9(i). This implies $\bar n_2^* < N - \bar n$ because $\bar n_1^* + \bar n_2^* = N$. We thus have $\pi_2(\bar\Lambda_m(N-\bar n), N-\bar n) > \pi_2(\bar\Lambda_m(\bar n_2^*), \bar n_2^*) = \pi_2(\bar\lambda_2^*, \bar n_2^*)$ due to $\pi_2(\bar\Lambda_m(n), n) = \pi(\bar\Lambda_m(n), n)$ is increasing in n from Lemma OS.6. Together with $\pi_2(\bar\Lambda_m(N-\bar n), N-\bar n) \le P_t$, we get $\pi_2(\lambda_2^*, n_2^*) = P_t > \pi_2(\bar\lambda_2^*, \bar n_2^*)$.

If $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ satisfies the conditions in Lemma OA.3(ii), then it is the optimal solution to problem $P^{\bar t}$ by the "*necessity*". This implies $\pi_2(\bar\Lambda_m(N-\bar n), N-\bar n) = P_{\bar t}$. If $\pi_1(\lambda_1^*, n_1^*) = t \le \bar t = \pi_1(\bar\Lambda_m(\bar n), \bar n)$, then $\pi_2(\lambda_2^*, n_2^*) = P_t \ge \pi_2(\bar\Lambda_m(N-\bar n), N-\bar n) = P_{\bar t}$ due to P_t is decreasing in t from Lemma OS.10.

Subcase II.2: $N \ge 2\underline{N}$ and $\underline{\Lambda}'(N) \le \Lambda \le \bar\Lambda(N)$. In this case, $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ satisfies the conditions in Lemma OA.3(ii). With the similar analysis in Subcase II.1, we get the desired results.

Subcase II.3: $N \ge 2\underline{N}$ and $\underline{\Lambda}(N) \le \Lambda < \underline{\Lambda}'(N)$. In this case, $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ satisfies the conditions in Lemma OA.3(ii) or (iii). Without of loss generality, we assume $\bar\lambda_2^* \le \Lambda/2$.

If $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ satisfies the conditions in Lemma OA.3(ii), we can verify it with the similar analysis in last paragraph of Subcase II.1.

If $(\bar\lambda_1^*, \bar n_1^*; \bar\lambda_2^*, \bar n_2^*)$ satisfies the conditions in Lemma OA.3(iii), then $\bar n_i^* = \underline{\Lambda}_m^{-1}(\bar\lambda_i^*)$. We will show $\pi_2(\lambda_2^*, n_2^*) = P_t > \pi_2(\bar\lambda_2^*, \bar n_2^*)$. If $\lambda_2^* > \Lambda/2$, $\lambda_2^* > \Lambda/2 \ge \bar\lambda$, if $\lambda_2^* \le \Lambda/2$, $\lambda_2^* \ge \bar\lambda$ by Lemma OS.9(ii). This implies $\lambda_1^* < \Lambda - \bar\lambda$ because $\lambda_1^* + \lambda_2^* = \Lambda$. We let $\pi_1(\Lambda-\bar\lambda, \underline{\Lambda}_m^{-1}(\Lambda-\bar\lambda)) = \bar t$. Thus, $t = \pi_1(\lambda_1^*, n_1^*) \le \pi_1(\lambda_1^*, \underline{\Lambda}_m^{-1}(\lambda_1^*)) < \pi_1(\Lambda-\bar\lambda, \underline{\Lambda}_m^{-1}(\Lambda-\bar\lambda)) = \bar t$, where the first inequality is due to $\pi_1(\lambda, n) = \pi(\lambda, n)$ is maximized at $n = \underline{\Lambda}_m^{-1}(\lambda)$ from Lemma OS.6, the second inequality is due to $\pi_1(\lambda, \underline{\Lambda}_m^{-1}(\lambda)) = \pi(\lambda, \underline{\Lambda}_m^{-1}(\lambda))$ is increasing in λ (Lemma OS.6) and $\lambda_1^* < \Lambda - \bar\lambda$ (Lemma OS.9(ii)). We thus have $\pi_2(\bar\lambda, \underline{\Lambda}_m^{-1}(\bar\lambda)) \le P_{\bar t} < P_t$, where the second inequality is due to $t < \bar t$ and P_t is decreasing from Lemma OS.10. Without of loss generality, we assume $\bar\lambda_2^* \le \Lambda/2$. By Lemma OS.9(ii), $\bar\lambda_2^* < \bar\lambda$. We thus have $\pi_2(\bar\lambda, \underline{\Lambda}_m^{-1}(\bar\lambda)) > \pi_2(\bar\lambda_2^*, \underline{\Lambda}_m^{-1}(\bar\lambda_2^*)) = \pi_2(\bar\lambda_2^*, \bar n_2^*)$ due to $\pi_2(\lambda, \underline{\Lambda}_m^{-1}(\lambda)) = \pi(\lambda, \underline{\Lambda}_m^{-1}(\lambda))$ is increasing in λ from Lemma OS.6. Together with $\pi_2(\bar\lambda, \underline{\Lambda}_m^{-1}(\bar\lambda)) < P_t$, we get $\pi_2(\lambda_2^*, n_2^*) = P_t > \pi_2(\bar\lambda_2^*, \bar n_2^*)$.

Case III: Limited demand and ample supply. We show the strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ that satisfying $n_i^* = \underline{\Lambda}_m^{-1}(\lambda_i^*)$, $\lambda_i^* \ge \Lambda_0$ and $\lambda_1^* + \lambda_2^* = \Lambda$ is Pareto efficient. In other words, any strategy profile in Lemma OA.3(iii) is Pareto efficient. Similar to Case I, we establish the proof by contradiction. Note that $\lambda_1' + \lambda_2' \le \Lambda$, then we have either $\lambda_1^* \ge \lambda_1'$ or $\lambda_2^* \ge \lambda_2'$. Without loss of generality, we assume $\lambda_1^* \ge \lambda_1'$. We have $\pi_1' = \pi_1(\lambda_1', n_1') = \pi(\lambda_1', n_1') \le \pi(\lambda_1', \underline{\Lambda}_m^{-1}(\lambda_1')) \le \pi(\lambda_1^*, \underline{\Lambda}_m^{-1}(\lambda_1^*)) = \pi_1(\lambda_1^*, \underline{\Lambda}_m^{-1}(\lambda_1^*)) = \pi_1^*$, where the first inequality holds because $\pi(\lambda, n)$ is maximized at $n = \underline{\Lambda}_m^{-1}(\lambda)$

from Lemma OS.6, the second inequality holds because $\pi(\lambda, \underline{\Lambda}_m^{-1}(\lambda))$ increases in λ from Lemma OS.6 and $\lambda_1' \le \lambda_1^*$. This contradicts to $\pi_1' > \pi_1^*$. □

Proof of Proposition 3. We establish the proof by contradiction. Suppose that there exists a splitting Nash equilibrium, where both platforms can win some customers and workers. Denote the splitting Nash equilibrium as $(\tilde{p}_1^*, \tilde{w}_1^*; \tilde{p}_2^*, \tilde{w}_2^*)$, and the customers and workers' joining equilibrium as $(\tilde{\lambda}_1^*, \tilde{n}_1^*; \tilde{\lambda}_2^*, \tilde{n}_2^*)$. Then, the splitting Nash equilibrium implies that $(\tilde{\lambda}_i^*, \tilde{n}_i^*)$ should be the best response to $(\tilde{\lambda}_j^*, \tilde{n}_j^*)$, for $i, j = 1, 2$ and $i \neq j$. Thus, for $i = 1, 2$, we have the first order condition regarding w_i satisfy $-\tilde{\lambda}_i^* + (\tilde{p}_i^* - \tilde{w}_i^*)\frac{\partial \lambda_i}{\partial w_i}\big|_{\{p_i = \tilde{p}_i^*, w_i = \tilde{w}_i^*\}} = 0$, which implies $\frac{\partial \lambda_i}{\partial w_i}\big|_{\{p_i = \tilde{p}_i^*, w_i = \tilde{w}_i^*\}} > 0$. This implies that $\tilde{\lambda}_i^*$ increases in $\tilde{w}_i^*$ given that $\tilde{p}_i^*$, $\tilde{p}_j^*$, and $\tilde{w}_j^*$ are fixed, where $i, j = 1, 2$, $i \neq j$. We then establish the contradiction by using this monotonicity and the conditions (c1)-(c3), and we consider the four cases regarding $(\tilde{\lambda}_1^*, \tilde{n}_1^*; \tilde{\lambda}_2^*, \tilde{n}_2^*)$.

Case I: $\tilde{\lambda}_1^* + \tilde{\lambda}_2^* < \Lambda$ and $\tilde{n}_1^* + \tilde{n}_2^* < N$. From (c2), for $i = 1, 2$, we have

$$\frac{h}{\tilde{n}_i^* \mu - \tilde{\lambda}_i^*} = \frac{h}{\tilde{n}_i^*(\mu - \tilde{\lambda}_i^*/\tilde{n}_i^*)} = v - \tilde{p}_i^*. \tag{OA.4}$$

Recall that $\tilde{\lambda}_i^*$ increases in $\tilde{w}_i^*$. Thus, if $\tilde{w}_i^*$ increases, then $\tilde{n}_i^*$ increases (from the first equality in (OA.4)), and $\tilde{\lambda}_i^*/\tilde{n}_i^*$ also increases (from the second equality in (OA.4)). This contradicts to "$\frac{\tilde{\lambda}_i^* \tilde{w}_i^*}{\tilde{n}_i^*} = r$" as required by (c1).

Case II: $\tilde{\lambda}_1^* + \tilde{\lambda}_2^* < \Lambda$ and $\tilde{n}_1^* + \tilde{n}_2^* = N$. In this case, (OA.4) holds. Recall that $\tilde{\lambda}_i^*$ increases in $\tilde{w}_i^*$. Thus, if $\tilde{w}_i^*$ increases, then $\tilde{n}_i^*$ and $\tilde{\lambda}_i^*/\tilde{n}_i^*$ increases. Hence, $\tilde{\lambda}_j^*/\tilde{n}_j^*$ also increases due to $\frac{\tilde{\lambda}_i^* \tilde{w}_i^*}{\tilde{n}_i^*} = \frac{\tilde{\lambda}_j^* \tilde{w}_j^*}{\tilde{n}_j^*}$. Consequently, $\tilde{n}_j^*$ increases due to the second equality in (OA.4), which contradicts to $\tilde{n}_i^* + \tilde{n}_j^* = N$.

Case III: $\tilde{\lambda}_1^* + \tilde{\lambda}_2^* = \Lambda$ and $\tilde{n}_1^* + \tilde{n}_2^* < N$. From (c1), for $i, j = 1, 2$, we have

$$\frac{\tilde{\lambda}_j^* \tilde{w}_j^*}{\tilde{n}_j^*} = \frac{\tilde{\lambda}_i^* \tilde{w}_i^*}{\tilde{n}_i^*} = r. \tag{OA.5}$$

Recall that $\tilde{\lambda}_i^*$ increases in $\tilde{w}_i^*$. Thus, if $\tilde{w}_i^*$ increases, then $\tilde{n}_i^*$ increase and $\tilde{\lambda}_i^*/\tilde{n}_i^*$ decreases (from the second equality in (OA.5)), and $\tilde{\lambda}_j^*/\tilde{n}_j^*$ is fixed by (OA.5). Consequently, $\tilde{n}_j^*$ increases due to $v - \tilde{p}_i^* - \frac{h}{\tilde{n}_i^* \mu - \tilde{\lambda}_i^*} = v - \tilde{p}_i^* - \frac{h}{\tilde{n}_i^*(\mu - \tilde{\lambda}_i^*/\tilde{n}_i^*)} = v - \tilde{p}_j^* - \frac{h}{\tilde{n}_j^* \mu - \tilde{\lambda}_j^*} = v - \tilde{p}_j^* - \frac{h}{\tilde{n}_j^*(\mu - \tilde{\lambda}_j^*/\tilde{n}_j^*)}$. Hence, $\tilde{\lambda}_j^*$ increases due to $\tilde{\lambda}_j^*/\tilde{n}_j^*$ is fixed, which contradicts to $\tilde{\lambda}_i^* + \tilde{\lambda}_j^* = \Lambda$.

Case IV: $\tilde{\lambda}_1^* + \tilde{\lambda}_2^* = \Lambda$ and $\tilde{n}_1^* + \tilde{n}_2^* = N$. Recall that $\tilde{\lambda}_i^*$ increases in $\tilde{w}_i^*$. If $\tilde{n}_i^*$ decreases in $\tilde{w}_i^*$, then $\frac{\tilde{\lambda}_i^* \tilde{w}_i^*}{\tilde{n}_i^*}$ increases in $\tilde{w}_i^*$, and $\tilde{\lambda}_j^*/\tilde{n}_j^*$ increases in $\tilde{w}_i^*$ due to $\frac{\tilde{\lambda}_i^* \tilde{w}_i^*}{\tilde{n}_i^*} = \frac{\tilde{\lambda}_j^* \tilde{w}_j^*}{\tilde{n}_j^*}$ by (c2). However, $\tilde{\lambda}_j^*$ decreases and $\tilde{n}_j^*$ increases in $\tilde{w}_i^*$ due to $\tilde{\lambda}_i^* + \tilde{\lambda}_j^* = \Lambda$ and $\tilde{n}_i^* + \tilde{n}_j^* = N$, which implies $\tilde{\lambda}_j^*/\tilde{n}_j^*$ decreases in $\tilde{w}_i^*$. Now we consider $\tilde{n}_i^*$ increases in $\tilde{w}_i^*$. If $\tilde{\lambda}_i^*/\tilde{n}_i^*$ decreases in $\tilde{w}_i^*$, then $\tilde{n}_i^* \mu - \tilde{\lambda}_i^* = \tilde{n}_i^*(\mu - \tilde{\lambda}_i^*/\tilde{n}_i^*)$ increases in $\tilde{w}_i^*$. Consequently, $\tilde{n}_j^* \mu - \tilde{\lambda}_j^*$ increases in $\tilde{w}_i^*$ due to $v - \tilde{p}_i^* - \frac{h}{\tilde{n}_i^* \mu - \tilde{\lambda}_i^*} = v - \tilde{p}_j^* - \frac{h}{\tilde{n}_j^* \mu - \tilde{\lambda}_j^*}$ from (c2), which contradicts to $\tilde{n}_i^* \mu - \tilde{\lambda}_i^* + \tilde{n}_j^* \mu - \tilde{\lambda}_j^* = N\mu - \Lambda$. If $\tilde{\lambda}_i^*/\tilde{n}_i^*$ increases in $\tilde{w}_i^*$, then $\tilde{\lambda}_j^*/\tilde{n}_j^*$ increases in $\tilde{w}_i^*$ due to $\frac{\tilde{\lambda}_i^* \tilde{w}_i^*}{\tilde{n}_i^*} = \frac{\tilde{\lambda}_j^* \tilde{w}_j^*}{\tilde{n}_j^*}$ from (c1). Consequently, $\tilde{n}_j^* \mu - \tilde{\lambda}_j^* = \tilde{n}_j^*(\mu - \tilde{\lambda}_j^*/\tilde{n}_j^*)$ decreases in $\tilde{w}_i^*$, which implies $\tilde{n}_i^* \mu - \tilde{\lambda}_i^*$ decreases in $\tilde{w}_i^*$ due to $v - \tilde{p}_i^* - \frac{h}{\tilde{n}_i^* \mu - \tilde{\lambda}_i^*} = v - \tilde{p}_j^* - \frac{h}{\tilde{n}_j^* \mu - \tilde{\lambda}_j^*}$ from (c1). This contradicts to $\tilde{n}_i^* \mu - \tilde{\lambda}_i^* + \tilde{n}_j^* \mu - \tilde{\lambda}_j^* = N\mu - \Lambda$.

For now we have shown that all the joining customers and workers will choose the same platform. Without loss of generality, we assume they choose platform 1. Suppose that $\tilde{p}_1^* - \tilde{w}_1^* > 0$, and we then consider any $\epsilon > 0$ such that $\tilde{p}_1^* - \tilde{w}_1^* > 2\epsilon > 0$. Platform 2 can set $\tilde{p}_2^* = \tilde{p}_1^* - \epsilon$ and $\tilde{w}_2^* = \tilde{w}_1^* + \epsilon$, with which $\tilde{p}_2^* - \tilde{w}_2^* > 0$ and all the joining customers and workers will switch to platform 2. So on so forth, platform 1 has to set $\tilde{p}_1^* = \tilde{w}_1^*$ to win the customers and workers. $\square$

Proof of Corollary 1. For the farsightedly stable outcomes, the results are valid from Lemma 1(i). We next establish the proof for Nash equilibrium outcomes by contradiction. Suppose that all the Nash equilibrium can fully extract both customers and workers' surplus. Consider any Nash equilibrium, say $(\tilde{p}_1^*, \tilde{w}_1^*; \tilde{p}_2^*, \tilde{w}_2^*)$ with $(\tilde{\lambda}_1^*, \tilde{n}_1^*; \tilde{\lambda}_2^*, \tilde{n}_2^*)$ as customers and worker's joining equilibrium. Note the platforms will set the wage be same as the price, and all the customers and workers will join the same platform. Without loss of generality, we assume $\tilde{p}_1^* = \tilde{w}_1^*$ and all the customers and workers choose platform 1. Then, the full extraction of customers and workers' utility implies that

$$\frac{\tilde{\lambda}_1^* \tilde{w}_1^*}{\tilde{n}_1^*} = r, \tag{OA.6}$$

$$v - \tilde{p}_1^* - \frac{h}{\tilde{n}_1^* \mu - \tilde{\lambda}_1^*} = 0. \tag{OA.7}$$

Since all the customers and workers join platform 1, we have either $\tilde{n}_1^* = N$ or $\tilde{\lambda}_1^* = \Lambda$. If $\tilde{n}_1^* = N$, then from (OA.6)-(OA.7), $\frac{Nr}{\tilde{\lambda}_1^*} = \tilde{w}_1^* = \tilde{p}_1^* = v - \frac{h}{N\mu - \tilde{\lambda}_1^*}$, from which we obtain $\tilde{\lambda}_1^* = \frac{1}{2v}\big(rN + vN\mu - h \pm \sqrt{(rN + vN\mu - h)^2 - 4rvN^2\mu}\big)$. Recall that $\tilde{\lambda}_1^* \le \Lambda$. Thus, for those demand size Λ such that $\Lambda < \frac{1}{2v}\big(rN + vN\mu - h - \sqrt{(rN + vN\mu - h)^2 - 4rvN^2\mu}\big)$, at least one of (OA.6) and (OA.7) cannot be valid. This is a contradiction. If $\tilde{\lambda}_1^* = \Lambda$, then from (OA.6)-(OA.7), $\frac{\tilde{n}_1^* r}{\Lambda} = \tilde{w}_1^* = \tilde{p}_1^* = v - \frac{h}{\tilde{n}_1^* \mu - \Lambda}$, from which we obtain $\tilde{n}_1^* = \frac{1}{2r\mu}\big(r\Lambda + v\Lambda\mu \pm \sqrt{(r\Lambda + v\Lambda\mu)^2 - 4v\Lambda^2 - 4h\Lambda}\big)$. Recall that $\tilde{n}_1^* \le N$. Thus, for those supply size N such that $N < \frac{1}{2r\mu}\big(r\Lambda + v\Lambda\mu - \sqrt{(r\Lambda + v\Lambda\mu)^2 - 4v\Lambda^2 - 4h\Lambda}\big)$, at least one of (OA.6) and (OA.7) cannot hold. This is also a contradiction. To summarize, Nash equilibria may leave customers or workers positive surplus. $\square$

Proof of Corollary 2. We first consider the farsightedly stable outcomes in this special case. From Proposition 2, it suffices to show that the farsightedly stable outcomes in Table 2 can be obtained by applying $r \to 0^+$. By letting $r \to 0^+$, we have $\Lambda_0 = 4hr\mu/(v\mu - r)^2 \to 0^+$, $\underline{\Lambda}'(N) = \frac{1}{4}\Big(\sqrt{\frac{h\mu}{r} + 4\Big(N\mu - \frac{2h\mu(v\mu + r)}{(v\mu - r)^2}\Big)} - \sqrt{\frac{h\mu}{r}}\Big)^2 + \frac{4hr\mu}{(v\mu - r)^2} \to 0^+$, $\underline{\Lambda}(N) = \frac{1}{4}\Big(\sqrt{\frac{2h\mu}{r} + 4N\mu} - \sqrt{\frac{2h\mu}{r}}\Big)^2 \to 0^+$, $\hat{\Lambda}(N) = \frac{1}{2v}\big(rN + vN\mu - 2h - \sqrt{(rN + vN\mu - 2h)^2 - 4rvN^2\mu}\big) \to 0^+$, and $\underline{N} = h/(\sqrt{v\mu} - \sqrt{r})^2 \to h/(v\mu)$. Based on Proposition 2, if $\Lambda \le \bar{\Lambda}(N)$ and $N > 2h/(v\mu)$ ("balanced demand and supply"), then the farsightedly stable outcomes satisfy $w_i^* = \frac{rn_i^*}{\lambda_i^*} \to 0^+$, $p_i^* = v - \frac{h}{n_i^*\mu - \lambda_i^*} = v - \frac{2h}{N\mu - \Lambda}$, $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \cup_{0 \le t \le P_0} \mathcal{F}(t)$ with $\mathcal{F}(t) = \{(\lambda_1, n_1; \lambda_2, n_2) | \pi_1(\lambda_1, n_1) = t$ and (λ_2, n_2) optimizes problem $P^t\}$; if $\Lambda > \bar{\Lambda}(N)$ and $N > 2h/(v\mu)$ ("ample demand and limited supply"), then we have $w_i^* = \frac{rn_i^*}{\lambda_i^*} \to 0^+$, $p_i^* = v - \frac{h}{n_i^*\mu - \lambda_i^*} = v - \sqrt{\frac{hv}{n_i^*\mu}}$, $n_i^* > h/(v\mu)$ and $n_1^* + n_2^* = N$, $\lambda_i^* = \bar{\Lambda}_m(n_i^*)$.

We then consider the myopically stable outcomes in this special case, which are the Nash equilibria discussed in Wu et al. (2020). Specifically, if we let $S_0 = N$, $S_i = n_i$, and $T(\lambda, n) = \frac{h}{n\mu - \lambda}$, it is easy to see that our model reduces to the simultaneous case in section 5 of Wu et al. (2020). Applying the definitions of $D(n_i, p_i)$ and $\tilde{p}$ in Wu et al. (2020) to our model, we have $D(N, p) := \max\{\lambda \ge 0 : v - p - \frac{h}{N\mu - \lambda} \ge 0\} = N\mu - \frac{h}{v - p}$, and $\tilde{p} := \arg\max_p\{pD(N, p)\} = \arg\max_p\{p\big(N\mu - \frac{h}{v - p}\big)\} = v - \sqrt{\frac{hv}{N\mu}}$. Thus, $D(N, \tilde{p}) = \big(1 - \sqrt{\frac{h}{vN\mu}}\big)N\mu = \bar{\Lambda}_m(N)$, where the last equality holds due to $N \ge 2\underline{N} = 2h/(\sqrt{v\mu} - \sqrt{r})^2 \to 2h/(v\mu)$ as $r \to 0^+$. Then, the myopically stable outcomes are obtained directly from Theorem 2 of Wu et al. (2020). We next consider customers and workers' utilities. From Table 2, if $\Lambda \le \bar\Lambda_m(N)$, then $\tilde w_1^* = \tilde p_1^* = v - \frac{h}{N\mu-\Lambda}$, $\tilde n_1^* = N$, and $\tilde\lambda_1^* = \Lambda$. In this case, we have customers and workers' utilities given by $v - \tilde p_1^* - \frac{h}{N\mu-\tilde\lambda_1^*} = v - \left(v - \frac{h}{N\mu-\Lambda}\right) - \frac{h}{N\mu-\Lambda} = 0$, and $\frac{\lambda_1^* w_1^*}{n_1^*} = \frac{\Lambda\left(v-\frac{h}{N\mu-\Lambda}\right)}{N} \ge \frac{\Lambda\left(v-\frac{h}{N\mu-\bar\Lambda_m(N)}\right)}{N} = \frac{\Lambda\left(v-\sqrt{\frac{hv}{N\mu}}\right)}{N} > \frac{\Lambda\left(v-\frac{\sqrt{2}v}{2}\right)}{N} > 0$, where the first inequality is due to $\Lambda \le \bar\Lambda_m(N)$, and the second inequality is due to $N > 2h/(v\mu)$. If $\Lambda > \bar\Lambda_m(N)$, then $\tilde w_1^* = \tilde p_1^* = v - \sqrt{\frac{hv}{N\mu}}$, $\tilde n_1^* = N$, and $\tilde\lambda_1^* = \bar\Lambda_m(N) = N\mu\big(1 - \sqrt{\frac{h}{vN\mu}}\big)$. In this case, we have customers and workers' utilities given by $v - \tilde p_1^* - \frac{h}{N\mu-\tilde\lambda_1^*} = v - \left(v - \sqrt{\frac{hv}{N\mu}}\right) - \frac{h}{N\mu-\bar\Lambda_m(N)} = 0$, and $\frac{\lambda_1^* w_1^*}{n_1^*} = \frac{\bar\Lambda_m(N)\left(v-\sqrt{\frac{hv}{N\mu}}\right)}{N} = \frac{N\mu\left(1-\sqrt{\frac{h}{vN\mu}}\right)\left(v-\sqrt{\frac{hv}{N\mu}}\right)}{N} > \frac{N\mu\left(1-\frac{\sqrt{2}}{2}\right)\left(v-\frac{\sqrt{2}v}{2}\right)}{N} > 0$, where the first inequality is due to $N > 2h/(v\mu)$. $\square$

Proof of Lemma 2. We show the total profit of any merged system is dominated by a monopoly system. That is $(p_1^m - w_1^m)\lambda_1^m + (p_2^m - w_2^m)\lambda_2^m \le \left(v - \frac{h}{n_1^m\mu-\lambda_1^m}\right) - rn_1^m + \left(v - \frac{h}{n_2^m\mu-\lambda_2^m}\right) - rn_2^m \le \left(v - \frac{h}{(n_1^m+n_2^m)\mu-(\lambda_1^m+\lambda_2^m)}\right) - rn_1^m + \left(v - \frac{h}{(n_1^m+n_2^m)\mu-(\lambda_1^m+\lambda_2^m)}\right) - rn_2^m = \left(v - \frac{h}{(n_1^m+n_2^m)\mu-(\lambda_1^m+\lambda_2^m)}\right) - r(n_1^m + n_2^m) = \left(v - \frac{h}{n\mu-\lambda}\right) - rn$, where $\lambda = \lambda_1^m + \lambda_2^m$, $n = n_1^m + n_2^m$, and the first inequality is due to $\frac{\lambda_1^m w_1^m}{n_1^m} = \frac{\lambda_2^m w_2^m}{n_2^m} \ge r$ and $v - p_1^m - \frac{h}{n_1^m\mu-\lambda_1^m} = v - p_2^m - \frac{h}{n_2^m\mu-\lambda_2^m} \ge 0$. Let $v - p - \frac{h}{n\mu-\lambda} = 0$ and $\frac{\lambda w}{n} = r$, then $(p-w)\lambda = \left(v - \frac{h}{n\mu-\lambda}\right) - rn$ is the profit of a monopoly with price p and wage w. $\square$

Proof of Proposition 4. By Lemma OA.4, we can transform problem "P_m'" with price and wage as decision variables into an equivalent problem "P_m" with the effective arrival rate λ_m and the effective participating number of workers n_m. By Lemma OA.5, the optimal solution satisfies either $n_m^* = N$ or $\lambda_m^* = \Lambda$. We consider the following cases to establish the proof.

Case I: $n_m^* = N$ and $\lambda_m^* < \Lambda$. In this case, we have $\eta_2 = 0$ from (OS.30). Besides, by using $n_m^* = N$ and $\eta_2 = 0$, we can rewrite the necessary conditions (OS.27), (OS.28), and (OS.31) as follows:

$$\frac{\partial L_m}{\partial n_m}\Big|_{n_m=N,\lambda_m=\lambda_m^*} = \frac{h\lambda_m^*\mu}{(N\mu-\lambda_m^*)^2} - r - \eta_1 - \eta_3\Big[\frac{r}{\lambda_m^*} - \frac{h\mu}{(N\mu-\lambda_m^*)^2}\Big] = 0, \tag{OA.8}$$

$$\frac{\partial L_m}{\partial \lambda_m}\Big|_{n_m=N,\lambda_m=\lambda_m^*} = v - \frac{hN\mu}{(N\mu-\lambda_m^*)^2} + \eta_3\Big[\frac{rN}{\lambda_m^{*2}} - \frac{h}{(N\mu-\lambda_m^*)^2}\Big] = 0, \tag{OA.9}$$

$$\eta_3\Big(v - \frac{h}{N\mu-\lambda_m^*} - \frac{rN}{\lambda_m^*}\Big) = 0. \tag{OA.10}$$

We then discuss two subcases as follows.

Subcase (a): $v - \frac{h}{N\mu-\lambda_m^*} - \frac{rN}{\lambda_m^*} > 0$. In this case, from (OA.10) we have $\eta_3 = 0$. By applying $\eta_3 = 0$ to (OA.9), we get $v = \frac{hN\mu}{(N\mu-\lambda_m^*)^2}$, i.e., $N\mu - \lambda_m^* = \sqrt{\frac{hN\mu}{v}}$, which implies

$$\lambda_m^* = \Big(1 - \sqrt{\frac{h}{vN\mu}}\Big)N\mu = \bar\Lambda_m(N). \tag{OA.11}$$

Next, we verify the condition $v - \frac{h}{N\mu-\lambda_m^*} - \frac{rN}{\lambda_m^*} > 0$ by using (OA.11). Let $v - \frac{h}{N\mu-\lambda_m^*} - \frac{rN}{\lambda_m^*} = \big(1 - \sqrt{\frac{h}{vN\mu}}\big)v - \frac{r}{\mu - \sqrt{\frac{h\mu}{vN}}} = \frac{v\mu[\left(1-\sqrt{\frac{h}{vN\mu}}\right)^2 - \frac{r}{v\mu}]}{\mu[1-\sqrt{\frac{h}{vN\mu}}]} > 0$. Note that $\lambda_m^* > 0$. Thus, from (OA.11), in this case we have $1 - \sqrt{\frac{h}{vN\mu}} > 0$, with which the above inequality implies $1 - \sqrt{\frac{h}{vN\mu}} > \sqrt{\frac{r}{v\mu}}$, or equivalently, $N > \frac{h}{(\sqrt{v\mu}-\sqrt{r})^2} = \underline{N}$. Next, we show that the Lagrangian multiplier $\eta_1 \ge 0$. Substituting $\eta_3 = 0$ and λ_m^* given in (OA.11) into (OA.8), we get $\eta_1 = \frac{h\lambda_m^*\mu}{(N\mu-\lambda_m^*)^2} - r = \frac{h\left(1-\sqrt{\frac{h}{vN\mu}}\right)N\mu^2}{\left(\sqrt{\frac{h}{vN\mu}}N\mu\right)^2} - r = \big(1 - \sqrt{\frac{h}{vN\mu}}\big)v\mu -$

$r > \big(1-\sqrt{\frac{h}{\frac{hv\mu}{(\sqrt{v\mu}-\sqrt{r})^2}}}\big)v\mu - r = \big(1-\sqrt{\frac{(\sqrt{v\mu}-\sqrt{r})^2}{v\mu}}\big)v\mu - r = \big(1-\frac{\sqrt{v\mu}-\sqrt{r}}{\sqrt{v\mu}}\big)v\mu - r = \sqrt{v\mu r}-r = \big(\sqrt{v}-\sqrt{\frac{r}{\mu}}\big)\sqrt{r\mu} > 0$, where the first inequality holds from $N > \frac{h}{(\sqrt{v\mu}-\sqrt{r})^2}$, and the second inequality holds from $v > r/\mu$. Hence, if $N > \underline{N}$ and $\Lambda > \lambda_m^* = \bar{\Lambda}_m(N)$, the optimal solution of problem P_m is the unique KKT solution: $n^* = N$, $\lambda_m^* = \bar{\Lambda}_m(N)$.

