Online Appendix

Here we only provide proofs for main theorems and propositions. Sections, corollaries, and remarks marked with a "*" can be found in the full version of the paper via `https://papers.ssrn.com/abstract=3477339`.

EC.1. Proofs for Section 3

Proof of Theorem 1.

We first introduce the *Fixed Job Scheduling Problem* (FJSP) as follows.

Definition EC.1 (Fixed Job Scheduling Problem). Consider D jobs, each of which starts at time s_d and ends at time e_d, and N machines, each of which has availabilities from time a_k to b_k. A *schedule* is an assignment of jobs to machines such that for the jobs d assigned to the same machine k, the corresponding intervals $[s_d, e_d)$ do not overlap and $a_k \leqslant s_d < e_d \leqslant b_k$ for all these jobs. The task of FJSP is to determine whether a *schedule* exists.

In Brucker and Nordmann (1994), the authors showed that FJSP is NP-Hard. In the following, to show Problem (1) is NP-Hard, we construct a polynomial time reduction from FJSP to Problem (1).

Given an instance of the FJSP described above, we first sort all endpoints $\{s_d\}$, $\{e_d\}$ and $\{a_j\}$, $\{b_j\}$ in an increasing manner and delete duplicate values. This can be done in $O((D+N)\log(D+N))$ time. Set M to be the length of such sequence minus 1 (note that $M \leq 2(D+N)$). Then we can rewrite the sequence as $t_0 < \cdots < t_M$ and the sequence partitions $[t_0, t_M)$ into M consecutive intervals. We regard each interval $[t_{\ell-1}, t_\ell)$ as the ℓ^{th} leg and each endpoint t_ℓ as the ℓ^{th} stop. We regard each job as a request of itinerary $i \to j$ if and only if it occupies legs from i to j. We also represent each machine by a $\{0,1\}^{1\times M}$ vector indicating whether it is occupied in each leg, and thus treat each machine as a seat. Let N be the total number of seats and d_{ij} be the total number of requests of type $i \to j$. Let $C \in \{0,1\}^{N\times M}$ be the capacity matrix consisting of all the seats. Then we have conducted a reduction to our setting in polynomial time.

With such a reduction, it is easy to see that, given any positive prices $\{p_{ij}\}$, there exists a *schedule* in FJSP if and only if the optimal value in the reduced problem is $\sum_{i,j} p_{ij} d_{ij}$. Thus, Problem (1) is NP-Hard.

□

Proof of Theorem 2. Before presenting the details, we describe the main ideas and steps. The proof consists of four steps. In the first step, we show that any strongly NSE matrix can be decomposed into non-overlapping groups in polynomial time, with the maximal sequences in each group having a specific structure. In the second step, we show that when the maximal sequences in a capacity matrix satisfy the specific structure, we can solve the aggregate optimization problem (3) and recover a seat assignment from the solution of (3) in polynomial time. The assignment is constructed by assigning the accepted requests in a particular order. In the third step, we show the reverse direction. That is, if C is not strongly NSE, then there always exists a set of demands $\{d_{ij}\}$ such that there is no way to assign the solution of (3). In the last step, we show that given a capacity matrix C that is NSE, we can add a dummy leg between each pair of consecutive legs, converting the problem to one with a strongly NSE capacity matrix. Thus, we see that an optimal solution can be found in polynomial time.

Step 1. Strongly NSE matrices could be decomposed into non-overlapping groups.

In this first step, we prove that, for any strongly NSE matrix C, we can decompose its maximal sequences into W groups such that in the w^{th} group, there is a *dominating* maximal sequence $[u_w, v_w] \sim C$, of which any other $[u, v] \sim C$ in this group satisfy either $u = u_w$ or $v = v_w$. In addition, for each w and the corresponding $[u_w, v_w]$, there exists $\ell_w (u_w \leqslant \ell_w \leqslant v_w)$ such that $C_{k\ell_w} = 1$ if and only if $[u_w, v_w] \sim C_k$. Furthermore, different groups have no intersection in *stop* (not only leg), which means $c_{u_w - 1} = c_{v_w + 1} = 0$.

Figure EC.1 gives all possible types of configurations for a single group. In the first configuration, all maximal sequences start from u_2 and $l_2 = v_2$. In the second configuration, all maximal sequences end at v_3 and $l_3 = u_3$. The last configuration is the most general one where all maximal sequences either start from u_4 or end with v_4. l_4 is the fourth leg such that any seat that has l_4 available must have entire $[u_4, v_4]$ unoccupied.

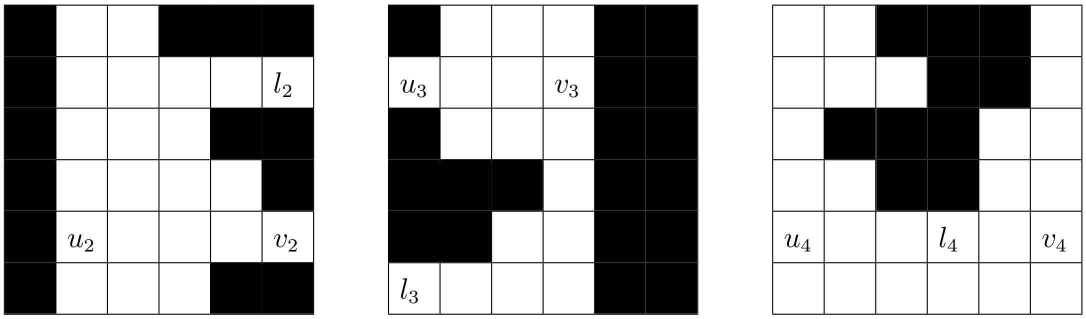

Figure EC.1 Examples of Possible Structures in a Single Group. Black Boxes Indicate Occupied Seats and White Boxes Indicate Available Seats.

Now we show the maximal sequences of a strongly NSE matrix can be decomposed into such groups. Let C be a strongly NSE matrix. Let $w = 1$ and $\tau = 0$. Let

$$u_w = \min\{u > \tau | \exists v : [u, v] \sim C\} \text{ and } v_w = \max\{v | [u_w, v] \sim C\}.$$

If u_w does not exist, then we terminate. Otherwise, for any $[u, v] \sim C$ with $u > \tau$, because C is strongly NSE, one of the followings must be true:

1. $u_w \leqslant v_w < u - 1$
2. $u_w = u \leqslant v \leqslant v_w$
3. $u_w \leqslant u \leqslant v = v_w$

We consider the latter two types of maximal sequences. Let

$$V_w = \{v < v_w | [u_w, v] \sim C\} \quad \text{and} \quad U_w = \{u > u_w | [u, v_w] \sim C\}.$$

Let $v'_w = \max V_w$ and $u'_w = \min U_w$. If $V_w = \emptyset$, we let $v'_w = u_w - 1$. If $U_w = \emptyset$, we let $u'_w = v_w + 1$. Since C is strongly NSE, we must have $v'_w + 1 < u'_w$. Let $\ell_w = v'_w + 1$, then $u_w \leqslant \ell_w \leqslant v_w$ and l_w is available in some seat k only if $[u_w, v_w] \sim C_k$. We represent group w as (u_w, ℓ_w, v_w). Now we let $\tau = v_w + 1$, $w \leftarrow w + 1$, and repeat the procedures above until we terminate. Note that the above procedures consume at most $O(M^3)$ time in total.

Step 2. If C is *strongly NSE*, then any integral optimal solution of (3) could be transformed into an assignment of (1) in polynomial time.

Suppose we have obtained an optimal integral solution $\{x^*_{ij}\}$ of (3) and a characterization of C in Step 2: $\{(u_1, \ell_1, v_1), ..., (u_W, \ell_W, v_W)\}$. In the following, we show that $\{x^*_{ij}\}$ can be turned into a feasible solution $\{x^*_{k,ij}\}$ of (1) with the same objective value in polynomial time. The detailed algorithm is shown in Algorithm 6.

The idea of Algorithm 6 is to assign the requests sequentially according to a particular order of the legs. More concretely, in each group w, we divide all requests $i \to j$ into 3 cases: (i) $i \leqslant \ell_w \leqslant j$, (ii) $\ell_w < i \leqslant j$, (iii) $i \leqslant j < \ell_w$. Note that in all these cases we always have $u_w \leqslant i \leqslant j \leqslant v_w$. We first start from leg ℓ_w. We allocate the requests in case (i) with an arbitrary order. Each request $i \to j$ occupies leg ℓ_w and thus is located on different seats. Then we continue with other legs by gradually moving from leg ℓ_w to two endpoints u_w or v_w. For case (ii), we allocate the requests according to the ascending order of $i = \ell_w + 1, ..., v_w$. For case (iii), we allocate the requests according to the

Algorithm 6: Assignment Algorithm

for $w=1,...,W$ **do**
    `/* Carry out Assignment in the` w^{th} `group. */`
    for $(i,j): i \leqslant \ell_w \leqslant j$ **do**
        `/* Assign the` x^*_{ij} `requests that occupy leg` ℓ_w`. */`
        Retrieve $\mathcal{K}_{ij}$ as a subset of $\mathcal{M}_{u_w v_w}(C)$, such that $|\mathcal{K}_{ij}| = x^*_{ij}$;
        Assign x^*_{ij} requests of type $i \to j$ to seats in $\mathcal{K}_{ij}$. Let $x^*_{k,ij} = 1, \forall k \in \mathcal{K}_{ij}$;
        $\mathcal{M}_{u_w v_w}(C) \leftarrow \mathcal{M}_{u_w v_w}(C) \setminus \mathcal{K}_{ij}$;
        if $u_w < i$ **then** $\mathcal{M}_{u_w(i-1)}(C) \leftarrow \mathcal{M}_{u_w(i-1)}(C) \cup \mathcal{K}_{ij}$;
        if $v_w > j$ **then** $\mathcal{M}_{(j+1)v_w}(C) \leftarrow \mathcal{M}_{(j+1)v_w}(C) \cup \mathcal{K}_{ij}$;
    end
    for $i = \ell_w + 1, ..., v_w$ **do**
        `/* Assign the` $\sum_{j:j \geqslant i} x^*_{ij}$ `requests that start with leg` i`. */`
        for $j: j \geqslant i$ **do**
            for $z = 1, ..., x^*_{ij}$ **do**
                Retrieve an element k from $\cup_{i':i' \leqslant i} \mathcal{M}_{i' v_w}(C)$. Suppose $[u, v_w] \sim C_k$, then $u \leqslant i$;
                Assign a request $i \to j$ to seat k. Let $x^*_{k,ij} = 1$;
                $\mathcal{M}_{u v_w}(C) \leftarrow \mathcal{M}_{u v_w}(C) \setminus \{k\}$;
                if $u < i$ **then** $\mathcal{M}_{u(i-1)}(C) \leftarrow \mathcal{M}_{u(i-1)}(C) \cup \{k\}$;
                if $v_w > j$ **then** $\mathcal{M}_{(j+1)v_w}(C) \leftarrow \mathcal{M}_{(j+1)v_w}(C) \cup \{k\}$;
            end
        end
    end
    for $j = \ell_w - 1, ..., u_w$ **do**
        `/* Assign the` $\sum_{i:i \leqslant j} x^*_{ij}$ `requests that end with leg` j`. */`
        for $i: i \leqslant j$ **do**
            for $z = 1, ..., x^*_{ij}$ **do**
                Retrieve an element k from $\cup_{j':j' \geqslant \ell} \mathcal{M}_{u_w j'}(C)$. Suppose $[u_w, v] \sim C_k$, then $v \geqslant j$;
                Assign a request $i \to j$ to seat k. Let $x^*_{k,ij} = 1$;
                $\mathcal{M}_{u_w v}(C) \leftarrow \mathcal{M}_{u_w v}(C) \setminus \{k\}$;
                if $u_w < i$ **then** $\mathcal{M}_{u_w(i-1)}(C) \leftarrow \mathcal{M}_{u_w(i-1)}(C) \cup \{k\}$;
                if $v > j$ **then** $\mathcal{M}_{(j+1)v}(C) \leftarrow \mathcal{M}_{(j+1)v}(C) \cup \{k\}$;
            end
        end
    end
end

