Online Appendix to "Managing The Personalized Order-Holding Problem in Online Retailing"

Appendix

A. Proofs

LEMMA 2 (PARTIALLY-HOLDING IS SUBOPTIMAL). *It is always suboptimal to hold only part of consecutive orders in the current pending orders and send others to the 3PL.*

Proof of Lemma 2. Denote π^* as the optimal order-holding policy, then the optimal total expected cost after taking the partially-holding policy in the current period, denoted as V_1, can be obtained by applying the optimal policy π^* in subsequent periods. To show it is suboptimal, it suffices to construct another policy whose total expected cost, denoted as V_2, is less than V_1 which is the least cost that the partially-holding policy can achieve.

Denote the set of orders held and sent according to the partially-holding policy in the current period as B and A respectively. The constructed policy is to hold the whole pending orders in the current period, which only triggers cost at h in the current period compared to $c+h$ for the partially-holding counterpart. The subsequent actions for the constructed policy will hold all orders in A and B until the beginning of period t when some orders, denoted as B', in B will be sent to the 3PL according to the optimal policy π^*. Then, the constructed policy at the beginning of period t is to send all orders in A and B' to the 3PL and hold the rest of orders if there remains some. After period t, the subsequent decisions for the constructed policy follow the optimal counterpart π^*. Due to the cost structure, the subsequent actions constructed above after the current period will lead to the same cost in subsequent periods as the optimal policy π^*, so the elaborated policy, which leads to less cost in the current period, has less cost than partially-holding counterpart which completes this proof. □

Proof of Lemma 1. Firstly, we show that the optimal order-holding decision at states $(\mathbf{x}_l, T, v=1)$ and $(\mathbf{x}_L, 0, v=1)$ must be to send the current pending orders to the 3PL immediately. We only provide proof for the states $(\mathbf{x}_l, T, v=1)$, and the proof for the state $(\mathbf{x}_L, 0, v=1)$ is similar, so ignored. To show this phenomenon is true, recall that the only subsequent state of $(\mathbf{x}_l, w)$ is $(\mathbf{x}_l, w+1)$ for $w \geq T$, so $p((\mathbf{x}_l, w), (\mathbf{x}_l, w+1)) = 1$ (the consumer has left the market). Then, problem (3) for $w \geq T$ will turns to

$$\begin{aligned} V(\mathbf{x}_l, w, v=1) &= \min_{z\in\{0,1\}} z\{h + p((\mathbf{x}_l, w), (\mathbf{x}_l, w+1)) \cdot V(\mathbf{x}_l, w+1, v=1)\} \qquad (24) \\ &\quad + (1-z)\{c + p((\mathbf{x}_l, w), (\mathbf{x}_l, w+1)) \cdot V(\mathbf{x}_l, w+1, v=0)\} \\ &= \min_{z\in\{0,1\}} z\{h + V(\mathbf{x}_l, w+1, v=1)\} + (1-z)\cdot c \qquad (25) \end{aligned}$$

where the second equality comes from the fact that $p((\mathbf{x}_l, w), (\mathbf{x}_l, w+1)) = 1$ and $V(\mathbf{x}_l, w+1, v=0) = 0$ (since $(\mathbf{x}_l, w+1, v=0)$ indicates that the current pending orders have been sent to the 3PL and no new order will come). Prove by contradiction. If the optimal decision at state $(\mathbf{x}_l, T, v=1)$ is $z^*=1$, then we have $V(\mathbf{x}_l, T, v=1) = h + V(\mathbf{x}_l, T+1, v=1) < c$ which implies that $V(\mathbf{x}_l, T+1, v=1) < c-h$. By (25), the optimal decision at state $(\mathbf{x}_l, T+1, v=1)$ must be $z^*=1$ otherwise $V(\mathbf{x}_l, T+1, v=1) = c$ which contradicts $V(\mathbf{x}_l, T+1, v=1) < c-h$. So the optimal decision at state $(\mathbf{x}_l, T+1, v=1)$ must be $z^*=1$ which implies that $V(\mathbf{x}_l, T+1, v=1) = h + V(\mathbf{x}, T+2, v=1) < c-h$. So we have $V(\mathbf{x}, T+2, v=1) < c-2h$. Follow the same procedures, we will show that $V(\mathbf{x}_l, w, v=1) < c-(w-T)\cdot h$ for $w \geq T$. Here comes the contradiction because $V(\mathbf{x}_l, w, v=1)$ is negative for large enough w. So, the optimal decision at state $(\mathbf{x}_l, T, v=1)$ is $z^*=0$, that is $V(\mathbf{x}_l, T, v=1) = c$ which is exactly (8). So we have proved that the optimal policy at these states must be to send the current to 3PL immediately, now we turn to characterize their optimal total expected cost.