Subcase (b): $v - \frac{h}{N\mu-\lambda_m^*} - \frac{rN}{\lambda_m^*} = 0$, i.e.,

$$v = \frac{rN}{\lambda_m^*} + \frac{h}{N\mu - \lambda_m^*}. \tag{OA.12}$$

Consider the necessary condition (OA.9). If the coefficient of η_3 is non-zero, i.e., $\frac{h}{(N\mu-\lambda_m^*)^2} - \frac{rN}{\lambda_m^{*2}} \neq 0$, then by using (OA.12), we have $\eta_3 = \frac{v-\frac{hN\mu}{(N\mu-\lambda_m^*)^2}}{\frac{h}{(N\mu-\lambda_m^*)^2}-\frac{rN}{\lambda_m^{*2}}} = \frac{\frac{rN}{\lambda_m^*}+\frac{h}{N\mu-\lambda_m^*}-\frac{hN\mu}{(N\mu-\lambda_m^*)^2}}{\frac{h}{(N\mu-\lambda_m^*)}-\frac{rN}{\lambda_m^{*2}}} = \frac{\frac{rN}{\lambda_m^*}-\frac{h\lambda_m^*}{(N\mu-\lambda_m^*)^2}}{\frac{h}{(N\mu-\lambda_m^*)^2}-\frac{rN}{\lambda_m^{*2}}} = -\lambda_m^* < 0$, which is a contradiction to the requirement "$\eta_3 \geq 0$" of the KKT solution. Thus, the coefficient of η_3 in (OA.9) must be zero, i.e.,

$$\frac{h}{(N\mu-\lambda_m^*)^2} - \frac{rN}{\lambda_m^{*2}} = 0. \tag{OA.13}$$

Solving (OA.13), we obtain

$$\lambda_m^* = \frac{\sqrt{rN}N\mu}{\sqrt{h}+\sqrt{rN}} = \big(1 - \frac{\sqrt{h}}{\sqrt{h}+\sqrt{rN}}\big)N\mu. \tag{OA.14}$$

Plugging it in (OA.12), we obtain $v = \frac{rN}{\lambda_m^*} + \frac{h}{N\mu-\lambda_m^*} = \frac{\sqrt{rN}(\sqrt{h}+\sqrt{rN})}{N\mu} + \frac{h}{\frac{\sqrt{h}N\mu}{\sqrt{h}+\sqrt{rN}}} = \frac{\sqrt{rN}(\sqrt{h}+\sqrt{rN})}{N\mu} + \frac{\sqrt{h}(\sqrt{h}+\sqrt{rN})}{N\mu} = \frac{(\sqrt{h}+\sqrt{rN})^2}{N\mu}$, i.e., $\sqrt{v} = \frac{\sqrt{h}+\sqrt{rN}}{\sqrt{N\mu}}$, which is equivalent to $\sqrt{v\mu N} = \sqrt{h}+\sqrt{rN}$. Thus, $N = \frac{h}{(\sqrt{v\mu}-\sqrt{r})^2} = \underline{N}$. Note also that we assume $v \geq r/\mu$, which implies $\sqrt{v\mu}-\sqrt{r} > 0$. Thus,

$$\sqrt{\underline{N}} = \frac{\sqrt{h}}{\sqrt{v\mu}-\sqrt{r}}. \tag{OA.15}$$

Substituting (OA.15) to (OA.14), we get $\lambda_m^* = \big(1-\frac{\sqrt{h}}{\sqrt{h}+\frac{\sqrt{hr}}{\sqrt{v\mu}-\sqrt{r}}}\big)\underline{N}\mu = \big(1-\frac{\sqrt{v\mu}-\sqrt{r}}{\sqrt{v\mu}}\big)\underline{N}\mu = \big(1-\sqrt{\frac{h}{\underline{N}}}\cdot\frac{1}{\sqrt{v\mu}}\big)\underline{N}\mu = \big(1-\sqrt{\frac{h}{v\underline{N}\mu}}\big)\underline{N}\mu = \bar{\Lambda}_m(\underline{N})$. We also need to verify the Lagrangian multiplier $\eta_1 > 0$. Substituting $\lambda_m^* = \big(1-\sqrt{\frac{h}{v\underline{N}\mu}}\big)\underline{N}\mu$ and $N = \frac{h}{(\sqrt{v\mu}-\sqrt{r})^2} = \underline{N}$ into (OA.8), we get $\eta_1 = \frac{h\lambda_m^*\mu}{(N\mu-\lambda_m^*)^2} - r - \eta_3\big[\frac{r}{\lambda_m^*} - \frac{h\mu}{(N\mu-\lambda_m^*)^2}\big] = (\lambda_m^*+\eta_3)\big[\frac{h\mu}{(N\mu-\lambda_m^*)^2} - \frac{r}{\lambda_m^*}\big] = (\lambda_m^*+\eta_3)\big[\frac{h\mu}{(\sqrt{\frac{h}{vN\mu}}\cdot N\mu)^2} - \frac{r}{\big(1-\sqrt{\frac{h}{vN\mu}}\big)N\mu}\big] = (\lambda_m^*+\eta_3)\big[\frac{v}{N} - \frac{r}{\big(1-\sqrt{\frac{h}{vN\mu}}\big)N\mu}\big] = \frac{\lambda_m^*+\eta_3}{N}\big[v - \frac{r}{\big(1-\sqrt{\frac{h}{vN\mu}}\big)\mu}\big] = \frac{\lambda_m^*+\eta_3}{N}\big[v - \frac{r}{\big(1-\sqrt{\frac{(\sqrt{v\mu}-\sqrt{r})^2}{v\mu}}\big)\mu}\big] = \frac{\lambda_m^*+\eta_3}{N}\big[v - \frac{r}{\big(1-\frac{\sqrt{v\mu}-\sqrt{r}}{\sqrt{v\mu}}\big)\mu}\big] = \frac{\lambda_m^*+\eta_3}{N}\big(v - \sqrt{\frac{vr}{\mu}}\big) > 0$, where the inequality is due to $v > r/\mu$. Hence, if $N = \underline{N}$ and $\Lambda > \lambda_m^* = \bar{\Lambda}_m(N)$, the optimal solution to problem P_m is: $n_m^* = N$, $\lambda_m^* = \bar{\Lambda}_m(N)$.

Summarize the above two subcases. If $N \geq \underline{N}$ and $\Lambda > \bar{\Lambda}_m(N)$, $n_m^* = N$, $\lambda_m^* = \bar{\Lambda}_m(N)$ is the unique solution satisfying the necessary KKT condition, and thus optimal.

Case II: $n_m^* = N$ and $\lambda_m^* = \Lambda$. In this case, we can rewrite the necessary conditions (OS.27), (OS.28), and (OS.31) as follows:

$$\frac{\partial L_m}{\partial n_m}\Big|_{n_m=N,\lambda_m=\Lambda} = \frac{h\Lambda\mu}{(N\mu-\Lambda)^2} - r - \eta_1 - \eta_3\Big[\frac{r}{\Lambda} - \frac{h\mu}{(N\mu-\Lambda)^2}\Big] = 0, \tag{OA.16}$$

$$\frac{\partial L_m}{\partial \lambda_m}\Big|_{n_m=N,\lambda_m=\Lambda} = v - \frac{hN\mu}{(N\mu-\Lambda)^2} - \eta_2 + \eta_3\Big[\frac{rN}{\Lambda^2} - \frac{h}{(N\mu-\Lambda)^2}\Big] = 0, \tag{OA.17}$$
$$\eta_3\big(v - \frac{h}{N\mu-\Lambda} - \frac{rN}{\Lambda}\big) = 0.$$

We first prove $\underline{\Lambda}_m(N) \le \Lambda \le \bar{\Lambda}_m(N)$. By (OA.16), we have $\eta_1 = (\Lambda+\eta_3)\big[\frac{h\mu}{(N\mu-\Lambda)^2} - \frac{r}{\Lambda}\big]$. Since $\eta_1 \ge 0$ and $\eta_3 \ge 0$, then $\frac{h\mu}{(N\mu-\Lambda)^2} \ge \frac{r}{\Lambda}$, i.e., $\sqrt{\frac{h\mu\Lambda}{r}} \ge N\mu - \Lambda$, which implies $\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} \ge N$. Consider also that $\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} - N = \frac{1}{\mu}\big(\Lambda + \frac{h\Lambda\mu}{r} - N\mu\big) = \frac{1}{\mu}\big(\sqrt{\Lambda} + \frac{\sqrt{\frac{h\mu}{r}+4N\mu}+\sqrt{\frac{h\mu}{r}}}{2}\big)\big(\sqrt{\Lambda} - \frac{\sqrt{\frac{h\mu}{r}+4N\mu}-\sqrt{\frac{h\mu}{r}}}{2}\big)$. Thus $\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} \ge N$ equals to $\sqrt{\Lambda} \ge \frac{\sqrt{\frac{h\mu}{r}+4N\mu}-\sqrt{\frac{h\mu}{r}}}{2}$, i.e., $\Lambda \ge \underline{\Lambda}_m(N) = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}+4N\mu} - \sqrt{\frac{h\mu}{r}}\big)^2$. Note $\frac{rN}{\Lambda^2} - \frac{h}{(N\mu-\Lambda)^2} \le \frac{v-\frac{h}{N\mu-\Lambda}}{\Lambda} - \frac{h}{(N\mu-\Lambda)^2} = \frac{v}{\Lambda} - \frac{h}{\Lambda(N\mu-\Lambda)} - \frac{h}{(N\mu-\Lambda)^2} = \frac{v}{\Lambda} - \frac{hN\mu}{\Lambda(N\mu-\Lambda)^2}$, where the inequality is due to $v - \frac{h}{N\mu-\Lambda} - \frac{rN}{\Lambda} \ge 0$. Then we rewrite (OA.17) as follows: $0 = v - \frac{hN\mu}{(N\mu-\Lambda)^2} - \eta_2 + \eta_3\big[\frac{rN}{\Lambda^2} - \frac{h}{(N\mu-\Lambda)^2}\big] \le v - \frac{hN\mu}{(N\mu-\Lambda)^2} - \eta_2 + \eta_3\big[\frac{v}{\Lambda} - \frac{hN\mu}{\Lambda(N\mu-\Lambda)^2}\big] = \big(1+\frac{\eta_3}{\lambda_m}\big)\big[v - \frac{hN\mu}{(N\mu-\Lambda)^2}\big] - \eta_2$, i.e., $\big(1+\frac{\eta_3}{\lambda_m}\big)\big[v - \frac{hN\mu}{(N\mu-\Lambda)^2}\big] \ge \eta_2 \ge 0$. Hence, $v \ge \frac{hN\mu}{(N\mu-\Lambda)^2}$ due to $\eta_3 \ge 0$. Note $v \ge \frac{hN\mu}{(N\mu-\Lambda)^2}$ is equivalent to $N\mu - \Lambda \ge \sqrt{\frac{hN\mu}{v}}$, i.e., $\Lambda \le N\mu - \sqrt{\frac{hN\mu}{v}} = \big(1-\sqrt{\frac{h}{vN\mu}}\big)N\mu = \bar{\Lambda}_m(N)$.

We have $v - \frac{h}{N\mu-\Lambda} - \frac{rN}{\Lambda} = \pi(\Lambda,N)/\Lambda \ge 0$ if and only if $\Lambda \ge \hat{\Lambda}_m(N)$ and $N \ge \underline{N}$ from Lemma OS.7(i). By Lemma OS.1, we have $\underline{\Lambda}_m(N) \le \hat{\Lambda}_m(N) \le \bar{\Lambda}_m(N)$ for $\underline{N} \le N \le \bar{N}$ and $\hat{\Lambda}_m(N) < \underline{\Lambda}_m(N) < \bar{\Lambda}_m(N)$ for $N > \bar{N}$.

In summary, if $\underline{N} \le N \le \bar{N}$ and $\hat{\Lambda}_m(N) \le \Lambda \le \bar{\Lambda}_m(N)$, or $N > \bar{N}$ and $\underline{\Lambda}_m(N) \le \Lambda \le \bar{\Lambda}_m(N)$, then $n_m^* = N$, $\lambda_m^* = \Lambda$ is the unique solution satisfying the necessary KKT condition, and thus optimal.

Case III: $n_m^* < N$ and $\lambda_m^* = \Lambda$. In this case, we have $\eta_1 = 0$ from (OS.29). Besides, by using $\lambda_m^* = \Lambda$ and $\eta_1 = 0$, we can rewrite the necessary conditions (OS.27), (OS.28), and (OS.31) as follows:

$$\frac{\partial L_m}{\partial n_m}\Big|_{n_m=n_m^*,\lambda_m=\Lambda} = \frac{h\Lambda\mu}{(n_m^*\mu-\Lambda)^2} - r - \eta_3\Big[\frac{r}{\Lambda} - \frac{h\mu}{(n_m^*\mu-\Lambda)^2}\Big] = 0, \tag{OA.18}$$
$$\frac{\partial L_m}{\partial \lambda_m}\Big|_{n_m=n_m^*,\lambda_m=\Lambda} = v - \frac{hn_m^*\mu}{(n_m^*\mu-\Lambda)^2} - \eta_2 + \eta_3\Big[\frac{rn_m^*}{\Lambda^2} - \frac{h}{(n_m^*\mu-\Lambda)^2}\Big] = 0, \tag{OA.19}$$
$$\eta_3\big(v - \frac{h}{n_m^*\mu-\Lambda} - \frac{rn_m^*}{\Lambda}\big) = 0.$$

Note (OA.18) can be simplified as $\Lambda[\frac{h\mu}{(n_m^*\mu-\Lambda)^2} - \frac{r}{\Lambda}] - \eta_3[\frac{r}{\Lambda} - \frac{h\mu}{(n_m^*\mu-\Lambda)^2}] = (\Lambda+\eta_3)[\frac{h\mu}{(n_m^*\mu-\Lambda)^2} - \frac{r}{\Lambda}] = 0$. From $\Lambda > 0$ and $\eta_3 \ge 0$, we have $\frac{h\mu}{(n_m^*\mu-\Lambda)^2} = \frac{r}{\Lambda}$, i.e., $\frac{\sqrt{h\mu}}{n_m^*\mu-\Lambda} = \sqrt{\frac{r}{\Lambda}}$, which implies

$$n_m^* = \big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} = \underline{\Lambda}_m^{-1}(\Lambda). \tag{OA.20}$$

Then, $n_m^* < N$ if and only if $\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} < N$. Consider that $\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} - N = \frac{1}{\mu}\big(\Lambda + \frac{h\Lambda\mu}{r} - N\mu\big) = \frac{1}{\mu}\big(\sqrt{\Lambda} + \frac{\sqrt{\frac{h\mu}{r}+4N\mu}+\sqrt{\frac{h\mu}{r}}}{2}\big)\big(\sqrt{\Lambda} - \frac{\sqrt{\frac{h\mu}{r}+4N\mu}-\sqrt{\frac{h\mu}{r}}}{2}\big)$. Thus, $\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} < N$ if and only if $\sqrt{\Lambda} < \frac{\sqrt{\frac{h\mu}{r}+4N\mu}-\sqrt{\frac{h\mu}{r}}}{2}$, i.e., $\Lambda < \underline{\Lambda}_m(N) = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}+4N\mu} - \sqrt{\frac{h\mu}{r}}\big)^2$. We have $v - \frac{h}{n_m^*\mu-\Lambda} - \frac{rn_m^*}{\Lambda} = \pi(\Lambda, \underline{\Lambda}_m^{-1}(\Lambda))/\Lambda \ge 0$ if and only if $\Lambda \ge \Lambda_0$ from Lemma OS.7(ii). And, $\Lambda_0 < \underline{\Lambda}_m(N)$ if and only if $N > \bar{N}$ from Lemma OS.7(iii).

We then show that the Lagrangian multiplier $\eta_2 > 0$. Substituting (OA.20) into (OA.19), we obtain $\eta_2 = v - \frac{hn_m^*\mu}{(n_m^*\mu-\Lambda)^2} + \eta_3\big[\frac{rn_m^*}{\Lambda^2} - \frac{h}{(n_m^*\mu-\Lambda)^2}\big] = v - \frac{h\Lambda\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)}{\big(\sqrt{\frac{h\mu}{r\Lambda}}\cdot\Lambda\big)^2} + \eta_3\big[\frac{r}{\mu\Lambda}\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big) - \frac{h}{\big(\sqrt{\frac{h\mu}{r\Lambda}}\cdot\Lambda\big)^2}\big] =$

$v-\frac{r}{\mu}\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)+\eta_3\big[\frac{r}{\mu\Lambda}\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)-\frac{r}{\mu\Lambda}\big]=v-\frac{r}{\mu}\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)+\frac{r\eta_3}{\mu\Lambda}\sqrt{\frac{h\mu}{r\Lambda}}$. The first two terms satisfy $v-\frac{r}{\mu}\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\geq v-\frac{r}{\mu}\big(1+\sqrt{\frac{h\mu}{r\frac{4hr\mu}{(v\mu-r)^2}}}\big)=v-\frac{r}{\mu}\big(1+\frac{v\mu-r}{2r}\big)=v-\frac{v\mu+r}{2\mu}=\frac{v\mu-r}{2\mu}>0$, where the first inequality holds because $\Lambda\geq 4hr\mu/(v\mu-r)^2$ and the second inequality holds from $v>r/\mu$.

In summary, if $N>\bar{N}$ and $\Lambda_0\leq\Lambda<\underline{\Lambda}_m(N)$, $n_m^*=\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu}=\underline{\Lambda}_m^{-1}(\Lambda)$, $\lambda_m^*=\Lambda$ is the unique solution satisfying the necessary KKT condition, and thus optimal.

Summarizing all the above cases, Proposition 4 is thus proved. $\square$

Proof of Corollaries 3 and 4. To establish the proof, we consider the three cases for the competition case as in Table 1. Then, for each case, we further discuss the three possible equilibria for the merging case.

Case I: $N\geq 2\underline{N}$ and $\Lambda>\bar{\Lambda}(N)$. From Proposition 2, a symmetric farsightedly stable outcome satisfies $n_i^*=N/2$ and $\lambda_i^*=\bar{\Lambda}_m(n_i^*)=\big(1-\sqrt{\frac{h}{vn_i^*\mu}}\big)n_i^*\mu$, $i=1,2$. Based on (1) and (2), the price and wage are $p_i^*=v-\frac{h}{n_i^*\mu-\lambda_i^*}=v-\frac{h}{\sqrt{\frac{h}{vn_i^*\mu}}n_i^*\mu}=v-\sqrt{\frac{hv}{n_i^*\mu}}$, and $w_i^*=\frac{rn_i^*}{\lambda_i^*}=\frac{r}{\big(1-\sqrt{\frac{h}{vn_i^*\mu}}\big)\mu}$. Thus, the total profit from the two platforms is $\pi_1^*+\pi_2^*=(p_1^*-w_1^*)\lambda_1^*+(p_2^*-w_2^*)\lambda_2^*=p_i^*(\lambda_1^*+\lambda_2^*)-r(n_1^*+n_2^*)=p_i^*(\lambda_1^*+\lambda_2^*)-rN$.

Regarding the merged system, three possible cases are all possible. Hence, we derive the optimal solutions and the profit for the merging case in three subcases, and then conduct the comparison.

Subcase (a): Ample demand and limited supply. By Proposition 4, $n_m^*=N=n_1^*+n_2^*$ and $\lambda_m^*=\bar{\Lambda}_m(N)=\big(1-\sqrt{\frac{h}{vN\mu}}\big)N\mu$. Based on (OA.2) and (OA.3), $p_m^*=v-\frac{h}{n_m^*\mu-\lambda_m^*}=v-\frac{h}{\sqrt{\frac{h}{vN\mu}}N\mu}=v-\sqrt{\frac{hv}{N\mu}}>v-\sqrt{\frac{hv}{n_i^*\mu}}=p_i^*$, and $w_m^*=\frac{rn_m^*}{\lambda_m^*}=\frac{r}{\big(1-\sqrt{\frac{h}{vN\mu}}\big)\mu}<\frac{r}{\big(1-\sqrt{\frac{h}{vn_i^*\mu}}\big)\mu}=w_i^*$. Then, we have $\lambda_m^*-(\lambda_1^*+\lambda_2^*)=\lambda_m^*-2\lambda_i^*=\big(1-\sqrt{\frac{h}{vN\mu}}\big)N\mu-\big(1-\sqrt{\frac{2h}{vN\mu}}\big)N\mu=\big(\sqrt{\frac{2h}{vN\mu}}-\sqrt{\frac{h}{vN\mu}}\big)N\mu>0$. Thus, $\pi_m^*=(p_m^*-w_m^*)\lambda_m^*=p_m^*\lambda_m^*-rN>p_i^*(\lambda_1^*+\lambda_2^*)-rN=\pi_1^*+\pi_2^*$, where the inequality holds from $p_m^*>p_i^*$ and $\lambda_m^*>\lambda_1^*+\lambda_2^*$.

Subcase(b): Balanced demand and supply. By Proposition 4, $n_m^*=N=n_1^*+n_2^*$ and $\lambda_1^*+\lambda_2^*<\lambda_m^*=\Lambda\leq\bar{\Lambda}_m(N)$. Based on (OA.2) and (OA.3), $p_m^*=v-\frac{h}{n_m^*\mu-\lambda_m^*}=v-\frac{h}{N\mu-\Lambda}\geq v-\frac{h}{N\mu-\bar{\Lambda}_m(N)}=v-\frac{h}{\sqrt{\frac{h}{vN\mu}}N\mu}=v-\sqrt{\frac{hv}{N\mu}}>v-\sqrt{\frac{hv}{n_i^*\mu}}=p_i^*$, and $w_m^*=\frac{rn_m^*}{\lambda_m^*}=\frac{rN}{\Lambda}<\frac{r}{\big(1-\sqrt{\frac{2h}{vN\mu}}\big)\mu}=\frac{r}{\big(1-\sqrt{\frac{h}{vn_i^*\mu}}\big)\mu}=w_i^*$, where the inequality holds because $\Lambda>\bar{\Lambda}(N)=\big(1-\sqrt{\frac{2h}{vN\mu}}\big)N\mu$ from Proposition 2, and the third equality holds from $n_i^*=N/2$. Thus, $\pi_m^*=(p_m^*-w_m^*)\lambda_m^*=p_m^*\lambda_m^*-rN>p_i^*\bar{\Lambda}(N)-rN=2\big(1-\sqrt{\frac{h}{vn_i^*\mu}}\big)^2vn_i^*\mu-rN=\pi_1^*+\pi_2^*$, where the inequality is due to $p_m^*>p_i^*=v-\sqrt{\frac{hv}{n_i^*\mu}}$ and $\Lambda>\bar{\Lambda}(N)=\big(1-\sqrt{\frac{2h}{vN\mu}}\big)N\mu$.

Subcase(c): Limited demand and ample supply. By Proposition 4, $n_m^*=\big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu}<n_1^*+n_2^*=N$ and $\lambda_m^*=\Lambda>\lambda_1^*+\lambda_2^*$. Based on (OA.2), $p_m^*=v-\frac{h}{n_m^*\mu-\lambda_m^*}=v-\frac{h}{\sqrt{\frac{h\mu}{r\Lambda}}\Lambda}=v-\sqrt{\frac{hr}{\mu\Lambda}}>v-\sqrt{\frac{hr}{\mu\bar{\Lambda}(N)}}=v-\sqrt{\frac{hr}{\big(1-\sqrt{\frac{2h}{vN\mu}}\big)N\mu^2}}\geq v-\sqrt{\frac{hr}{\sqrt{\frac{r}{v\mu}}N\mu^2}}=v-\sqrt{\frac{h\sqrt{v\mu r}}{N\mu^2}}>v-\sqrt{\frac{hv}{N\mu}}>v-\sqrt{\frac{hv}{n_i^*\mu}}=p_i^*$, where the first inequality is due to $\Lambda>\bar{\Lambda}(N)$, the second inequality is due to $N\geq 2\underline{N}=2h/(\sqrt{v\mu}-\sqrt{r})^2$, the third inequality is due to $v>r/\mu$, and the last inequality is due to $n_i^*=N/2$. By (OA.3), $w_m^*=\frac{rn_m^*}{\lambda_m^*}<\frac{rN}{\Lambda}<\frac{rN}{2\lambda_i^*}=\frac{rn_i^*}{\lambda_i^*}=w_i^*$, where the inequality is due to $\Lambda>\lambda_1^*+\lambda_2^*=2\lambda_i^*$ in the ample demand and limited supply case for duopoly case. Thus, $\pi_m^* = (p_m^* - w_m^*)\lambda_m^* = p_m^*\lambda_m - rn_m^* > p_i^*(\lambda_1^* + \lambda_2^*) - rN = \pi_1^* + \pi_2^*$, where the inequality is due to $p_m^* > p_i^*$, $n_m^* < N$ and $\lambda_m^* > \lambda_1^* + \lambda_2^*$.

Case II. (1) $2\underline{N} \le N \le 2\bar{N}$ and $\hat{\Lambda}(N) \le \Lambda \le \bar{\Lambda}(N)$, or (2) $N \ge 2\bar{N}$ and $\underline{\Lambda}(N) \le \Lambda \le \bar{\Lambda}(N)$. The duopoly system reaches balanced demand and supply. From Proposition 2, a symmetric solution satisfies $n_i^* = N/2$ and $\lambda_i^* = \Lambda/2$, $i = 1, 2$. Based on (1) and (2), $p_i^* = v - \frac{h}{n_i^*\mu - \lambda_i^*} = v - \frac{2h}{N\mu - \Lambda}$, $w_i^* = \frac{rn_i^*}{\lambda_i^*} = \frac{rN}{\Lambda}$. Consequently, $\pi_1^* + \pi_2^* = (p_1^* - w_1^*)\lambda_1^* + (p_2^* - w_2^*)\lambda_2^* = p_i^*(\lambda_1^* + \lambda_2^*) - r(n_1^* + n_2^*) = p_i^*\Lambda - rN$.

Regarding the merged system, three possible cases are all possible. Hence, we derive the optimal solutions and the profit for the merging case in three subcases, and then conduct the comparison.

Subcase (a): Ample demand and limited supply under merging case. By Proposition 4, $n_m^* = N$ and $\lambda_m^* = \bar{\Lambda}_m(N) < \Lambda$. Note that in this case we have $\Lambda < \bar{\Lambda}(N)$ from Proposition 2, and $\bar{\Lambda}_m(N) = \big(1 - \sqrt{\frac{h}{vN\mu}}\big)N\mu > \bar{\Lambda}(N) = \big(1 - \sqrt{\frac{2h}{vN\mu}}\big)N\mu$. This is a contradiction.

Subcase (b): Balanced demand and supply. By Proposition 4, $n_m^* = N = n_1^* + n_2^*$ and $\lambda_m^* = \Lambda = \lambda_1^* + \lambda_2^*$. Based on (OA.2) and (OA.3), $p_m^* = v - \frac{h}{n_m^*\mu - \lambda_m^*} = v - \frac{h}{N\mu - \Lambda} > v - \frac{2h}{N\mu - \Lambda} = p_i^*$, and $w_m^* = \frac{rn_m^*}{\lambda_m^*} = \frac{rN}{\Lambda} = w_i^*$. Thus, $\pi_m^* = (p_m^* - w_m^*)\lambda_m^* = p_m^*\lambda^* - rN > p_i^*\Lambda - rN = \pi_1^* + \pi_2^*$.

Subcase (c): Limited demand and ample supply. By Proposition 4, $n_m^* = (1 + \sqrt{\frac{h\mu}{r\Lambda}})\frac{\Lambda}{\mu} < n_1^* + n_2^* = N$ and $\lambda_m^* = \Lambda = \lambda_1^* + \lambda_2^*$. Based on (OA.2), $p_m^* = v - \frac{h}{n_m^*\mu - \lambda_m^*} = v - \frac{h}{\sqrt{\frac{h\mu}{r\Lambda}}\Lambda} = v - \sqrt{\frac{hr}{\mu\Lambda}}$. Consider that $p_m^* - p_i^* = \frac{2h}{N\mu - \Lambda} - \sqrt{\frac{hr}{\mu\Lambda}} = \frac{\sqrt{hr}}{(N\mu - \Lambda)\sqrt{\mu\Lambda}}\big[2\sqrt{\frac{h\mu\Lambda}{r}} - (N\mu - \Lambda)\big]$. Thus, $p_m^* - p_i^* > 0$ if and only if $2\sqrt{\frac{h\mu\Lambda}{r}} + \Lambda - N\mu > 0$. By Proposition 2, balanced demand and supply under duopoly case implies either (1)$2\underline{N} \le N \le 2\bar{N}$, $\hat{\Lambda}(N) \le \Lambda \le \bar{\Lambda}(N)$, or (2) $N > 2\bar{N}$, $\underline{\Lambda}(N) \le \Lambda \le \bar{\Lambda}(N)$. Note $\hat{\Lambda}(N) \ge \underline{\Lambda}(N)$ for $N \le 2\bar{N}$ due to Lemma OS.1(iii) ($\underline{\Lambda}(N)$ and $\hat{\Lambda}(N)$ follow the expressions for $\underline{\Lambda}_m(N)$ and $\hat{\Lambda}_m(N)$ by replacing "h" with "$2h$"). Hence, we have $\Lambda \ge \underline{\Lambda}(N)$. Therefore, $2\sqrt{\frac{h\mu\Lambda}{r}} + \Lambda - N\mu \ge 2\sqrt{\frac{h\mu\underline{\Lambda}(N)}{r}} + \underline{\Lambda}(N) - N\mu = \sqrt{\frac{h\mu}{r}}\big(\sqrt{\frac{2h\mu}{r} + 4N\mu} - \sqrt{\frac{2h\mu}{r}}\big) + \frac{1}{4}\big(\frac{4h\mu}{r} + 4N\mu - 2\sqrt{\frac{2h\mu}{r}}\sqrt{\frac{2h\mu}{r} + 4N\mu}\big) - N\mu = \big(1 - \frac{\sqrt{2}}{2}\big)\sqrt{\frac{h\mu}{r}}\sqrt{\frac{2h\mu}{r} + 4N\mu} - (\sqrt{2} - 1)\frac{h\mu}{r} > \big(1 - \frac{\sqrt{2}}{2}\big)\sqrt{\frac{h\mu}{r}}\sqrt{\frac{2h\mu}{r}} - (\sqrt{2} - 1)\frac{h\mu}{r} = 0$. By (OA.3), $w_m^* = \frac{rn_m^*}{\lambda_m^*} = \frac{rn_m^*}{\Lambda} < \frac{rN}{\Lambda} = w_i^*$, where the inequality is due to $N > n^*$ in the ample demand and limited supply case for merging case. Thus, $\pi_m^* = (p_m^* - w_m^*)\lambda_m^* = p_m^*\Lambda - rn_m^* > p_i^*\Lambda - rN = \pi_1^* + \pi_2^*$, where the inequality is due to $p_m^* > p_i^*$ and $n_m^* < N$.