descending order of $j = \ell_w - 1, ..., u_w$. In Algorithm 6, the seemingly complex operation on $\{\mathcal{M}_{uv}(C)\}$ is to track the dynamics of maximal sequences in C.

In the following, we demonstrate that each time we assign requests that include leg ℓ, there are enough seats to assign those requests, or in other words, the procedure in Algorithm 6 is valid. Since different groups are disjoint in stop, there will be no requests that cross two groups. Hence we only need to show within each group, we can assign all requests.

For the w^{th} group, we first assign $\sum_{(i,j):i\leqslant \ell_w\leqslant j} x^*_{ij}$ requests that occupy leg ℓ_w to different seats (Line 2-9 in Algorithm 6). Since $c_{u_w-1}=c_{v_w+1}=0$, any request of type $i\to j$ with $i\leqslant \ell_w\leqslant j$ must satisfy $u_w\leqslant i\leqslant j\leqslant v_w$. Since $\sum_{(i,j):i\leqslant \ell_w\leqslant j} x^*_{ij}\leqslant c_{\ell_w}$, and by the construction in Step 1 in the proof, any seat with leg l_w unoccupied must contain $[u_w,v_w]\sim C$, thus this step is valid.

For $i>\ell_w$ and any seat k, if the i^{th} leg is not occupied before the corresponding iteration, then $C_{ki}=\cdots=C_{kv_w}=1$ and $C_{k(v_w+1)}=c_{v_w+1}=0$. Any request of type $i\to j$ must satisfy $j\leqslant v_w$ and thus could be assigned to seat k. Therefore, we only need to verify that each time when we move to leg i (Line 9-18 in Algorithm 6), the number of requests that start with leg i is no more than the number of seats that still have leg i unoccupied at that iteration. Note that the number of requests that start with i is $\sum_{j:j\geqslant i} x^*_{ij}$, and the number of seats that still have leg i unoccupied is $c_i-\sum_{(i',j):i'<i\leqslant j} x^*_{i'j}$. By the second group of constraints in (3), we have $c_i\geqslant \sum_{(i',j):i'\leqslant i\leqslant j} x^*_{i'j}=\sum_{j:j\geqslant i} x^*_{ij}+\sum_{(i',j):i'<i\leqslant j} x^*_{i'j}$. Therefore, all requests that start with leg i can be assigned.

For $j<\ell_w$ and any seat k, if the j^{th} leg is not occupied before the corresponding iteration, then $C_{kj}=\cdots=C_{ku_w}=1$ and $C_{k(u_w-1)}=c_{u_w-1}=0$. Any request of type $i\to j$ must satisfy $i\geqslant u_w$ and thus could be assigned to seat k. Therefore, we only need to verify that each time when we move to leg j (Line 21-29 in Algorithm 6), the number of requests that end with leg j is no more than the number of seats that still have j unoccupied at that iteration. Note that the number of requests that end with j is $\sum_{i:i\leqslant j} x^*_{ij}$, and the number of seats that still have leg j unoccupied is $c_j-\sum_{(i,j'):i\leqslant j<j'} x^*_{ij'}$. By the second group of constraints in (3), we have $c_j\geqslant \sum_{(i,j'):i\leqslant j\leqslant j'} x^*_{ij'}=\sum_{i:i\leqslant j} x^*_{ij}+\sum_{(i,j'):i\leqslant j<j'} x^*_{ij'}$. Therefore, all requests that end with leg j can be assigned.

Hence, when C is strongly NSE, for any integral optimal solution of (3) we can construct a feasible solution of (1) with the same objective value. Meanwhile, it is easy to see that the optimal value of (3) offers an upper bound to that of (1). Thus, when C is strongly NSE, (1) and (3) have the same optimal value, and (1) can be solved in polynomial time.

Step 3. For any fixed positive $\{p_{ij}\}$, if (1) and (3) have the same optimal value for any nonnegative integers $\{d_{ij}\}$, then C must be strongly NSE.

Suppose the number of $[u,v]\sim C$ is $m_{uv}=|\mathcal{M}_{uv}(C)|$. We construct $\{d_{ij}\}_{i\leqslant j}$ as follows.

$$d_{ij}=\begin{cases} m_{ij}+1, & \text{if } (i,j)=(u_1,v_2) \text{ or } (u_2,v_1)\\ m_{ij}-1, & \text{if } (i,j)=(u_1,v_1) \text{ or } (u_2,v_2)\\ m_{ij}, & \text{otherwise.}\end{cases}$$

It is easy to verify that $x^*_{ij}=d_{ij}$ is an integral optimal solution of (3), which means that, without the assign-to-seat restriction, we can accept all the requests, and the optimal objective value of (3) is $\sum_{i\leqslant j} p_{ij}d_{ij}$.

Now we show that if C is not strongly NSE, then the optimal value of (1) is different from $\sum_{i\leqslant j} p_{ij}d_{ij}$. To show this, suppose C is not strongly NSE, then by definition, there must exist $[u_1,v_1]\sim C$ and $[u_2,v_2]\sim C$ such that one of the following holds (the three cases are illustrated in Figure EC.2):

1. $u_1\leqslant v_1=u_2-1\leqslant v_2-1$
2. $u_1<u_2\leqslant v_1<v_2$
3. $u_1<u_2\leqslant v_2<v_1$

Now we consider each of the three cases. For the first and second cases, consider

$$\sum_k \sum_{(i,j):i\leqslant u_1<v_2\leqslant j} x_{k,ij},$$

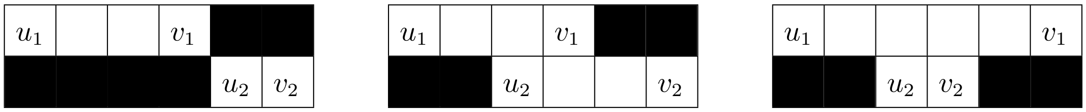

Figure EC.2 Illustrations of the Three Cases When C is not Strongly NSE.

which is the total possible number of accepted requests that occupy leg u_1 to v_2. Such requests can only be assigned to seats with leg u_1 to v_2 unoccupied, and each seat could only be assigned at most one request of these types. Therefore,

$$\sum_k \sum_{\substack{(i,j):\\ i\leqslant u_1<v_2\leqslant j}} x_{k,ij} \leqslant \sum_{\substack{(i,j):\\ i\leqslant u_1<v_2\leqslant j}} m_{ij} = \sum_{\substack{(i,j):\\ i\leqslant u_1<v_2\leqslant j}} d_{ij} - 1,$$

where the last inequality follows from $d_{ij} = m_{ij} + 1$ when $(i,j) = (u_1, v_2)$. Therefore, we could not accept all the requests, and thus (1) has strictly smaller optimal value than that of (3).

For the third case, we consider

$$\sum_k \sum_{\substack{(i,j):\\ i\leqslant u_1<v_2\leqslant j\\ \text{or}\\ i\leqslant u_2<v_1\leqslant j}} x_{k,ij},$$

which is the total possible number of accepted requests that occupy leg u_1 to v_2 or leg u_2 to v_1. Such requests can only be assigned to seats with leg u_1 to v_2 or leg u_2 to v_1 unoccupied, and since $[u_1, v_2] \cap [u_2, v_1] \neq \emptyset$, each seat could only be assigned at most one request of these types. Therefore,

$$\sum_k \sum_{\substack{(i,j):\\ i\leqslant u_1<v_2\leqslant j\\ \text{or}\\ i\leqslant u_2<v_1\leqslant j}} x_{k,ij} \leqslant \sum_{\substack{(i,j):\\ i\leqslant u_1<v_2\leqslant j\\ \text{or}\\ i\leqslant u_2<v_1\leqslant j}} m_{ij} = \sum_{\substack{(i,j):\\ i\leqslant u_1<v_2\leqslant j\\ \text{or}\\ i\leqslant u_2<v_1\leqslant j}} d_{ij} - 1,$$

where the last inequality follows from $d_{ij} = m_{ij} + 1$ when $(i,j) = (u_1, v_2)$ or (u_2, v_1), and $d_{ij} = m_{ij} - 1$ when $(i,j) = (u_1, v_1)$. Therefore, we could not accept all the requests, and thus again, (1) has strictly smaller objective value than that of (3).

Therefore, we showed that if C is not strongly NSE, then for any positive prices $\{p_{ij}\}$, there exist a set of nonnegative integers $\{d_{ij}\}$ such that (1) and (3) have different optimal objective values.

Step 4. If C is NSE, then problem (1) can be solved in polynomial time.