By the optimal policy at states $(\mathbf{x}_l, T, v=1)$ and $(\mathbf{x}_L, 0, v=1)$, it is easy to check their corresponding optimal total expected cost. Firstly, at state $(\mathbf{x}_l, T, v=0)$, the current pending orders have been sent to the 3PL and no new order will come, so $V(\mathbf{x}_l, T, v=0)=0$. At states $(\mathbf{x}_l, T, v=1)$ and $(\mathbf{x}_L, 0, v=1)$, since the optimal decision is to send the current pending orders to the 3PL immediately, by (25), their corresponding optimal cost must equal to c. So we complete the proof of Lemma 1. □

Proof of Proposition 1. Recall the large scale MDP problem (5) with holding state $v=1$, subtract $V(\mathbf{x}_l, w, v=0)$ from both side of the first equation in (5), we have

$$\begin{aligned}
&V(\mathbf{x}_l, w, v=1) - V(\mathbf{x}_l, w, v=0) \\
&= \min_{z\in\{0,1\}} z\left\{h + \sum_{(\mathbf{x}',w')\in\sigma((\mathbf{x}_l,w))} p((\mathbf{x}_l,w),(\mathbf{x}',w')) \begin{pmatrix} V(\mathbf{x}',w',v'=1) \\ -V(\mathbf{x}',w',v'=\mathcal{I}((\mathbf{x}',w')\neq(\mathbf{x}_l,w+1))) \end{pmatrix}\right\} \\
&+ (1-z)\{c\}, \forall w \in \langle T-1\rangle. \\
&= \min_{z\in\{0,1\}} z\{h + p((\mathbf{x}_t,w),(\mathbf{x}_l,w+1))\left(V(\mathbf{x}_l,w+1,v=1) - V(\mathbf{x}_l,w+1,v=0)\right)\} \\
&+ (1-z)\cdot c. \qquad (26)
\end{aligned}$$

Since subtracting $V(\mathbf{x}_l, w, v=0)$ has no effect on the optimal decision of the original optimization problem in (5), problem (26) must share the same optimal decision as the original problem (5) at state $(\mathbf{x}_l, w, v=1)$. Define $\Delta_{\mathbf{x}_l}(w) = V(\mathbf{x}_l, w, v=1) - V(\mathbf{x}_l, w, v=0)$ (hence $\Delta_{\mathbf{x}_l}(T) = V(\mathbf{x}, T, v=1) - V(\mathbf{x}_l, T, v=0) = c$ by (7) and (8)), then (26) turns to the following single-dimensional MDP problem (27):

$$\begin{aligned}
&\Delta_{\mathbf{x}_l}(w) = \min_{z\in\{0,1\}} z\{h + p((\mathbf{x}_t,w),(\mathbf{x}_l,w+1))\cdot\Delta_{\mathbf{x}_l}(w+1)\} + (1-z)c, \forall 0\le w\le T-1 \qquad (27)\\
&\Delta_{\mathbf{x}_l}(T) = c, \text{ for } w=T
\end{aligned}$$

whose optimal decision at state w, $z^*_{\mathbf{x}_l}(w)$, coincides with the optimal order-holding decision of the original problem (5) at state $(\mathbf{x}_l, w, v=1)$, $z^*(\mathbf{x}_l, w, v=1)$. This completes the proof of Proposition 1. □

Proof of Theorem 1. By Proposition 1, for each cluster $\mathfrak{C}_{\mathbf{x}_l} = \{(\mathbf{x}_l, w) : 0 \le w \le T\}$, we know that the optimal order-holding policy at state $(\mathbf{x}_l, w, v=1)$ with $(\mathbf{x}_l, w) \in \mathfrak{C}_{\mathbf{x}_l}$ made by the online retailer can be characterized by the single-dimensional MDP problem as follows:

$$\begin{aligned}
&\Delta_{\mathbf{x}_l}(w) = \min_{z\in\{0,1\}} z\{h + p((\mathbf{x}_l,w),(\mathbf{x}_l,w+1))\cdot\Delta_{\mathbf{x}_l}(w+1)\} + (1-z)c, \forall 0\le w\le T-1 \qquad (28)\\
&\Delta_{\mathbf{x}_l}(T) = c, \text{ for } w=T.
\end{aligned}$$

Firstly, we characterize the optimal decision of the above single-dimensional MDP problem, and then using their connections stated in Proposition 1 to characterize the optimal policy for the original MDP problem (5). Denote $S(\mathbf{x}_l)$ as the first $w \in \langle T-1\rangle$ such that $h + p((\mathbf{x}_l, w), (\mathbf{x}_l, w+1)) \cdot \Delta_{\mathbf{x}_l}(w+1) \ge c$, that is $S(\mathbf{x}_l) = \min\{w \in \langle T-1\rangle | h + p((\mathbf{x}_l, w), (\mathbf{x}_l, w+1)) \cdot \Delta_{\mathbf{x}_l}(w+1) \ge c\}$ (set to T if the set is empty). Then, for any $w < S(\mathbf{x}_l)$, we must have $h + p((\mathbf{x}_l, w), (\mathbf{x}_l, w+1)) \cdot \Delta_{\mathbf{x}_l}(w+1) < c$ which implies that the optimal decision is $z^* = 1$. While for $w = S(\mathbf{x}_l)$, the optimal decision is $z^* = 0$.