Case III. $N > 2\bar{N}$ and $2\Lambda_0 \le \Lambda < \underline{\Lambda}'(N)$. In this case, the duopoly setting reaches limited demand and ample supply. From Proposition 2, a symmetric solution shall satisfy $n_i^* = (1 + \sqrt{\frac{h\mu}{r\lambda_i^*}})\frac{\lambda_i^*}{\mu}$ and $\lambda_i^* = \Lambda/2$. Based on (1) and (2), $p_i^* = v - \frac{h}{n_i^*\mu - \lambda_i^*} = v - \frac{h}{\sqrt{\frac{h\mu}{r\lambda_i^*}}\lambda_i^*} = v - \sqrt{\frac{hr}{\mu\lambda_i^*}}$, and $w_i^* = \frac{rn_i^*}{\lambda_i^*} = \big(1 + \sqrt{\frac{h\mu}{r\lambda_i^*}}\big)\frac{r}{\mu}$. Consequently, $\pi_1^* + \pi_2^* = (p_1^* - w_1^*)\lambda_1^* + (p_2^* - w_2^*)\lambda_2^* = p_i^*(\lambda_1^* + \lambda_2^*) - r(n_1^* + n_2^*) = p_i^*\Lambda - r(n_1^* + n_2^*)$.

Regarding the merging setting, we first show that in this case the merging platform will reach limited demand and ample supply by showing that $\underline{\Lambda}_m(N) - \underline{\Lambda}'(N) \ge 0$ for $N \ge 2\bar{N}$. From the definition of $\underline{\Lambda}_m(N)$ and $\underline{\Lambda}'(N)$, we have $\underline{\Lambda}_m(N) - \underline{\Lambda}'(N) = \frac{1}{4}\big(\frac{2h\mu}{r} + 4N\mu - 2\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r} + 4N\mu}\big) - \frac{1}{4}\big(\frac{2h\mu}{r} + 4N\mu - \frac{8h\mu(v\mu+r)}{(v\mu-r)^2} - 2\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r} + 4N\mu - \frac{8h\mu(v\mu+r)}{(v\mu-r)^2}}\big) - \frac{4hr\mu}{(v\mu-r)^2} = \frac{2h\mu(v\mu+r)}{(v\mu-r)^2} - \frac{4hr\mu}{(v\mu-r)^2} - \frac{1}{2}\sqrt{\frac{h\mu}{r}}\big(\sqrt{\frac{h\mu}{r} + 4N\mu} - \sqrt{\frac{h\mu}{r} + 4N\mu - \frac{8h\mu(v\mu+r)}{(v\mu-r)^2}}\big) = \frac{2h\mu}{v\mu-r} - \frac{1}{2}\sqrt{\frac{h\mu}{r}}\frac{\frac{8h\mu(v\mu+r)}{(v\mu-r)^2}}{\sqrt{\frac{h\mu}{r}+4N\mu}+\sqrt{\frac{h\mu}{r}+4N\mu-\frac{8h\mu(v\mu+r)}{(v\mu-r)^2}}} = \frac{2h\mu}{v\mu-r}\big[1 - 2\sqrt{\frac{h\mu}{r}}\frac{v\mu+r}{(v\mu-r)\big(\sqrt{\frac{h\mu}{r}+4N\mu}+\sqrt{\frac{h\mu}{r}+4N\mu-\frac{8h\mu(v\mu+r)}{(v\mu-r)^2}}\big)}\big] =$

$\frac{4h\mu(v\mu+r)\sqrt{\frac{h\mu}{r}}}{(v\mu-r)^2}\Big[\frac{v\mu-r}{2(v\mu+r)\sqrt{\frac{h\mu}{r}}} - \frac{1}{(v\mu-r)\Big(\sqrt{\frac{h\mu}{r}+4N\mu}+\sqrt{\frac{h\mu}{r}+4N\mu-\frac{8h\mu(v\mu+r)}{(v\mu-r)^2}}\Big)}\Big]$. Consider the terms in the bracket, we have $\sqrt{\frac{h\mu}{r}+4N\mu} + \sqrt{\frac{h\mu}{r}+4N\mu - \frac{8h\mu(v\mu+r)}{(v\mu-r)^2}} - 2\sqrt{\frac{h\mu}{r}}\frac{v\mu+r}{v\mu-r} \geq \sqrt{\frac{h\mu}{r} + \frac{8h\mu(v\mu+r)}{(v\mu-r)^2}} + \sqrt{\frac{h\mu}{r}} - 2\sqrt{\frac{h\mu}{r}}\frac{v\mu+r}{v\mu-r} = \sqrt{\frac{h\mu(v\mu-r)^2+8hr\mu(v\mu-r)}{r(v\mu-r)^2}} - \sqrt{\frac{h\mu}{r}}\frac{v\mu+3r}{v\mu-r} = \sqrt{h\mu\frac{(v\mu+3r)^2}{r(v\mu-r)^2}} - \sqrt{\frac{h\mu}{r}}\frac{v\mu+3r}{v\mu-r} = 0$, where the inequality is due to $N \geq \frac{2h\mu(v\mu+r)}{(v\mu-r)^2}$. Thus, $N \geq 2\bar{N} > \bar{N}$, $\Lambda_0 < 2\Lambda_0 \leq \Lambda < \underline{\Lambda}'(N) \leq \underline{\Lambda}_m(N)$. By Proposition 4, the merging platform reaches limited demand and ample supply. Then, by Proposition 4, we have $n_m^* = (1+\sqrt{\frac{h\mu}{r\Lambda}})\frac{\Lambda}{\mu}$ and $\lambda_m^* = \Lambda = \lambda_1^* + \lambda_2^*$. Based on (OA.2) and (OA.3), $p_m^* = v - \frac{h}{n_m^*\mu-\lambda_m^*} = v - \frac{h}{\sqrt{\frac{h\mu}{r\Lambda}}\Lambda} = v - \sqrt{\frac{hr}{\mu\Lambda}} > v - \sqrt{\frac{hr}{\mu\lambda_i^*}} = p_i^*$, and $w_m^* = \frac{rn_m^*}{\lambda_m^*} = \big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{r}{\mu} < \big(1+\sqrt{\frac{h\mu}{r\lambda_i^*}}\big)\frac{r}{\mu} = w_i^*$. Then, we have $n_m^* - (n_1^*+n_2^*) = n_m^* - 2n_1^* = \big(1+\sqrt{\frac{h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} - \big(1+\sqrt{\frac{2h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} = \big(\sqrt{\frac{h\mu}{r\Lambda}} - \sqrt{\frac{2h\mu}{r\Lambda}}\big)\frac{\Lambda}{\mu} < 0$. Thus, $\pi_m^* = (p_m^* - w_m^*)\lambda_m^* = p_m^*\Lambda - rn_m^* > p_i^*\Lambda - r(n_1^*+n_2^*) = \pi_1^* + \pi_2^*$, where the inequality is due to $p_m^* > p_i^*$ and $n_m^* < n_1^* + n_2^*$. $\square$

Proof of Proposition 5. By theorem 4.8 in Bloch and van den Nouweland (2021), we consider the three cases of Nash equilibria in Chen and Wan (2003), and identify the farsightedly stable outcomes by selecting the Pareto efficient outcomes, given in Lemma OA.6, indirectly dominate any other feasible strategy profile.

Case I: $\Lambda < \underline{\Lambda}^o$. In this case, the Nash equilibrium is $\tilde{p}_i^o = \frac{4h\Lambda}{(2\mu-\Lambda)^2}$ with the effective arrival rate $\tilde{\lambda}_i^o = \Lambda/2$. Consider any Pareto efficient outcome $(\bar{p}_1^o = v - \frac{h}{\mu-\bar{\lambda}_1^o}; \bar{p}_2^o = v - \frac{h}{\mu-\bar{\lambda}_2^o})$ that satisfies $g_1(\bar{\lambda}_i^o, \Lambda) \geq 0$, where $(\bar{\lambda}_1^o; \bar{\lambda}_2^o)$ is the corresponding effective arrival rate. Since $\bar{\lambda}_1^o + \bar{\lambda}_2^o = \Lambda$, without of loss generality, we can assume $\bar{\lambda}_1^o \leq \Lambda/2 \leq \bar{\lambda}_2^o$. We first show $\tilde{p}_i^o < \max\{\bar{p}_1^o, \bar{p}_2^o\} = \bar{p}_1^o$. We have $\bar{p}_1^o = v - \frac{h}{\mu-\bar{\lambda}_1^o} \geq v - \frac{h}{\mu-\Lambda/2} > \tilde{p}_i^o$, where the first inequality is due to $v - \frac{h}{\mu-\bar{\lambda}_1^o}$ is decreasing in $\bar{\lambda}_1^o$ and $\bar{\lambda}_1^o \leq \Lambda/2$, and the second inequality is due to $v - \tilde{p}_i^o - \frac{h}{\mu-\tilde{\lambda}_i^o} = v - \frac{h}{\mu-\Lambda/2} - \tilde{p}_i^o > 0$.

Then we show there is a indirect dominance path to $(\bar{p}_1^o; \bar{p}_2^o)$ from any other strategy profile $(p_1^o; p_2^o) \neq (\bar{p}_1^o; \bar{p}_2^o)$. Since $(\bar{p}_1^o; \bar{p}_2^o)$ is Pareto efficient, $\tilde{\pi}_1^o(\tilde{p}_1^o; \tilde{p}_2^o) = \tilde{\pi}_2^o(\tilde{p}_1^o; \tilde{p}_2^o) < \max\{\tilde{\pi}_1^o(\bar{p}_1^o; \bar{p}_2^o), \tilde{\pi}_2^o(\bar{p}_1^o; \bar{p}_2^o)\} = \tilde{\pi}_2^o(\bar{p}_1^o; \bar{p}_2^o)$. Moreover, at least one of the two inequality $\tilde{\pi}_1^o(p_1^o; p_2^o) < \tilde{\pi}_1^o(\bar{p}_1^o, \bar{p}_2^o)$ and $\tilde{\pi}_2^o(p_1^o; p_2^o) < \tilde{\pi}_2^o(\bar{p}_1^o, \bar{p}_2^o)$ holds. Without of loss generality, we suppose $\tilde{\pi}_1^o(p_1^o; p_2^o) < \tilde{\pi}_1^o(\bar{p}_1^o, \bar{p}_2^o)$. Consider the path $(p_1^o; p_2^o) \to_1 (\tilde{p}_1^o; p_2^o) \to_2 (\tilde{p}_1^o; \tilde{p}_2^o) \to_2 (\tilde{p}_1^o; \bar{p}_2^o) \to_1 (\bar{p}_1^o; \bar{p}_2^o)$. We check all moves on this path to verify that it is a indirectly dominance path. We have $\tilde{\pi}_2^o(\tilde{p}_1^o; p_2^o) \leq \tilde{\pi}_2^o(\tilde{p}_1^o; \tilde{p}_2^o) < \tilde{\pi}_2^o(\bar{p}_1^o; \bar{p}_2^o)$, where the first inequality holds because $\tilde{p}_2^o$ is the service platform 2's best response when service platform 1 sets the price $\tilde{p}_1^o$. We next prove $\tilde{\pi}_1^o(\tilde{p}_1^o; \bar{p}_2^o) < \tilde{\pi}_1^o(\bar{p}_1^o; \bar{p}_2^o)$. Let $(\tilde{\lambda}_1^{o'}; \bar{\lambda}_2^{o'})$ be the corresponding effective arrival rate under strategy profile $(\tilde{p}_1^o; \bar{p}_2^o)$. We first prove $v - \tilde{p}_1^o - \frac{h}{\mu-\tilde{\lambda}_1^{o'}} = v - \bar{p}_2^o - \frac{h}{\mu-\bar{\lambda}_2^{o'}} > 0$ by contradiction. Suppose $v - \tilde{p}_1^o - \frac{h}{\mu-\tilde{\lambda}_1^{o'}} = v - \bar{p}_2^o - \frac{h}{\mu-\bar{\lambda}_2^{o'}} = 0$. Then, $\tilde{p}_1^o = v - \frac{h}{\mu-\tilde{\lambda}_1^{o'}} \geq v - \frac{h}{\mu-(\Lambda-\bar{\lambda}_2^{o'})} = v - \frac{h}{\mu-(\Lambda-\bar{\lambda}_2^o)} = v - \frac{h}{\mu-\bar{\lambda}_1^o} = \bar{p}_1^o$, where the first inequality is due to $v - \frac{h}{\mu-\lambda}$ is decreasing in λ and $\tilde{\lambda}_1^{o'} \leq \Lambda - \bar{\lambda}_2^{o'}$, the second equality is due to $\bar{\lambda}_2^{o'} = \bar{\lambda}_2^o$ implied by $v - \bar{p}_2^o - \frac{h}{\mu-\bar{\lambda}_2^{o'}} = v - \bar{p}_2^o - \frac{h}{\mu-\bar{\lambda}_2^o} = 0$, and the third equality is due to $\bar{\lambda}_1^o + \bar{\lambda}_2^o = \Lambda$. This contradicts to $\tilde{p}_1^o < \bar{p}_1^o$. Hence, $\tilde{\lambda}_1^{o'} + \bar{\lambda}_2^{o'} = \Lambda$, and $v - \tilde{p}_1^o - \frac{h}{\mu-\tilde{\lambda}_1^{o'}} = v - \frac{4h\Lambda}{(2\mu-\Lambda)^2} - \frac{h}{\mu-\tilde{\lambda}_1^{o'}} = \frac{h}{\mu-\bar{\lambda}_2^o} - \frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})} = v - \bar{p}_2^o - \frac{h}{\mu-\bar{\lambda}_2^{o'}}$, which is equal to $v - \frac{4h\Lambda}{(2\mu-\Lambda)^2} - \frac{h}{\mu-\tilde{\lambda}_1^{o'}} + \frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})} = \frac{h}{\mu-(\Lambda-\bar{\lambda}_1^o)}$. Thus, $g_1(\bar{\lambda}_1^o, \Lambda) \geq 0$ implies $\frac{4h\Lambda}{(2\mu-\Lambda)^2}\tilde{\lambda}_1^{o'} \leq \big(v - \frac{h}{\mu-\bar{\lambda}_1^o}\big)\bar{\lambda}_1^o$. Hence, $\tilde{\pi}_1^o(\tilde{p}_1^o; \bar{p}_2^o) = \tilde{p}_1^o\tilde{\lambda}_1^{o'} = \frac{4h\Lambda}{(2\mu-\Lambda)^2}\tilde{\lambda}_1^{o'} \leq \big(v - \frac{h}{\mu-\bar{\lambda}_1^o}\big)\bar{\lambda}_1^o = \bar{p}_1^o\bar{\lambda}_1^o = \tilde{\pi}_1^o(\bar{p}_1^o; \bar{p}_2^o)$ (with strict inequality when $\tilde{p}_1^o \neq \bar{p}_1^o$).

Case II: $\underline{\Lambda}^o \leq \Lambda < \bar{\Lambda}^o$. In this case, from the proof of theorem 2 in Chen and Wan (2003), the Nash equilibria are those $(\tilde{p}_1^o; \tilde{p}_2^o)$ satisfy (OS.32), (OS.34), and $\frac{h\tilde{\lambda}_i^o}{(\mu+\tilde{\lambda}_i^o-\Lambda)^2} + \frac{h\tilde{\lambda}_i^o}{(\mu-\tilde{\lambda}_i^o)^2} \geq \tilde{p}_i^o \geq \frac{h\tilde{\lambda}_i^o}{(\mu-\tilde{\lambda}_i^o)^2}$, where $i=1,2$. From (OS.32), the first inequality requires $g_2(\tilde{\lambda}_i^o,\Lambda)\geq v$, and the second inequality is equivalent to (OS.33). Thus, all the Nash equilibria are also Pareto efficient. By corollary 4.9 of Bloch and van den Nouweland (2021), if a Nash equilibrium is Pareto efficient, then it is a singleton vNM FSS. Consequently, any Nash equilibrium should be a singleton vNM FSS.

Case III: $\Lambda \geq \bar{\Lambda}^o$. In this case, from Chen and Wan (2003), the Nash equilibrium $\tilde{p}_i^o = v-\frac{h}{\mu-\tilde{\lambda}_i^o}$, $\tilde{\lambda}_i^o=\mu-\sqrt{\frac{h\mu}{v}}$, $i=1,2$, which is also Pareto efficient from Lemma OA.6. Thus, the unique Nash equilibrium is also Pareto efficient. By Corollary 4.9 of Bloch and van den Nouweland (2021), if a Nash equilibrium is Pareto efficient, then it is a singleton vNM FSS. Consequently, the unique Nash equilibrium should be a singleton vNM FSS. □

Proof of Corollary 5. (i) It comes from Table OA.1 directly.

(ii) Table OA.1 shows that under both the "moderate demand" and "ample demand" markets, all the myopically stable outcomes are also farsightedly stable. We thus focus on "scarce demand", that is $\Lambda<\underline{\Lambda}^o$. For "scarce demand" market, the Nash equilibrium is $\tilde{p}_i^o=\frac{4h\Lambda}{(2\mu-\Lambda)^2}$ with the effective arrival rate $\tilde{\lambda}_i^o=\Lambda/2$. Then the profit is $\tilde{\pi}_i^o=\tilde{p}_i^o\tilde{\lambda}_i^o$. Consider any farsightedly stable outcome $(p_1^o=v-\frac{h}{\mu-\lambda_1^o};p_2^o=v-\frac{h}{\mu-\lambda_2^o})$, where $(\lambda_1^o;\lambda_2^o)$ is the corresponding effective arrival rate. Since $\lambda_1^o+\lambda_2^o=\Lambda$, without of loss generality, we can assume $\lambda_1^o\leq\Lambda/2\leq\lambda_2^o$, which implies $\pi_1^o=\big(v-\frac{h}{\mu-\lambda_1^o}\big)\lambda_1^o\leq\pi_2^o=\big(v-\frac{h}{\mu-\lambda_2^o}\big)\lambda_2^o$. By Proposition 5(i), we have $g_1(\lambda_i^o,\Lambda)\geq 0$. From the Case I in the proof of Proposition 5(i), $g_1(\lambda_1^o,\Lambda)\geq 0$ implies $\pi_1^o=\big(v-\frac{h}{\mu-\lambda_1^o}\big)\lambda_1^o\geq\frac{4h\Lambda}{(2\mu-\Lambda)^2}\tilde{\lambda}_1^{o'}=\tilde{p}_i^o\tilde{\lambda}_1^{o'}$, where $\tilde{\lambda}_1^{o'}$ satisfies $v-\frac{4h\Lambda}{(2\mu-\Lambda)^2}-\frac{h}{\mu-\tilde{\lambda}_1^{o'}}+\frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})}=\frac{h}{\mu-(\Lambda-\lambda_1^o)}$. To show $\pi_1^o\geq\tilde{\pi}_i^o=\tilde{p}_i^o\tilde{\lambda}_i^o$, it is sufficiently to show $\tilde{\lambda}_1^{o'}\geq\tilde{\lambda}_i^o=\Lambda/2$.

Now we will show $\tilde{\lambda}_1^{o'}\geq\tilde{\lambda}_i^o=\Lambda/2$. Note $v-\frac{4h\Lambda}{(2\mu-\Lambda)^2}-\frac{h}{\mu-\tilde{\lambda}_1^{o'}}+\frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})}=\frac{h}{\mu-(\Lambda-\lambda_1^o)}$ implies

$$\frac{h}{\mu-\tilde{\lambda}_1^{o'}}-\frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})}=v-\frac{h}{\mu-(\Lambda-\lambda_1^o)}-\frac{4h\Lambda}{(2\mu-\Lambda)^2}. \tag{OA.21}$$

First, we show $\tilde{\lambda}_1^{o'}\geq\lambda_1^o$ by contradiction. Suppose $\tilde{\lambda}_1^{o'}<\lambda_1^o$, then $\frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})}>\frac{h}{\mu-(\Lambda-\lambda_1^o)}$, which implies $\frac{h}{\mu-\tilde{\lambda}_1^{o'}}>v-\frac{4h\Lambda}{(2\mu-\Lambda)^2}$ by (OA.21). It is easy to check $v-\frac{4h\Lambda}{(2\mu-\Lambda)^2}>\frac{h}{\mu-\Lambda/2}$ for $\Lambda<\underline{\Lambda}^o$, thus $\frac{h}{\mu-\tilde{\lambda}_1^{o'}}>\frac{h}{\mu-\Lambda/2}$, i.e., $\tilde{\lambda}_1^{o'}>\Lambda/2$. It contradicts to $\tilde{\lambda}_1^{o'}<\lambda_1^o\leq\Lambda/2$. Second, we show $\frac{h}{\mu-\tilde{\lambda}_1^{o'}}\geq\frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})}$, i.e., $\tilde{\lambda}_1^{o'}\geq\Lambda/2$. That is $\big(\frac{h}{\mu-\tilde{\lambda}_1^{o'}}-\frac{h}{\mu-(\Lambda-\tilde{\lambda}_1^{o'})}\big)\tilde{\lambda}_1^{o'}=\big(v-\frac{h}{\mu-(\Lambda-\lambda_1^o)}\big)\tilde{\lambda}_1^{o'}-\frac{4h\Lambda}{(2\mu-\Lambda)^2}\tilde{\lambda}_1^{o'}\geq\big(v-\frac{h}{\mu-(\Lambda-\lambda_1^o)}\big)(\tilde{\lambda}_1^{o'}-\lambda_1^o)\geq 0$, where the equality is (OA.21) multiplied by $\tilde{\lambda}_1^{o'}$, the first inequality is due to $\big(v-\frac{h}{\mu-\lambda_1^o}\big)\lambda_1^o\geq\frac{4h\Lambda}{(2\mu-\Lambda)^2}\tilde{\lambda}_1^{o'}$, and the second inequality is due to $\tilde{\lambda}_1^{o'}\geq\lambda_1^o$. □

Proof of Proposition 6. First, we show $p_i^m=v-\frac{h}{\mu-\lambda_i^m}$ by contradiction. Suppose $p_i^m<v-\frac{h}{\mu-\lambda_i^m}$, then we can always increase the total profit $p_1^m\lambda_1^m+p_2^m\lambda_2^m$ by increasing p_i^m. We verify the proposition by considering the following two cases.

Case I: $\Lambda\geq\bar{\Lambda}^o=2\big(\mu-\sqrt{\frac{h\mu}{v}}\big)$. Recall it is optimal to let $\lambda=\mu-\sqrt{\frac{h\mu}{v}}$ and $p=v-\frac{h}{\mu-\lambda}$ under a monopoly case. Hence, the total profit $p_1^m\lambda_1^m+p_2^m\lambda_2^m$ can be maximized by $\lambda_i^m=\mu-\sqrt{\frac{h\mu}{v}}$, $p_i^m=v-\frac{h}{\mu-\lambda_i^m}$. By Table OA.1, we have $\lambda_1^o=\lambda_2^o=\mu-\sqrt{\frac{h\mu}{v}}$, $p_1^o=p_2^o=v-\frac{h}{\mu-\lambda_i^o}$, and $\pi_1^o=\pi_2^o=p_1^o\lambda_1^o=p_2^o\lambda_2^o$. We thus get the desired results.

Case II: $\Lambda<\bar{\Lambda}^o=2\big(\mu-\sqrt{\frac{h\mu}{v}}\big)$. First, we show $\lambda_1^m+\lambda_2^m=\Lambda$ by contradiction. In this case, $\lambda_1^m\geq\mu-\sqrt{\frac{h\mu}{v}}$ and $\lambda_2^m\geq\mu-\sqrt{\frac{h\mu}{v}}$ cannot hold together. Without of loss generality, we suppose

$\lambda_1^m < \mu - \sqrt{\frac{h\mu}{v}}$, then $\pi_1^m = p_1^m \lambda_1^m = \big(v - \frac{h}{\mu-\lambda_1^m}\big)\lambda_1^m$ is increasing in λ_1^m. Thus, we can increase the total profit $\pi_1^m + \pi_2^m$ by increasing λ_1^m and fixing λ_2^m. Second, we show $\lambda_1^m = \lambda_2^m = \Lambda/2$. Note $\pi_1^m + \pi_2^m = p_1^m \lambda_1^m + p_2^m \lambda_2^m = \big(v - \frac{h}{\mu-\lambda_1^m}\big)\lambda_1^m + \big(v - \frac{h}{\mu-\lambda_2^m}\big)\lambda_2^m$ and $\lambda_1^m + \lambda_2^m = \Lambda$, we have $\frac{d\pi_1^m+\pi_2^m}{d\lambda_1^m} = \frac{h\mu}{(\mu-\lambda_2^m)^2} - \frac{h\mu}{(\mu-\lambda_1^m)^2}$, which is no more than zero if $\lambda_1^m \le \Lambda/2 \le \lambda_2^m$. Hence, $\pi_1^m + \pi_2^m$ is maximized by $\lambda_1^m = \lambda_2^m = \Lambda/2$. Consequently, $p_1^o = p_2^o = v - \frac{h}{\mu-\Lambda/2}$ and $\pi_1^m = \pi_2^m = p_1^m \lambda_1^m = p_2^m \lambda_2^m = \big(v - \frac{h}{\mu-\Lambda/2}\big)\Lambda/2$. By Table OA.1, we have $\lambda_1^o = \lambda_2^o = \Lambda/2$, $p_1^o = p_2^o = v - \frac{h}{\mu-\Lambda/2}$, and $\pi_1^o = \pi_2^o = p_1^o \lambda_1^o = p_2^o \lambda_2^o = \big(v - \frac{h}{\mu-\Lambda/2}\big)\Lambda/2$. We thus get the desired results. $\square$

Proof of Proposition 7. By proposition 3 of Chwe (1994), any vNM FSS is a subset of LCS. Together with Proposition 1, we have $\mathcal{P} \subset$ LCS. Let $L = \{s | \pi_i(s) \ge \min_{s'\in\mathcal{P}} \pi_i(s'),\ i = 1,2\}$ and its complementary set $\bar{L} = \{s|\pi_1(s) < \min_{s'\in\mathcal{P}} \pi_1(s'),\ or\ \pi_2(s) < \min_{s'\in\mathcal{P}} \pi_2(s')\}$. It is easy to see $\mathcal{P} \subset L$ and $L \cap \bar{L} = \emptyset$.

First, we show $L \subset$ LCS, i.e., $a \in$ LCS for any $a \in L$. There exists $e \in \mathcal{P}$ satisfying $\pi_i(a) \ge \pi_i(e)$ by the definition of L. Note we have proved any Pareto effcient outcome can indirectly dominate all other strategy profiles in proof of Proposition 1. Hence, $\forall\ d,\ i \in \{1,2\}$ such that $a \to_i d$, d is indirectly dominated by e and $e \in$ LCS. That is $a \in$ LCS by Definition 2.

Second, we show $\min_{s'\in\mathcal{P}} \pi_i(s') = 0$ for $i = 1,2$. In other words, L contains all feasible strategy profiles, which implies LCS contains all feasible strategy profiles because we have proved $L \subset$ LCS. It is equivalent to find a Pareto effcient outcome $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ satisfies $\pi_1(\lambda_1^*, n_1^*) = 0$ (if it exists, then $\pi_2(\lambda_1^*, n_1^*) = 0$ by the symmetry). Consider the following three cases.

Case I: $N \ge 2\underline{N}$ and $\Lambda > \bar{\Lambda}'(N)$. Let $n_1^* = \underline{N}$ and $n_2^* = N - \underline{N}$, then $\lambda_1^* = \bar{\Lambda}_m(\underline{N})$ and $\lambda_2^* = \bar{\Lambda}_m(N - \underline{N})$. Recall $\bar{\Lambda}'(N) = \bar{\Lambda}_m(\underline{N}) + \bar{\Lambda}_m(N - \underline{N})$ by Lemma OS.3(i), which implies $\lambda_1^* + \lambda_2^* < \Lambda$. By Proposition 2(i), $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ is a Pareto effcient outcome. Moreover, it is easy to check $\pi_1(\bar{\Lambda}(\underline{N}), \underline{N}) = \pi_m(\bar{\Lambda}_m(\underline{N}), \underline{N}) = 0$.

Case II: $N \ge 2\underline{N}$ and $\underline{\Lambda}'(N) \le \Lambda \le \bar{\Lambda}'(N)$. By Proposition 2(i), $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \mathcal{F}(0)$ is Pareto efficient, i.e., $\pi_1(\lambda_1^*, n_1^*) = 0$.

Case III: $N \ge 2\underline{N}$ and $\Lambda < \underline{\Lambda}'(N)$. Let $\Lambda_1^* = \Lambda_0$ and $\Lambda_2^* = \Lambda - \Lambda_0$, then $n_1^* = \underline{\Lambda}_m^{-1}(\Lambda_0)$ and $n_2^* = \underline{\Lambda}_m^{-1}(\Lambda - \Lambda_0)$. Recall $n_1^* + n_2^* = \underline{\Lambda}_m^{-1}(\Lambda_0) + \underline{\Lambda}_m^{-1}(\Lambda - \Lambda_0) < N$ is equivalent to $\Lambda > \underline{\Lambda}'(N)$ in the proof of Lemma OA.3. By Proposition 2(i), $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ is a Pareto effcient outcome. Moreover, it is easy to check $\pi_1(\Lambda_0, \underline{\Lambda}_m^{-1}(\Lambda_0)) = \pi(\Lambda_0, \underline{\Lambda}_m^{-1}(\Lambda_0)) = 0$. $\square$

Proof of Proposition 8. With the same proof of Lemma OA.1 by replacing v, μ with v_i, μ_i, respectively, we have that: for any feasible strategy profile $(p_1, w_1; p_2, w_2) \in S_h$, if $(p_1, w_1; p_2, w_2)$ is Pareto efficient, then (p_i, w_i) must be the optimal solution of problem $P'_{i_h}|_{(p_j,w_j)}$ for $i,j = 1,2$ and $i \ne j$. By substituting (OS.3) and (OS.4) into problem $P'_{i_h}|_{(p_j,w_j)}$, the problem can be rewritten as problem $P_{i_h}|_{(\lambda_j,n_j)}$. Therefore, we get Proposition 8(ii) by Lemmas OS.11 and OS.12 immediately. With the same proof of Proposition 1 by replacing v, μ with v_i, μ_i, we get Proposition 8(i). $\square$

Proof of Lemma 3. (i) If $y_1 + y_2 < 1$, $\frac{\lambda_1 w_1}{n_1} - l y_1 = \frac{\lambda_2 w_2}{n_2} - l(1 - y_2) = r$. Otherwise, $\frac{\lambda_1 w_1}{n_1} - l y_1 = \frac{\lambda_2 w_2}{n_2} - l(1 - y_2) \ge r$. Note $y_1 + y_2 = 1$ if and only if $n_1 + n_2 = N$, $y_1 = \frac{n_1}{N}$ and $1 - y_2 = \frac{n_2}{N}$.