Given a capacity matrix C that is NSE, we construct a new capacity matrix $\tilde{C}$ together with new requests and prices. The idea of the construction is to add a *dummy* leg between each pair of neighboring legs in the original problem and to keep the maximal sequence structure. More precisely, we construct $\tilde{C} \in \{0,1\}^{N\times(2M-1)}$, where the number of seats is N but the number of legs is $2M-1$. For any $[u,v] \sim C_k$, we let $\tilde{C}_{k\ell} = 1$ for $2u - 1 \leq \ell \leq 2v - 1$. For the rest of $\tilde{C}_{k\ell}$, we let them be zero. Then for any request of type $i \to j$, we construct a new request of type $\tilde{i} \to \tilde{j}$ with $\tilde{i} = 2i - 1$ and $\tilde{j} = 2j - 1$. This means that in $\tilde{C}$, there are only requests that both start and end with an odd leg.

We claim that if C is NSE, then $\tilde{C}$ must be strongly NSE. To prove the claim, for any two disjoint $[\tilde{u}_1, \tilde{v}_1] \sim \tilde{C}$ and $[\tilde{u}_2, \tilde{v}_2] \sim \tilde{C}$, we have all of $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{v}_1$, and $\tilde{v}_2$ are odd numbers, thus if $\tilde{u}_2 > \tilde{v}_1$ ($\tilde{u}_1 > \tilde{v}_2$, resp.), then it must be that $\tilde{u}_2 > \tilde{v}_1 + 1$ ($\tilde{u}_1 > \tilde{v}_2 + 1$, resp.). Also, a request of type $i \to j$ can be assigned to a $[u,v] \sim C_k$ if and only if $\tilde{i} \to \tilde{j}$ can be assigned to a $[\tilde{u}, \tilde{v}] \sim \tilde{C}_k$.

For all $(\tilde{i}, \tilde{j}) : 1 \leqslant \tilde{i} \leqslant \tilde{j} \leqslant 2M - 1$, if both $\tilde{i}$ and $\tilde{j}$ are odd numbers, then we let

$$\tilde{p}_{\tilde{i}\tilde{j}} = p_{(\frac{\tilde{i}+1}{2})(\frac{\tilde{j}+1}{2})}, \tilde{d}_{\tilde{i}\tilde{j}} = d_{(\frac{\tilde{i}+1}{2})(\frac{\tilde{j}+1}{2})}, \tilde{\mathcal{M}}_{\tilde{i}\tilde{j}}(\tilde{C}) = \mathcal{M}_{(\frac{\tilde{i}+1}{2})(\frac{\tilde{j}+1}{2})}(C).$$

Otherwise, we let $\tilde{p}_{\tilde{i}\tilde{j}}=1$, $\tilde{d}_{\tilde{i}\tilde{j}}=0$ and $\tilde{\mathcal{M}}_{\tilde{i}\tilde{j}}(C)=\emptyset$. To construct an assignment, we first solve

$$\begin{aligned}
\text{maximize} \quad & \sum_{\tilde{i},\tilde{j}} \tilde{p}_{\tilde{i}\tilde{j}}\tilde{x}_{\tilde{i}\tilde{j}} \\
\text{subject to} \quad & 0 \leqslant \tilde{x}_{\tilde{i}\tilde{j}} \leqslant \tilde{d}_{\tilde{i}\tilde{j}}, & \forall (\tilde{i},\tilde{j}): 1 \leqslant \tilde{i} \leqslant \tilde{j} \leqslant 2M-1, \\
& \sum_{(\tilde{i},\tilde{j}):\tilde{i}\leqslant\tilde{\ell}\leqslant\tilde{j}} \tilde{x}_{\tilde{i}\tilde{j}} \leqslant \sum_k \tilde{C}_{k\tilde{\ell}} = \tilde{c}_{\tilde{\ell}}, & \forall \tilde{\ell} \in [2M-1],
\end{aligned}$$

and obtain an integral optimal solution $\{\tilde{x}^*_{\tilde{i}\tilde{j}}\}$. We then can obtain an assignment $\{\tilde{x}^*_{k,\tilde{i}\tilde{j}}\}$ using Step 1 and Algorithm 6 with $\{\tilde{x}^*_{\tilde{i}\tilde{j}}\}$ and $\{\tilde{\mathcal{M}}_{\tilde{u}\tilde{v}}(\tilde{C})\}$. During the process, we update $x^*_{k,ij}$ and $\tilde{x}^*_{k,\tilde{i}\tilde{j}}$ simultaneously so that $x^*_{k,(\frac{\tilde{i}+1}{2})(\frac{\tilde{j}+1}{2})}=\tilde{x}^*_{k,\tilde{i}\tilde{j}}$. As a result, we get an integral optimal solution of (1) in polynomial time.

Combining the four steps, we have shown that if C is NSE, then (1) can be solved in polynomial time. Furthermore, for any fixed positive prices $\{p_{ij}\}$, (1) and (3) have the same optimal value for any nonnegative integers $\{d_{ij}\}$ if and only if C is strongly NSE. Thus Theorem 2 is proved.

□

Proof of Corollary 1.*

EC.2. Equivalent Formulations and Approximation Algorithms*

EC.3. Booking Limit Control Policy*

EC.4. Appendix for Section 4

EC.4.1. Appendix for Section 4.1*

DP based on the capacity matrix.* Derivation of (12).*

Proof of Theorem 3. Fix any capacity matrix C. For simplicity, we abuse some notations. We let (6) represent the objective value of (6). We also let (12) represent the objective value of (12) when $A=f(C)$.

We first testify that $\{\beta^{\dagger t}_{uv}\}$ defined in (13) together with $\{z^{\dagger}_{ij}\}=\{z_{ij}\}$ in (12) satisfy the constraints in (12). That is, the objective value of (12) is no larger than that of (6). It is easy to see that $\beta^{\dagger t}_{uv}$ is nonnegative. For fixed $u_0 \leqslant v_0$, we let $k_0 \in [N]$ be an integer such that

$$\sum_{\ell=u_0}^{v_0} \beta^t_{k_0\ell} = \min_{k'\in[N]}\{\sum_{\ell=u_0}^{v_0}\beta^t_{k'\ell}\}$$

Then

$$\begin{aligned}
&\beta^{\dagger t}_{u_0v_0} - \beta^{\dagger t}_{u_0(i-1)} - \beta^{\dagger t}_{(j+1)v_0} \\
=&\sum_{\ell=u_0}^{v_0}\beta^t_{k_0\ell} - \min_{k'\in[N]}\{\sum_{\ell=u_0}^{i-1}\beta^t_{k'\ell}\} - \min_{k'\in[N]}\{\sum_{\ell=j+1}^{v_0}\beta^t_{k'\ell}\} \\
\geqslant&\sum_{\ell=u_0}^{v_0}\beta^t_{k_0\ell} - \sum_{\ell=u_0}^{i-1}\beta^t_{k_0\ell} - \sum_{\ell=j+1}^{v_0}\beta^t_{k_0\ell} \\
=&\sum_{\ell=i}^{j}\beta^t_{k_0\ell} \geqslant p_{ij} - z^{\dagger}_{ij},
\end{aligned}$$

Thus, (13) is feasible. Moreover, from (13) we can infer that the objective value of (6) is no less than that of (12).

We then point out that $\{z^{\dagger}_{ij}\}$ together with $\{\beta^{\dagger t}_{uv}\}$ form an optimal solution of (12). To prove this, we prove a stronger result: the objective value of (6) is no larger than that of (12). This is done by constructing a feasible solution of (6) from an optimal solution of (12). Let $\{z^{\dagger}_{ij}\}$ and $\{\beta^{\dagger t}_{uv}\}$ be an optimal solution of (12) such that $c=\#\{(u,v): u\leqslant v, \beta^{\dagger t}_{uv} < \beta^{\dagger t}_{(u+1)v)}\}$ achieves its minimum. We will show that $c=0$. Otherwise, let $(u_0,v_0)\in\arg\min_v\{(u,v): u\leqslant v, \beta^{\dagger t}_{uv} < \beta^{\dagger t}_{(u+1)v}\}$. Then for any $u_0+1\leqslant i\leqslant j\leqslant v_0$, we have

$$\begin{aligned}
z^{\dagger}_{ij} + \beta^{\dagger t}_{(u_0+1)v_0} &\geqslant p_{ij} + \beta^{\dagger t}_{(u_0+1)(i-1)} + \beta^{\dagger t}_{(j+1)v_0}, \\
z^{\dagger}_{ij} + \beta^{\dagger t}_{u_0v_0} &\geqslant p_{ij} + \beta^{\dagger t}_{u_0(i-1)} + \beta^{\dagger t}_{(j+1)v_0} \geqslant p_{ij} + \beta^{\dagger t}_{(u_0+1)(i-1)} + \beta^{\dagger t}_{(j+1)v_0}.
\end{aligned}$$

Now we decrease $\beta^{\dagger t}_{(u_0+1)v_0}$ to $\beta^{\dagger t}_{u_0 v_0}$, then the constraints are still not violated, but the objective value will not increase. This is a contradiction.

We let $\{z_{ij}\} = \{z^{\dagger}_{ij}\}$ and construct $\{\beta^t_{k\ell}\}$ as follows,

$$\beta^t_{k\ell} = \begin{cases} \beta^{\dagger t}_{\ell v} - \beta^{\dagger t}_{(\ell+1)v} \geqslant 0, & \text{if } u \leqslant \ell \leqslant v, [u,v] \sim C_k, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then for any $k \in [N]$ and $i \leqslant j$, if there is some $\ell \in [i,j]$ such that $C_{k\ell} = 0$, then we must have

$$z_{ij} + \sum_{\ell : i \leqslant \ell \leqslant j} \beta^t_{k\ell} \geqslant p_{ij}.$$

Else, there must exist a $[u,v] \sim C_k$ such that $u \leqslant i \leqslant j \leqslant v$, and we have

$$z_{ij} + \sum_{\ell : i \leqslant \ell \leqslant j} \beta^t_{k\ell} = z^{\dagger}_{ij} + \beta^{\dagger t}_{iv} - \beta^{\dagger t}_{(j+1)v} \geqslant p_{ij}.$$

In addition,

$$\begin{aligned} & \sum_{i \leqslant j} \lambda_{ij} z_{ij} + \sum_{k \in [N]} \sum_{\ell \in [M]} C_{k\ell} \beta^t_{k\ell} \\ = & \sum_{i \leqslant j} \lambda_{ij} z^{\dagger}_{ij} + \sum_{k \in [N]} \sum_{\substack{(u,v): \\ [u,v] \sim C_k}} \sum_{\ell : u \leqslant \ell \leqslant v} \beta^t_{k\ell} \\ = & \sum_{i \leqslant j} \lambda_{ij} z^{\dagger}_{ij} + \sum_{k \in [N]} \sum_{\substack{(u,v): \\ [u,v] \sim C_k}} \beta^{\dagger t}_{uv} \\ = & \sum_{i \leqslant j} \lambda_{ij} z^{\dagger}_{ij} + \sum_{u \leqslant v} f(C)_{uv} \beta^{\dagger t}_{uv} = (12). \end{aligned}$$

Thus we've completed the proof.