With the $S(\mathbf{x}_l)$ defined above characterizing the optimal decision of problem (28), we are ready to characterize the structure of optimal policy for the original MDP problem (5). We prove the structure of the optimal policy by considering the following two cases stated in Theorem 1.

Case 1. If no new order is received for $S(\mathbf{x}_l)$ period. When receiving the order, the current pending orders are being held, so the initial state is $(\mathbf{x}_l, 0, v=1)$. By Proposition 1, the optimal order-holding decision at state $(\mathbf{x}_l, 0, v=1)$ is the same as the counterpart of problem (28) at state

0. So, the optimal order-holding decision at state $(\mathbf{x}, 0, v=1)$ is to hold the current pending orders since $0 < S(\mathbf{x}_l)$. Since no new order is received for $S(\mathbf{x}_l)$ period, the state in the next period is $(\mathbf{x}_l, 1, v=1)$. By the same statement, the pending orders will be held until the beginning of period $S(\mathbf{x}_l)$ in which the state is $(\mathbf{x}_l, S(\mathbf{x}_l), v=1)$. The optimal decision of problem (28) at state $S(\mathbf{x}_l)$ is $z^* = 0$, so does the optimal decision for the original MDP problem (5) at state $(\mathbf{x}_l, S(\mathbf{x}_l), v=1)$ by Proposition 1. That is to say the pending orders will be sent to the 3PL after holding $S(\mathbf{x}_l)$ period.

Case 2. A new order comes in the first $S(\mathbf{x}_l)$ period. Similar to the statement in Case 1, the current pending orders will be held until the period receiving the new order. After receiving the new consecutive order, the system will enter another cluster with different order features $\mathbf{x}_{l+1}$ whose optimal threshold $S(\mathbf{x}_{l+1})$ can be characterized and begin a new cycle. So we have completed the proof of Theorem 1. $\square$

Proof of Proposition 2. By the reformulation of $\bar{q}_{w,M}(\mathbf{x}_l;\Theta,N)$ in (44), we have

$$\bar{q}_{w,M}(\mathbf{x}_l;\Theta,N) = e^{aw}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N), \forall w \in \langle T-1\rangle. \tag{29}$$

According to the Markovian logit model, the probability of making consecutive orders is

$$\begin{aligned}
&\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) + \bar{q}_{1,M}(\mathbf{x}_l;\Theta,N) + \cdots + \bar{q}_{T-1,M}(\mathbf{x}_l;\Theta,N)\\
&= \bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) + e^{a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) + \cdots + e^{(T-1)a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\\
&= \frac{1-e^{aT}}{1-e^{a}}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)
\end{aligned}$$

where the first equality follows (29) and the second equality is by simple calculation. This completes the proof of Proposition 2. $\square$

Proof of Theorem 2. Before starting the proof, let's first reformulate the transition probabilities $p((\mathbf{x}_l, w),(\mathbf{x}_l, w+1))$ using the choice probabilities of making another order $\{\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)\}_{\tau\in\langle T-1\rangle}$. Recall the original formulation of $p((\mathbf{x}_l, w),(\mathbf{x}_l, w+1)) = \mathbb{P}(w+1;\mathbf{x}_l,\Theta,N)/\mathbb{P}(w;\mathbf{x}_l,\Theta,N)$ in (16) where $\mathbb{P}(w;\mathbf{x}_l,\Theta,N)$ defined in (15) denotes the probability of the event that no new order by time w. Due to the structure of the choice network N, it can be reformulated as $\mathbb{P}(w;\mathbf{x}_l,\Theta,N) = 1 - \sum_{\tau=0}^{w-1}\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)$, which leads to the reformulation of transition probabilities as

$$p((\mathbf{x}_l, w),(\mathbf{x}_l, w+1)) = \frac{1-\sum_{\tau=0}^{w}\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)}{1-\sum_{\tau=0}^{w-1}\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)} = 1 - \frac{\bar{q}_{w,M}(\mathbf{x}_l;\Theta,N)}{1-\sum_{\tau=0}^{w-1}\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)}. \tag{30}$$