(ii) If $x_1 + x_2 < 1$, $v - p_1 - \frac{h}{n_1\mu - \lambda_1} - t x_1 = v - p_2 - \frac{h}{n_2\mu - \lambda_2} - t(1 - x) = 0$. Otherwise, $v - p_1 - \frac{h}{n_1\mu - \lambda_1} - t x_1 = v - p_2 - \frac{h}{n_2\mu - \lambda_2} - t(1 - x) \ge 0$. Note $x_1 + x_2 = 1$ if and only if $\lambda_1 + \lambda_2 = \Lambda$, $x_1 = \frac{\lambda_1}{\Lambda}$ and $1 - x_2 = \frac{\lambda_2}{\Lambda}$. $\square$

Proof of Proposition 9. First, similarly to Lemma OA.1, we verify that a Pareto-efficient strategy profile $s=(p_1,w_1;p_2,w_2)$ must be the optimal solution to problem $\bar{P}_1'\&\bar{P}_2'$, where problem $\bar{P}_i'|_{(p_j,w_j)}$ is given as follows:

$$
\begin{aligned}
\bar{P}_i'|_{(p_j,w_j)}: \quad \max_{p_i,w_i} \quad & \pi_i=(p_i-w_i)\lambda_i \\
& \frac{\lambda_1 w_1}{n_1}-\frac{ln_1}{N}=\frac{\lambda_2 w_2}{n_2}-\frac{ln_2}{N}\geq r, \\
& v-p_1-\frac{h}{n_1\mu-\lambda_1}-\frac{t\lambda_1}{\Lambda}=v-p_2-\frac{h}{n_2\mu-\lambda_2}-\frac{t\lambda_2}{\Lambda}\geq 0, \\
& n_1+n_2\leq N, \quad \lambda_1+\lambda_2\leq\Lambda, \\
& n_i>0, \quad \lambda_i>0, \quad p_i>0, \quad w_i>0, \quad p_i-w_i\geq 0, \quad i=1,2.
\end{aligned}
$$

It suffices to show that if a feasible strategy profile $s=(p_1,w_1;p_2,w_2)$ is not the optimal solution of problem $\bar{P}_1'\&\bar{P}_2'$, then s cannot be Pareto efficient, i.e., dominated by some strategy profile $s'\in S$. Suppose that s is not optimal of problem $\bar{P}_1'\&\bar{P}_2'$. In other words, there exists a $i\in\{1,2\}$ such that (p_i,w_i) is not an optimal solution of problem $\bar{P}_i'|_{(p_j,w_j)}$, where $j\neq i$. We then construct an alternative strategy profile $s'\in S$ which dominates s for the three cases, respectively.

Case I: $\frac{\lambda_1 w_1}{n_1}-\frac{ln_1}{N}=\frac{\lambda_2 w_2}{n_2}-\frac{ln_2}{N}>r$. In this case, we let $s'=(p_1,w_1';p_2,w_2')$ which satisfies $\frac{\lambda_1 w_1'}{n_1}-\frac{ln_1}{N}=\frac{\lambda_2 w_2'}{n_2}-\frac{ln_2}{N}=r$. It is easy to check that $w_1'<w_1$ and $w_2'<w_2$. Then, we have $\pi_i(s')=(p_i-w_i')\lambda_i>(p_i-w_i)\lambda_i=\pi_i(s)$ for $i=1,2$. Thus, the strategy profile s is dominated by s'.

Case II: $v-p_1-\frac{h}{n_1\mu-\lambda_1}-\frac{t\lambda_1}{\Lambda}=v-p_2-\frac{h}{n_2\mu-\lambda_2}-\frac{t\lambda_2}{\Lambda}>0$. In this case, we let $s'=(p_1',w_1;p_2',w_2)$ which satisfies $v-p_1'-\frac{h}{n_1\mu-\lambda_1}-\frac{t\lambda_1}{\Lambda}=v-p_2'-\frac{h}{n_2\mu-\lambda_2}-\frac{t\lambda_2}{\Lambda}=0$. It is easy to check that $p_1'>p_1$ and $p_2'>p_2$. Then, we have $\pi_i(s')=(p_i'-w_i)\lambda_i>(p_i-w_i)\lambda_i=\pi_i(s)$. Thus, the strategy profile s is dominated by s'.

Case III: $\frac{\lambda_1 w_1}{n_1}-\frac{ln_1}{N}=\frac{\lambda_2 w_2}{n_2}-\frac{ln_2}{N}=r$, $v-p_1-\frac{h}{n_1\mu-\lambda_1}-\frac{t\lambda_1}{\Lambda}=v-p_2-\frac{h}{n_2\mu-\lambda_2}-\frac{t\lambda_2}{\Lambda}=0$, and there exists a $i\in\{1,2\}$ such that (p_i,w_i) is not optimal solution of problem $\bar{P}_i'|_{(p_j,w_j)}$, where $j\neq i$. Without loss of generality, we assume (p_1,w_1) is not optimal solution of problem $\bar{P}_1'|_{(p_2,w_2)}$. Denote (p_1^*,w_1^*) as the optimal solution of problem $\bar{P}_1'|_{(p_2,w_2)}$. In this case, we let $s'=(p_1^*,w_1^*;p_2,w_2)$. Then, $\pi_1(s')=(p_1^*-w_1^*)\lambda_1^*>(p_1-w_1)\lambda_1=\pi_1(s)$, and $\pi_2(s')=(p_2-w_2)\lambda_2=\pi_2(s)$. Thus, the strategy profile s is dominated by s'.

The proof of Proposition 9(i) is as same as that of Proposition 1 (because the proof only requires that Lemma OA.1(i) holds, and we have verified it). We now prove Proposition 9(ii). It suffices to prove a Pareto-efficient strategy profile $s=(p_1,w_1;p_2,w_2)$ should satisfy $\frac{\lambda_i w_i}{n_i}-\frac{ln_i}{N}=r$ and $v-p_i-\frac{h}{n_i\mu-\lambda_i}-\frac{t\lambda_i}{\Lambda}=0$. If the former one doesn't hold, from the analysis in Case I, s is not Pareto efficient, if the latter one doesn't hold, from the analysis in Case II, s is not Pareto efficient. □

Proof of Proposition 10. With the same proof of Lemma OA.1 by replacing $\frac{1}{n\mu-\lambda}$ with $W(\lambda,n)$ (because the proof doesn't depend on the scheme of the expected waiting time), we have that: for any feasible strategy profile $(p_1,w_1;p_2,w_2)\in S_w$, if $(p_1,w_1;p_2,w_2)$ is Pareto efficient, then (p_i,w_i) must be the optimal solution of problem $P_{i_w}'|_{(p_j,w_j)}$ for $i,j=1,2$ and $i\neq j$. By substituting (OS.3) and (OS.4) into problem $P_{i_w}'|_{(p_j,w_j)}$, the problem can be rewritten as problem $P_{i_w}|_{(\lambda_j,n_j)}$. Therefore, we get Proposition 10(ii) by Lemmas OS.13 and OS.14 immediately.

With the same proof of Proposition 1 by replacing $\frac{1}{n\mu-\lambda}$ with $W(\lambda,n)$ (because the proof doesn't depend on the scheme of the expected waiting time), we get Proposition 10(i). □

Online Supplement to "Farsighted Stability in Service Platforms Competition"

OS.1. Preliminaries for the Auxiliary Functions in Appendix OA.1

In Appendix OA.1, we define the notations and the auxiliary functions, now we present some preliminaries for the auxiliary functions.

LEMMA OS.1. *(i) $\bar{\Lambda}_m(N)$ and $\underline{\Lambda}_m(N)$ increases in N on the interval $[N_0,+\infty)$, $\underline{\Lambda}_m(N)\le\bar{\Lambda}_m(N)$ if and only if $N\ge N_0$; (ii) $\bar{\Lambda}_m(N)$ intersects $\hat{\Lambda}_m(N)$ at $N=\underline{N}$, and $\underline{\Lambda}_m(N)$ intersects $\hat{\Lambda}_m(N)$ at $N=\bar{N}$; (iii) $\hat{\Lambda}_m(N)$ is convex in N with the minima attained at $\bar{N}$, and $\hat{\Lambda}_m(N)\le\underline{\Lambda}_m(N)$ if and only if $N\ge\bar{N}$.*

LEMMA OS.2. *(i) $\bar{\Lambda}(N)$ and $\underline{\Lambda}(N)$ increase in N for $N\ge 2N_0$, $\bar{\Lambda}'(N)$ increases in N for $N\ge 2\underline{N}$, $\underline{\Lambda}'(N)$ increases in N for $2\bar{N}$, $\hat{\Lambda}(N)$ is convex in N, with the minima attained at $2\bar{N}$; (ii) $\underline{\Lambda}'(N)\ge\underline{\Lambda}(N)$, and $\underline{\Lambda}'(N)$, $\underline{\Lambda}(N)$ and $\hat{\Lambda}(N)$ intersect at $N=2\bar{N}$; (iii) $\bar{\Lambda}'(N)\ge\bar{\Lambda}(N)$, and $\bar{\Lambda}'(N)$, $\bar{\Lambda}(N)$ and $\hat{\Lambda}(N)$ intersect at $N=2\underline{N}$.*

LEMMA OS.3. *(i) $\bar{\Lambda}_m(N/2)+\bar{\Lambda}_m(N/2)=\bar{\Lambda}(N)$ and $\bar{\Lambda}_m(\underline{N})+\bar{\Lambda}_m(N-\underline{N})=\bar{\Lambda}'(N)$; (ii) $\hat{\Lambda}_m(N/2)+\hat{\Lambda}_m(N/2)=\hat{\Lambda}(N)$; (iii) $\underline{\Lambda}_m(N/2)+\underline{\Lambda}_m(N/2)=\underline{\Lambda}(N)$; (iv) $\underline{\Lambda}_m^{-1}(\Lambda/2)+\underline{\Lambda}_m^{-1}(\Lambda/2)=\underline{\Lambda}^{-1}(\Lambda)$ and $\underline{\Lambda}_m^{-1}(\Lambda_0)+\underline{\Lambda}_m^{-1}(\Lambda-\Lambda_0)=\underline{\Lambda}'^{-1}(\Lambda)$.*

LEMMA OS.4. *(i) $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)\ge\bar{\Lambda}(N)$, and $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)\le\bar{\Lambda}'(N)$ if $\underline{N}\le n\le N-\underline{N}$ for $N\ge 2\underline{N}$; (ii) $\hat{\Lambda}_m(n)+\hat{\Lambda}_m(N-n)\ge\hat{\Lambda}(N)$; (iii) $\underline{\Lambda}_m(n)+\underline{\Lambda}_m(N-n)\ge\underline{\Lambda}(N)$; (iv) $\underline{\Lambda}_m^{-1}(\lambda)+\underline{\Lambda}_m^{-1}(\Lambda-\lambda)\ge\underline{\Lambda}'^{-1}(\Lambda)$ if $\Lambda_0\le\lambda\le\Lambda-\Lambda_0$ for $\Lambda\ge 2\Lambda_0$.*

LEMMA OS.5. *(i) If $N\ge 2\underline{N}$ and $\bar{\Lambda}(N)<\Lambda\le\bar{\Lambda}'(N)$, there exists $\underline{N}\le n<N/2$ such that $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)=\Lambda$; (ii) If $N\ge 2\underline{N}$ and $\underline{\Lambda}(N)\le\Lambda<\underline{\Lambda}'(N)$, there exists $\Lambda_0<\lambda\le\Lambda/2$ such that $\underline{\Lambda}_m^{-1}(\lambda)+\underline{\Lambda}_m^{-1}(\Lambda-\lambda)=N$.*

LEMMA OS.6. *For any given n, $\pi(\lambda,n)$ is maximized at $\lambda=\bar{\Lambda}_m(n)$, for any given λ, $\pi(\lambda,n)$ is maximized at $n=\underline{\Lambda}_m^{-1}(\lambda)$. Moreover, $\pi(\bar{\Lambda}_m(n),n)$ increases in n, and $\pi\big(\lambda,\underline{\Lambda}_m^{-1}(\lambda)\big)$ increases in λ.*

LEMMA OS.7. *(i) $\pi(\Lambda,N)\ge 0$ if and only if $\Lambda\ge\hat{\Lambda}_m(N)$ and $N\ge\underline{N}$; (ii) $\pi(\Lambda,\underline{\Lambda}_m^{-1}(\Lambda))\ge 0$ if and only if $\Lambda\ge\Lambda_0$; (iii) $\Lambda_0<\underline{\Lambda}_m(N)$ if and only if $N>\bar{N}$.*

OS.2. Auxiliary Lemmas for the Proof of Proposition 2

LEMMA OS.8. *Let $\mathcal{P}$ be the set contains all Pareto efficient outcomes. A feasible strategy profile s is Pareto efficient if and only if there doesn't exist $s'\in\mathcal{P}$ such that $\pi_i(s')\ge\pi_i(s)$ for $i=1,2$, and $\pi_i(s')>\pi_i(s)$ for some $i=1,2$.*

LEMMA OS.9. *For $i,j=1,2$ and $i\ne j$, let (λ_i^*,n_i^*) be the optimal solution of problem $P_i|_{(\lambda_j^*,n_j^*)}$.*

(i) When $N\ge 2\underline{N}$ and $\bar{\Lambda}(N)<\Lambda\le\bar{\Lambda}'(N)$, $(\lambda_1^*,n_1^*;\lambda_2^*,n_2^*)$ satisfies the conditions in Lemma OA.3(i) or (ii). For $n_1^*\le N/2$, it is given in Lemma OA.3(ii) if and only if $n_1^*\le\bar{n}$, where $\bar{n}$ satisfies $\underline{N}\le\bar{n}<N/2$ and $\bar{\Lambda}_m(\bar{n})+\bar{\Lambda}_m(N-\bar{n})=\Lambda$.

(ii) When $N\ge 2\underline{N}$ and $\underline{\Lambda}(N)\le\Lambda<\underline{\Lambda}'(N)$, $(\lambda_1^*,n_1^*;\lambda_2^*,n_2^*)$ satisfies the conditions in Lemma OA.3(ii) or (iii). For $\lambda_1^*\le\Lambda/2$, it is given in Lemma OA.3(ii) if and only if $\lambda_1^*\ge\bar{\lambda}$, where $\bar{\lambda}$ satisfies $\Lambda_0<\bar{\lambda}\le\Lambda/2$ and $\underline{\Lambda}_m^{-1}(\bar{\lambda})+\underline{\Lambda}_m^{-1}(\Lambda-\bar{\lambda})=N$.

Lemma OS.10. *For any strategy profile* $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ *that satisfying the conditions in Lemma OA.3(ii), if* $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \mathcal{F}(t)$*, then* $P_t = \pi_2(\lambda_2^*, n_2^*)$ *is decreasing in* t*, where* P_t *is optimal objective value of problem* P^t*. Moreover, we have* $\pi_1(\lambda_1^*, n_1^*) \le P_0$.

OS.3. Supplementary Results of Section 6

OS.3.1. Supplementary Results of Section 6.2

We define some auxiliary functions as follows:

$$\bar{\Lambda}_{m_i}(N) = \Big(1 - \sqrt{\frac{h}{v_i N \mu_i}}\Big) N\mu_i, \quad \underline{\Lambda}_{m_i}(N) = \frac{1}{4}\Big(\sqrt{\frac{h\mu_i}{r} + 4N\mu_i} - \sqrt{\frac{h\mu_i}{r}}\Big)^2,$$

$$\underline{\Lambda}_{m_i}^{-1}(\Lambda) = \Big(1 + \sqrt{\frac{h\mu_i}{r\Lambda}}\Big)\frac{\Lambda}{\mu_i}, \quad \hat{\Lambda}_{m_i}(N) = \frac{1}{2v_i}\big(rN + v_i N\mu_i - h - \sqrt{(rN + v_i N \mu_i - h)^2 - 4rv_i N^2 \mu_i}\big).$$

We define some notations as follows:

$$\underline{N}_i = h/(\sqrt{v_i\mu_i} - \sqrt{r})^2, \quad \Lambda_{0_i} = 4hr\mu_i/(v_i\mu_i - r)^2.$$

Lemma OS.11. *For* $i, j = 1, 2$ *and* $i \neq j$*, if* (p_i^*, w_i^*) *is the optimal solution of problem* $P'_{i_h}|_{(p_j^*, w_j^*)}$*, then we have*

$$p_i^* = v_i - \frac{h}{n_i^* \mu_i - \lambda_i^*}, \tag{OS.1}$$

$$w_i^* = \frac{rn_i^*}{\lambda_i^*}. \tag{OS.2}$$

Moreover, problem $P'_{i_h}|_{(p_j, w_j)}$ *is defined as follows:*

$$\begin{aligned} P'_{i_h}|_{(p_j, w_j)}: \quad & \max \quad \pi_i = (p_i - w_i)\lambda_i \\ & s.t. \quad (p_i, w_i) \in S_{i_w}|_{(p_j, w_j)}, \end{aligned}$$

where $S_h = \{(p_1, w_1; p_2, w_2) | (c1), (c2_h), (c3)\}$*,* $S_{1_h}|_{(p_2, w_2)} = \{(p, w) | (p, w; p_2, w_2) \in S_h\}$ *and* $S_{2_h}|_{(p_1, w_1)} = \{(p, w) | (p_1, w_1; p, w) \in S_h\}$.

Lemma OS.12. *For* $i, j = 1, 2$ *and* $i \neq j$*, if* (λ_i^*, n_i^*) *is the optimal solution of problem* $P_{i_h}|_{(\lambda_j^*, n_j^*)}$*, then* $n_1^* + n_2^* = N$*, or* $\lambda_1^* + \lambda_2^* = \Lambda$*, or both hold. Moreover, problem* $P_{i_h}|_{(\lambda_j, n_j)}$ *is defined as follows:*

$$\begin{aligned} P_{i_h}|_{(\lambda_j, n_j)}: \quad & \max_{\lambda_i > 0, n_i > 0} \quad \pi_i = \Big(v - \frac{h}{n_i\mu - \lambda_i}\Big)\lambda_i - rn_i \\ & s.t. \quad N - n_j \ge n_i, \quad \Lambda - \lambda_j \ge \lambda_i, \quad v_i - \frac{h}{n_i\mu_i - \lambda_i} - \frac{rn_i}{\lambda_i} \ge 0. \end{aligned}$$

Proposition OS.1. *Consider the two-sided platform competition where the platforms are heterogeneous. Any vNM FSS* $\{s = (\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)\}$ *belongs to one of the following three cases.*

(i) $n_i^* \ge \underline{N}_i$*,* $n_1^* + n_2^* = N$*,* $\lambda_i^* = \bar{\Lambda}_{m_i}(n_i^*)$ *and* $\lambda_1^* + \lambda_2^* < \Lambda$.

(ii) $n_i^* \ge \underline{N}_i$*,* $n_1^* + n_2^* = N$*,* $\max\{\hat{\Lambda}_{m_i}(n_i^*), \underline{\Lambda}_{m_i}(n_i^*)\} \le \lambda_i^* \le \bar{\Lambda}_{m_i}(n_i^*)$ *and* $\lambda_1^* + \lambda_2^* = \Lambda$*. Moreover,* $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \cup_{0 \le t \le P_{0_1}} \mathcal{F}_2(t)$ *with* $\mathcal{F}_2(t) = \{(\lambda_1, n_1; \lambda_2, n_2) | \pi_1(\lambda_1, n_1) = t$ *and* (λ_2, n_2) *optimizes problem* $P_2^t\}$*, where*

$$\begin{aligned} P_i^t: \quad & \max \quad \pi_i(\lambda_i, n_i) = \Big(v_i - \frac{h}{n_i\mu_i - \lambda_i}\Big)\lambda_i - rn_i \\ & s.t. \quad \pi_j(\lambda_j, n_j) = \Big(v_j - \frac{h}{n_j\mu_j - \lambda_j}\Big)\lambda_j - rn_j = t, \end{aligned}$$

and P_{0_i} *is the optimal objective value of problem* P_i^0.

(iii) $n_i^* = \underline{\Lambda}_{m_i}^{-1}(\lambda_i^*)$, $n_1^* + n_2^* < N$, $\lambda_i^* \geq \Lambda_{0_i}$ *and* $\lambda_1^* + \lambda_2^* = \Lambda$.

OS.3.2. Supplementary Results of Section 6.4

We define some auxiliary functions as follows: $\bar{\Lambda}_{m_w}(N)$ satisfies

$$v - hW(\bar{\Lambda}_{m_w}(N), N) - h\bar{\Lambda}_{m_w}(N)\frac{\partial W(\Lambda, N)}{\partial \Lambda}|_{\Lambda=\bar{\Lambda}_{m_w}(N)} = 0,$$

$\underline{\Lambda}_{m_w}^{-1}(\Lambda)$ satisfies

$$h\Lambda\frac{\partial W(\Lambda, N)}{\partial N}|_{N=\underline{\Lambda}_{m_w}^{-1}(\Lambda)} + r = 0,$$

$\hat{\Lambda}_{m_w}(N)$ satisfies

$$(v - hW(\hat{\Lambda}_{m_w}(N), N))\hat{\Lambda}_{m_w}(N) - rN = 0.$$

We define some notations as follows: $\underline{N}_w$ satisfies

$$(v - hW(\bar{\Lambda}_{m_w}(\underline{N}_w), \underline{N}_w))\bar{\Lambda}_{m_w}(\underline{N}_w) - r\underline{N}_w = 0,$$

$\bar{N}_w$ satisfies

$$(v - hW(\underline{\Lambda}_{m_w}(\bar{N}_w), \bar{N}_w))\underline{\Lambda}_{m_w}(\bar{N}_w) - r\bar{N}_w = 0,$$

Λ_{0_w} satisfies

$$(v - hW(\Lambda_{0_w}, \underline{\Lambda}_{m_w}^{-1}(\Lambda_{0_w})))\Lambda_{0_w} - r\underline{\Lambda}_{m_w}^{-1}(\Lambda_{0_w}) = 0.$$

LEMMA OS.13. *For* $i, j = 1, 2$ *and* $i \neq j$, *if* (p_i^*, w_i^*) *is the optimal solution of problem* $P'_{i_w}|_{(p_j^*, w_j^*)}$, *then we have*

$$p_i^* = v - hW(\lambda_i^*, n_i^*), \tag{OS.3}$$
$$w_i^* = \frac{rn_i^*}{\lambda_i^*}. \tag{OS.4}$$

Moreover, problem $P'_{i_w}|_{(p_j, w_j)}$ *is defined as follows:*

$$\begin{aligned} P'_{i_w}|_{(p_j,w_j)}: \quad & \max \quad \pi_i = (p_i - w_i)\lambda_i \\ & s.t. \quad (p_i, w_i) \in S_{i_w}|_{(p_j,w_j)}, \end{aligned}$$

where $S_w = \{(p_1, w_1; p_2, w_2)|(c1), (c2_w), (c3)\}$, $S_{1_w}|_{(p_2,w_2)} = \{(p, w)|(p, w; p_2, w_2) \in S_w\}$ *and* $S_{2_w}|_{(p_1,w_1)} = \{(p, w)|(p_1, w_1; p, w) \in S_w\}$.

LEMMA OS.14. *For* $i, j = 1, 2$ *and* $i \neq j$, *if* (λ_i^*, n_i^*) *is the optimal solution of problem* $P_{i_w}|_{(\lambda_j^*, n_j^*)}$, *then* $n_1^* + n_2^* = N$, *or* $\lambda_1^* + \lambda_2^* = \Lambda$, *or both hold. Moreover, problem* $P_{i_w}|_{(\lambda_j, n_j)}$ *is defined as follows:*

$$\begin{aligned} P_{i_w}|_{(\lambda_j,n_j)}: \quad & \max_{\lambda_i>0, n_i>0} \quad \pi_i = \Big(v - \frac{h}{n_i\mu - \lambda_i}\Big)\lambda_i - rn_i \\ & s.t. \quad N - n_j \geq n_i, \quad \Lambda - \lambda_j \geq \lambda_i, \quad v - hW(\lambda_i, n_i) - \frac{rn_i}{\lambda_i} \geq 0. \end{aligned}$$

PROPOSITION OS.2. *Consider the two-sided platform competition with the waiting time formulated by the form* (3). *Any vNM FSS* $\{s = (\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)\}$ *belongs to one of the following three cases.*

(i) $n_i^* \geq \underline{N}_w$, $n_1^* + n_2^* = N$, $\lambda_i^* = \bar{\Lambda}_{m_w}(n_i^*)$ *and* $\lambda_1^* + \lambda_2^* < \Lambda$.

(ii) $n_i^* \geq \underline{N}_w$, $n_1^*+n_2^*=N$, $\max\{\hat{\Lambda}_{m_w}(n_i^*), \underline{\Lambda}_{m_w}(n_i^*)\} \leq \lambda_i^* \leq \bar{\Lambda}_{m_w}(n_i^*)$ *and* $\lambda_1^*+\lambda_2^*=\Lambda$. *Moreover,* $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \cup_{0\leq t\leq P_{0_w}} \mathcal{F}_w(t)$ *with* $\mathcal{F}_w(t)=\{(\lambda_1,n_1;\lambda_2,n_2)|\pi_1(\lambda_1,n_1)=t$ *and* (λ_2,n_2) *optimizes problem* $P_w^t\}$, *where*

$$\begin{aligned} P_w^t: \qquad & \max \qquad \pi_2(\lambda_2,n_2)=(v-hW(\lambda_2,n_2))\lambda_2-rn_2 \\ & s.t. \qquad \pi_1(\lambda_1,n_1)=(v-hW(\lambda_1,n_1))\lambda_1-rn_1=t, \end{aligned}$$

and P_{0_w} *is the optimal objective value of problem* P_w^0.

(iii) $n_i^*=\underline{\Lambda}_{m_w}^{-1}(\lambda_i)$, $n_1^*+n_2^*<N$, $\lambda_i^*\geq\Lambda_{0_w}$ *and* $\lambda_1^*+\lambda_2^*=\Lambda$.

OS.3.3. Proofs of Proposition 11 and Corollary 6

Proof of Proposition 11. (i) We first show $W(\lambda_1^m,n_1^m)\lambda_1^m+W(\lambda_2^m,n_2^m)\lambda_2^m \geq W(\lambda_1^m+\lambda_2^m, n_1^m+n_2^m))(\lambda_1^m+\lambda_2^m)$. Let $\rho_1=\lambda_1^m/(n_1^m\mu)$, $\rho_2=\lambda_2^m/(n_2^m\mu)$ and $\rho=\lambda/(n\mu)$. We have $W(\lambda_1^m,n_1^m)\lambda_1^m+W(\lambda_2^m,n_2^m))\lambda_2^m=\frac{\rho_1^{\sqrt{2(n_1^m+1)}}}{1-\rho_1}+\frac{\lambda_1^m}{\mu}+\frac{\rho_2^{\sqrt{2(n_2^m+1)}}}{1-\rho_2}+\frac{\lambda_2^m}{\mu}\geq\frac{n_1^m}{n}\frac{\rho_1^{\sqrt{2(n+1)}}}{1-\rho_1}+\frac{n_2^m}{n}\frac{\rho_2^{\sqrt{2(n+1)}}}{1-\rho_2}+\frac{\lambda_1^m+\lambda_2^m}{\mu}\geq\frac{\rho^{\sqrt{2(n+1)}}}{1-\rho}+\frac{\lambda_1^m+\lambda_2^m}{\mu}=W(\lambda_1^m+\lambda_2^m,n_1^m+n_2^m))(\lambda_1^m+\lambda_2^m)$, where the first and last equalities are due to (3), the first inequality is due to $\rho_1<1$, $\rho_2<1$, $n_1^m\leq n$, $n_2^m\leq n$, the second inequality is due to Jensen's inequality (we can verify $\frac{\rho^{\sqrt{2(n+1)}}}{1-\rho}$ is convex about ρ) and $\frac{n_1^m\rho_1}{n}+\frac{n_2^m\rho_2}{n}=\rho$.

We now show the total profit of any merged system is dominated by a monopoly system. That is $(p_1^m-w_1^m)\lambda_1^m+(p_2^m-w_2^m)\lambda_2^m\leq(v-hW(\lambda_1^m,n_1^m))\lambda_1^m-rn_1^m+(v-hW(\lambda_2^m,n_2^m))\lambda_2^m-rn_2^m\leq(v-hW(\lambda,n))\lambda-rn$, where the first inequality is due to $\frac{\lambda_1^m w_1^m}{n_1^m}=\frac{\lambda_2^m w_2^m}{n_2^m}\geq r$ and $v-p_1^m-hW(\lambda_1^m,n_1^m)=v-p_2^m-hW(\lambda_2^m,n_2^m)\geq 0$, and the second inequality is due to $\lambda=\lambda_1^m+\lambda_2^m$, $n=n_1^m+n_2^m$, and $W(\lambda_1^m,n_1^m)\lambda_1^m+W(\lambda_2^m,n_2^m))\lambda_2^m\geq W(\lambda_1^m+\lambda_2^m,n_1^m+n_2^m))(\lambda_1^m+\lambda_2^m)$. Let $v-p-hW(\lambda,n)=0$ and $\frac{\lambda w}{n}=r$, then $(p-w)\lambda=(v-hW(\lambda,n))\lambda-rn$ is the profit of a monopoly with price p and wage w.

(ii) With the same proof of Proposition 4 by replacing $\frac{1}{n\mu-\lambda}$ with $W(\lambda,n)$ (because the proof doesn't depend on the scheme of the expected waiting time), we get Proposition 11(ii). The Table 4 is obtained by replacing $\bar{\Lambda}_m(N)$, $\underline{\Lambda}_m(N)$, $\hat{\Lambda}_m(N)$, $\underline{N}$, $\bar{N}$, Λ_0 in Table 3 with $\bar{\Lambda}_{m_w}(N)$, $\underline{\Lambda}_{m_w}(N)$, $\hat{\Lambda}_{m_w}(N)$, $\underline{N}_w$, $\bar{N}_w$, Λ_{0_w}, respectively. $\square$

Proof of Corollary 6. To establish the proof, we consider the three cases for the competition case as in Proposition OS.2. Then, for each case, we further discuss the three possible equilibria for the merging case.

We first prove $W(\bar{\Lambda}_{m_w}(n),n)$ is decreasing in n. Let $a(n)=\sqrt{2(n+1)}$ and $\rho=\lambda/(n\mu)$, then $W(\lambda,n)=\frac{\rho^{a(n)-1}}{n\mu(1-\rho)}+\frac{1}{\mu}$ by (3). Consequently, $(v-hW(\lambda,n))\lambda=n\mu\rho(v-h/\mu)-\frac{h\rho^{a(n)}}{1-\rho}$, which is maximized at $\rho=\rho(n)=\bar{\Lambda}_{m_w}(n)/(n\mu)$. Furthermore, the FOC is $\frac{\rho^{a(n)-1}[a(n)-(a(n)-1)\rho(n)]}{n(1-\rho(n))^2}=v\mu/h-1$, where the leaf hand of it is decreasing in n and increasing in $\rho(n)$. Then, we get $\rho(n)$ is increasing in n. Plugging the FOC into $W(\lambda,n)=\frac{\rho^{a(n)-1}}{n\mu(1-\rho)}+\frac{1}{\mu}$, we get $W(\bar{\Lambda}_{m_w}(n),n)=\frac{v/h-1/\mu}{a(n)+\frac{\rho(n)}{1-\rho(n)}}+\frac{1}{\mu}$, which is decreasing in n because $a(n)=\sqrt{2(n+1)}$ is increasing in n and we can also prove $\rho(n)$ is increasing in n by some algebra.

We then prove $W(\lambda,\underline{\Lambda}_{m_w}^{-1}(\lambda))$ is decreasing in λ. The FOC of $(v-hW(\lambda,n))\lambda-rn$ with respect to n is $h\mu\rho\frac{\rho^{a(n)-1}}{n\mu(1-\rho)}\left(a(n)+\frac{\rho}{1-\rho}-\frac{n\ln\rho}{a(n)}\right)=r$. Plugging the FOC into $W(\lambda,n)=\frac{\rho^{a(n)-1}}{n\mu(1-\rho)}+\frac{1}{\mu}$, we get $W(\lambda,\underline{\Lambda}_{m_w}^{-1}(\lambda))=\frac{r}{h\mu\rho(\lambda)\left(a(\underline{\Lambda}_{m_w}^{-1}(\lambda))+\frac{\rho(\lambda)}{1-\rho(\lambda)}-\frac{\underline{\Lambda}_{m_w}^{-1}(\lambda)\ln\rho(\lambda)}{a(\underline{\Lambda}_{m_w}^{-1}(\lambda))}\right)}+\frac{1}{\mu}$, where $\rho(\lambda)=\lambda/(\underline{\Lambda}_{m_w}^{-1}(\lambda)\mu)$. We have $W(\lambda,\underline{\Lambda}_{m_w}^{-1}(\lambda))$ is decreasing in λ because we can prove $a(\underline{\Lambda}_{m_w}^{-1}(\lambda))$ and $\rho(\lambda)$ are increasing in λ by some algebra.