□

EC.4.2. Appendix for Section 4.2

On (14) may not imply (15).*

Proof of Lemma 1. (14) is actually the dual of (12). From Theorem 3, (2) $\geqslant$ (14). Therefore, we only need to prove that (2) $\leqslant$ (14) + (15), which in turn implies our results.

Step 1. In our first step, we show that the optimal objective value of (1), the IP version of (2), is no more than that of (EC.1), the IP version of (14)+(15).

$$\begin{aligned} \text{maximize}_{\gamma} \quad & \sum_{i \leqslant j} \left(p_{ij} \sum_{\substack{(u,v): \\ u \leqslant i \leqslant j \leqslant v}} \gamma_{uijv} \right), && \text{(EC.1)} \\ \text{subject to} \quad & \sum_{\substack{(u,v): \\ u \leqslant i \leqslant j \leqslant v}} \gamma_{uijv} \leqslant d_{ij}, && \forall i \leqslant j, \\ & \sum_{\substack{(i,j): \\ u \leqslant i \leqslant j \leqslant v}} \gamma_{uijv} \leqslant \sum_{\substack{(k,\ell): \\ v+1 \leqslant k \leqslant \ell}} \gamma_{u(v+1)k\ell} + \sum_{\substack{(k,\ell): \\ \ell \leqslant k \leqslant u-1}} \gamma_{\ell k (u-1) v} + A_{uv}, && \forall u \leqslant v, \\ & \gamma_{u i^t j^t v} \leqslant A_{uv}, && \forall u \leqslant i^t \leqslant j^t \leqslant v, \\ & \gamma_{uijv} \in \mathbb{N}, && \forall u \leqslant i \leqslant j \leqslant v. \end{aligned}$$

For any optimal solution $\{x^*_{k,ij}\}$ of (1), we impose an index on all the allocated requests. Here we put the smallest indexes on all $i^t \to j^t$. Now we consider to take out all requests and assign the requests again, but in a *sequential* manner, i.e., we allocate the requests one by one according to the index. During this procedure, we will arrange a tuple $(\cdot,\cdot,\cdot,\cdot)$ to each ticket and track the dynamics.

Suppose we have allocated $t-1$ requests and we are allocating the t^{th} request $i\to j$ into seat k, then seat k must own a unique *maximal sequence* $[u,v]$ such that $u\leqslant i\leqslant j\leqslant v$. Then the tuple given to the t^{th} seat is $g(t)=(u,i,j,v)$. In fact, the middle two numbers indicate the type of this request ($i\to j$), and the other two numbers represent the "environment" *when it is allocated* (*maximal sequence* $[u,v]$). When we've finished allocating the t^{th} ticket $i\to j$ into $[u,v]\sim C_k$, we track the dynamics as in (8).

Since we first allocate all $i^t\to j^t$ into C, we must have

$$\gamma_{ui^tj^tv}\leqslant f(C)_{uv},\quad \forall u\leqslant i^t\leqslant j^t\leqslant v.$$

For any given $u\leqslant i\leqslant j\leqslant v$, denote γ_{uijv} as the total number of requests that are arranged with the tuple (u,i,j,v), i.e.,

$$\gamma^*_{uijv}=\sum_{t=1}^{d}\mathbb{1}\left\{g(t)=(u,i,j,v)\right\}.$$

Now let's examine the procedures stated above and explore some necessary conditions for $\{\gamma^*_{uijv}\}$. First, the total number of allocated request $i\to j$ is clearly

$$\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}}\gamma^*_{uijv}=\sum_k x^*_{k,ij},$$

which should be no larger than d_{ij}.

Second, during the above process, the number of times that $[u,v]$ splits is

$$\sum_{\substack{(i,j):\\ u\leqslant i\leqslant j\leqslant v}}\gamma^*_{uijv}.$$

The number of times that $[u,v]$ initially exists or is generated by other *maximal sequence*s is

$$\left(\sum_{\substack{(k,\ell):\\ v+1\leqslant k\leqslant \ell}}\gamma^*_{u(v+1)k\ell}\right)+\left(\sum_{\substack{(k,\ell):\\ \ell\leqslant k\leqslant u-1}}\gamma^*_{\ell k(u-1)v}\right)+f(C)_{uv}.$$

Thus $\{\gamma^*_{uijv}\}$ must satisfy

$$\begin{aligned}
&\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}}\gamma^*_{uijv}\leqslant d_{ij}, && \forall i\leqslant j,\\
&\sum_{\substack{(i,j):\\ u\leqslant i\leqslant j\leqslant v}}\gamma^*_{uijv}\leqslant\left(\sum_{\substack{(k,l):\\ v+1\leqslant k\leqslant l}}\gamma^*_{u(v+1)kl}\right)+\left(\sum_{\substack{(k,l):\\ l\leqslant k\leqslant u-1}}\gamma^*_{lk(u-1)v}\right)+f(C)_{uv}, && \forall u\leqslant v,\\
&\gamma_{ui^tj^tv}\leqslant f(C)_{uv}, && \forall u\leqslant i^t\leqslant j^t\leqslant v,\\
&\gamma^*_{uijv}\in\mathbb{N}, && \forall u\leqslant i\leqslant j\leqslant v,
\end{aligned}\tag{EC.2}$$

which is exactly the constraints of (EC.1). Thus the optimal value of (EC.1) is at least as large as the optimal value of (1).

Step 2. Now we prove the relationship between (2) and (14)+(15). We will prove that the optimal objective value of (2) is no more than that of (14)+(15). Since the coefficients of (2) are all rational, there exists an optimal solution $\{x^*_{k,ij}\}$ such that all the variables are rational. Let θ_1 be a positive integer such that $\theta_1 x^*_{k,ij}\in\mathbb{N}$. We consider to *copy* the capacity matrix C by θ_1 times as $C(\theta_1)$. Now for given $k\in[N]$, $\{\theta_1 x^*_{k,ij}\}_{i\leqslant j}$ satisfy

$$\sum_{\substack{(i,j):\\ i\leqslant \ell\leqslant j}}\theta_1 x^*_{k,ij}\leqslant\theta_1 C_{k\ell}=\sum_{s=0}^{\theta_1-1}C_{(k+sN)\ell},\quad \forall k\in[N],\ell\in[M],$$

which is the constraint at an aggregation level. Note that for any k, the matrix formed by $\{C_{k+sN}\}_s$ is a strongly NSE matrix. Therefore, by Theorem 2, there exists $\{\tilde{x}_{k,ij}\}$ such that

$$\begin{aligned}
&\sum_{s=0}^{\theta_1-1} \tilde{x}_{k+sN,ij} = \theta_1 x^*_{k,ij}, && \forall k \in [N], i \leqslant j,\\
&\sum_{\substack{(i,j):\\ i\leqslant \ell \leqslant j}} \tilde{x}_{k,ij} \leqslant C(\theta_1)_{k\ell}, && \forall k \in [\theta_1 N], \ell \in [M],\\
&\tilde{x}_{k,ij} \in \{0,1\}, && \forall k \in [\theta_1 N], i \leqslant j.
\end{aligned}$$

Thus we have $\sum_{k=1}^{\theta_1 N} \tilde{x}_{k,ij} = \sum_{k=1}^{N} \theta_1 x^*_{k,ij} \leqslant \theta_1 d_{ij}$. From Step 1, there exist $\{\tilde{\gamma}_{uijv}\}$ such that

$$\begin{aligned}
&\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \tilde{\gamma}_{uijv} = \sum_{k=1}^{\theta_1 N} \tilde{x}_{k,ij} \leqslant \theta_1 d_{ij}, && \forall 1 \leqslant i \leqslant j \leqslant M,\\
&\sum_{\substack{(i,j):\\ u\leqslant i\leqslant j\leqslant v}} \tilde{\gamma}_{uijv} \leqslant \sum_{\substack{(k,l):\\ v+1\leqslant k\leqslant l}} \tilde{\gamma}_{u(v+1)kl} + \sum_{\substack{(k,l):\\ l\leqslant k\leqslant u-1}} \tilde{\gamma}_{lk(u-1)v} + \theta_1 f(C)_{uv}, && \forall u \leqslant v,\\
&\tilde{\gamma}_{ui^t j^t v} \leqslant \theta_1 f(C)_{uv}, && \forall u \leqslant i^t \leqslant j^t \leqslant v,\\
&\tilde{\gamma}_{uijv} \in \mathbb{N}, && \forall u \leqslant i \leqslant j \leqslant v.
\end{aligned}$$

Set $\gamma^*_{uijv} = \frac{1}{\theta_1}\tilde{\gamma}_{uijv}$ completes our Step 2. Thus, the proof is completed.

□

To proceed with our discussions and proof, we need some new notations. We consider the following variant of (14).

$$\begin{aligned}
\text{maximize}_\gamma \quad & \sum_{i\leqslant j}\left(p_{ij}\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \gamma_{uijv}\right), && \text{(EC.3)}\\
\text{subject to} \quad & \sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \gamma_{uijv} + \gamma_{0ij0} = d_{ij}, && \forall i \leqslant j,\\
& \sum_{\substack{(i,j):\\ u\leqslant i\leqslant j\leqslant v}} \gamma_{uijv} \leqslant \sum_{\substack{(k,\ell):\\ v+1\leqslant k\leqslant \ell}} \gamma_{u(v+1)k\ell} + \sum_{\substack{(k,\ell):\\ \ell\leqslant k\leqslant u-1}} \gamma_{\ell k(u-1)v} + A_{uv}, && \forall u \leqslant v,\\
& \gamma_{uijv} \geqslant \hat{\gamma}_{uijv}, \quad \forall u \leqslant i \leqslant j \leqslant v, \qquad \gamma_{0ij0} \geqslant \hat{\gamma}_{0ij0}, \quad \forall i \leqslant j.
\end{aligned}$$

Note that comparing to (14), we impose a set of constraints $\gamma_{uijv} \geqslant \hat{\gamma}_{uijv}$ for all $u \leqslant i \leqslant j \leqslant v$. We denote $\mathrm{OPT}(A, d, \hat{\gamma})$ as the problem instance/objective value of (EC.3), where $A = \{A_{uv}\}$, $d = \{d_{ij}\}$, and $\hat{\gamma} = \{\hat{\gamma}_{uijv}\}$ are non-negative. It is clear that (14) equals $\mathrm{OPT}(A, d, 0)$.

Fix a sample w. In the following, we adopt $\gamma^{[t_1,t_2]}_{uijv} = \gamma^{[t_1,t_2+1)}_{uijv}$ to represent the number of requests $i \to j$ that are put into $[u, v]$ during the time interval $[t_1, t_2]$ by the RDP policy. To be precise,

$$\gamma^{[t_1,t_2]}_{uijv} = \sum_{t_1\leqslant t\leqslant t_2} \mathbb{1}\left\{(u^t, i^t, j^t, v^t) = (u, i, j, v)\right\}.$$

We define $d^{[t,t)} = \gamma^{[t,t)} = 0$ for all $t \geqslant 1$. We also let $A^t = f(C^t)$ be the state of the maximal sequences at time t, where f is defined in (7). To avoid confusion, we note that any variable with a time stamp on the upper right corner is associated with the realized sample path.

The following lemma, Lemma EC.1, shows an important property of (EC.3).

LEMMA EC.1. *Fix any nonnegative $\hat{d} = \{\hat{d}_{ij}\}$. For any $1 \leqslant t_1 \leqslant t_2 \leqslant T+1$, we have*

$$\mathrm{OPT}(A^1, \hat{d} + d^{[1,t_2)}, \gamma^{[1,t_2)}) = \sum_{i\leqslant j}\left(p_{ij}\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \gamma^{[1,t_1)}_{uijv}\right) + \mathrm{OPT}(A^{t_1}, \hat{d} + d^{[t_1,t_2)}, \gamma^{[t_1,t_2)}).$$

Proof of Lemma EC.1. The proof is based on the following equalities. For any $t \geqslant 1$, we have

$$d_{ij}^{[1,t)} = \sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \gamma_{uijv}^{[1,t)} + \gamma_{0ij0}^{[1,t)}, \quad \forall i \leqslant j,$$

$$A_{uv}^{t} - A_{uv}^{1} = \sum_{\substack{(k,\ell):\\ v+1\leqslant k\leqslant \ell}} \gamma_{u(v+1)k\ell}^{[1,t)} + \sum_{\substack{(k,\ell):\\ \ell\leqslant k\leqslant u-1}} \gamma_{\ell k(u-1)v}^{[1,t)} - \sum_{\substack{(i,j):\\ u\leqslant i\leqslant j\leqslant v}} \gamma_{uijv}^{[1,t)}, \quad \forall u \leqslant v.$$

Then for any optimal solution γ^* of $\mathrm{OPT}(A^{t_1}, \hat{d} + d^{[t_1,t_2)}, \gamma^{[t_1,t_2)})$, $\gamma^* + \gamma^{[1,t_1)}$ is a feasible solution of $\mathrm{OPT}(A^1, \hat{d} + d^{[1,t_2)}, \gamma^{[1,t_2)})$. Also, for any optimal solution γ^* of $\mathrm{OPT}(A^1, \hat{d} + d^{[1,t_2)}, \gamma^{[1,t_2)})$, $\gamma^* - \gamma^{[1,t_1)}$ is a feasible solution of $\mathrm{OPT}(A^{t_1}, \hat{d} + d^{[t_1,t_2)}, \gamma^{[t_1,t_2)})$ because $\gamma^* - \gamma^{[1,t_1)} \geqslant \gamma^{[1,t_2)} - \gamma^{[1,t_1)} = \gamma^{[t_1,t_2)}$. The proof is completed.

□

Fix a sample path ω. From Lemma 1, the Hindsight Optimum under ω is $\mathrm{OPT}\left(A^1, d^{[1,T]}, 0\right)$. From Lemma EC.1, by setting $\hat{d} = 0$ and $t_1 = t_2 = T+1$, we can see that the total revenue collected under ω is $\mathrm{OPT}\left(A^1, d^{[1,T]}, \gamma^{[1,T]}\right)$. Therefore, the loss incurred by RDP for ω can be written as

$$\begin{aligned}&\mathrm{OPT}\left(A^1, d^{[1,T]}, 0\right) - \mathrm{OPT}\left(A^1, d^{[1,T]}, \gamma^{[1,T]}\right)\\ =&\sum_{t=1}^{T}\left[\mathrm{OPT}\left(A^1, d^{[1,T]}, \gamma^{[1,t)}\right) - \mathrm{OPT}\left(A^1, d^{[1,T]}, \gamma^{[1,t+1)}\right)\right].\end{aligned} \tag{EC.4}$$

(EC.4) shows that the loss between RDP and HO can be decomposed into T increments, with each increment characterized by the gap between two "adjacent" OPTs. Lemma EC.2 shows that such gap can be uniformly bounded from above.

LEMMA EC.2. *There exists some $l > 0$ only dependent on $\{p_{ij}\}$ such that*

$$0 \leqslant \mathrm{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t)}) - \mathrm{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t+1)}) \leqslant 2l$$

for any A^1, $d^{[1,T]}$ and $t \in [1,T]$.

Proof of Lemma EC.2. The left-hand side is trivial, since $\gamma^{[1,t+1)} \geqslant \gamma^{[1,t)}$ leads to a smaller feasible region of $\mathrm{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t+1)})$ than that of $\mathrm{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t)})$. In the following we consider the right-hand side.

Step 1. We first prove that $\mathrm{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t)})$ is equivalent to solving (EC.5),

$$\text{maximize}_{\gamma} \quad \sum_{i\leqslant j}\left(p_{ij}\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}}\gamma_{uijv}\right) - \sum_{u\leqslant v}\beta_{uv}\left(\sum_{\substack{(i,j):\\ u\leqslant i\leqslant j\leqslant v}}\gamma_{uijv} - \sum_{\substack{(k,\ell):\\ v+1\leqslant k\leqslant \ell}}\gamma_{u(v+1)k\ell} - \sum_{\substack{(k,\ell):\\ \ell\leqslant k\leqslant u-1}}\gamma_{\ell k(u-1)v} - A_{uv}^1\right)^+, \tag{EC.5}$$

$$\begin{aligned}\text{subject to} \quad & \sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}}\gamma_{uijv} + \gamma_{0ij0} = d_{ij}^{[1,T]}, \quad \forall i \leqslant j,\\ & \gamma_{uijv} \geqslant \gamma_{uijv}^{[1,t)}, \quad \forall u \leqslant i \leqslant j \leqslant v; \qquad \gamma_{0ij0} \geqslant \gamma_{0ij0}^{[1,t)}, \quad \forall i \leqslant j\end{aligned}$$

with fixed and appropriately chosen β such that

$$\beta_{uv} > \beta_{u(i-1)} + p_{ij} + \beta_{(j+1)v}, \quad \forall u \leqslant i \leqslant j \leqslant v.$$

Here, such β can be obtained by setting

$$\beta_{ii} = p_{ii}, \quad \forall i,$$

and inductively setting

$$\beta_{uv} = \max_{(i,j):u\leqslant i\leqslant j\leqslant v}\left\{\beta_{u(i-1)} + p_{ij} + \beta_{(j+1)v}\right\} + \epsilon, \quad \forall u \leqslant v$$

for some $\epsilon > 0$. We denote (EC.5) as $\widetilde{\mathrm{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t)})$. We only need to prove that the optimal solution of $\widetilde{\mathrm{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t)})$ must satisfy the second group of constraints of $\mathrm{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t)})$.

We prove by contradiction. Suppose this is not the case, then there exists an optimal solution $\tilde{\gamma}^{*,t}$ of $\widetilde{\mathrm{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t)})$ such that

$$\sum_{\substack{(i,j):\\ u'\leqslant i\leqslant j\leqslant v'}} \tilde{\gamma}^{*,t}_{u'ijv'} - \sum_{\substack{(k,\ell):\\ v'<k\leqslant \ell}} \tilde{\gamma}^{*,t}_{u'(v'+1)k\ell} - \sum_{\substack{(k,\ell):\\ \ell\leqslant k<u'}} \tilde{\gamma}^{*,t}_{\ell k(u'-1)v'} - A^1_{u'v'} > 0$$

holds for some (u', v'). Then we have

$$\begin{aligned}
\sum_{\substack{(i,j):\\ u'\leqslant i\leqslant j\leqslant v'}} \tilde{\gamma}^{*,t}_{u'ijv'} &> \sum_{\substack{(k,\ell):\\ v'<k\leqslant \ell}} \tilde{\gamma}^{*,t}_{u'(v'+1)k\ell} + \sum_{\substack{(k,\ell):\\ \ell\leqslant k<u'}} \tilde{\gamma}^{*,t}_{\ell k(u'-1)v'} + A^1_{u'v'} \\
&\overset{\text{(a)}}{\geqslant} \sum_{\substack{(k,\ell):\\ v'<k\leqslant \ell}} \gamma^{[1,t)}_{u'(v'+1)k\ell} + \sum_{\substack{(k,\ell):\\ \ell\leqslant k<u'}} \gamma^{[1,t)}_{\ell k(u'-1)v'} + A^1_{u'v'} \overset{\text{(b)}}{\geqslant} \sum_{\substack{(i,j):\\ u'\leqslant i\leqslant j\leqslant v'}} \gamma^{[1,t)}_{u'ijv'}.
\end{aligned}$$

(a) holds from the second group of constraints in (EC.5). (b) holds because up to any time period t, the number of $[u', v'] \sim C$ generated is no less than those depleted. Therefore, there exists some (i', j') such that

$$\tilde{\gamma}^{*,t}_{u'i'j'v'} > \gamma^{[1,t)}_{u'i'j'v'}.$$

Now we show the contradiction. Let $f(\gamma; A^1, \beta)$ denote the objective function in $\widetilde{\mathrm{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t)})$, then f is uniformly Lipschitz continuous in γ under the L_1 norm with Lipschitz constant

$$l = \max_{u\leqslant i\leqslant j\leqslant v} \left\{\beta_{u(i-1)} + p_{ij} + \beta_{(j+1)v}, \beta_{uv}\right\},$$

regardless of the value of A^1. Note that l depends only on $\{p_{ij}\}$. Since $\partial f/\partial \tilde{\gamma}^{*,t}_{u'i'j'v'} \leqslant \beta_{u'(i'-1)} + p_{i'j'} + \beta_{(j'+1)v'} - \beta_{u'v'} < 0$ by the choice of β, we can decrease $\tilde{\gamma}^{*,t}_{u'i'j'v'}$ and increase $\tilde{\gamma}^{*,t}_{0i'j'0}$ by an identical small constant such that f strictly increases while the constraints of $\widetilde{\mathrm{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t)})$ are still satisfied. This contradicts with the optimality of γ^*. Thus, OPT and $\widetilde{\mathrm{OPT}}$ have the same optimal value.