Case I: $n_i^* \geq \underline{N}_w$, $n_1^*+n_2^*=N$, $\lambda_i^*=\bar{\Lambda}_{m_w}(n_i^*)$. From Proposition OS.2(i), a symmetric farsightedly stable outcome satisfies $n_i^*=N/2$ and $\lambda_i^*=\bar{\Lambda}_{m_w}(N/2)$, $i=1,2$. Based on (OS.3) and (OS.4), the price and wage are $p_i^*=v-hW(\bar{\Lambda}_{m_w}(N/2),N/2)$, and $w_i^*=\frac{rN/2}{\bar{\Lambda}_{m_w}(N/2)}$. Thus, the total profit from the two platforms is $\pi_1^*+\pi_2^*=(p_1^*-w_1^*)\lambda_1^*+(p_2^*-w_2^*)\lambda_2^*=p_i^*(\lambda_1^*+\lambda_2^*)-r(n_1^*+n_2^*)=p_i^*(\lambda_1^*+\lambda_2^*)-rN$.

Regarding the merged system, three possible cases are all possible. Hence, we derive the optimal solutions and the profit for the merging case in three subcases, and then conduct the comparison.

Subcase (a): Ample demand and limited supply. By Table 4, $n_m^*=N=n_1^*+n_2^*$ and $\lambda_m^*=\bar{\Lambda}_{m_w}(N)$. Moreover, $p_m^*=v-hW(\bar{\Lambda}_{m_w}(N),N)$, and $w_m^*=\frac{rN}{\bar{\Lambda}_{m_w}(N)}$. Then, $p_m^*=v-hW(\bar{\Lambda}_{m_w}(N),N)>v-hW(\bar{\Lambda}_{m_w}(N/2),N/2)=p_i^*$ because we have proved $W(\bar{\Lambda}_{m_w}(n),n)$ is decreasing in n in former analysis. Moreover, $\lambda_i^*=n_i^*\mu\rho(n_i^*)=N\mu\rho(N/2)/2<N\mu\rho(N)/2$ due to $\rho(n)$ is increasing in n. We get $\lambda_m^*=N\mu\rho(N)>\lambda_1^*+\lambda_2^*$ immediately. Thus, $\pi_m^*=(p_m^*-w_m^*)\lambda_m^*=p_m^*\lambda_m^*-rN>p_i^*(\lambda_1^*+\lambda_2^*)-rN=\pi_1^*+\pi_2^*$, where the inequality holds from $p_m^*>p_i^*$ and $\lambda_m^*>\lambda_1^*+\lambda_2^*$.

Subcase(b): Balanced demand and supply. By Table 4, $n_m^*=N=n_1^*+n_2^*$ and $\lambda_1^*+\lambda_2^*<\lambda_m^*=\Lambda\leq\bar{\Lambda}_{m_w}(N)$. Moreover, $p_m^*=v-hW(\Lambda,N)\geq v-hW(\bar{\Lambda}_{m_w}(N),N)>v-hW(\bar{\Lambda}_{m_w}(N/2),N/2)=p_i^*$, where the first inequality is due to $W(\lambda,n)$ is increasing in λ and $\Lambda\leq\bar{\Lambda}_{m_w}(N)$, the second inequality is due to $W(\bar{\Lambda}_{m_w}(n),n)$ is decreasing in n. And, $w_m^*=\frac{rn_m^*}{\lambda_m^*}=\frac{rN}{\Lambda}<\frac{rN/2}{\lambda_i^*}=\frac{rn_i^*}{\lambda_i^*}=w_i^*$, where the inequality holds because $\Lambda>2\lambda_i^*$ and $N=2n_i^*$. Thus, $\pi_m^*=(p_m^*-w_m^*)\lambda_m^*=p_m^*\lambda_m^*-rN>p_i^*(\lambda_1^*+\lambda_2^*)-rN=\pi_1^*+\pi_2^*$, where the inequality holds from $p_m^*>p_i^*$ and $\lambda_m^*>\lambda_1^*+\lambda_2^*$.

Subcase(c): Limited demand and ample supply. By Table 4, $n_m^*=\underline{\Lambda}_{m_w}^{-1}(\Lambda)<n_1^*+n_2^*=N$ and $\lambda_m^*=\Lambda>\lambda_1^*+\lambda_2^*$. Moreover, $p_m^*=v-hW(\Lambda,\underline{\Lambda}_{m_w}^{-1}(\Lambda))>v-hW(\bar{\Lambda}_{m_w}(N/2),\underline{\Lambda}_{m_w}^{-1}(\bar{\Lambda}_{m_w}(N/2)))\geq v-hW(\bar{\Lambda}_{m_w}(N/2),N/2)=p_i^*$, where the first inequality is due to $W(\lambda,\underline{\Lambda}_{m_w}^{-1}(\lambda))$ is decreasing in λ and $\lambda_i^*=\bar{\Lambda}_{m_w}(N/2)<\Lambda$, the second inequality is due to $W(\lambda,n)$ is decreasing in n, $\bar{\Lambda}_{m_w}(n)\geq\underline{\Lambda}_{m_w}(n)$ for $n\geq\underline{N}_w$ and $N/2=n_i^*\geq\underline{N}_w$. And, $w_m^*=\frac{rn_m^*}{\lambda_m^*}<\frac{rN}{\Lambda}<\frac{rN}{2\lambda_i^*}=\frac{rn_i^*}{\lambda_i^*}=w_i^*$, where the inequality is due to $\Lambda>\lambda_1^*+\lambda_2^*=2\lambda_i^*$. Thus, $\pi_m^*=(p_m^*-w_m^*)\lambda_m^*=p_m^*\lambda_m-rn_m^*>p_i^*(\lambda_1^*+\lambda_2^*)-rN=\pi_1^*+\pi_2^*$, where the inequality is due to $p_m^*>p_i^*$, $n_m^*<N$ and $\lambda_m^*>\lambda_1^*+\lambda_2^*$.

Case II. $n_i^*\geq\underline{N}_w$, $n_1^*+n_2^*=N$, $\max\{\hat{\Lambda}_{m_w}(n_i^*),\underline{\Lambda}_{m_w}(n_i^*)\}\leq\lambda_i^*\leq\bar{\Lambda}_{m_w}(n_i^*)$ and $\lambda_1^*+\lambda_2^*=\Lambda$. From Proposition OS.2(ii), a symmetric solution satisfies $n_i^*=N/2$ and $\lambda_i^*=\Lambda/2$, $i=1,2$. Based on (OS.3) and (OS.4), $p_i^*=v-hW(\lambda_i^*,n_i^*)=v-hW(\Lambda/2,N/2)$, $w_i^*=\frac{rn_i^*}{\lambda_i^*}=\frac{rN}{\Lambda}$. Consequently, $\pi_1^*+\pi_2^*=(p_1^*-w_1^*)\lambda_1^*+(p_2^*-w_2^*)\lambda_2^*=p_i^*(\lambda_1^*+\lambda_2^*)-r(n_1^*+n_2^*)=p_i^*\Lambda-rN$.

Regarding the merged system, three possible cases are all possible. Hence, we derive the optimal solutions and the profit for the merging case in three subcases, and then conduct the comparison.

Subcase (a): Ample demand and limited supply under merging case. By Table 4, $n_m^*=N$ and $\lambda_m^*=\bar{\Lambda}_{m_w}(N)<\Lambda$. Note that in this case we have $\Lambda=\lambda_1^*+\lambda_2^*<2\bar{\Lambda}_{m_w}(N/2)$, and we can prove $\bar{\Lambda}_{m_w}(N)>2\bar{\Lambda}_{m_w}(N/2)$ by some algebra. This is a contradiction.

Subcase (b): Balanced demand and supply. By Table 4, $n_m^*=N=n_1^*+n_2^*$ and $\lambda_m^*=\Lambda=\lambda_1^*+\lambda_2^*$. Moreover, $p_m^*=v-hW(\Lambda,N)=v-h\frac{(\Lambda/(N\mu))^{\sqrt{2(N+1)}}}{\Lambda(1-\Lambda/(N\mu))}-\frac{h}{\mu}>v-h\frac{2(\Lambda/(N\mu))^{\sqrt{2(N/2+1)}}}{\Lambda(1-\Lambda/(N\mu))}-\frac{h}{\mu}=v-hW(\Lambda/2,N/2)=p_i^*$, due to $\Lambda<N\mu$, and $w_m^*=\frac{rN}{\Lambda}=w_i^*$. Thus, $\pi_m^*=(p_m^*-w_m^*)\lambda_m^*=p_m^*\lambda^*-rN>p_i^*\Lambda-rN=\pi_1^*+\pi_2^*$.

Subcase (c): Limited demand and ample supply. By Table 4, $n_m^*=\underline{\Lambda}_{m_w}^{-1}(\Lambda)<n_1^*+n_2^*=N$ and $\lambda_m^*=\Lambda=\lambda_1^*+\lambda_2^*$. Moreover, $p_m^*=v-hW(\lambda_m^*,n_m^*)=v-hW(\Lambda,\underline{\Lambda}_{m_w}^{-1}(\Lambda))>v-hW(\Lambda/2,\underline{\Lambda}_{m_w}^{-1}(\Lambda/2))\geq v-hW(\Lambda/2,N/2)=p_i^*$, where the first inequality is due to $W(\lambda,\underline{\Lambda}_{m_w}^{-1}(\lambda))$ is decreasing in λ, the second inequality is due to $W(\lambda,n)$ is decreasing in n, $\lambda_i^*\geq\underline{\Lambda}_{m_w}(n_i^*)$, $\lambda_i^*=\Lambda/2$

and $n_i^* = N/2$. And $w_m^* = \frac{rn_m^*}{\lambda_m^*} = \frac{r\underline{\Lambda}_{mw}^{-1}(\Lambda)}{\Lambda} < \frac{rN}{\Lambda} = w_i^*$ due to $\Lambda < \underline{\Lambda}_{mw}(N)$ from Table 4. Thus, $\pi_m^* = (p_m^* - w_m^*)\lambda_m^* = p_m^*\Lambda - rn_m^* > p_i^*\Lambda - rN = \pi_1^* + \pi_2^*$, due to $p_m^* > p_i^*$ and $n_m^* < N$.

Case III. $n_i^* = \underline{\Lambda}_{mw}^{-1}(\lambda_i)$, $n_1^* + n_2^* < N$, $\lambda_i^* \geq \Lambda_{0_w}$ and $\lambda_1^* + \lambda_2^* = \Lambda$. From Proposition OS.2(iii), a symmetric solution shall satisfy $n_i^* = \underline{\Lambda}_{mw}^{-1}(\Lambda/2)$ and $\lambda_i^* = \Lambda/2$, $i = 1, 2$. Based on (OS.3) and (OS.4), $p_i^* = v - hW(\lambda_i^*, n_i^*) = v - hW(\Lambda/2, \underline{\Lambda}_{mw}^{-1}(\Lambda/2))$, $w_i^* = \frac{rn_i^*}{\lambda_i^*} = \frac{2r\underline{\Lambda}_{mw}^{-1}(\Lambda/2)}{\Lambda}$. Consequently, $\pi_1^* + \pi_2^* = (p_1^* - w_1^*)\lambda_1^* + (p_2^* - w_2^*)\lambda_2^* = p_i^*(\lambda_1^* + \lambda_2^*) - r(n_1^* + n_2^*) = p_i^*\Lambda - r(n_1^* + n_2^*)$.

Regarding the merging setting, we first show that in this case the merging platform will reach limited demand and ample supply. By Table 4, it is equivalent to show $N > \bar{N}_w$ and $\Lambda_{0w} \leq \Lambda < \underline{\Lambda}_{mw}(N)$. First, it is easy to see $\Lambda > \lambda_i^* \geq \Lambda_{0_w}$. Second, $N > n_1^* + n_2^* = 2\underline{\Lambda}_{mw}^{-1}(\Lambda/2)$. We can show $\underline{\Lambda}_{mw}^{-1}(\Lambda) < 2\underline{\Lambda}_{mw}^{-1}(\Lambda/2)$ by some algebra. Thus, we have $N > \underline{\Lambda}_{mw}^{-1}(\Lambda)$, which implies $\Lambda < \underline{\Lambda}_{mw}(N)$. Last, $\Lambda_{0w} < \underline{\Lambda}_{mw}(N)$ implies $N > \bar{N}_w$.

Then we have $n_m^* = \underline{\Lambda}_{mw}^{-1}(\lambda_m^*) = \underline{\Lambda}_{mw}^{-1}(\Lambda)$ and $\lambda_m^* = \Lambda = \lambda_1^* + \lambda_2^*$ by Table 4. Moreover, $p_m^* = v - hW(\lambda_m^*, n_m^*) = v - hW(\Lambda, \underline{\Lambda}_{mw}^{-1}(\Lambda)) > v - hW(\Lambda/2, \underline{\Lambda}_{mw}^{-1}(\Lambda/2)) = p_i^*$ due to $W(\lambda, \underline{\Lambda}_{mw}^{-1}(\lambda))$ is decreasing in λ. Because $\underline{\Lambda}_{mw}^{-1}(\Lambda) < 2\underline{\Lambda}_{mw}^{-1}(\Lambda/2)$, $n_m^* = \underline{\Lambda}_{mw}^{-1}(\Lambda) < 2\underline{\Lambda}_{mw}^{-1}(\Lambda/2) = n_1^* + n_2^*$. And $w_m^* = \frac{rn_m^*}{\lambda_m^*} = \frac{rn_m^*}{\Lambda} < \frac{2rn_i^*}{\Lambda} = w_i^*$. Thus, $\pi_m^* = (p_m^* - w_m^*)\lambda_m^* = p_m^*\Lambda - rn_m^* > p_i^*\Lambda - r(n_1^* + n_2^*) = \pi_1^* + \pi_2^*$, where the inequality is due to $p_m^* > p_i^*$ and $n_m^* < n_1^* + n_2^*$. $\square$

OS.4. Proofs in Online Appendix

Proof of Lemma OA.1. We first prove Lemma OA.1(ii). Similar to the proof of Lemma OA.4, we first construct a problem by relaxing the "binding" constraints in problem $P_i'|_{(p_j, w_j)}$ as follows:

$$
\begin{aligned}
\bar{P}_i'|_{(p_j,w_j)}: \quad \max_{p_i, w_i} \quad & \pi_i = (p_i - w_i)\lambda_i \\
& \frac{\lambda_1 w_1}{n_1} = \frac{\lambda_2 w_2}{n_2} \geq r, \\
& v - p_1 - \frac{h}{n_1\mu - \lambda_1} = v - p_2 - \frac{h}{n_2\mu - \lambda_2} \geq 0, \\
& n_1 + n_2 \leq N, \quad \lambda_1 + \lambda_2 \leq \Lambda, \\
& n_i > 0, \quad \lambda_i > 0, \quad p_i > 0, \quad w_i > 0, \quad p_i - w_i \geq 0, \quad i = 1, 2.
\end{aligned}
\tag{OS.5}
$$

Then, it suffices to show that the optimal solutions of the problem $\bar{P}_i'|_{(p_j,w_j)}$ satisfy Lemma OA.1.

Without loss of generality, we consider the case with p_2 and w_2 given and show the optimal solution of problem $\bar{P}_1'|_{(p_2,w_2)}$ satisfy the conditions in Lemma OA.1. It is sufficient to show that the first-order necessary condition of problem $\bar{P}_1'|_{(p_2,w_2)}$ satisfy the conditions in Lemma OA.1. We define the Lagrangian function for the problem $\bar{P}_1'|_{(p_2,w_2)}$ as follows:

$$
\begin{aligned}
L_d = & (p_1 - w_1)\lambda_1 + \eta_1\Big(\frac{\lambda_1 w_1}{n_1} - \frac{\lambda_2 w_2}{n_2}\Big) + \eta_2\Big(p_1 + \frac{h}{n_1\mu - \lambda_1} - p_2 - \frac{h}{n_2\mu - \lambda_2}\Big) + \eta_3\Big(\frac{\lambda_1 w_1}{n_1} - r\Big) \\
& + \eta_4\Big(v - \frac{h}{n_1\mu - \lambda_1} - p_1\Big) + \eta_5(N - n_1 - n_2) + \eta_6(\Lambda - \lambda_1 - \lambda_2) + \eta_7(p_1 - w_1),
\end{aligned}
$$

where $\eta_3 \geq 0$, $\eta_4 \geq 0$, $\eta_5 \geq 0$, $\eta_6 \geq 0$, $\eta_7 \geq 0$. The optimal solution of problem $\bar{P}_1'|_{(p_2,w_2)}$, denoted as (p_1^*, w_1^*), shall satisfy the first-order conditions of the Lagrangian function, i.e.,

$$
\frac{\partial L_d}{\partial p_1}\Big|_{p_1 = p_1^*, w_1 = w_1^*} = \lambda_1^* + \eta_2 - \eta_4 + \eta_7 = 0,
$$

$$
\begin{aligned}
\frac{\partial L_d}{\partial w_1}\Big|_{p_1=p_1^*,w_1=w_1^*} &= -\lambda_1^* + (\eta_1+\eta_3)\frac{\lambda_1^*}{n_1^*} - \eta_7 = 0,\\
\frac{\partial L_d}{\partial n_1}\Big|_{p_1=p_1^*,w_1=w_1^*} &= -(\eta_1+\eta_3)\frac{\lambda_1^* w_1^*}{n_1^{*2}} - (\eta_2-\eta_4)\frac{h\mu}{(n_1^*\mu-\lambda_1^*)^2} - \eta_5 = 0,\\
\frac{\partial L_d}{\partial n_2^*}\Big|_{p_1=p_1^*,w_1=w_1^*} &= \eta_1\frac{\lambda_2^* w_2}{n_2^{*2}} + \eta_2\frac{h\mu}{(n_2^*\mu-\lambda_2^*)^2} - \eta_5 = 0,\\
\frac{\partial L_d}{\partial \lambda_1}\Big|_{p_1=p_1^*,w_1=w_1^*} &= p_1^* - w_1^* + (\eta_1+\eta_3)\frac{w_1^*}{n_1^*} + (\eta_2-\eta_4)\frac{h}{(n_1^*\mu-\lambda_1^*)^2} - \eta_6 = 0,\\
\frac{\partial L_d}{\partial \lambda_2}\Big|_{p_1=p_1^*,w_1=w_1^*} &= -\eta_1\frac{w_2}{n_2^*} - \eta_2\frac{h}{(n_2^*\mu-\lambda_2^*)^2} - \eta_6 = 0.
\end{aligned}
$$

Note that in the above equations, n_i^* and λ_i^* are the corresponding values given p_1^* and w_1^*. Solving the above equation set, we obtain the Lagrangian multipliers satisfying the following equations.

$$\eta_1 = -\frac{n_2^{*2}\mu}{w_2(n_2^*\mu-\lambda_2^*)}\big(\frac{\eta_5}{\mu}+\eta_6\big), \tag{OS.6}$$

$$\eta_2 = \frac{n_2^*(n_2^*\mu-\lambda_2^*)}{h}\big(\eta_5+\frac{\lambda_2^*}{n_2^*}\eta_6\big), \tag{OS.7}$$

$$\eta_3 = \frac{(\lambda_1+\eta_7)n_1^*}{\lambda_1^*} - \eta_1 \geq 0, \tag{OS.8}$$

$$\eta_4 = \lambda_1^* + \eta_2 + \eta_7 \geq 0, \tag{OS.9}$$

$$\eta_5 = -(\lambda_1^*+\eta_7)\big(\frac{w_1^*}{n_1^*} - \frac{h\mu}{(n_1^*\mu-\lambda_1^*)^2}\big) \geq 0, \tag{OS.10}$$

$$\eta_6 = p_1^* - w_1^* + (\lambda_1^*+\eta_7)\big(\frac{w_1^*}{\lambda_1^*} - \frac{h}{(n_1^*\mu-\lambda_1^*)^2}\big) \geq 0. \tag{OS.11}$$

Besides the first-order necessary conditions, the following complementary slackness conditions shall also be satisfied. $\eta_3\big(\frac{\lambda_1^* w_1^*}{n_1^*} - r\big) = 0$, $\eta_4\big(v - \frac{h}{n_1^*\mu-\lambda_1^*} - p_1^*\big) = 0$. Thus, to establish the proof, it is equivalent to show that $\eta_3 > 0$ and $\eta_4 > 0$.

We first consider η_3. In (OS.8), the Lagrangian multiplier $\eta_7 \geq 0$, and an optimal solution requires $\lambda_1^* > 0$ and $n_1^* > 0$. Thus, we have $\eta_3 > 0$ if $\eta_1 \leq 0$, which is valid from (OS.6), where $\eta_5 \geq 0$, $\eta_6 \geq 0$, and $n_2^*\mu > \lambda_2^*$. We then consider η_4. In (OS.9), the Lagrangian multiplier $\eta_7 \geq 0$, and an optimal solution requires $\lambda_1^* > 0$. Thus, we have $\eta_4 > 0$ if $\eta_2 \geq 0$, which is valid from (OS.7), where $\eta_5 \geq 0$, $\eta_6 \geq 0$, and $n_2^*\mu > \lambda_2^*$.

We then prove Lemma OA.1(i). For a feasible strategy profile $s = (p_1, w_1; p_2, w_2)$, it suffices to show that if (p_i, w_i) is not the optimal solution of problem $P_i'|_{(p_j,w_j)}$ for $i, j = 1, 2$ and $i \neq j$, then s cannot be Pareto efficient, i.e., dominated by some strategy profile $s' \in S$. Suppose that s is not optimal of problem $P_1'\&P_2'$, then from Lemma OA.1(ii), we have neither (1) nor (2) holds, or both (1) and (2) hold but there exists a $i \in \{1, 2\}$ such that (p_i, w_i) is not an optimal solution of problem $P_i'|_{(p_j,w_j)}$, where $j \neq i$. We then construct an alternative strategy profile $s' \in S$ which dominates s for the three cases, respectively. For ease of presentation, in this proof we denote $\pi_i(x)$, $i = 1, 2$, as platform i's profit given that the strategy profile is x.

Case I: $\frac{\lambda_1 w_1}{n_1} = \frac{\lambda_2 w_2}{n_2} > r$. In this case, we let $s' = (p_1, w_1'; p_2, w_2')$ which satisfies $\frac{\lambda_1 w_1'}{n_1} = \frac{\lambda_2 w_2'}{n_2} = r$. It is easy to check that $w_1' < w_1$ and $w_2' < w_2$. Then, we have $\pi_i(s') = (p_i - w_i')\lambda_i > (p_i - w_i)\lambda_i = \pi_i(s)$ for $i = 1, 2$. Thus, the strategy profile s is dominated by s'.

Case II: $v-p_1-\frac{h}{n_1\mu-\lambda_1}=v-p_2-\frac{h}{n_2\mu-\lambda_2}>0$. In this case, we let $s'=(p_1',w_1;p_2',w_2)$ which satisfies $v-p_1'-\frac{h}{n_1\mu-\lambda_1}=v-p_2'-\frac{h}{n_2\mu-\lambda_2}=0$. It is easy to check that $p_1'>p_1$ and $p_2'>p_2$. Then, we have $\pi_i(s')=(p_i'-w_i)\lambda_i>(p_i-w_i)\lambda_i=\pi_i(s)$. Thus, the strategy profile s is dominated by s'.

Case III: $\frac{\lambda_1 w_1}{n_1}=\frac{\lambda_2 w_2}{n_2}=r$, $v-p_1-\frac{h}{n_1\mu-\lambda_1}=v-p_2-\frac{h}{n_2\mu-\lambda_2}=0$, and there exists a $i\in\{1,2\}$ such that (p_i,w_i) is not optimal solution of problem $P_i'|_{(p_j,w_j)}$, where $j\neq i$. Without loss of generality, we assume (p_1,w_1) is not optimal solution of problem $P_1'|_{(p_2,w_2)}$. Denote (p_1^*,w_1^*) as the optimal solution to problem $P_1'|_{(p_2,w_2)}$. We let $s'=(p_1^*,w_1^*;p_2,w_2)$. Then, $\pi_1(s')=(p_1^*-w_1^*)\lambda_1^*>(p_1-w_1)\lambda_1=\pi_1(s)$, and $\pi_2(s')=(p_2-w_2)\lambda_2=\pi_2(s)$. Thus, the strategy profile s is dominated by s'. □

Proof of Lemma OA.2. We establish the proof by contradiction. First recall the Lagrangian function in the proof of Lemma OA.1, the complementary slackness conditions require

$$\eta_5(N-n_1-n_2)=0, \tag{OS.12}$$
$$\eta_6(\Lambda-\lambda_1-\lambda_2)=0. \tag{OS.13}$$

Suppose $n_1^*+n_2^*<N$ and $\lambda_1^*+\lambda_2^*<\Lambda$. Then, from (OS.12) and (OS.13), we must have $\eta_5=0$ and $\eta_6=0$. Substituting $\eta_5=0$ to (OS.10), we obtain $\frac{w_1^*}{n_1^*\mu}=\frac{h}{(n_1^*\mu-\lambda_1^*)^2}$. Substituting $\eta_6=0$ to (OS.11) and using the above equation, we get $p_1^*=w_1^*-(\lambda_1^*+\eta_7)\left[\frac{w_1^*}{\lambda_1^*}-\frac{h}{(n_1^*\mu-\lambda_1^*)^2}\right]=w_1^*-(\lambda_1^*+\eta_7)\left(\frac{w_1^*}{\lambda_1^*}-\frac{w_1^*}{n_1^*\mu}\right)=w_1^*-\frac{(\lambda_1^*+\eta_7)w_1^*}{\lambda_1^*n_1^*\mu}(n_1^*\mu-\lambda_1^*)<w_1^*$, where the last inequality holds from $n_1^*\mu-\lambda_1^*>0$. This is a contradiction. Therefore, we get $n_1^*+n_2^*=N$ or $\lambda_1^*+\lambda_2^*=\Lambda$. □

Proof of Lemma OA.3. We apply the results in Proposition 4 to establish the proof.

Case I: $n_1^*+n_2^*=N$ and $\lambda_1^*+\lambda_2^*<\Lambda$, i.e., $n_i^*=N-n_j^*$ and $\lambda_i^*<\Lambda-\lambda_j^*$. This case corresponds to Case I in the proof of Proposition 4. Then, from Case I in the proof of Proposition 4, we have the following results: $\lambda_i^*=\bar\Lambda_m(N-n_j^*)=\bar\Lambda_m(n_i^*)$, and

$$n_i^*=N-n_j^*\geq \underline{N}, \tag{OS.14}$$
$$\Lambda-\lambda_j^*>\bar\Lambda_m(N-n_j^*)=\bar\Lambda_m(n_i^*)=\lambda_i^*. \tag{OS.15}$$

Furthermore, we verify the above inequalities lead to $N\geq 2\underline{N}$ and $\Lambda>\bar\Lambda(N)$. Regarding N, from (OS.14), we have $N=n_1^*+n_2^*\geq 2\underline{N}$. Regarding Λ, from (OS.15), we have $\Lambda>\lambda_1^*+\lambda_2^*=\bar\Lambda_m(n_1^*)+\bar\Lambda_m(n_2^*)=\bar\Lambda_m(n_1^*)+\bar\Lambda_m(N-n_1^*)$. By Lemma OS.4(i), $\bar\Lambda_m(n_1^*)+\bar\Lambda_m(N-n_1^*)\geq\bar\Lambda(N)$. Therefore, if $N\geq 2\underline{N}$ and $\Lambda>\bar\Lambda(N)$, then $n_1^*+n_2^*=N$, $n_i^*\geq\underline{N}$, $\lambda_i^*=\bar\Lambda_m(n_i^*)$.

Case II: $n_1^*+n_2^*=N$ and $\lambda_1^*+\lambda_2^*=\Lambda$, i.e., $n_i^*=N-n_j^*$ and $\lambda_i^*=\Lambda-\lambda_j^*$. This case corresponds to Case II in the proof of Proposition 4. Then, from Case II in the proof of Proposition 4, we have either

$$\underline{N}\leq n_i^*=N-n_j^*\leq\bar N, \tag{OS.16}$$
$$\hat\Lambda_m(n_i^*)=\hat\Lambda_m(N-n_j^*)\leq\lambda_i^*=\Lambda-\lambda_j^*\leq\bar\Lambda_m(N-n_j^*)=\bar\Lambda_m(n_i^*), \tag{OS.17}$$

or

$$n_i^*=N-n_j^*>\bar N, \tag{OS.18}$$
$$\underline{\Lambda}_m(n_i^*)=\underline{\Lambda}_m(N-n_j^*)\leq\lambda_i^*=\Lambda-\lambda_j^*\leq\bar\Lambda_m(N-n_j^*)=\bar\Lambda_m(n_i^*). \tag{OS.19}$$

By Lemma OS.1, (OS.17) and (OS.19) imply $\max\{\hat\Lambda_m(n_i^*),\underline{\Lambda}_m(n_i^*)\}\leq\lambda_i^*\leq\bar\Lambda_m(n_i^*)$. Furthermore, we verify (OS.16) and (OS.17) lead to $2\underline{N}\leq N\leq 2\overline{N}$, $\hat\Lambda(N)\leq\Lambda\leq\bar\Lambda'(N)$, (OS.18) and (OS.19) lead to $N > 2\overline{N}$, $\underline{\Lambda}(N) \le \Lambda \le \bar{\Lambda}'(N)$. From (OS.16), we have $2\underline{N} = 2\underline{N} \le N = n_1^* + n_2^* \le 2\underline{N}$. From (OS.17), we have $\hat{\Lambda}_m(n_1^*) + \hat{\Lambda}_m(N - n_1^*) \le \Lambda \le \bar{\Lambda}_m(n_1^*) + \bar{\Lambda}_m(N - n_1^*)$. Hence, $\hat{\Lambda}(N) \le \Lambda \le \bar{\Lambda}'(N)$ by Lemma OS.4(i) and (ii).

Now we show (OS.18) and (OS.19) lead to $N > 2\overline{N}$, $\underline{\Lambda}(N) \le \Lambda \le \bar{\Lambda}'(N)$. From (OS.18), we have $N = n_1^* + n_2^* > 2\underline{N}$. From (OS.19), we have $\underline{\Lambda}_m(n_1^*) + \underline{\Lambda}_m(N - n_1^*) \le \Lambda \le \bar{\Lambda}_m(n_1^*) + \bar{\Lambda}_m(N - n_1^*)$. Hence, $\underline{\Lambda}(N) \le \Lambda \le \bar{\Lambda}'(N)$ by Lemma OS.4(i) and (iii).