Step 2. Now we construct a feasible solution $\tilde{\gamma}^{\mathrm{fea},t+1}$ of $\widetilde{\mathrm{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t+1)})$ from the optimal solution $\tilde{\gamma}^{*,t}$ of $\widetilde{\mathrm{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t)})$ such that $\|\tilde{\gamma}^{\mathrm{fea},t+1} - \tilde{\gamma}^{*,t}\|_1 \leqslant 2$. Notice that $\gamma^{[1,t)}$ and $\gamma^{[1,t+1)}$ differs at exactly one component with

$$\gamma^{[1,t+1)}_{uijv} = \begin{cases} \gamma^{[1,t)}_{uijv} + 1, & \text{if } (u,i,j,v) = (u^t, i^t, j^t, v^t) \\ \gamma^{[1,t)}_{uijv}, & \text{otherwise.} \end{cases} \tag{EC.6}$$

We let

$$\tilde{\gamma}^{\mathrm{fea},t+1}_{uijv} = \begin{cases} \tilde{\gamma}^{*,t}_{uijv} + \max\{\gamma^{[1,t+1)}_{uijv} - \tilde{\gamma}^{*,t}_{uijv}, 0\}, & \text{if } (u,i,j,v) = (u^t, i^t, j^t, v^t) \\ \tilde{\gamma}^{*,t}_{uijv} - (\tilde{\gamma}^{*,t}_{uijv} - \gamma^{[1,t)}_{uijv})\epsilon, & \text{if } (i,j) = (i^t, j^t), (u,v) \neq (u^t, v^t), \\ \tilde{\gamma}^{*,t}_{uijv}, & \text{otherwise,} \end{cases}$$

where

$$\epsilon = \frac{\max\{\gamma^{[1,t+1)}_{u^t i^t j^t v^t} - \tilde{\gamma}^{*,t}_{u^t i^t j^t v^t}, 0\}}{\sum_{(u,v)\neq(u^t,v^t)} (\tilde{\gamma}^{*,t}_{u i^t j^t v} - \gamma^{[1,t)}_{u i^t j^t v})}.$$

Now we show that $\{\tilde{\gamma}^{\mathrm{fea},t+1}_{uijv}\}$ satisfies the constraints in (EC.5). First, we show the first group of constraints. When $(i,j) \neq (i^t, j^t)$, we have

$$\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \tilde{\gamma}^{\mathrm{fea},t+1}_{uijv} + \tilde{\gamma}^{\mathrm{fea},t+1}_{0ij0} = \sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \tilde{\gamma}^{*,t}_{uijv} + \tilde{\gamma}^{*,t}_{0ij0} = d^{[1,T]}_{ij}.$$

When $(i,j)=(i^t,j^t)$, if $\gamma^{[1,t+1)}_{u^ti^tj^tv^t} \leqslant \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}$, we have

$$\sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \tilde{\gamma}^{\text{fea},t+1}_{uijv} + \tilde{\gamma}^{\text{fea},t+1}_{0ij0} = \sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \tilde{\gamma}^{*,t}_{uijv} + \tilde{\gamma}^{*,t}_{0ij0} = d^{[1,T]}_{ij}.$$

If $\gamma^{[1,t+1)}_{u^ti^tj^tv^t} > \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}$, we have

$$\begin{aligned}\sum_{\substack{(u,v):\\ u\leqslant i^t\leqslant j^t\leqslant v}} \tilde{\gamma}^{\text{fea},t+1}_{ui^tj^tv} + \tilde{\gamma}^{\text{fea},t+1}_{0i^tj^t0} &= \sum_{(u,v)} \tilde{\gamma}^{*,t}_{ui^tj^tv} + \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t} + \gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t} - \sum_{(u,v)\neq(u^t,v^t)} (\tilde{\gamma}^{*,t}_{ui^tj^tv} - \gamma^{[1,t)}_{ui^tj^tv})\epsilon \\ &= d^{[1,T]}_{ij} + (\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}) - (\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}) = d^{[1,T]}_{ij}.\end{aligned}$$

Then, we show the second group of constraints. As an intermediate step, we show that $\epsilon \leqslant 1$. This is because

$$\begin{aligned}\frac{\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}}{\sum_{(u,v)\neq(u^t,v^t)} (\tilde{\gamma}^{*,t}_{ui^tj^tv} - \gamma^{[1,t)}_{ui^tj^tv})} &\overset{\text{(a)}}{=} \frac{\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}}{d^{[1,T]}_{i^tj^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t} - \sum_{(u,v)\neq(u^t,v^t)} \gamma^{[1,t)}_{ui^tj^tv}} \\ &\overset{\text{(b)}}{=} \frac{\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}}{d^{[1,T]}_{i^tj^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t} - \sum_{(u,v)\neq(u^t,v^t)} \gamma^{[1,t+1)}_{ui^tj^tv}} \\ &\overset{\text{(c)}}{=} \frac{d^{[1,t+1)}_{i^tj^t} - \sum_{(u,v)\neq(u^t,v^t)} \gamma^{[1,t+1)}_{ui^tj^tv} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t}}{d^{[1,T]}_{i^tj^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t} - \sum_{(u,v)\neq(u^t,v^t)} \gamma^{[1,t+1)}_{ui^tj^tv}} \leqslant 1.\end{aligned}$$

(a) follows because by the constraints in (EC.5),

$$d^{[1,T]}_{i^tj^t} = \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t} + \sum_{(u,v)\neq(u^t,v^t)} \tilde{\gamma}^{*,t}_{ui^tj^tv}.$$

(b) follows from (EC.6). (c) follows from the definition of $\gamma^{[1,t+1)}$.

Now, $\epsilon \leqslant 1$ ensures that $\gamma^{\text{fea},t+1}_{uijv} \geqslant \tilde{\gamma}^{[1,t)}_{uijv} = \gamma^{[1,t+1)}_{uijv}$ for all $(u,i,j,v) \neq (u^t,i^t,j^t,v^t)$. As a result, $\tilde{\gamma}^{\text{fea},t+1}$ is a feasible solution of $\widetilde{\text{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t+1)})$. Therefore,

$$\begin{aligned}&\text{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t)}) - \text{OPT}(A^1, d^{[1,T]}, \gamma^{[1,t+1)}) \\ &= \widetilde{\text{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t)}) - \widetilde{\text{OPT}}(A^1, d^{[1,T]}, \gamma^{[1,t+1)}) \\ &\leqslant f(\gamma^{*,t}; A^1, \beta) - f(\tilde{\gamma}^{\text{fea},t+1}; A^1, \beta) \\ &\leqslant l\|\gamma^{*,t} - \tilde{\gamma}^{\text{fea},t+1}\|_1 \\ &= l\left(\tilde{\gamma}^{\text{fea},t+1}_{u^ti^tj^tv^t} - \tilde{\gamma}^{*,t}_{u^ti^tj^tv^t} + \sum_{(u,v)\neq(u^tv^t)} \left(\tilde{\gamma}^{*,t}_{ui^tj^tv} - \tilde{\gamma}^{\text{fea},t+1}_{ui^tj^tv}\right)\right) \\ &\overset{\text{(a)}}{=} l\left(\max\{\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \gamma^{*,t}_{u^ti^tj^tv^t}, 0\} + \sum_{(u,v)\neq(u^tv^t)} \left(\tilde{\gamma}^{*,t}_{ui^tj^tv} - \gamma^{[1,t)}_{ui^tj^tv}\right)\epsilon\right) \\ &= l\cdot 2\max\{\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \gamma^{*,t}_{u^ti^tj^tv^t}, 0\} \\ &\leqslant l\cdot 2\max\{\gamma^{[1,t+1)}_{u^ti^tj^tv^t} - \gamma^{[1,t)}_{u^ti^tj^tv^t}, 0\} \leqslant 2l.\end{aligned}$$

Here, (a) follows from the definition of $\{\tilde{\gamma}^{\text{fea},t+1}_{ui^tj^tv}\}$. The last inequality follows from (EC.6). Thus, the lemma is proved. $\square$

Before proceeding to the formal proof of Theorem 4, we introduce Lemma EC.3, which is a special case of Theorem 2.4 in Mangasarian and Shiau (1987). It indicates the sensitivity of optimal solutions when the right-hand side of (14) changes.

LEMMA EC.3. *There exists some $\delta > 0$ that only depends on the constraint matrix of (14), and independent of A^t, $d \geqslant 0$, such that for any optimal solution $\gamma^{*,t}$ of $\text{OPT}(A^t, d, 0)$, there exists an optimal solution $\tilde{\gamma}^{*,t}$ of $\text{OPT}(A^t, \tilde{d}, 0)$ with $\|\gamma^{*,t} - \tilde{\gamma}^{*,t}\|_\infty \leqslant \delta\|d - \tilde{d}\|_\infty$.*

Proof of Theorem 4.