Case III: $n_1^* + n_2^* < N$ and $\lambda_1^* + \lambda_2^* = \Lambda$, i.e., $n_i^* < N - n_j^*$ and $\lambda_i^* = \Lambda - \lambda_j^*$. This case corresponds to Case III in the proof of Proposition 4. Then, from Case III in the proof of Proposition 4, we have the following results:

$$n_i^* = \underline{\Lambda}_m^{-1}(\Lambda - \lambda_j^*) = \underline{\Lambda}_m^{-1}(\lambda_i^*), \tag{OS.20}$$

$$n_i^* > \bar{N}, \tag{OS.21}$$

and

$$\Lambda_0 \le \lambda_i^* = \Lambda - \lambda_j^* < \underline{\Lambda}_m(N - n_j^*). \tag{OS.22}$$

Furthermore, we verify the above inequalities lead to $N > 2\bar{N}$ and $2\Lambda_0 \le \Lambda < \underline{\Lambda}'(N)$. Regarding N, from (OS.21), we have $N > n_1^* + n_2^* > 2\bar{N}$. From the first inequality in (OS.22), we have $\Lambda = \lambda_1^* + \lambda_2^* \ge 2\Lambda_0$. By (OS.20), we have $n_1^* + n_2^* = \underline{\Lambda}_m^{-1}(\lambda_1^*) + \underline{\Lambda}_m^{-1}(\Lambda - \lambda_1^*) \ge \underline{\Lambda}'^{-1}(\Lambda)$, where the inequality is due to Lemma OS.4(iv). Hence, we get $N > \underline{\Lambda}'^{-1}(\Lambda)$, i.e., $\Lambda < \underline{\Lambda}'(N)$. $\square$

Proof of Lemma OA.4. In problem P'_m, the requirements "at least one inequality binding" are not typical for constraint-optimization problems. To tackle this issue, we first relax such constraints by constructing the following problem.

$$\begin{aligned}
\bar{P}'_m: \quad \max_{p\ge 0, w\ge 0} \quad & \pi_m = (p_m - w_m)\lambda_m \\
s.t. \quad & \frac{\lambda w_m}{n_m} \ge r, \\
& v - p_m - \frac{h}{n_m\mu - \lambda_m} \ge 0, \\
& 0 < n_m \le N, \quad 0 < \lambda_m \le \Lambda, \quad p_m - w_m \ge 0, \quad \lambda_m < n_m\mu.
\end{aligned}$$

Problem $\bar{P}'_m$ differs from problem P'_m only in relaxing the constraints on being binding of the equalities. Thus, the optimal solution to problem P'_m can be obtained by refining the optimal solutions to $\bar{P}'_m$. Then it suffices to show the optimal solution to problem $\bar{P}'_m$ satisfies Lemma OA.4.

Consider the necessary conditions to problem $\bar{P}'_m$. Define the Lagrangian function by

$$L'_m = (p_m - w_m)\lambda + \eta_1\Big(\frac{\lambda w_m}{n_m} - r\Big) + \eta_2\Big(v - p_m - \frac{h}{n_m\mu - \lambda_m}\Big) + \eta_3(N - n_m) + \eta_4(\Lambda - \lambda_m) + \eta_5(p_m - w_m),$$

where $\eta_1 \ge 0$, $\eta_2 \ge 0$, $\eta_3 \ge 0$, $\eta_4 \ge 0$ and $\eta_5 \ge 0$ are the Lagrangian parameters. Then, the optimal solution of problem $\bar{P}'_m$, denoted as (p^*, w^*), with the corresponding effective arrival rate λ_m^* and the participating number of workers n_m^*, shall satisfy the first-order necessary conditions and the complementary slackness conditions, given as follows:

$$\frac{\partial L'_m}{\partial p_m}\Big|_{p_m = p_m^*, w_m = w_m^*} = \lambda_m^* - \eta_2 + \eta_5 = 0, \tag{OS.23}$$

$$\frac{\partial L'_m}{\partial w_m}\Big|_{p_m = p_m^*, w_m = w_m^*} = -\lambda_m^* + \eta_1\frac{\lambda_m^*}{n_m^*} - \eta_5 = 0, \tag{OS.24}$$

$$\eta_1\big(\frac{\lambda_m^* w_m^*}{n_m^*} - r\big) = 0, \tag{OS.25}$$

$$\eta_2\big(v - p_m^* - \frac{h}{n_m^*\mu - \lambda_m^*}\big) = 0. \tag{OS.26}$$

In the above conditions, (OS.23)-(OS.24) are the first-order necessary conditions, and (OS.25)-(OS.26) are the complementary slackness conditions.

We then establish the proof by contradiction. Suppose that there exists an optimal solution that $\frac{\lambda_m^* w_m^*}{n_m^*} > r$ or $v - p_m^* - \frac{h}{n_m^*\mu-\lambda_m^*} > 0$. If $\frac{\lambda_m^* w_m^*}{n_m^*} > r$, then (OS.25) implies that $\eta_1 = 0$, applying which to (OS.24) derives $\eta_5 = -\lambda_m^* < 0$. This contradicts "$\eta_5 \geq 0$". If $v - p_m^* - \frac{h}{n_m^*\mu-\lambda_m^*} > 0$, then (OS.26) implies that $\eta_2 = 0$, applying which to (OS.23) derives $\eta_5 = -\lambda_m^* < 0$, which contradicts "$\eta_5 \geq 0$". Lemma OA.4 is thus proved. □

Proof of Lemma OA.5. From Lemma OA.4, we can rewrite P'_m by using (OA.2) and (OA.3) to replace p_m and w_m. We obtain problem P_m. We then establish the proof by contradiction. Suppose that there exists an optimal solution that $n_m^* < N$ and $\lambda_m^* < \Lambda$. We then investigate the necessary conditions of problem P_m. Define L_m as the Lagrangian function. We have

$$L_m = \big(v - \frac{h}{n_m\mu - \lambda_m}\big)\lambda_m - n_m r + \eta_1(N - n_m) + \eta_2(\Lambda - \lambda_m) + \eta_3\big(v - \frac{h}{n_m\mu - \lambda_m} - \frac{r n_m}{\lambda_m}\big),$$

where $\eta_1 \geq 0$, $\eta_2 \geq 0$ and $\eta_3 \geq 0$ are the Lagrangian parameters. Then, the optimal solution of problem P_m, denoted as (n_m^*, λ_m^*), shall satisfy the conditions as follows:

$$\frac{\partial L_m}{\partial n_m}\Big|_{n_m=n_m^*,\lambda_m=\lambda_m^*} = \frac{h\lambda_m^*\mu}{(n_m^*\mu - \lambda_m^*)^2} - r - \eta_1 - \eta_3\big[\frac{r}{\lambda_m^*} - \frac{h\mu}{(n_m^*\mu - \lambda_m^*)^2}\big] = 0, \tag{OS.27}$$

$$\frac{\partial L_m}{\partial \lambda}\Big|_{n_m=n_m^*,\lambda_m=\lambda_m^*} = v - \frac{h n_m^*\mu}{(n_m^*\mu - \lambda_m^*)^2} - \eta_2 + \eta_3\big[\frac{r n_m^*}{\lambda_m^{*2}} - \frac{h}{(n_m^*\mu - \lambda_m^*)^2}\big] = 0, \tag{OS.28}$$

$$\eta_1(N - n_m^*) = 0, \tag{OS.29}$$

$$\eta_2(\Lambda - \lambda_m^*) = 0, \tag{OS.30}$$

$$\eta_3\big(v - \frac{h}{n_m^*\mu - \lambda_m^*} - \frac{r n_m^*}{\lambda_m^*}\big) = 0. \tag{OS.31}$$

The conditions (OS.27)-(OS.28) are the first-order necessary conditions, and (OS.29)-(OS.31) are the complementary slackness conditions.

Recall that we assume there exists an optimal solution with $n_m^* < N$ and $\lambda_m^* < \Lambda$. Then from (OS.29) and (OS.30), we have $\eta_1 = 0$ and $\eta_2 = 0$, respectively. By using $\eta_1 = 0$, we can rewrite (OS.27) as $\big(1 + \frac{\eta_3}{\lambda_m^*}\big)r = \frac{h\mu(\lambda_m^*+\eta_3)}{(n_m^*\mu-\lambda_m^*)^2}$, which implies $r = \frac{h\lambda_m^*\mu}{(n_m^*\mu-\lambda_m^*)^2}$. Substituting the above equation to (OS.28) and using $\eta_2 = 0$, we get $v = \frac{h n_m^*\mu}{(n_m^*\mu-\lambda_m^*)^2} + \eta_3\big[\frac{h\lambda_m^*\mu}{(n_m^*\mu-\lambda_m^*)^2} \cdot \frac{n_m^*}{\lambda_m^{*2}} - \frac{h}{(n_m^*\mu-\lambda_m^*)^2}\big] = \frac{h n_m^*\mu}{(n_m^*\mu-\lambda_m^*)^2} - \frac{h\eta_3}{(n_m^*\mu-\lambda_m^*)^2}\big(\frac{n_m^*\mu}{\lambda_m^*} - 1\big) = \frac{h n_m^*\mu}{(n_m^*\mu-\lambda_m^*)^2} - \frac{h\eta_3}{(n_m^*\mu-\lambda_m^*)\lambda_m^*}$. By using the above two expressions for r and v, together with (OA.2) and (OA.3), we have $p_m^* - w_m^* = v - \frac{h}{n_m^*\mu-\lambda_m^*} - \frac{r n_m^*}{\lambda_m^*} = \frac{h n_m^*\mu}{(n_m^*\mu-\lambda_m^*)^2} - \frac{h\eta_3}{(n_m^*\mu-\lambda_m^*)\lambda_m^*} - \frac{h}{n_m^*\mu-\lambda_m^*} - \frac{h n_m^*\mu}{(n_m^*\mu-\lambda_m^*)^2} = -\frac{h\eta_3}{(n_m^*\mu-\lambda_m^*)\lambda_m^*} - \frac{h}{n_m^*\mu-\lambda_m^*} < 0$, where the last inequality holds from $\lambda_m^* < n_m^*\mu$ and $\eta_3 \geq 0$. The inequality $p_m^* - w_m^* < 0$ is a contradiction to the optimal solution. Lemma OA.5 is thus proved. □

Proof of Lemma OA.6. Case I: $\Lambda < \bar{\Lambda}^o$. Lemma OA.6 requires the following constraints:

$$\lambda_i^o = \mu - \frac{h}{v - p_i^o}, \quad i = 1, 2, \tag{OS.32}$$

$$\lambda_i^o \le \mu - \sqrt{\frac{h\mu}{v}}, \quad i = 1, 2. \tag{OS.33}$$

$$\lambda_1^o + \lambda_2^o = \Lambda, \tag{OS.34}$$

We establish the proof by contradiction. We first show that any feasible strategy profile satisfying (OS.32)-(OS.34) is Pareto efficient. Suppose $(p_1^o; p_2^o)$ satisfies (OS.32)-(OS.34), and the corresponding effective arrival rates are $(\lambda_1^o; \lambda_2^o)$ and the profits for the two service platforms are $(\pi_1^o; \pi_2^o)$. Consider any feasible solution $(p_1'; p_2') \in S^o$ with $(\lambda_1'; \lambda_2')$ and $(\pi_1'; \pi_2')$ as the effective arrival rates and the profits. Suppose that $(p_1^o; p_2^o)$ is dominated by $(p_1'; p_2')$, then $\pi_1^o < \pi_1'$ and $\pi_2^o < \pi_2'$. Note that $(p_1'; p_2') \in S^o$ implies that $v - p_i' - \frac{h}{\mu - \lambda_i'} \ge 0$, $i = 1, 2$. It is easy to see that the strategy profile $(p_1''; p_2'')$, where $v - p_i'' - \frac{h}{\mu - \lambda_i'} = 0$, with the effective arrival rate λ_i', $i = 1, 2$, dominates the strategy profile $(p_1'; p_2')$. Thus, it suffices to consider those $(p_1'; p_2') \in S^o$ such that $v - p_i' - \frac{h}{\mu - \lambda_i'} = 0$, $i = 1, 2$. Note that the function $f(x) = x(\mu - \frac{h}{v - x})$ is concave and decreasing in x for $x \ge v - \sqrt{\frac{hv}{\mu}}$. By (OS.32) and (OS.33), we have $\pi_i^o \ge v - \sqrt{\frac{hv}{\mu}}$. Thus, $\pi_i^o < \pi_i'$ implies that $p_i' < p_i^o$, where $i = 1, 2$. This, together with (OS.34), implies that $\Lambda = 2\mu - \frac{h}{v - p_1^o} - \frac{h}{v - p_2^o} < 2\mu - \frac{h}{v - p_1'} - \frac{h}{v - p_2'} = \lambda_1' + \lambda_2' \le \Lambda$, where the first equality holds from (OS.34), the second inequality is valid because $p_i' < p_i^o$, $i = 1, 2$, and the last equality holds because we restrict our attention to those $(p_1'; p_2') \in S^o$ such that $v - p_i' - \frac{h}{\mu - \lambda_i'} = 0$, $i = 1, 2$. We reach a contradiction.

We then prove that any Pareto efficient outcomes must satisfy (OS.32)-(OS.34), or equivalently, any feasible strategy profile $s^o = (p_1^o; p_2^o)$ which does not satisfy one of (OS.32)-(OS.34) cannot be Pareto efficient. (i) Suppose $(p_1^o; p_2^o)$ does not satisfy (OS.32), i.e., $v - p_i^o - \frac{h}{\mu - \lambda_i^o} > 0$, $i = 1, 2$. Then, the strategy profile $(p_1'; p_2')$, where $v - p_i' - \frac{h}{\mu - \lambda_i^o} = 0$, with the effective arrival rate λ_i^o, $i = 1, 2$, dominates the strategy profile $(p_1^o; p_2^o)$. This is a contradiction. (ii) If $(p_1^o; p_2^o)$ satisfies (OS.32), then the profit $\pi_i^o = \left(v - \frac{h}{\mu - \lambda_i^o}\right)\lambda_i^o$, which is concave in λ_i^o and maximized at $\lambda_i^o = \mu - \sqrt{\frac{h\mu}{v}}$. Under this condition, we further consider either (OS.33) or (OS.34) is not satisfied by $(p_1^o; p_2^o)$. If (OS.33) is not satisfied, i.e., $\lambda_1^o > \mu - \sqrt{\frac{h\mu}{v}}$ or $\lambda_2^o > \mu - \sqrt{\frac{h\mu}{v}}$. It is easy to see that the strategy profile $(p_1'; p_2^o)$, where $p_1' = v - \frac{h}{\mu - \left(\mu - \sqrt{\frac{h\mu}{v}}\right)} = 0$, dominates the strategy profile $(p_1^o; p_2^o)$. It is a contradiction. If (OS.34) is not satisfied, i.e., $\lambda_1^o + \lambda_2^o < \Lambda$. It is easy to see that the strategy profile $(p_1'; p_2^o)$, where $p_1' = v - \frac{h}{\mu - (\Lambda - \lambda_2^o)} = 0$, dominates the strategy profile $(p_1^o; p_2^o)$. This is also a contradiction.

Case II: $\Lambda \ge \bar{\Lambda}^o$. In this case, by theorem 1 of Chen and Wan (2003), $p_i^o = v - \frac{h}{\mu - \lambda_i^o}$, where $\lambda_i^o = \mu - \sqrt{\frac{h\mu}{v}}$, $i = 1, 2$, is the unique Nash equilibrium, where customer's utility is zero and the each service platform can earn the same profit as the monopoly. Therefore, $p_i^o = v - \frac{h}{\mu - \lambda_i^o}$, where $\lambda_i^o = \mu - \sqrt{\frac{h\mu}{v}}$, $i = 1, 2$, is also the unique Pareto efficient outcome. □

OS.5. Proofs in Online Supplement

Proof of Lemma OS.1. (i) Taking derivatives of $\underline{\Lambda}_m(N)$ and $\bar{\Lambda}_m(N)$ with respect to N respectively, we have

$$\frac{d\underline{\Lambda}_m(N)}{dN} = \frac{\mu\left(\sqrt{\frac{h\mu}{r} + 4N\mu} - \sqrt{\frac{h\mu}{r}}\right)}{\sqrt{\frac{h\mu}{r} + 4N\mu}} = \mu\left(1 - \sqrt{\frac{1}{1 + \sqrt{\frac{4Nr}{h}}}}\right), \tag{OS.35}$$

and $\frac{d\bar{\Lambda}_m(N)}{dN} = \big(1-\frac{1}{2}\sqrt{\frac{h}{vN\mu}}\big)\mu$. It is easy to check that both of the two derivatives are increasing in N. Thus, $\underline{\Lambda}_m(N)$ and $\bar{\Lambda}_m(N)$ are convex in N.

Regarding $\underline{\Lambda}_m(N)$, it is easy to see from (OS.35) that $\frac{d\underline{\Lambda}_m(N)}{dN} > 0$. Regarding $\bar{\Lambda}_m(N)$, solving the zero root of $\frac{d\bar{\Lambda}_m(N)}{dN} = 0$ derives $\frac{h}{4v\mu}$. Since $\bar{\Lambda}_m(N)$ is convex, we have $\bar{\Lambda}_m(N)$ increases in N over the interval $[\frac{h}{4v\mu}, +\infty)$. From the definition of N_0, it is easy to check that $N_0 > \frac{h}{4v\mu}$. Thus, $\bar{\Lambda}_m(N)$ also increases in N for $N \geq N_0$.

From the definition of $\underline{\Lambda}_m(N)$ and $\bar{\Lambda}_m(N)$, we have $\underline{\Lambda}_m(N) - \bar{\Lambda}_m(N) = \frac{1}{2}\frac{h\mu}{r} + \sqrt{\frac{hN\mu}{v}} - \frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}$, where $\frac{1}{2}\frac{h\mu}{r} + \sqrt{\frac{hN\mu}{v}} \geq 0$ and $\frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu} \geq 0$. Thus, to show that $\underline{\Lambda}_m(N) - \bar{\Lambda}_m(N) \leq 0$, it is equivalent to show that $\big(\frac{1}{2}\frac{h\mu}{r} + \sqrt{\frac{hN\mu}{v}}\big)^2 - \big(\frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}\big)^2 \leq 0$. Note that $\big(\frac{1}{2}\frac{h\mu}{r} + \sqrt{\frac{hN\mu}{v}}\big)^2 - \big(\frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}\big)^2 = \frac{1}{4}\big(\frac{h\mu}{r}\big)^2 + \frac{hN\mu}{v} + \frac{h\mu}{r}\sqrt{\frac{hN\mu}{v}} - \frac{1}{4}\frac{h\mu}{r}\big(\frac{h\mu}{r}+4N\mu\big) = \frac{hN\mu}{v} + \frac{h\mu}{r}\sqrt{\frac{hN\mu}{v}} - \frac{h\mu}{r}N\mu = \frac{h\mu\sqrt{N}}{r}\big[\sqrt{\frac{h\mu}{v}} - \sqrt{N}\big(\mu - \frac{r}{v}\big)\big]$, the sign of which is determined by the term in the bracket. It is easy to see that the term in the bracket decreases in N, and the zero root of which is attained at $N = hv\mu/(v\mu-r)^2$. Thus, we have $\underline{\Lambda}_m(N) - \bar{\Lambda}_m(N) \leq 0$ if and only if $N \geq N_0$ from the definition of N_0.

(ii) We verify $\bar{\Lambda}_m(\underline{N}) = \hat{\Lambda}_m(\underline{N})$ directly. Recall that $\underline{N} = h/(\sqrt{v\mu}-\sqrt{r})^2$. First consider $\bar{\Lambda}_m(\underline{N})$. We have $\bar{\Lambda}_m(\underline{N}) = \big(1-\sqrt{\frac{h}{vN\mu}}\big)N\mu = \big(1-\sqrt{\frac{(\sqrt{v\mu}-\sqrt{r})^2}{v\mu}}\big)\cdot\frac{h\mu}{(\sqrt{v\mu}-\sqrt{r})^2} = \frac{\sqrt{r}}{\sqrt{v\mu}}\cdot\frac{h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}$. Next, we consider $\hat{\Lambda}_m(\underline{N})$. We have $\hat{\Lambda}_m(\underline{N}) = \frac{1}{2v}\big(rN + vN\mu - h - \sqrt{(rN+vN\mu-h)^2 - 4rvN^2\mu}\big) = \frac{1}{2v}\big(\frac{(v\mu+r)h}{(\sqrt{v\mu}-\sqrt{r})^2} - h - \sqrt{\big(\frac{(v\mu+r)h}{(\sqrt{v\mu}-\sqrt{r})^2}-h\big)^2 - \frac{4v\mu rh^2}{(\sqrt{v\mu}-\sqrt{r})^4}}\big) = \frac{1}{2v}\big(\frac{(v\mu+r)-(\sqrt{v\mu}-\sqrt{r})^2}{(\sqrt{v\mu}-\sqrt{r})^2}h - \sqrt{\big(\frac{(v\mu+r)-(\sqrt{v\mu}-\sqrt{r})^2}{(\sqrt{v\mu}-\sqrt{r})^2}h\big)^2 - \frac{4v\mu rh^2}{(\sqrt{v\mu}-\sqrt{r})^4}}\big) = \frac{1}{2v}\big(\frac{2\sqrt{v\mu r}h}{(\sqrt{v\mu}-\sqrt{r})^2} - \sqrt{\big(\frac{2\sqrt{v\mu r}h}{(\sqrt{v\mu}-\sqrt{r})^2}\big)^2 - \frac{4v\mu rh^2}{(\sqrt{v\mu}-\sqrt{r})^4}}\big) = \frac{\sqrt{r}}{\sqrt{v\mu}}\cdot\frac{h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}$. Comparing the above two expressions, we have $\bar{\Lambda}_m(\underline{N}) = \hat{\Lambda}_m(\underline{N})$.

Now we verify $\underline{\Lambda}_m(\bar{N}) = \hat{\Lambda}_m(\bar{N})$. Recall that $\bar{N} = 2h(v\mu+r)/(v\mu-r)^2$. First consider $\underline{\Lambda}_m(\bar{N})$. We have $\underline{\Lambda}_m(\bar{N}) = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}+4N\mu} - \sqrt{\frac{h\mu}{r}}\big)^2 = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}+\frac{8h\mu(v\mu+r)}{(v\mu-r)^2}} - \sqrt{\frac{h\mu}{r}}\big)^2 = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}\big(1+\frac{8(v\mu+r)r}{(v\mu-r)^2}\big)} - \sqrt{\frac{h\mu}{r}}\big)^2 = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}\frac{(v\mu-r)^2+8(v\mu+r)r}{(v\mu-r)^2}} - \sqrt{\frac{h\mu}{r}}\big)^2 = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}\cdot\frac{v^2\mu^2+6v\mu r+9r^2}{(v\mu-r)^2}} - \sqrt{\frac{h\mu}{r}}\big)^2 = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}\cdot\frac{(v\mu+3r)^2}{(v\mu-r)^2}} - \sqrt{\frac{h\mu}{r}}\big)^2 = \frac{1}{4}\big(\sqrt{\frac{h\mu}{r}}\cdot\big(\frac{v\mu+3r}{v\mu-r}-1\big)\big)^2 = \frac{1}{4}\big(\frac{h\mu}{r}\cdot\frac{16r^2}{(v\mu-r)^2}\big) = \frac{4h\mu r}{(v\mu-r)^2}$. Next, we consider $\hat{\Lambda}_m(\bar{N})$. We have $\hat{\Lambda}_m(\bar{N}) = \frac{1}{2v}\big(rN + vN\mu - h - \sqrt{(rN+vN\mu-h)^2 - 4rvN^2\mu}\big) = \frac{1}{2v}\big(2h\frac{(v\mu+r)^2}{(v\mu-r)^2} - h - \sqrt{\big(2h\frac{(v\mu+r)^2}{(v\mu-r)^2}-h\big)^2 - \frac{16v\mu rh^2(v\mu+r)^2}{(v\mu-r)^4}}\big) = \frac{1}{2v}\big(\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}h - \sqrt{\big(\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}h\big)^2 - \frac{16v\mu rh^2(v\mu+r)^2}{(v\mu-r)^4}}\big) = \frac{1}{2v}\big(\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}h - \sqrt{\big(\frac{(v\mu+r)^2-4v\mu r}{(v\mu-r)^2}h\big)^2}\big) = \frac{1}{2v}\big(\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}h - h\big) = \frac{1}{2v}\cdot\frac{8v\mu r}{(v\mu-r)^2}h = \frac{4h\mu r}{(v\mu-r)^2}$. Comparing the above two expressions, we have $\underline{\Lambda}_m(\bar{N}) = \hat{\Lambda}_m(\bar{N})$.

(iii) Taking the first- and second-order derivatives of $\hat{\Lambda}_m(N)$ with respect to N, we have $\frac{d\hat{\Lambda}_m(N)}{dN} = \frac{r+v\mu}{2v} - \frac{(rN+vN\mu-h)(r+v\mu)-4rvN\mu}{2v\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}}$, and $\frac{d^2\hat{\Lambda}_m(N)}{dN^2} = -\frac{\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}[(r+v\mu)^2-4rv\mu] - \frac{[(ry+vN\mu-h)(r+v\mu)-4rvN\mu]^2}{\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}}}{2v[(rN+vN\mu-h)^2-4rvN^2\mu]} = -\frac{[(rN+vN\mu-h)^2-4rvN^2\mu][(r+v\mu)^2-4rv\mu]-[(rN+vN\mu-h)(r+v\mu)-4rvN\mu]^2}{2v[(rN+vN\mu-h)^2-4rvN^2\mu]\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}} > 0$, where the last inequality holds because the numerator satisfies $[(rN+vN\mu-h)^2-4rvN^2\mu][(r+v\mu)^2-4rv\mu] - [(rN+vN\mu-h)(r+v\mu)-4rvN\mu]^2 = [(r-v\mu)^2N^2 - 2(r+v\mu)hN + h^2](r-v\mu)^2 - [(r-v\mu)^2N - (r+v\mu)h]^2 = (r-v\mu)^4N^2 - 2(r-v\mu)^2(r+v\mu)hN + (r-v\mu)^2h^2 - [(r-v\mu)^4N^2 - 2(r-v\mu)^2(r+v\mu)hN + (r+v\mu)^2h^2] = (r-v\mu)^2h^2 - (r+v\mu)^2h^2 = -4v\mu rh^2 < 0$. Thus, $\hat{\Lambda}_m(N)$ is convex in N.

Next, we show that the minima of $\hat{\Lambda}_m(N)$ is attained at $N=\bar{N}$. It is sufficient to show that $\frac{d\hat{\Lambda}_m(N)}{dN}$ attains zero at $\bar{N}$. We have $\frac{d\hat{\Lambda}_m(N)}{dN}\Big|_{N=\bar{N}} = \frac{r+v\mu}{2v} - \frac{\left[\frac{2(v\mu+r)^2}{(v\mu-r)^2}h-h\right](r+v\mu)-4rv\mu\frac{2(v\mu+r)h}{(v\mu-r)^2}}{2v\sqrt{\left[\frac{2(v\mu+r)^2}{(v\mu-r)^2}h-h\right]^2-4rv\mu\frac{4(v\mu+r)^2h^2}{(v\mu-r)^4}}} = \frac{r+v\mu}{2v} - \frac{r+v\mu}{2v}\frac{\left[2(v\mu+r)^2-(v\mu-r)^2\right]h-8rv\mu h}{\sqrt{\left[2(v\mu+r)^2-(v\mu-r)^2\right]^2h^2-16rv\mu(v\mu+r)^2h^2}} = \frac{r+v\mu}{2v} - \frac{r+v\mu}{2v}\frac{\left[(v\mu+r)^2+4v\mu r\right]h-8rv\mu h}{\sqrt{\left[(v\mu+r)^2+4v\mu r\right]^2h^2-16rv\mu(v\mu+r)^2h^2}} = \frac{r+v\mu}{2v} - \frac{r+v\mu}{2v}\frac{\left[(v\mu+r)^2-4v\mu r\right]h}{\sqrt{\left[(v\mu+r)^4-8rv\mu(v\mu+r)^2+16(v\mu r)^2\right]h^2}} = \frac{r+v\mu}{2v} - \frac{r+v\mu}{2v}\frac{\left[(v\mu+r)^2-4v\mu r\right]h}{\sqrt{\left[(v\mu+r)^2-4v\mu r\right]^2h^2}} = 0.$

Last, recall that $\hat{\Lambda}_m(N)$ is convex in N and the minima is attained at $\bar{N}$, $\underline{\Lambda}_m(N)$ increases in N for $N\geq\bar{N}>N_0$, and $\underline{\Lambda}_m(\bar{N})=\hat{\Lambda}_m(\bar{N})$. Thus, to show $\hat{\Lambda}_m(N)\leq\underline{\Lambda}_m(N)$ if and only if $N\geq\bar{N}$, it suffices to show that $\hat{\Lambda}_m(N)\leq\underline{\Lambda}_m(N)$ for $N\geq\bar{N}$. From the definition of $\hat{\Lambda}_m(N)$ and $\underline{\Lambda}(N)$, we have $\hat{\Lambda}_m(N)-\underline{\Lambda}_m(N)=\frac{1}{2v}\big(rN+vN\mu-h-\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\big)-\frac{1}{2}\frac{h\mu}{r}-y\mu+\frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}=\frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}-\frac{1}{2v}\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}-\frac{1}{2v}\big(h+\frac{hv\mu}{r}+(v\mu-r)N\big)$, where we can check the last term $\frac{1}{2v}\big(h+\frac{hv\mu}{r}+(v\mu-r)N\big)\geq 0$ from $v>r/\mu$. The fact $v>r/\mu$ is due to $r\leq\frac{\lambda w}{n}<\frac{n\mu w}{n}=w\mu\leq p\mu<v\mu$. Thus, if the sum of the first two terms is non-positive, i.e., $\frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}-\frac{1}{2v}\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\leq 0$, then $\hat{\Lambda}_m(N)-\underline{\Lambda}_m(N)\leq 0$. Otherwise, to show that $\hat{\Lambda}_m(N)-\underline{\Lambda}_m(N)\leq 0$, it is equivalent to show that

$$
\begin{aligned}
&\Big(\frac{1}{2}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}-\frac{1}{2v}\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\Big)^2-\frac{1}{4v^2}\Big[x+\frac{xv\mu}{r}+(v\mu-r)y\Big]^2\\
=&\frac{1}{4}\frac{h\mu}{r}\Big(\frac{h\mu}{r}+4N\mu\Big)+\frac{1}{4v^2}\Big[(rN+vN\mu-h)^2-4rvN^2\Big]\\
&-\frac{1}{2v}\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}-\frac{1}{4v^2}\Big[h+\frac{hv\mu}{r}+(v\mu-r)N\Big]^2\\
=&\frac{1}{2v}\Big[\frac{\mu}{r}hN(v\mu-r)-\frac{\mu}{r}h^2-\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\Big]\\
=&\frac{1}{2v}\sqrt{\frac{h\mu}{r}}\Big[\sqrt{\frac{h\mu}{r}}(N(v\mu-r)-h)-\sqrt{\frac{h\mu}{r}+4N\mu}\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\Big]\leq 0. \qquad \text{(OS.36)}
\end{aligned}
$$

For $N\geq\bar{N}$, we can check that $\sqrt{\frac{h\mu}{r}}(N(v\mu-r)-h)\geq 0$ from $v>r/\mu$, and $\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4N\mu}\sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\geq 0$. Thus, to have the validity of (OS.36), it is equivalent to show $\frac{h\mu}{r}(N(v\mu-r)-h)^2-\big(\frac{h\mu}{r}+4N\mu\big)[(rN+vN\mu-h)^2-4rvN^2\mu]=\frac{h\mu}{r}[N^2(v\mu-r)^2+h^2-2(v\mu-r)hN]-\big(\frac{h\mu}{r}+4N\mu\big)[N^2(v\mu-r)^2+h^2-2(v\mu+r)hN]=4h^2N\mu-4N\mu[N^2(v\mu-r)^2+h^2-2(v\mu+r)hN]=4N\mu[2(v\mu+r)hN-N^2(v\mu-r)^2]=4N^2\mu[2(v\mu+r)h-N(v\mu-r)^2]\leq 0$. Note that the term in the bracket decreases in N and the zero root is attained at $2h(v\mu+r)/(v\mu-r)^2$, i.e., $\bar{N}$. Thus, if $N\geq\bar{N}$, then the inequality (OS.36) is valid, and $\hat{\Lambda}_m(N)\leq\underline{\Lambda}_m(N)$. □

Proof of Lemma OS.2. (i) Recall that $\bar{\Lambda}(N)$, $\underline{\Lambda}(N)$ and $\hat{\Lambda}(N)$ can be obtained by replacing "h" with "$2h$" in the $\bar{\Lambda}_m(N)$, $\underline{\Lambda}_m(N)$ and $\hat{\Lambda}_m(N)$, respectively. Thus, we can get the announced results for the three functions by similar analysis in the proof of Lemma OS.1.

We take derivative of $\underline{\Lambda}'(N)$ and $\bar{\Lambda}'(N)$ with respect to N. We have $\frac{d\underline{\Lambda}'(N)}{dN}=\Big(\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}-\sqrt{\frac{h\mu}{r}}\Big)\frac{\mu}{\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}}=\Big(1-\frac{\sqrt{\frac{h\mu}{r}}}{\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}}\Big)\mu>0$, which is increasing in N. Then $\underline{\Lambda}'(N)$ is convex and increasing in N. We also have $\frac{d\bar{\Lambda}'(N)}{dN}=\mu-$

$\frac{h\mu}{2v\sqrt{\frac{h\mu[(\sqrt{v\mu}-\sqrt{r})^2N-h]}{(\sqrt{v\mu}-\sqrt{r})^2v}}}$, which is increasing in N. Let $\frac{d\bar{\Lambda}'(N)}{dN}=0$, $N=\frac{h}{(\sqrt{v\mu}-\sqrt{r})^2v}+\frac{h}{4v^2\mu}<\frac{2h}{(\sqrt{v\mu}-\sqrt{r})^2v}=2\underline{N}$. Thus $\bar{\Lambda}'(N)$ is increasing in N for $N\geq 2\underline{N}$.

(ii) First, we verify $\underline{\Lambda}'(N)\geq\underline{\Lambda}(N)$. From the definition of $\underline{\Lambda}'(N)$ and $\underline{\Lambda}(N)$, we have $\underline{\Lambda}'(N)-\underline{\Lambda}(N)=\frac{1}{4}\big(\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}-\sqrt{\frac{h\mu}{r}}\big)^2+\frac{4hr\mu}{(v\mu-r)^2}-\frac{1}{4}\big(\sqrt{\frac{2h\mu}{r}+4N\mu}-\sqrt{\frac{2h\mu}{r}}\big)^2=\frac{1}{4}\big[\frac{2h\mu}{r}+4\bigg(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)-2\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}\big]+\frac{4hr\mu}{(v\mu-r)^2}-\frac{1}{4}\big(\frac{4h\mu}{r}+4y\mu-2\sqrt{\frac{2h\mu}{r}}\sqrt{\frac{2h\mu}{r}+4N\mu}\big)=\frac{1}{2}\big[\sqrt{\frac{2h\mu}{r}}\sqrt{\frac{2h\mu}{r}+4y\mu}-\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}\big]-\big(\frac{h\mu}{2r}+\frac{2h\mu}{v\mu-r}\big)=\frac{1}{2}\big[\sqrt{\frac{2h\mu}{r}}\sqrt{\frac{2h\mu}{r}+4N\mu}-\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}\big]-\frac{h\mu(v\mu+3r)}{2r(v\mu-r)}$, where $\sqrt{\frac{2h\mu}{r}}\sqrt{\frac{2h\mu}{r}+4N\mu}-\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4\Big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\Big)}>0$ and $\frac{h\mu(v\mu+3r)}{2r(v\mu-r)}>0$. To show $\underline{\Lambda}'(N)-\underline{\Lambda}(N)\geq 0$, it is equivalent to show $\big[\sqrt{\frac{2h\mu}{r}}\sqrt{\frac{2h\mu}{r}+4N\mu}-\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}\big]^2-\frac{h^2\mu^2(v\mu+3r)^2}{r^2(v\mu-r)^2}>0$. Note that $\big[\sqrt{\frac{2h\mu}{r}}\sqrt{\frac{2h\mu}{r}+4y\mu}-\sqrt{\frac{h\mu}{r}}\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}\big]^2-\frac{h^2\mu^2(v\mu+3r)^2}{r^2(v\mu-r)^2}=\frac{4h^2\mu^2}{r^2(v\mu-r)^2}\big[v^2\mu^2-6v\mu r-3r^2+\frac{3rN}{h}(v\mu-r)^2\big]-\frac{2\sqrt{2}h\mu}{r}\sqrt{\frac{2h\mu}{r}+4N\mu}\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}=\frac{2\sqrt{2}h\mu}{r}\big[\frac{\sqrt{2}h\mu}{r(v\mu-r)^2}\big[v^2\mu^2-6v\mu r-3r^2+\frac{3ry}{h}(v\mu-r)^2\big]-\sqrt{\frac{2h\mu}{r}+4N\mu}\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}\big]$, where $v^2\mu^2-6v\mu r-3r^2+\frac{3ry}{x}(v\mu-r)^2>v^2\mu^2-6v\mu r-3r^2+12r(v\mu+r)>v^2\mu^2-6v\mu r-3r^2+12r^2=(v\mu-3r)^2\geq 0$, where the first inequality is due to $N\geq 2\bar{N}=\frac{4h(v\mu+r)}{(v\mu-r)^2}$, and $\frac{2\sqrt{2}h\mu}{r}\sqrt{\frac{2h\mu}{r}+4N\mu}\sqrt{\frac{h\mu}{r}+4\Big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\Big)}>0$. Thus, it is equivalent to show $\frac{2h^2\mu^2}{r^2(v\mu-r)^4}\big[v^2\mu^2-6v\mu r-3r^2+\frac{3rN}{h}(v\mu-r)^2\big]^2-\big(\frac{2h\mu}{r}+4N\mu\big)\big(\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)\big)\geq 0$. We can check that $\frac{2h^2\mu^2}{r^2(v\mu-r)^4}\big[v^2\mu^2-6v\mu r-3r^2+\frac{3rN}{x}(v\mu-r)^2\big]^2-\big(\frac{2h\mu}{r}+4N\mu\big)\big(\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)\big)=\frac{2h^2\mu^2}{r^2(v\mu-r)^4}\big[(v^2\mu^2-6v\mu r-3r^2)^2+\frac{6rN}{h}(v\mu-r)^2(v^2\mu^2-6v\mu r-3r^2)+\frac{9r^2N^2}{h^2}(v\mu-r)^4\big]-\frac{2h^2\mu^2}{r^2}\cdot\frac{v^2\mu^2-10v\mu r-7r^2}{(v\mu-r)^2}-\frac{4hN\mu^2}{r}\cdot\frac{3v^2\mu^2-14v\mu r-5r^2}{(v\mu-r)^2}-16N^2\mu^2=\frac{32h^2\mu^2(v\mu+r)^2}{(v\mu-r)^4}-\frac{16hN\mu^2(v\mu+r)}{(v\mu-r)^2}+2N^2\mu^2=2\big[N\mu-\frac{4h\mu(v\mu+r)}{(v\mu-r)^2}\big]^2\geq 0$.

Next, we verify $\underline{\Lambda}'(2\bar{N})=\underline{\Lambda}(2\bar{N})=\hat{\Lambda}(2\bar{N})$. From the definition $\bar{N}=2h(v\mu+r)/(v\mu-r)^2$, we have $\underline{\Lambda}'(2\bar{N})=\frac{1}{4}\big(\sqrt{\frac{h\mu}{r}+4\big(N\mu-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}-\sqrt{\frac{h\mu}{r}}\big)^2+\frac{4hr\mu}{(v\mu-r)^2}=\frac{1}{4}\Big(\sqrt{\frac{h\mu}{r}+4\big(\frac{4h\mu(v\mu+r)}{(v\mu-r)^2}-\frac{2h\mu(v\mu+r)}{(v\mu-r)^2}\big)}-\sqrt{\frac{h\mu}{r}}\Big)^2+\frac{4hr\mu}{(v\mu-r)^2}=\frac{1}{4}\Big(\sqrt{\frac{h\mu}{r}\big(1+\frac{8r(v\mu+r)}{(v\mu-r)^2}\big)}-\sqrt{\frac{h\mu}{r}}\Big)^2+\frac{4hr\mu}{(v\mu-r)^2}=\frac{1}{4}\Big(\sqrt{\frac{h\mu}{r}\cdot\frac{v^2\mu^2+6v\mu r+9r^2}{(v\mu-r)^2}}-\sqrt{\frac{h\mu}{r}}\Big)^2+\frac{4hr\mu}{(v\mu-r)^2}=\frac{1}{4}\frac{h\mu}{r}\big(\frac{v\mu+3r}{v\mu-r}-1\big)^2+\frac{4hr\mu}{(v\mu-r)^2}=\frac{1}{4}\frac{h\mu}{r}\big(\frac{4r}{v\mu-r}\big)^2+\frac{4hr\mu}{(v\mu-r)^2}=\frac{8hr\mu}{(v\mu-r)^2}$, $\underline{\Lambda}(2\bar{N})=\frac{1}{4}\big(\sqrt{\frac{2h\mu}{r}+4N\mu}-\sqrt{\frac{2h\mu}{r}}\big)^2=\frac{1}{4}\big(\sqrt{\frac{2h\mu}{r}+\frac{16h\mu(v\mu+r)}{(v\mu-r)^2}}-\sqrt{\frac{2h\mu}{r}}\big)^2=\frac{1}{4}\big(\sqrt{\frac{2h\mu}{r}\big(1+\frac{8r(v\mu+r)}{(v\mu-r)^2}\big)}-\sqrt{\frac{2h\mu}{r}}\big)^2=\frac{1}{4}\big(\sqrt{\frac{2h\mu}{r}\cdot\frac{v^2\mu^2+6v\mu r+9r^2}{(v\mu-r)^2}}-\sqrt{\frac{2h\mu}{r}}\big)^2=\frac{1}{2}\frac{h\mu}{r}\big(\frac{v\mu+3r}{v\mu-r}-1\big)^2=\frac{1}{2}\frac{h\mu}{r}\big(\frac{4r}{v\mu-r}\big)^2=\frac{8hr\mu}{(v\mu-r)^2}$, and $\hat{\Lambda}(2\bar{N})=\frac{1}{2v}\big(rN+vN\mu-2h-\sqrt{(rN+vN\mu-2h)^2-4rvN^2\mu}\big)=\frac{1}{2v}\Big(4h\frac{(v\mu+r)^2}{(v\mu-r)^2}-2h-\sqrt{\big(4h\frac{(v\mu+r)^2}{(v\mu-r)^2}-2h\big)^2-\frac{64v\mu rh^2(v\mu+r)^2}{(v\mu-r)^4}}\Big)=\frac{1}{2v}\big(2h\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}-\sqrt{\big(2h\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}\big)^2-\frac{64v\mu rh^2(v\mu+r)^2}{(v\mu-r)^4}}\big)=\frac{1}{2v}\big(2h\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}-\sqrt{\big(2h\frac{(v\mu+r)^2-4v\mu r}{(v\mu-r)^2}\big)^2}\big)=\frac{1}{2v}\big(2h\frac{(v\mu+r)^2+4v\mu r}{(v\mu-r)^2}-2h\big)=\frac{1}{2v}\cdot\frac{16hv\mu r}{(v\mu-r)^2}=\frac{8hr\mu}{(v\mu-r)^2}$. Comparing the above three expressions, we have $\underline{\Lambda}'(2\bar{N})=\underline{\Lambda}(2\bar{N})=\hat{\Lambda}(2\bar{N})$.

Thus, we get the announced results.

(iii) First, we verify $\bar{\Lambda}'(N)\geq\bar{\Lambda}(N)$. From the definition of $\bar{\Lambda}'(N)$ and $\bar{\Lambda}(N)$, we have $\bar{\Lambda}'(N)-\bar{\Lambda}(N)=N\mu-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}-\sqrt{\frac{h\mu\big[(\sqrt{v\mu}-\sqrt{r})^2N-h\big]}{(\sqrt{v\mu}-\sqrt{r})^2v}}-\Big(1-\sqrt{\frac{2h}{vN\mu}}\Big)N\mu=\sqrt{\frac{2hN\mu}{v}}-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}-$

$\sqrt{\frac{h\mu\left[(\sqrt{v\mu}-\sqrt{r})^2N-h\right]}{(\sqrt{v\mu}-\sqrt{r})^2v}}$, where $\sqrt{\frac{2hN\mu}{v}}-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}>0$ for $N\geq 2\underline{N}$ and $\sqrt{\frac{h\mu\left[(\sqrt{v\mu}-\sqrt{r})^2N-h\right]}{(\sqrt{v\mu}-\sqrt{r})^2v}}>0$. To show $\bar{\Lambda}'(N)-\bar{\Lambda}(N)\geq 0$, it is equivalent to show $\big(\sqrt{\frac{2hN\mu}{v}}-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}\big)^2-\frac{h\mu\left[(\sqrt{v\mu}-\sqrt{r})^2N-h\right]}{(\sqrt{v\mu}-\sqrt{r})^2v}\geq 0$. We can check that $\big(\sqrt{\frac{2hN\mu}{v}}-\frac{x\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}\big)^2-\frac{h\mu\left[(\sqrt{v\mu}-\sqrt{r})^2N-h\right]}{(\sqrt{v\mu}-\sqrt{r})^2v}=\frac{h\mu}{v}\big[2N-\frac{2\sqrt{h}\sqrt{N}}{\sqrt{v\mu}-\sqrt{r}}+\frac{h}{(\sqrt{v\mu}-\sqrt{r})^2}\big]-\frac{h\mu}{v}\big[N-\frac{h}{(\sqrt{v\mu}-\sqrt{r})^2}\big]=\frac{h\mu}{v}\big[N+\frac{2h}{(\sqrt{v\mu}-\sqrt{r})^2}-\frac{2\sqrt{h}\sqrt{N}}{\sqrt{v\mu}-\sqrt{r}}\big]=\frac{h\mu}{v}\big(\sqrt{N}-\frac{\sqrt{2h}}{\sqrt{v\mu}-\sqrt{r}}\big)^2\geq 0$.

Next, we verify $\bar{\Lambda}'(2\underline{N})=\bar{\Lambda}(2\underline{N})=\hat{\Lambda}(2\underline{N})$. From $\underline{N}=h/(\sqrt{v\mu}-\sqrt{r})^2$, we have $\bar{\Lambda}'(2\underline{N})=N\mu-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}-\sqrt{\frac{h\mu[(\sqrt{v\mu}-\sqrt{r})^2N-h]}{(\sqrt{v\mu}-\sqrt{r})^2v}}=\frac{2h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}-\sqrt{\frac{h\mu(2h-h)}{(\sqrt{v\mu}-\sqrt{r})^2v}}=\frac{2h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}-\frac{h\sqrt{\mu}}{(\sqrt{v\mu}-\sqrt{r})\sqrt{v}}=\frac{2h\mu\sqrt{v}-2h\sqrt{\mu}(\sqrt{v\mu}-\sqrt{r})}{(\sqrt{v\mu}-\sqrt{r})^2\sqrt{v}}=\frac{\sqrt{r}}{\sqrt{v\mu}}\cdot\frac{2h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}$, $\bar{\Lambda}(2\underline{N})=\big(1-\sqrt{\frac{2h}{vN\mu}}\big)N\mu=\big(1-\sqrt{\frac{(\sqrt{v\mu}-\sqrt{r})^2}{v\mu}}\big)\cdot\frac{2h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}=\frac{\sqrt{r}}{\sqrt{v\mu}}\cdot\frac{2h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}$, and $\hat{\Lambda}(2\underline{N})=\frac{1}{2v}\big(rN+vN\mu-2h-\sqrt{(rN+vN\mu-2h)^2-4rvN^2\mu}\big)=\frac{1}{2v}\big(\frac{2(v\mu+r)h}{(\sqrt{v\mu}-\sqrt{r})^2}-2h-\sqrt{\big(\frac{2(v\mu+r)h}{(\sqrt{v\mu}-\sqrt{r})^2}-2h\big)^2-\frac{16v\mu rh^2}{(\sqrt{v\mu}-\sqrt{r})^4}}\big)=\frac{1}{2v}\big(2h\frac{(v\mu+r)-(\sqrt{v\mu}-\sqrt{r})^2}{(\sqrt{v\mu}-\sqrt{r})^2}-\sqrt{\big(2h\frac{(v\mu+r)-(\sqrt{v\mu}-\sqrt{r})^2}{(\sqrt{v\mu}-\sqrt{r})^2}\big)^2-\frac{16v\mu rh^2}{(\sqrt{v\mu}-\sqrt{r})^4}}\big)=\frac{1}{2v}\big(\frac{4\sqrt{v\mu r}h}{(\sqrt{v\mu}-\sqrt{r})^2}-\sqrt{\big(\frac{4\sqrt{v\mu r}h}{(\sqrt{v\mu}-\sqrt{r})^2}\big)^2-\frac{16v\mu rh^2}{(\sqrt{v\mu}-\sqrt{r})^4}}\big)=\frac{\sqrt{r}}{\sqrt{v\mu}}\cdot\frac{2h\mu}{(\sqrt{v\mu}-\sqrt{r})^2}$. Comparing the above three expressions, we have $\bar{\Lambda}'(2\underline{N})=\bar{\Lambda}(2\underline{N})=\hat{\Lambda}(2\underline{N})$.

Thus, we get the announced results. □

Proof of Lemma OS.3. (i) We can check that $\bar{\Lambda}_m(N/2)+\bar{\Lambda}_m(N/2)=\big(1-\sqrt{\frac{2h}{vN\mu}}\big)\frac{N}{2}\mu+\big(1-\sqrt{\frac{2h}{vN\mu}}\big)\frac{N}{2}\mu=\big(1-\sqrt{\frac{2h}{vN\mu}}\big)N\mu=\bar{\Lambda}(N)$ and $\bar{\Lambda}_m(\underline{N})+\bar{\Lambda}_m(N-\underline{N})=\big(1-\sqrt{\frac{h}{v\underline{N}\mu}}\big)\underline{N}\mu+\big(1-\sqrt{\frac{h}{v(N-\underline{N})\mu}}\big)(N-\underline{N})\mu=\bar{\Lambda}'(N)$.

(ii) We can check that $\hat{\Lambda}_m(N/2)+\hat{\Lambda}_m(N/2)=\hat{\Lambda}_m(N/2)+\hat{\Lambda}_m(N/2)=2\hat{\Lambda}_m(N/2)=\hat{\Lambda}(N)$ by the definitions of $\hat{\Lambda}_m(N)$ and $\hat{\Lambda}(N)$.

(iii) We can check that $\underline{\Lambda}_m(N/2)+\underline{\Lambda}_m(N/2)=2\underline{\Lambda}_m(N/2)=\underline{\Lambda}(N)$ by the definitions of $\underline{\Lambda}_m(N)$ and $\underline{\Lambda}(N)$.

(iv) We can check that $\underline{\Lambda}_m^{-1}(\Lambda_0)+\underline{\Lambda}_m^{-1}(\Lambda-\Lambda_0)=\big(1+\sqrt{\frac{h\mu}{r\Lambda_0}}\big)\frac{\Lambda_0}{\mu}+\big(1+\sqrt{\frac{h\mu}{r(\Lambda-\Lambda_0)}}\big)\frac{\Lambda-\Lambda_0}{\mu}=\frac{\Lambda}{\mu}+\sqrt{\frac{h\Lambda_0}{r\mu}}+\sqrt{\frac{h(\Lambda-\Lambda_0)}{r\mu}}=\underline{\Lambda}'^{-1}(\Lambda)$. □

Proof of Lemma OS.4. (i) We first show $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)\geq\bar{\Lambda}(N)$. We have $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)=\big(1-\sqrt{\frac{h}{vn\mu}}\big)n\mu+\big(1-\sqrt{\frac{h}{v(N-n)\mu}}\big)(N-n)\mu$. Taking the first-order derivative of $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)$ over n, we get $\frac{d(\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n))}{dn}=\big(1-\frac{1}{2}\sqrt{\frac{h}{vn\mu}}\big)\mu-\big(1-\frac{1}{2}\sqrt{\frac{h}{v(N-n)\mu}}\big)\mu=\frac{1}{2}\big(\sqrt{\frac{h}{v(N-n)\mu}}-\sqrt{\frac{h}{vn\mu}}\big)\mu$, which is increasing in n. This implies that $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)$ attains minimum at the first order condition, i.e., $n=N/2$. Lemma OS.3(i), $(\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n))\big|_{n=N/2}=\bar{\Lambda}(N)$.

We next show $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)\leq\bar{\Lambda}'(N)$ if $\underline{N}\leq n\leq N-\underline{N}$ for $N\geq 2\underline{N}$. From the former paragraph, we have $\frac{d(\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n))}{dn}$ is increasing in n and $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)$ is minimized at $n=N/2$, which implies $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)$ is convex in n, decreasing in $n\leq N/2$ and increasing in $n\geq N/2$. Note $N\geq 2\underline{N}$, then $\underline{N}\leq N/2\leq N-\underline{N}$. Thus the maximum of $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)$ can be attainted at $n=\underline{N}$ or $n=N-\underline{N}$ because $\underline{N}\leq n\leq N-\underline{N}$. By Lemma OS.3, $\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)\big|_{n=\underline{N}}=\bar{\Lambda}_m(n)+\bar{\Lambda}_m(N-n)\big|_{n=N-\underline{N}}=\bar{\Lambda}'(N)$.

(ii) Taking the first-order derivative of $\hat{\Lambda}_m(n)+\hat{\Lambda}_m(N-n)$ over n, we have $\frac{d(\hat{\Lambda}_m(n)+\hat{\Lambda}_m(N-n))}{dn}=\frac{r+v\mu}{2v}-\frac{(rn+vn\mu-h)(r+v\mu)-4rvn\mu}{2v\sqrt{(rn+vn\mu-h)^2-4rvn^2\mu}}-\frac{r+v\mu}{2v}+\frac{(r(N-n)+v(N-n)\mu-h)(r+v\mu)-4rv(N-n)\mu}{2v\sqrt{(r(N-n)+v(N-n)\mu-h)^2-4rv(N-n)^2\mu}}=$

$\frac{(r(N-n)+v(N-n)\mu-h)(r+v\mu)-4rv(N-n)\mu}{2v\sqrt{(r(N-n)+v(N-n)\mu-h)^2-4rv(N-n)^2\mu}} - \frac{(rn+vn\mu-h)(r+v\mu)-4rvn\mu}{2v\sqrt{(rn+vn\mu-h)^2-4rvn^2\mu}}$. Solving $\frac{d(\hat{\Lambda}_m(n)+\hat{\Lambda}_m(N-n))}{dn} = 0$, we have $n = N/2$. Since $\hat{\Lambda}_m(N)$ is convex (see Lemma OS.1(iii)), $\hat{\Lambda}_m(n) + \hat{\Lambda}_m(N-n)$ is convex in n. This implies that $\hat{\Lambda}_m(n) + \hat{\Lambda}_m(N-n)$ attains minimum at the first order condition, i.e., $n = N/2$. By Lemma OS.3(ii), $(\hat{\Lambda}_m(n) + \hat{\Lambda}_m(N-n))\big|_{n=N/2} = \hat{\Lambda}(N)$.

(iii) We take first-order derivative of $\underline{\Lambda}_m(n) + \underline{\Lambda}_m(N-n)$ over n, $\frac{d(\underline{\Lambda}_m(n)+\underline{\Lambda}_m(N-n))}{dn} = \mu\big(1 - \sqrt{\frac{1}{1+\sqrt{\frac{4nr}{h}}}}\big) - \mu\big(1 - \sqrt{\frac{1}{1+\sqrt{\frac{4(N-n)r}{h}}}}\big) = \mu\big(\sqrt{\frac{1}{1+\sqrt{\frac{4(N-n)r}{h}}}} - \sqrt{\frac{1}{1+\sqrt{\frac{4nr}{h}}}}\big)$, which is increasing in n, thus $\underline{\Lambda}_m(n) + \underline{\Lambda}_m(N-n)$ is convex in n. Let $\frac{d(\underline{\Lambda}_m(n)+\underline{\Lambda}_m(N-n))}{dn} = 0$, we have $n = N/2$. This implies that $\underline{\Lambda}_m(n) + \underline{\Lambda}_m(N-n)$ attains minimum at the first order condition, i.e., $n = N/2$. Lemma OS.3, $(\underline{\Lambda}_m(n) + \underline{\Lambda}_m(N-n))\big|_{n=N/2} = \underline{\Lambda}(N)$.

(iv) Taking first-order derivative of $\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda-\lambda)$, we have $\frac{d(\underline{\Lambda}_m^{-1}(\lambda)+\underline{\Lambda}_m^{-1}(\Lambda-\lambda))}{d\lambda} = \big(1 + \frac{1}{2}\sqrt{\frac{h\mu}{r\lambda}}\big)\frac{1}{\mu} - \big(1 - \frac{1}{2}\sqrt{\frac{h\mu}{r(\Lambda-\lambda)}}\big)\frac{1}{\mu} = \frac{1}{2\mu}\big(\sqrt{\frac{h\mu}{r\lambda}} - \sqrt{\frac{h\mu}{r(\Lambda-\lambda)}}\big)$, which is decreasing in λ. Thus, $\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda-\lambda)$ is concave. Solving $\frac{d(\underline{\Lambda}_m^{-1}(\lambda)+\underline{\Lambda}_m^{-1}(\Lambda-\lambda))}{d\lambda} = 0$, we obtain $\lambda = \Lambda/2$. Since $\Lambda \geq 2\Lambda_0$, we have $\Lambda_0 \leq \Lambda/2 \leq \Lambda - \Lambda_0$. Thus, the minimum of $\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda-\lambda)$ can be attainted at $\lambda = \Lambda_0$ or $\lambda = \Lambda - \Lambda_0$. By Lemma OS.3(iv), $(\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda-\lambda))\big|_{\lambda=\Lambda_0} = (\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda-\lambda))\big|_{\lambda=\Lambda-\Lambda_0} = \underline{\Lambda}'^{-1}(\Lambda)$. $\square$

Proof of Lemma OS.5. (i) On the one hand, from the proof of Lemma OS.4(i), $\bar{\Lambda}_m(n) + \bar{\Lambda}_m(N-n)$ is decreasing in $\underline{N} \leq n \leq N/2$. On the other hand, by Lemma OS.3(i), $\bar{\Lambda}_m(N/2) + \bar{\Lambda}_m(N/2) = \bar{\Lambda}(N)$ and $\bar{\Lambda}_m(\underline{N}) + \bar{\Lambda}_m(N-\underline{N}) = \bar{\Lambda}'(N)$. Because $\bar{\Lambda}(N) < \Lambda \leq \bar{\Lambda}'(N)$, we thus have there exists $\underline{N} \leq n < N/2$ such that $\bar{\Lambda}_m(n) + \bar{\Lambda}_m(N-n) = \Lambda$.

(ii) On the one hand, from the proof of Lemma OS.4(iv), $\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda-\lambda)$ is increasing in $\Lambda_0 \leq \lambda \leq \Lambda/2$. On the other hand, by Lemma OS.3(iv), $\underline{\Lambda}_m^{-1}(\Lambda_0) + \underline{\Lambda}_m^{-1}(\Lambda-\Lambda_0) = \bar{\Lambda}'^{-1}(\Lambda)$. and $\underline{\Lambda}_m^{-1}(\Lambda/2) + \underline{\Lambda}_m^{-1}(\Lambda/2) = \underline{\Lambda}^{-1}(\Lambda)$. Because $\underline{\Lambda}(N) \leq \Lambda < \underline{\Lambda}'(N)$, i.e., $\bar{\Lambda}'^{-1}(\Lambda) < N \leq \underline{\Lambda}^{-1}(\Lambda)$, we thus have there exists $\Lambda_0 < \lambda \leq \Lambda/2$ such that $\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda-\lambda) = N$. $\square$

Proof of Lemma OS.6. Take the derivatives of $\pi(\lambda,n)$ with respect to λ and n, respectively. We have $\frac{\partial\pi(\lambda,n)}{\partial\lambda} = v - \frac{h}{n\mu-\lambda} - \frac{h\lambda}{(n\mu-\lambda)^2} = v - \frac{hn\mu}{(n\mu-\lambda)^2}$, and $\frac{\partial^2\pi(\lambda,n)}{\partial\lambda^2} = -\frac{2hn\mu}{(n\mu-\lambda)^3} < 0$. Thus, for any given n, $\pi(\lambda,n)$ is concave in λ, and the maximum value is uniquely attained at $\lambda = \big(1 - \sqrt{\frac{h}{vn\mu}}\big)n\mu = \bar{\Lambda}_m(n)$. We also have $\frac{\partial\pi(\lambda,n)}{\partial n} = \frac{h\mu\lambda}{(n\mu-\lambda)^2} - r$ and $\frac{\partial^2\pi(\lambda,n)}{\partial n^2} = \frac{-2h\mu^2\lambda}{(n\mu-\lambda)^3} < 0$. Thus, for any given λ, $\pi(\lambda,n)$ is concave in n, and the maximum value is uniquely attained at $n = \big(1 + \sqrt{\frac{h\mu}{r\lambda}}\big)\frac{\lambda}{\mu} = \underline{\Lambda}_m^{-1}(\lambda)$. From (OA.1), we have $\pi(\bar{\Lambda}_m(n),n) = \big(v - \frac{h}{n\mu-\big(1-\sqrt{\frac{h}{vn\mu}}\big)n\mu}\big)\big(1 - \sqrt{\frac{h}{vn\mu}}\big)n\mu - rn = \big(v - \sqrt{\frac{hv}{n\mu}}\big)\big(1 - \sqrt{\frac{h}{vn\mu}}\big)n\mu - rn = \big[\big(1 - \sqrt{\frac{h}{vn\mu}}\big)^2 v\mu - r\big]n$, and $\pi\big(\lambda, \underline{\Lambda}_m^{-1}(\lambda)\big) = \big[v - \sqrt{\frac{hr}{\mu\lambda}} - \big(1 + \sqrt{\frac{h\mu}{r\lambda}}\big)\frac{r}{\mu}\big]\lambda = \big(v - \frac{r}{\mu} - 2\sqrt{\frac{hr}{\mu\lambda}}\big)\lambda$. It is easy to see that $\pi(\bar{\Lambda}_m(n),n)$ increases in n, and $\pi\big(\lambda, \underline{\Lambda}_m^{-1}(\lambda)\big)$ increases in λ. $\square$

Proof of Lemma OS.7. (i) Denote $f(\Lambda) = v\Lambda^2 - (rN + v\mu N - h)\Lambda + rN^2\mu$. It is easy to check that $\pi(\Lambda,N) \geq 0$ if and only if $f(\Lambda) \leq 0$. Based on Lemma OS.1(i), $\underline{\Lambda}_m(N) \leq \bar{\Lambda}_m(N)$ if and only if $N \geq N_0$. Since we have proved $\underline{\Lambda}_m(N) \leq \Lambda \leq \bar{\Lambda}_m(N)$ in this case, we just need to consider $N \geq N_0$. By checking the discriminant of $f(\Lambda)$, we obtain $f(\Lambda) > 0$ if $N_0 \leq N < \underline{N}$. Thus, we only need to show that $f(\Lambda) \leq 0$ if and only if $\Lambda \geq \hat{\Lambda}_m(N)$ over $\Lambda \in (0, \bar{\Lambda}_m(N)]$. Note $f(\Lambda) = v\big(\Lambda - \frac{1}{2v}\big(rN + v\mu N - h + \sqrt{(rN+vN\mu-h)^2 - 4rvN^2\mu}\big)\big) \cdot \big(\Lambda - \frac{1}{2v}\big(rN + v\mu N - h - \sqrt{(rN+vN\mu-h)^2 - 4rvN^2\mu}\big)\big)$, and we will show $\frac{1}{2v}\big(rN + v\mu N - h + \sqrt{(rN+vN\mu-h)^2 - 4rvN^2\mu}\big) \leq \bar{\Lambda}_m(N)$. We have $\frac{1}{2v}\big(rN + v\mu N - h + \sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\big) - \bar{\Lambda}_m(N) = \frac{1}{2v}\big(rN - v\mu N - h + 2\sqrt{hvN\mu} + \sqrt{(rN+vN\mu-h)^2-4rvN^2\mu}\big) = \sqrt{(\sqrt{v\mu}-\sqrt{r})^2N-h}\sqrt{(\sqrt{v\mu}+\sqrt{r})^2N-h} - (\sqrt{vN\mu}-\sqrt{h})^2 + rN = \sqrt{(\sqrt{v\mu}-\sqrt{r})^2N-h}\sqrt{(\sqrt{v\mu}+\sqrt{r})^2N-h} - ((\sqrt{v\mu}-\sqrt{r})\sqrt{N} - \sqrt{h})((\sqrt{v\mu}+\sqrt{r})\sqrt{N}-\sqrt{h}) \geq 0$, where the inequality holds because $\sqrt{(\sqrt{v\mu}-\sqrt{r})^2N-h} = \sqrt{(\sqrt{v\mu}-\sqrt{r})\sqrt{N}-\sqrt{h}}\sqrt{(\sqrt{v\mu}-\sqrt{r})\sqrt{N}+\sqrt{h}} \geq ((\sqrt{v\mu}-\sqrt{r})\sqrt{N}-\sqrt{h})$ and $\sqrt{(\sqrt{v\mu}+\sqrt{r})^2N-h} = \sqrt{(\sqrt{v\mu}+\sqrt{r})\sqrt{N}-\sqrt{h}}\sqrt{(\sqrt{v\mu}+\sqrt{r})\sqrt{N}+\sqrt{h}} \geq ((\sqrt{v\mu}+\sqrt{r})\sqrt{N}-\sqrt{h})$. Thus, we have $f(\Lambda)\leq 0$ if and only if $\Lambda \geq \hat{\Lambda}_m(N)$.