Let $T_{ij} = \sup\left\{t \leqslant T : \lambda_{ij}^t > 0\right\}$. Without loss of generality, we assume that $T_{ij} \geqslant 1$ ($\forall i \leqslant j$). As a preliminary step, we demonstrate that

$$\inf_{\substack{t,i,j:\\ t\in[\theta T_{ij}]}} \frac{\lambda_{ij}^{(t,\theta T_{ij}]}(\theta)}{\theta T_{ij} - t}$$

is lower bounded by some positive constant

$$\lambda_{\min} \triangleq \inf_{i\leqslant j} \frac{\lambda_{ij}^{T_{ij}}}{T_{ij}}$$

irrelevant to θ, where we let $\frac{0}{0} = 1 \geqslant \lambda_{\min}$. In fact, when $t \leqslant \theta(T_{ij} - 1)$,

$$\inf_{\substack{t,i,j:\\ t\in[\theta T_{ij}]}} \frac{\lambda_{ij}^{(t,\theta T_{ij}]}(\theta)}{\theta T_{ij} - t} \geqslant \inf_{i\leqslant j} \frac{\theta\lambda_{ij}^{T_{ij}}}{\theta T_{ij}} = \lambda_{\min}.$$

When $\theta(T_{ij} - 1) < t \leqslant \theta T_{ij}$,

$$\inf_{\substack{t,i,j:\\ t\in[\theta T_{ij}]}} \frac{\lambda_{ij}^{(t,\theta T_{ij}]}(\theta)}{\theta T_{ij} - t} \geqslant \inf_{i\leqslant j} \frac{(\theta T_{ij} - t)\lambda_{ij}^{T_{ij}}}{\theta T_{ij} - t} \geqslant \lambda_{\min}.$$

Therefore, we have

$$\inf_{\substack{t,i,j:\\ t\in[\theta T_{ij}]}} \frac{\lambda_{ij}^{(t,\theta T_{ij}]}(\theta)}{\theta T_{ij} - t} \geqslant \lambda_{\min} > 0. \tag{EC.7}$$

Now we analyze the loss of RDP. Let δ be defined as that in Lemma EC.3. Let $\theta > 2\lceil \frac{(1+\delta)(M+1)^2}{2\lambda_{\min}} \rceil$ be any scaling parameter. The loss can be upper bounded by

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,1)})\right] - \mathbb{E}\left[\mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,\theta T]})\right] \\
=& \sum_{t=1}^{\theta T} \mathbb{E}\left[\mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,t)}) - \mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,t+1)})\right] \\
\overset{\text{(a)}}{\leqslant}& 2l \sum_{t=1}^{\theta T} \mathbb{P}\left(\mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,t)}) - \mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,t+1)}) > 0\right) \\
\overset{\text{(b)}}{=}& 2l \sum_{t=1}^{\theta T} \mathbb{P}\left(\mathrm{OPT}(A^t, d^{[t,\theta T]}, 0) - \mathrm{OPT}(A^t, d^{[t,\theta T]}, \gamma^{[t,t+1)}) > 0\right).
\end{aligned} \tag{EC.8}$$

(a) follows from Lemma EC.2. Here, $\mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,t)})$ can be interpreted as the total reward obtained under a virtual "policy" where we first follow the RDP policy during $[1,t)$ and then from time t we follow the optimal solution assuming that we know the future demands. (b) follows from Lemma EC.1. To be more concrete, let $t_1 = t_2 = t$ and $\hat{d} = d^{[t,\theta T]}$, we have by Lemma EC.1,

$$\mathrm{OPT}(A^1, d^{[t,\theta T]} + d^{[1,t)}, \gamma^{[1,t)}) = \sum_{i\leqslant j} \left(p_{ij} \sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \gamma_{uijv}^{[1,t)} \right) + \mathrm{OPT}(A^t, d^{[t,\theta T]} + d^{[t,t)}, \gamma^{[t,t)}). \tag{EC.9}$$

Let $t_1 = t$, $t_2 = t+1$, and $\hat{d} = d^{[t+1,\theta T]}$, we have

$$\mathrm{OPT}(A^1, d^{[t+1,\theta T]} + d^{[1,t+1)}, \gamma^{[1,t+1)}) = \sum_{i\leqslant j} \left(p_{ij} \sum_{\substack{(u,v):\\ u\leqslant i\leqslant j\leqslant v}} \gamma_{uijv}^{[1,t)} \right) + \mathrm{OPT}(A^t, d^{[t+1,\theta T]} + d^{[t,t+1)}, \gamma^{[t,t+1)}). \tag{EC.10}$$

Subtracting (EC.9) from (EC.10) yields (b).

For each t, consider $\mathrm{OPT}(A^t, d, 0)$. In the RDP policy, $d = \lambda^{[t,\theta T]}$, while in the sample path HO, $d = d^{[t,\theta T]}$. By Lemma EC.3, we can choose $\gamma^{*,t}$ be an optimal solution of $\mathrm{OPT}(A^t, d^{[t,\theta T]}, 0)$ such that

$$\|\gamma^{*,t} - \gamma^{\mathrm{RDP},t}\|_\infty \leqslant \delta \|d^{[t,\theta T]} - \lambda^{[t,\theta T]}\|_\infty = \delta \|d^{[t,\theta T]} - \mathbb{E}[d^{[t,\theta T]}]\|_\infty. \tag{EC.11}$$

Now we show that (EC.8) can be further upper bounded as

$$2l \sum_{t=1}^{\theta T} \mathbb{P}\left(\mathrm{OPT}(A^t, d^{[t,\theta T]}, 0) - \mathrm{OPT}(A^t, d^{[t,\theta T]}, \gamma^{[t,t+1)}) > 0\right) \leqslant 2l \sum_{t=1}^{\theta T} \mathbb{P}\left(\gamma^{*,t}_{u^t i^t j^t v^t} < 1\right). \tag{EC.12}$$

The reason is as follows. If $\gamma^{*,t}_{u^t i^t j^t v^t} \geqslant 1$, then $\gamma^{*,t}$ is still feasible for $\mathrm{OPT}(A^t, d^{[t,\theta T]}, \gamma^{[t,t+1)})$, so $\mathrm{OPT}(A^t, d^{[t,\theta T]}, 0)$ and $\mathrm{OPT}(A^t, d^{[t,\theta T]}, \gamma^{[t,t+1)})$ in (EC.8) must be equal. Therefore, if the two OPTs are not equal, then we must have $\gamma^{*,t}_{u^t i^t j^t v^t} < 1$.

Now we analyze $\mathbb{P}\left(\gamma^{*,t}_{u^t i^t j^t v^t} < 1\right)$. In time period t, after realization of (i^t, j^t), based on the maximum choice of (u^t, v^t) in RDP, we have

$$\gamma^{\mathrm{RDP},t}_{u^t i^t j^t v^t} = \max_{(u,v)} \gamma^{\mathrm{RDP},t}_{u i^t j^t v} \geqslant \frac{\sum_{(u,v)} \gamma^{\mathrm{RDP},t}_{u i^t j^t v}}{\sum_{(u,v): u \leqslant i^t \leqslant j^t \leqslant v} 1} = \frac{\lambda^{[t,\theta T]}_{i^t j^t}}{i^t (M+1-j^t)} \geqslant \frac{\lambda^{[t,\theta T]}_{i^t j^t}}{(M+1)^2/4}.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\gamma^{*,t}_{u^t i^t j^t v^t} < 1\right) \\
&\leqslant \mathbb{P}\left(\gamma^{*,t}_{u^t i^t j^t v^t} < \gamma^{\mathrm{RDP},t}_{u^t i^t j^t v^t} + 1 - \frac{\lambda^{[t,\theta T]}_{i^t j^t}}{(M+1)^2/4}\right) \\
&= \mathbb{P}\left(\gamma^{\mathrm{RDP},t}_{u^t i^t j^t v^t} - \gamma^{*,t}_{u^t i^t j^t v^t} > \frac{\mathbb{E}[d^{[t,\theta T]}_{i^t j^t}] - (M+1)^2/4}{(M+1)^2/4}\right) \\
&\overset{(a)}{\leqslant} \mathbb{P}\left(\max_{i' \leqslant j'} \left|\mathbb{E}[d^{[t,\theta T]}_{i'j'}] - d^{[t,\theta T]}_{i'j'}\right| > \frac{\mathbb{E}[d^{[t,\theta T]}_{i^t j^t}] - (M+1)^2/4}{\delta (M+1)^2/4}\right) \\
&= \mathbb{P}\left(\bigcup_{i' \leqslant j'} \left\{\left|\mathbb{E}[d^{[t,\theta T]}_{i'j'}] - d^{[t,\theta T]}_{i'j'}\right| > \frac{\mathbb{E}[d^{[t,\theta T]}_{i^t j^t}] - (M+1)^2/4}{\delta (M+1)^2/4}\right\}\right) \\
&\leqslant \sum_{i' \leqslant j'} \mathbb{P}\left(\left|\mathbb{E}[d^{[t,\theta T]}_{i'j'}] - d^{[t,\theta T]}_{i'j'}\right| > \frac{\mathbb{E}[d^{[t,\theta T]}_{i^t j^t}] - (M+1)^2/4}{\delta (M+1)^2/4}\right) \\
&\overset{(b)}{=} \sum_{i' \leqslant j'} \sum_{i \leqslant j} \mathbb{P}\left(\left|\mathbb{E}[d^{[t,\theta T]}_{i'j'}] - d^{[t,\theta T]}_{i'j'}\right| > \frac{\mathbb{E}[d^{[t,\theta T]}_{ij}] - (M+1)^2/4}{\delta (M+1)^2/4} \middle| (i^t, j^t) = (i,j)\right) \mathbb{P}\left((i^t, j^t) = (i,j)\right) \\
&\leqslant \sum_{i \leqslant j} \sum_{i' \leqslant j'} \mathbb{P}\left(\left|\mathbb{E}[d^{[t,\theta T]}_{i'j'}] - d^{[t,\theta T]}_{i'j'}\right| > \frac{\mathbb{E}[d^{[t,\theta T]}_{ij}] - (M+1)^2/4}{\delta (M+1)^2/4} \middle| (i^t, j^t) = (i,j)\right) \mathbb{1}\left\{t \leqslant \theta T_{ij}\right\} \\
&\overset{(c)}{\leqslant} \sum_{i \leqslant j} \sum_{i' \leqslant j'} \mathbb{P}\left(\left|\mathbb{E}[d^{(t,\theta T]}_{i'j'}] - d^{(t,\theta T]}_{i'j'}\right| > \frac{\mathbb{E}[d^{(t,\theta T]}_{ij}] - (1+\delta)(M+1)^2/4}{\delta (M+1)^2/4} \middle| (i^t, j^t) = (i,j)\right) \mathbb{1}\left\{t \leqslant \theta T_{ij}\right\} \\
&\overset{(d)}{=} \sum_{i \leqslant j} \sum_{i' \leqslant j'} \mathbb{P}\left(\left|\mathbb{E}[d^{(t,\theta T]}_{i'j'}] - d^{(t,\theta T]}_{i'j'}\right| > \frac{\mathbb{E}[d^{(t,\theta T]}_{ij}] - (1+\delta)(M+1)^2/4}{\delta (M+1)^2/4}\right) \mathbb{1}\left\{t \leqslant \theta T_{ij}\right\}.
\end{aligned} \tag{EC.13}$$

(a) follows from Lemma EC.3 and (EC.11). (b) follows from Bayes formula. (c) follows because

$$\left|\mathbb{E}[d^{[t,t]}_{i'j'}] - d^{[t,t]}_{i'j'}\right| \leqslant 1.$$

(d) follows because of the arrival independence between different time periods.