(ii) By (OA.1) and $\underline{\Lambda}_m^{-1}(\Lambda) = \Big(1+\sqrt{\frac{h\mu}{r\Lambda}}\Big)\frac{\Lambda}{\mu}$, we have $\pi(\Lambda, \underline{\Lambda}_m^{-1}(\lambda)) = \Big(v - \frac{r}{\mu} - 2\sqrt{\frac{hr}{\Lambda\mu}}\Big)\Lambda$. Thus, $\pi(\Lambda, \underline{\Lambda}_m^{-1}(\lambda)) \geq 0$ if and only if $v - \frac{r}{\mu} \geq 2\sqrt{\frac{hr}{\Lambda\mu}}$, or equivalently, $\Lambda \geq \frac{4hr\mu}{(v\mu-r)^2} = \Lambda_0$.

(iii) Recall $\bar{N} = 2h(v\mu+r)/(v\mu-r)^2$, we have $\underline{\Lambda}_m(\bar{N}) = \frac{1}{4}\Big(\sqrt{\frac{h\mu}{r}+4\bar{N}\mu} - \sqrt{\frac{h\mu}{r}}\Big)^2 = \frac{1}{4}\Big(\sqrt{\frac{h\mu}{r}+\frac{8h\mu(v\mu+r)}{(v\mu-r)^2}} - \sqrt{\frac{h\mu}{r}}\Big)^2 = \frac{1}{4}\Big(\sqrt{\frac{h\mu(v\mu-r)^2+8h\mu r(v\mu+r)}{(v\mu-r)^2r}} - \sqrt{\frac{h\mu}{r}}\Big)^2 = \frac{1}{4}\Big(\sqrt{\frac{h\mu(v^2\mu^2+6v\mu r+9r^2)}{(v\mu-r)^2r}} - \sqrt{\frac{h\mu}{r}}\Big)^2 = \frac{1}{4}\Big(\sqrt{\frac{h\mu(v\mu+3r)^2}{(v\mu-r)^2r}} - \sqrt{\frac{h\mu}{r}}\Big)^2 = \frac{1}{4}\Big[\sqrt{\frac{h\mu}{r}}\Big(\frac{v\mu+3r}{v\mu-r}-1\Big)\Big]^2 = \frac{1}{4}\Big(\sqrt{\frac{h\mu}{r}}\frac{4r}{v\mu-r}\Big)^2 = \frac{4hr\mu}{(v\mu-r)^2} = \Lambda_0$. By Lemma OS.1(i), $\underline{\Lambda}_m(N)$ is increasing for $N \geq N_0$. Since $\bar{N} > N_0$, we have $\Lambda_0 < \underline{\Lambda}_m(N)$ if and only if $N > \bar{N}$. □

Proof of Lemma OS.8. To show Lemma OS.8 is equivalent to show: a feasible strategy profile s is not Pareto efficient if and only if there exists $s' \in \mathcal{P}$ such that $\pi_i(s') \geq \pi_i(s)$ for $i = 1,2$ and $\pi_i(s') > \pi_i(s)$ for some $i = 1,2$. The "*sufficiency*" is obvious by the definition of Pareto efficient. Next, we will show the "*necessity*". In other words, for a feasible strategy profile s is not Pareto efficient, we find $s' \in \mathcal{P}$ such that $\pi_i(s') \geq \pi_i(s)$ for $i = 1,2$ and $\pi_i(s') > \pi_i(s)$ for some $i = 1,2$.

Denote $B = \{\bar{s}|\pi_i(\bar{s}) \geq \pi_i(s)$ for $i=1,2$ and $\pi_i(\bar{s}) > \pi_i(s)$ for some $i=1,2\}$. Consider $s' \in B$ that satisfies $\pi_1(s') \geq \pi_1(\bar{s})$, for any $\bar{s} \in B$, and $\pi_2(s') \geq \pi_2(\bar{s})$ for any $\bar{s} \in \{\bar{s} \in B|\pi_1(\bar{s}) = \pi_1(s')\}$. Thus, s' must be Pareto efficient. Otherwise, we can find s'', which satisfies $\pi_1(s'') \geq \pi_1(s')$ and $\pi_2(s'') \geq \pi_2(s')$, and $\pi_1(s'') > \pi_1(s')$ or $\pi_2(s'') > \pi_2(s')$ holds. It is easy to see $s'' \in B$. If $\pi_1(s'') > \pi_1(s')$, which contradicts to $\pi_1(s') \geq \pi_1(\bar{s})$ for any $\bar{s} \in B$. If $\pi_1(s'') = \pi_1(s')$, then $\pi_2(s'') > \pi_2(s')$, which contradicts to $\pi_2(s') \geq \pi_2(\bar{s})$ for any $\bar{s} \in \{\bar{s} \in B|\pi_1(\bar{s}) = \pi_1(s')\}$. □

Proof of Lemma OS.9. (i) By Lemma OS.5(i), $\bar{n}$ exists. If a strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ satisfies the conditions in Lemma OA.3(ii) and $n_1^* \leq N/2$, we must have $n_1^* \leq \bar{n}$. Otherwise, we suppose $n_1^* > \bar{n}$. Then, $\bar{\Lambda}_m(n_1^*) + \bar{\Lambda}_m(n_2^*) = \bar{\Lambda}_m(n_1^*) + \bar{\Lambda}_m(N - n_1^*) < \bar{\Lambda}_m(\bar{n}) + \bar{\Lambda}_m(N - \bar{n}) = \Lambda$, where the inequality is due to $\bar{\Lambda}_m(n) + \bar{\Lambda}_m(N-n)$ is decreasing in $\underline{N} \leq n \leq N/2$ from the proof of Lemma OS.4(i). This contradicts to $\bar{\Lambda}_m(n_1^*) + \bar{\Lambda}_m(n_2^*) \geq \lambda_1^* + \lambda_2^* = \Lambda$.

(ii) By Lemma OS.5(ii), $\bar{\lambda}$ exists. If a strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ satisfies the conditions in Lemma OA.3(ii) and $\lambda_1^* \leq \Lambda/2$, we must have $\lambda_1^* \geq \bar{\lambda}$. Otherwise, we suppose $\lambda_1^* < \bar{\lambda}$. Then, $\underline{\Lambda}_m^{-1}(\lambda_1^*) + \underline{\Lambda}_m^{-1}(\lambda_2^*) = \underline{\Lambda}_m^{-1}(\lambda_1^*) + \underline{\Lambda}_m^{-1}(\Lambda - \lambda_1^*) < \underline{\Lambda}_m^{-1}(\bar{\lambda}) + \underline{\Lambda}_m^{-1}(\Lambda - \bar{\lambda}) = N$, where the inequality is due to $\underline{\Lambda}_m^{-1}(\lambda) + \underline{\Lambda}_m^{-1}(\Lambda - \lambda)$ is increasing in $\Lambda_0 \leq \lambda \leq \Lambda/2$ from the proof of Lemma OS.4(iv). This contradicts to $\underline{\Lambda}_m^{-1}(\lambda_1^*) + \underline{\Lambda}_m^{-1}(\lambda_2^*) \geq n_1^* + n_2^* = N$, where the inequality comes from $\underline{\Lambda}_m(n_i^*) \leq \lambda_i^*$, i.e., $n_i^* \leq \underline{\Lambda}_m^{-1}(\lambda_i^*)$. □

Proof of Lemma OS.10. We define problem $P_1^{t'}$ as follows:

$$P_1^{t'}: \qquad \max \qquad \pi_2(\lambda_2, n_2) = \Big(v - \frac{h}{n_2\mu - \lambda_2}\Big)\lambda_2 - rn_2$$
$$s.t. \qquad \pi_1(\lambda_1, n_1) \geq t \text{ and } (\lambda_1, n_1; \lambda_2, n_2) \in S'.$$

For any strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ that satisfying the conditions in Lemma OA.3(ii), if it is the optimal solution of problem $P_1^{t'}$, we show $\pi_1(\lambda_1^*, n_1^*) = t$ by contradiction. Suppose $\pi_1(\lambda_1^*, n_1^*) > t$. Recall $\lambda_1^* \geq \max\{\hat{\Lambda}_m(n_1^*), \underline{\Lambda}_m(n_1^*)\} \geq \hat{\Lambda}_m(n_1^*)$ from Lemma OA.3(ii) and $\pi_1(\hat{\Lambda}_m(n_1^*), n_1^*) = 0$. We thus have $\lambda_1^* > \hat{\Lambda}_m(n_1^*)$ because $\pi_1(\lambda_1^*, n_1^*) > t \geq 0$. We have $\lambda_2^* \leq \bar{\Lambda}_m(n_2^*)$ from Lemma OA.3(ii). Then we must have $\lambda_2^* = \bar{\Lambda}_m(n_2^*)$. If $\lambda_2^* < \bar{\Lambda}_m(n_2^*)$, we can always increase $\pi_2(\lambda_2^*, n_2^*)$ by increasing λ_2^* and decreasing λ_1^* because $\pi_2(\lambda_2^*, n_2^*)$ is increasing in λ_2^* if $\lambda_2^* < \bar{\Lambda}_m(n_2^*)$. Furthermore, We have $n_2^* \leq N - \underline{N}$ from Lemma OA.3(ii). Then we must have $n_2^* = N - \underline{N}$. If $n_2^* < N - \underline{N}$, we can always increase $\pi_2(\lambda_2^*, n_2^*)$ by increasing λ_2^* and n_2^*, and decreasing λ_1^* and n_1^*. Hence, $\Lambda = \lambda_1^* + \lambda_2^* > \hat{\Lambda}_m(n_1^*) + \bar{\Lambda}_m(n_2^*) = \hat{\Lambda}_m(\underline{N}) + \bar{\Lambda}_m(N - \underline{N}) = \bar{\Lambda}_m(\underline{N}) + \bar{\Lambda}_m(N - \underline{N}) = \bar{\Lambda}'(N)$, where the inequality is due to $\lambda_1^* > \hat{\Lambda}_m(n_1^*)$, the second equality is due to $n_1^* = N - n_2^* = \underline{N}$, and the third equality is due to $\hat{\Lambda}_m(\underline{N}) = \bar{\Lambda}_m(\underline{N})$ (Lemma OS.1(ii)), the last equality is due to $\bar{\Lambda}'(N) = \bar{\Lambda}_m(\underline{N}) + \bar{\Lambda}_m(N - \underline{N})$ from Lemma OS.3(i). However, $\Lambda \leq \bar{\Lambda}'(N)$ from Lemma OA.3(ii). This a contradiction.

For any strategy profile $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ that satisfying the conditions in Lemma OA.3(ii), if $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*) \in \mathcal{F}(t)$, then $(\lambda_1^*, n_1^*; \lambda_2^*, n_2^*)$ is also the optimal solution of problem $P_1^{t'}$ from the analysis in the first paragraph. In other words, the optimal objective value of problem $P_1^{t'}$ is equal to that of problem P^t. It is easy to see the optimal objective value of problem $P_1^{t'}$ is decreasing in t, that is $P_t = \pi_2(\lambda_2^*, n_2^*)$ is decreasing in t. Consequently, we get $\pi_2(\lambda_2^*, n_2^*) \leq P_0$. By the symmetry of $\pi_1(\lambda_1, n_1)$ and $\pi_2(\lambda_2, n_2)$, we also have $\pi_1(\lambda_1^*, n_1^*) \leq P_0$. □

Proof of Lemma OS.11. The proof is as same as that of Lemma OA.1 just by replacing v, μ with v_i, μ_i, respectively. □

Proof of Lemma OS.12. The proof is as same as that of Lemma OA.2 just by replacing v, μ with v_i, μ_i, respectively. □

Proof of Proposition OS.1. From the proof of Proposition 8, for any feasible strategy profile profile $(p_1, w_1; p_2, w_2) \in S_h$, if $(p_1, w_1; p_2, w_2)$ is Pareto efficient, then (p_i, w_i) must be the optimal solution of problem $P'_{i_h}|_{(p_j, w_j)}$ for $i, j = 1, 2$ and $i \neq j$. By substituting (OS.1) and (OS.2) into problem $P'_{i_h}|_{(p_j, w_j)}$, the problem can be rewritten as problem $P_{i_h}|_{(\lambda_j, n_j)}$. Then, we should identify Pareto-efficient strategy profiles from the optimal solution of problem $P_{i_h}|_{(\lambda_j, n_j)}$.

The proof is very similar to Lemma OA.3 and Proposition 2, just by replacing v, μ with v_i, μ_i, respectively. Hence, we get Proposition OS.1 by replacing $\bar{\Lambda}_m(N)$, $\underline{\Lambda}_m^{-1}(\Lambda)$, $\hat{\Lambda}_m(N)$, $\underline{N}$, Λ_0 in Proposition 2 with $\bar{\Lambda}_{m_i}(N)$, $\underline{\Lambda}_{m_i}^{-1}(\Lambda)$, $\hat{\Lambda}_{m_i}(N)$, $\underline{N}_i$, Λ_{0_i}, respectively. □

Proof of Lemma OS.13. Similar to the proof of Lemma OA.1, we first construct a problem by relaxing the "binding" constraints in problem $P'_{i_w}|_{(p_j, w_j)}$ as follows:

$$
\begin{aligned}
\bar{P}'_{i_w}|_{(p_j, w_j)}: \quad \max_{p_i, w_i} \quad & \pi_i = (p_i - w_i)\lambda_i \\
& \frac{\lambda_1 w_1}{n_1} = \frac{\lambda_2 w_2}{n_2} \geq r, \\
& v - p_1 - hW(\lambda_1, n_1) = v - p_2 - hW(\lambda_2, n_2) \geq 0, \\
& n_1 + n_2 \leq N, \quad \lambda_1 + \lambda_2 \leq \Lambda, \\
& n_i > 0, \quad \lambda_i > 0, \quad p_i > 0, \quad w_i > 0, \quad p_i - w_i \geq 0, \quad i = 1, 2.
\end{aligned}
$$

Then, it suffices to show that the optimal solutions of the problem $\bar{P}'_{i_w}|_{(p_j, w_j)}$ satisfy (OS.3) and (OS.4).

Without loss of generality, we consider the case with p_2 and w_2 given and show the optimal solution of problem $\bar{P}'_{1_w}|_{(p_2,w_2)}$ satisfies (OS.3) and (OS.4). It is sufficient to show that the first-order necessary condition of problem $\bar{P}'_{1_w}|_{(p_2,w_2)}$ satisfies (OS.3) and (OS.4). We define the Lagrangian function for the problem $\bar{P}'_{1_w}|_{(p_2,w_2)}$ as follows:

$$\begin{aligned}L_d =&(p_1-w_1)\lambda_1+\eta_1\big(\frac{\lambda_1 w_1}{n_1}-\frac{\lambda_2 w_2}{n_2}\big)+\eta_2\big(p_1+hW(\lambda_1,n_1)-p_2-hW(\lambda_2,n_2)\big)+\eta_3\big(\frac{\lambda_1 w_1}{n_1}-r\big)\\&+\eta_4\big(v-hW(\lambda_1,n_1)-p_1\big)+\eta_5(N-n_1-n_2)+\eta_6(\Lambda-\lambda_1-\lambda_2)+\eta_7(p_1-w_1),\end{aligned}$$

where $\eta_3\geq 0$, $\eta_4\geq 0$, $\eta_5\geq 0$, $\eta_6\geq 0$, $\eta_7\geq 0$. The optimal solution of problem $\bar{P}'_{1_w}|_{(p_2,w_2)}$, denoted as (p_1^*,w_1^*), shall satisfy the first-order conditions of the Lagrangian function, i.e.,

$$\frac{\partial L_d}{\partial p_1}\Big|_{p_1=p_1^*,w_1=w_1^*}=\lambda_1^*+\eta_2-\eta_4+\eta_7=0, \tag{OS.37}$$

$$\frac{\partial L_d}{\partial w_1}\Big|_{p_1=p_1^*,w_1=w_1^*}=-\lambda_1^*+(\eta_1+\eta_3)\frac{\lambda_1^*}{n_1^*}-\eta_7=0, \tag{OS.38}$$

$$\frac{\partial L_d}{\partial n_1}\Big|_{p_1=p_1^*,w_1=w_1^*}=-(\eta_1+\eta_3)\frac{\lambda_1^* w_1^*}{n_1^{*2}}+(\eta_2-\eta_4)h\frac{\partial W(\lambda_1,n_1)}{\partial n_1}\Big|_{\lambda_1=\lambda_1^*,n_1=n_1^*}-\eta_5=0, \tag{OS.39}$$

$$\frac{\partial L_d}{\partial n_2}\Big|_{p_1=p_1^*,w_1=w_1^*}=\eta_1\frac{\lambda_2^* w_2}{n_2^{*2}}-\eta_2 h\frac{\partial W(\lambda_2,n_2)}{\partial n_2}\Big|_{\lambda_2=\lambda_2^*,n_2=n_2^*}-\eta_5=0, \tag{OS.40}$$

$$\frac{\partial L_d}{\partial \lambda_1}\Big|_{p_1=p_1^*,w_1=w_1^*}=p_1^*-w_1^*+(\eta_1+\eta_3)\frac{w_1^*}{n_1^*}+(\eta_2-\eta_4)h\frac{\partial W(\lambda_1,n_1)}{\partial \lambda_1}\Big|_{\lambda_1=\lambda_1^*,n_1=n_1^*}-\eta_6=0, \tag{OS.41}$$

$$\frac{\partial L_d}{\partial \lambda_2}\Big|_{p_1=p_1^*,w_1=w_1^*}=-\eta_1\frac{w_2}{n_2^*}-\eta_2 h\frac{\partial W(\lambda_2,n_2)}{\partial \lambda_2}\Big|_{\lambda_2=\lambda_2^*,n_2=n_2^*}-\eta_6=0. \tag{OS.42}$$

Note that in the above equations, n_i^* and λ_i^* are the corresponding values given p_i^* and w_i^*. Besides the first-order necessary conditions, the following complementary slackness conditions shall also be satisfied. $\eta_3\big(\frac{\lambda_1^* w_1^*}{n_1^*}-r\big)=0$, $\eta_4\big(v-\frac{h}{n_1^*\mu-\lambda_1^*}-p_1^*\big)=0$. Thus, to establish the proof, it is equivalent to show that $\eta_3>0$ and $\eta_4>0$. By (OS.38), we have $\eta_3=\frac{(\lambda_1^*+\eta_7)n_1^*}{\lambda_1^*}-\eta_1\geq 0$. Thus, we have $\eta_3>0$ if $\eta_1\leq 0$ because $\eta_7\geq 0$, $\lambda_1^*>0$ and $n_1^*>0$. By (OS.37), we have $\eta_4=\lambda_1^*+\eta_2+\eta_7\geq 0$. Thus, we have $\eta_4>0$ if $\eta_2\geq 0$ because $\eta_7\geq 0$ and $\lambda_1^*>0$.

Now we will show $\eta_1\leq 0$ and $\eta_2\geq 0$ to complete the proof. Based on (3), we get

$$\frac{\partial W(\lambda_2,n_2)}{\partial \lambda_2}\Big|_{\lambda_2=\lambda_2^*,n_2=n_2^*}=\frac{[(n_2^*\mu-\lambda_2^*)(\sqrt{2(n_2^*+1)}-1)/\lambda_2^*+1](\lambda_2^*/(n_2^*\mu))^{\sqrt{2(n_2^*+1)}-1}}{(n_2^*\mu-\lambda_2^*)^2},$$

$$\begin{aligned}&\frac{\partial W(\lambda_2,n_2)}{\partial n_2}\Big|_{\lambda_2=\lambda_2^*,n_2=n_2^*}\\=&\frac{[(n_2^*\mu-\lambda_2^*)(-(\sqrt{2(n_2^*+1)}-1)/n^*+\ln(\lambda_2^*/(n_2^*\mu))/\sqrt{2(n_2^*+1)})-\mu](\lambda_2^*/(n_2^*\mu))^{\sqrt{2(n_2^*+1)}-1}}{(n_2^*\mu-\lambda_2^*)^2}.\end{aligned}$$

Denote $a(\lambda,n)=(n\mu-\lambda)(\sqrt{2(n+1)}-1)/\lambda+1$ and $b(\lambda,n)=(n\mu-\lambda)(-(\sqrt{2(n+1)}-1)/n+\ln(\lambda/(n\mu))/\sqrt{2(n+1)})-\mu$, then

$$\frac{\partial W(\lambda_2,n_2)}{\partial \lambda_2}\Big|_{\lambda_2=\lambda_2^*,n_2=n_2^*}=\frac{a(\lambda_2^*,n_2^*)(\lambda_2^*/(n_2^*\mu))^{\sqrt{2(n_2^*+1)}-1}}{(n_2^*\mu-\lambda_2^*)^2}, \tag{OS.43}$$

$$\frac{\partial W(\lambda_2,n_2)}{\partial n_2}\Big|_{\lambda_2=\lambda_2^*,n_2=n_2^*}=\frac{b(\lambda_2^*,n_2^*)(\lambda_2^*/(n_2^*\mu))^{\sqrt{2(n_2^*+1)}-1}}{(n_2^*\mu-\lambda_2^*)^2}. \tag{OS.44}$$

Plugging (OS.43) and (OS.44) into (OS.40) and (OS.42), we obtain the following equations.

$$\Big(\frac{\lambda_2^* a(\lambda_2^*, n_2^*)}{n_2^*} + b(\lambda_2^*, n_2^*)\Big)\frac{w_2}{n_2^*}\eta_1 = a\eta_5 - b\eta_6, \tag{OS.45}$$

$$\frac{h(\lambda_2^*/(n_2^*\mu))^{\sqrt{2(n_2^*+1)}-1}}{(n_2^*\mu - \lambda_2^*)^2}\Big(\frac{\lambda_2^* a(\lambda_2^*, n_2^*)}{n_2^*} + b(\lambda_2^*, n_2^*)\Big)\eta_2 = -\big(\eta_5 + \frac{\lambda_2^*}{n_2^*}\eta_6\big). \tag{OS.46}$$

By the definitions of a and b, we get

$$\begin{aligned}
&\frac{\lambda_2^* a(\lambda_2^*, n_2^*)}{n_2^*} + b(\lambda_2^*, n_2^*) \\
=&(n_2^*\mu - \lambda_2^*)(\sqrt{2(n_2^*+1)} - 1)/n_2^* + \lambda_2^*/n_2^* + (n_2^*\mu - \lambda_2^*)(-(\sqrt{2(n_2^*+1)} - 1)/n_2^* \\
&+ \ln(\lambda_2^*/(n_2^*\mu))/\sqrt{2(n_2^*+1)}) - \mu \\
=&\lambda_2^*/n_2^* - \mu + (n_2^*\mu - \lambda_2^*)\ln(\lambda_2^*/(n_2^*\mu))/\sqrt{2(n_2^*+1)} < 0,
\end{aligned}$$

where the inequality is due to $n_2^*\mu - \lambda_2^* > 0$ and $n_2^* > 0$. It is easy to check $a(\lambda_2^*, n_2^*) > 0$ and $b(\lambda_2^*, n_2^*) < 0$ for $n_2^*\mu - \lambda_2^* > 0$. Together with $\eta_5 \geq 0$ and $\eta_6 \geq 0$, we have $\eta_1 \leq 0$ and $\eta_2 \geq 0$ by (OS.45) and (OS.46). □

Proof of Lemma OS.14. We establish the proof by contradiction. Suppose $n_1^* + n_2^* < N$ and $\lambda_1^* + \lambda_2^* < \Lambda$. Then, from the complementary slackness conditions $\eta_5(N - n_1 - n_2) = 0$ and $\eta_6(\Lambda - \lambda_1 - \lambda_2) = 0$ in proof of Lemma OS.13, we must have $\eta_5 = 0$ and $\eta_6 = 0$. Substituting $\eta_5 = 0$ to (OS.45), we obtain $\eta_1 = 0$ because $\frac{\lambda_2^* a(\lambda_2^*, n_2^*)}{n_2^*} + b(\lambda_2^*, n_2^*) < 0$, $w_2 > 0$ and $n_2^* > 0$. Substituting $\eta_6 = 0$ to (OS.46), we obtain $\eta_1 = 0$ because $\frac{\lambda_2^* a(\lambda_2^*, n_2^*)}{n_2^*} + b(\lambda_2^*, n_2^*) < 0$. Substituting $\eta_1 = 0$, $\eta_2 = 0$ and $\eta_5 = 0$ to (OS.39), we get

$$\eta_3\frac{\lambda_1^* w_1^*}{n_1^{*2}} - \eta_4 h\frac{\partial W(\lambda_1, n_1)}{\partial n_1}\Big|_{\lambda_1=\lambda_1^*, n_1=n_1^*} = 0. \tag{OS.47}$$

Substituting $\eta_1 = 0$, $\eta_2 = 0$ and $\eta_6 = 0$ to (OS.41), we get

$$p_1^* - w_1^* + \eta_3\frac{w_1^*}{n_1^*} - \eta_4 h\frac{\partial W(\lambda_1, n_1)}{\partial \lambda_1}\Big|_{\lambda_1=\lambda_1^*, n_1=n_1^*} = 0. \tag{OS.48}$$

Similarly to (OS.43) and (OS.44), we get

$$\frac{\partial W(\lambda_1, n_1)}{\partial \lambda_1}\Big|_{\lambda_1=\lambda_1^*, n_1=n_1^*} = \frac{a(\lambda_1^*, n_1^*)(\lambda_1^*/(n_1^*\mu))^{\sqrt{2(n_1^*+1)}-1}}{(n_1^*\mu - \lambda_1^*)^2}, \tag{OS.49}$$

$$\frac{\partial W(\lambda_1, n_1)}{\partial n_1}\Big|_{\lambda_1=\lambda_1^*, n_1=n_1^*} = \frac{b(\lambda_1^*, n_1^*)(\lambda_1^*/(n_1^*\mu))^{\sqrt{2(n_1^*+1)}-1}}{(n_1^*\mu - \lambda_1^*)^2}. \tag{OS.50}$$

Plugging (OS.49) and (OS.50) into (OS.47) and (OS.48), we obtain

$$p_1^* - w_1^* = \frac{h(\lambda_1^*/(n_1^*\mu))^{\sqrt{2(n_1^*+1)}-1}}{(n_1^*\mu - \lambda_1^*)^2}\Big(\frac{\lambda_1^* a(\lambda_1^*, n_1^*)}{n_1^*} + b(\lambda_1^*, n_1^*)\Big)\eta_4. \tag{OS.51}$$

On the one hand, we can prove $\frac{\lambda_1^* a(\lambda_1^*, n_1^*)}{n_1^*} + b(\lambda_1^*, n_1^*) < 0$ (the analysis is as same as proving $\frac{\lambda_2^* a(\lambda_2^*, n_2^*)}{n_2^*} + b(\lambda_2^*, n_2^*) < 0$ in the former analysis). On the other hand, we have shown $\eta_4 > 0$ in the former analysis. We thus have $p_1^* - w_1^* < 0$ from (OS.51). This is a contradiction. Therefore, we get the announced results. □

Proof of Proposition OS.2. From the proof of Proposition 10, for any feasible strategy profile $(p_1, w_1; p_2, w_2) \in S_w$, if $(p_1, w_1; p_2, w_2)$ is Pareto efficient, then (p_i, w_i) must be the optimal solution of problem $P'_{i_w}|_{(p_j, w_j)}$ for $i, j = 1, 2$ and $i \neq j$. By substituting (OS.3) and (OS.4) into problem $P'_{i_w}|_{(p_j, w_j)}$, the problem can be rewritten as problem $P_{i_w}|_{(\lambda_j, n_j)}$. Then, we should identify Pareto-efficient strategy profiles from the optimal solution of problem $P_{i_w}|_{(\lambda_j, n_j)}$.

The proof is very similar to Lemma OA.3 and Proposition 2, just by replacing $\frac{1}{n\mu-\lambda}$ with $W(\lambda, n)$. We can check that $\bar{\Lambda}_m(N)$ satisfies

$$v - \frac{h}{N\mu - \bar{\Lambda}_m(N)} - h\bar{\Lambda}_m(N) \cdot \frac{\partial \frac{1}{N\mu - \Lambda}}{\partial \Lambda}\bigg|_{\Lambda = \bar{\Lambda}_m(N)} = 0,$$

$\underline{\Lambda}_m^{-1}(\Lambda)$ satisfies

$$h\Lambda \cdot \frac{\partial \frac{1}{N\mu - \Lambda}}{\partial N}\bigg|_{N = \underline{\Lambda}_m^{-1}(\Lambda)} + r = 0,$$

$\hat{\Lambda}_m(N)$ satisfies

$$\Big(v - \frac{h}{N\mu - \hat{\Lambda}_m(N)}\Big)\hat{\Lambda}_m(N) - rN = 0,$$

$\underline{N}$ satisfies

$$\Big(v - \frac{h}{\underline{N}\mu - \bar{\Lambda}_m(\underline{N})}\Big)\bar{\Lambda}_m(\underline{N}) - r\underline{N} = 0,$$

Λ_0 satisfies

$$\Big(v - \frac{h}{\underline{\Lambda}_m^{-1}(\Lambda_0)\mu - \Lambda_0}\Big)\Lambda_0 - r\underline{\Lambda}_m^{-1}(\Lambda_0) = 0.$$

Hence, we get Proposition OS.2 by replacing $\bar{\Lambda}_m(N)$, $\underline{\Lambda}_m^{-1}(\Lambda)$, $\hat{\Lambda}_m(N)$, $\underline{N}$, Λ_0 into Lemma OA.3 and Proposition 2 with $\bar{\Lambda}_{m_w}(N)$, $\underline{\Lambda}_{m_w}^{-1}(\Lambda)$, $\hat{\Lambda}_{m_w}(N)$, $\underline{N}_w$, Λ_{0_w}, respectively. $\square$