Let $T_0 = \lceil \frac{(1+\delta)(M+1)^2}{2\lambda_{\min}} \rceil < \frac{\theta}{2}$. From (EC.7), we have

$$\mathbb{E}[d^{(t,\theta T]}_{ij}] = \lambda^{(t,\theta T_{ij}]}_{ij} \geqslant \lambda_{\min}(\theta T_{ij} - t).$$

Then for $t \leqslant \theta T_{ij} - T_0$,

$$\frac{\mathbb{E}[d_{ij}^{(t,\theta T]}] - (1+\delta)(M+1)^2/4}{\delta(M+1)^2/4} \geqslant \frac{\lambda_{\min}(\theta T_{ij}-t) - \lambda_{\min}T_0/2}{\delta(M+1)^2/4} \geqslant 2\frac{\lambda_{\min}(\theta T_{ij}-t)}{\delta(M+1)^2}.$$

Thus, combining (EC.8), (EC.12), (EC.13) yields

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,1)})\right] - \mathbb{E}\left[\mathrm{OPT}(A^1, d^{[1,\theta T]}, \gamma^{[1,\theta T]})\right] \\
&\leqslant 2l \sum_{t=1}^{\theta T} \sum_{i \leqslant j} \sum_{i' \leqslant j'} \mathbb{P}\left(\left| \mathbb{E}[d_{i'j'}^{(t,\theta T]}] - d_{i'j'}^{(t,\theta T]} \right| > \frac{\mathbb{E}[d_{ij}(t,\theta T]] - (1+\delta)(M+1)^2/4}{\delta(M+1)^2/4} \right) \mathbb{1}\{t \leqslant \theta T_{ij}\} \\
&\leqslant 2l \sum_{i \leqslant j} \sum_{t=1}^{\theta T_{ij}} \sum_{i' \leqslant j'} \mathbb{P}\left(\left| \mathbb{E}[d_{i'j'}^{(t,\theta T]}] - d_{i'j'}^{(t,\theta T]} \right| > 2\frac{\lambda_{\min}(\theta T_{ij}-t)}{\delta(M+1)^2} \right) \\
&\overset{(a)}{\leqslant} 2l \sum_{i \leqslant j} \sum_{t=1}^{\theta T_{ij}-T_0} \sum_{i' \leqslant j'} 2\exp\left(-8\frac{(\lambda_{\min}(\theta T_{ij}-t))^2}{(\delta(M+1)^2)^2(\theta T - t)}\right) + 2l \sum_{i \leqslant j} \sum_{t=\theta T_{ij}-T_0+1}^{\theta T_{ij}} \sum_{i' \leqslant j'} 1 \\
&= 2l \sum_{i \leqslant j} \sum_{t=1}^{\theta T_{ij}-T_0} \sum_{i' \leqslant j'} 2\exp\left(-8\frac{\lambda_{\min}^2}{\delta^2(M+1)^4} \cdot \frac{(\theta T_{ij}-t)^2}{\theta T - t}\right) + O(1) \\
&\leqslant l(M+1)^2 \sum_{i \leqslant j} \sum_{t=1}^{\theta T_{ij}-T_0} 2\exp\left(-8\frac{\lambda_{\min}^2}{\delta^2(M+1)^4} \cdot \frac{(\theta T_{ij}-t)^2}{\theta T - t}\right) + O(1).
\end{aligned} \tag{EC.14}$$

(a) holds from Hoeffding's inequality. We can further upper bound (EC.14) by

$$\begin{aligned}
&l(M+1)^2 \sum_{i \leqslant j} \sum_{t=T_0}^{+\infty} 2\exp\left(-8\frac{\lambda_{\min}^2 t^2}{\delta^2(M+1)^4 \theta T}\right) + O(1) \\
&\leqslant l\frac{(M+1)^4}{2} \sum_{t=T_0}^{+\infty} 2\exp\left(-\left(8\frac{\lambda_{\min}^2}{\delta^2(M+1)^4 T}\right) \cdot \frac{t^2}{\theta}\right) + O(1) \\
&= O(\sqrt{\theta}),
\end{aligned}$$

where the last equality is because for given $A > 0$, $\sum_{t=1}^{+\infty} \exp\left(-A\frac{t^2}{\theta}\right) = O(\sqrt{\theta})$.

When the NEE property is satisfied, $T_{ij} = T$ for all $i \leqslant j$. (EC.14) can be alternatively bounded by

$$\begin{aligned}
&l(M+1)^2 \sum_{i \leqslant j} \sum_{t=T_0}^{+\infty} 2\exp\left(-8\frac{\lambda_{\min}^2 t^2}{\delta^2(M+1)^4 t}\right) + O(1) \\
&\leqslant l\frac{(M+1)^4}{2} \sum_{t=T_0}^{+\infty} 2\exp\left(-\left(8\frac{\lambda_{\min}^2}{\delta^2(M+1)^4}\right) \cdot t\right) + O(1) \\
&= O(1),
\end{aligned}$$

where the last equality is because for given $A > 0$, $\sum_{t=1}^{+\infty} \exp(-At) = O(1)$. The theorem is proved.

□

Proof of Proposition 1.

Hindsight Optimum (HO): Let $\mathrm{val}(x, d_1, d_2)$ be the objective value of the following problem:

$$\begin{aligned}
\text{maximize}_z \quad & p_1 z_1 + p_2 z_2 \\
\text{subject to} \quad & z_1 + z_2 \leqslant x \\
& 0 \leqslant z_1 \leqslant d_1, \\
& 0 \leqslant z_2 \leqslant d_2.
\end{aligned}$$

This is the hindsight optimum when the realized demand for ticket k is d_k and the number of unoccupied seats is x. We note that $d_k \sim \mathrm{Ber}(2\theta, 1/2)$ $(k \in \{1,2\})$. Here, we use $\mathrm{Ber}(n,p)$ to denote a Bernoulli random variable with parameters n and p. It's easy to obtain that

$$\mathrm{val}(x, d_1, d_2) = p_1 \min\{d_1, x - \min\{d_2, x\}\} + p_2 \min\{d_2, x\}.$$

Then the hindsight optimum for the θ^{th} problem is

$$V_\theta^{\mathrm{HO}}(\mathcal{I}_0) = \mathbb{E}_{d_1,d_2}\left[p_1 \min\{d_1, \theta - \min\{d_2, \theta\}\} + p_2 \min\{d_2, \theta\}\right].$$

Dynamic Programming (DP): Now we investigate DP. For the θ^{th} problem, instead of considering the problem in 2θ times, we consider a relaxed two-stage process: At time θ, we receive all requests for ticket 1, and accept a subset of them. At time 2θ, we receive all requests for ticket 2, and accept a subset of them. This is exactly Littlewood's two-class model, and an optimal policy is to set a threshold y_θ such that the number of requests 1 we accept is exactly $\min\{y_\theta, d_1\}$ (see, e.g., Talluri and van Ryzin 2006). Therefore, the revenue collected for the θth problem is

$$V_\theta^{\mathrm{DP}}(\mathcal{I}_0) = \mathbb{E}_{d_1,d_2}\left[p_1 \min\{y_\theta, d_1\} + p_2 \min\{d_2, \theta - \min\{y_\theta, d_1\}\}\right].$$

Bounding the gap between HO and DP: Let $\theta \geqslant 4$. We discuss about the value of y_θ.

Case 1: If $y_\theta \geqslant \sqrt{\theta}$, let A be the event such that

$$A = \{d_1 \geqslant \theta, d_2 \geqslant \theta\},$$

and we have

$$\begin{aligned}
\mathbb{E}\left[V_\theta^{\mathrm{HO}}(\mathcal{I}_0) - V_\theta^{\mathrm{DP}}(\mathcal{I}_0)\right] &\geqslant \mathbb{E}\left[V_\theta^{\mathrm{HO}}(\mathcal{I}_0) - V_\theta^{\mathrm{DP}}(\mathcal{I}_0) \middle| A\right]\mathbb{P}(A) \\
&\geqslant \mathbb{E}\left[p_2\theta - (p_1 y_\theta + p_2(\theta - y_\theta)) \middle| A\right]\frac{1}{4} \\
&= \frac{p_1}{4} y_\theta = \Omega(\sqrt{\theta}).
\end{aligned}$$

Case 2: If $y_\theta < \sqrt{\theta}$, let A be the event such that

$$A = \{d_1 \geqslant \theta, d_2 \leqslant \theta - 2\sqrt{\theta}\},$$

and from central limit theorem,

$$\mathbb{P}(A) \geqslant \frac{1}{2}\mathbb{P}\left(d_2 \leqslant \theta - 2\sqrt{\theta}\right) = \frac{1}{2}\mathbb{P}\left(\mathrm{Ber}(2\theta, 1/2) \leqslant \theta - 2\sqrt{\theta}\right) = \frac{1}{2}\mathbb{P}\left(\frac{\mathrm{Ber}(2\theta,1/2) - \theta}{\sqrt{\mathrm{Var}(\mathrm{Ber}(2\theta,1/2))}} \leqslant -4\right)$$

is lower bounded by a constant irrelevant with θ. Then

$$\begin{aligned}
\mathbb{E}\left[V_\theta^{\mathrm{HO}}(\mathcal{I}_0) - V_\theta^{\mathrm{DP}}(\mathcal{I}_0)\right] &\geqslant \mathbb{E}\left[V_\theta^{\mathrm{HO}}(\mathcal{I}_0) - V_\theta^{\mathrm{DP}}(\mathcal{I}_0) \middle| A\right]\mathbb{P}(A) \\
&= \mathbb{E}\left[p_1 \min\{d_1, \theta - d_2\} + p_2 d_2 - (p_1 y_\theta + p_2 \min\{d_2, \theta - y_\theta\}) \middle| A\right]\mathbb{P}(A) \\
&= \mathbb{E}\left[p_1(\theta - d_2 - y_\theta) \middle| A\right]\mathbb{P}(A) \\
&\geqslant p_1\sqrt{\theta}\,\mathbb{P}(A) = \Omega(\sqrt{\theta}).
\end{aligned}$$

In conclusion,

$$\mathbb{E}\left[V_\theta^{\mathrm{HO}}(\mathcal{I}_0) - V_\theta^{\mathrm{DP}}(\mathcal{I}_0)\right] = \Omega(\sqrt{\theta}).$$

□

EC.4.3. A Probabilistic Allocation Policy with A Uniformly Bounded Loss*

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