After the reformulation of transition probabilities, let's define an important function which will play a central role to characterize the close form of personalized thresholds, that is $\delta(w) := h + p((\mathbf{x}_l, w),(\mathbf{x}_l, w+1))\cdot c - c$. When substituting (30), it becomes

$$\delta(w) = h - \frac{\bar{q}_{w,M}(\mathbf{x}_l;\Theta,N)}{1-\sum_{\tau=0}^{w-1}\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)}c. \tag{31}$$

By the reformulation of $\bar{q}_{w,M}(\mathbf{x}_l;\Theta,N)$ in (29) and (44), we have

$$\begin{aligned}
\bar{q}_{w,M}(\mathbf{x}_l;\Theta,N) &= e^{aw}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N), \forall w \in \langle T-1\rangle\\
\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) &= \frac{\exp(\boldsymbol{\eta}\mathbf{x}_l + b)}{\sum_{\tau\in\langle T-1\rangle}\exp(\beta_\tau + a\tau) + \sum_{\tau\in\langle T-1\rangle} e^{a\tau}\cdot\exp(\boldsymbol{\eta}\mathbf{x}_l + b)}\\
&= \frac{\exp(\boldsymbol{\eta}\mathbf{x}_l + b)}{\sum_{\tau\in\langle T-1\rangle}\exp(\beta_\tau + a\tau) + \frac{1-e^{aT}}{1-e^{a}}\cdot\exp(\boldsymbol{\eta}\mathbf{x}_l + b)}.
\end{aligned} \tag{32}$$

With the reformulation (29) of $\bar{q}_{w,M}(\mathbf{x}_l;\Theta,N)$ above, we can finally reformulate the function $\delta(w)$ as

$$\delta(w) = h - \frac{e^{aw}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)}{1-\frac{1-e^{aw}}{1-e^{a}}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)}c = h - \frac{1}{\frac{1-\frac{1}{1-e^{a}}\cdot\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)}{e^{aw}\cdot\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)} + \frac{1}{1-e^{a}}}c. \tag{33}$$

With the reformulation of $\delta(w)$, we are ready to characterize close form of the personalized threshold $S(\mathbf{x}_l;\Theta,N)$ case by case (Case (18a), (18b), (19a), (19b) and (19c) in Theorem 2) as follows.

Case (18a) is with conditions that $h<c\cdot(1-e^a)$ and $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)<1-e^a$.

For this case we will verify that $S(\mathbf{x}_l;\Theta,N)=(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ with the nonlinear function $f_{nl}(\cdot)$ of the close form defined in (17). It is easy to check that $\delta(w)$ is increasing in w for $w\in\mathbb{R}$ by (33) when noting that $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)<1-e^a$ under **Case (18a)**. Define $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)):=\min\{w\in\mathbb{R}:\delta(w)\geq 0\}$ as the break point of the sign of $\delta(w)$, we will first verify that $S(\mathbf{x}_l;\Theta,N)=(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ (in **Step 1**) and then characterize the close form of the nonlinear function $f_{nl}(\cdot)$ (in **Step 2**).

Step 1. We verify that $S(\mathbf{x}_l;\Theta,N)=(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ by considering cases as follows depending on $S(\mathbf{x}_l;\Theta,N)=T$ (*Case A*), $S(\mathbf{x}_l;\Theta,N)=0$ (*Case B*) or $1\leq S(\mathbf{x}_l;\Theta,N)\leq T-1$ (*Case C*).

Case A that $S(\mathbf{x}_l;\Theta,N)=T$ which implies that $\delta(T-1)=h+p((\mathbf{x}_l,T-1),(\mathbf{x}_l,T))\cdot c-c=h+p((\mathbf{x}_l,T-1),(\mathbf{x}_l,T))\cdot\Delta_{\mathbf{x}_l}(T)-c<0$ by the characterization of personalized thresholds in (11). By definition, $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))>T-1$ which results in $(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0=T=S(\mathbf{x}_l;\Theta,N)$. So we have verified that $S(\mathbf{x}_l;\Theta,N)=(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ under *Case A*.

Case B that $S(\mathbf{x}_l;\Theta,N)=0$ which implies that $h+p((\mathbf{x}_l,0),(\mathbf{x}_l,1))\cdot\Delta_{\mathbf{x}_l}(1)-c\geq 0$ by the characterization of personalized thresholds in (11). The recursive definition of $\Delta_{\mathbf{x}_l}(w)$ in (10) implies that

$$\Delta_{\mathbf{x}_l}(w)\leq c,\forall w\in\langle T\rangle \tag{34}$$

which results in $\delta(0)=h+p((\mathbf{x}_l,0),(\mathbf{x}_l,1))\cdot c-c\geq h+p((\mathbf{x}_l,0),(\mathbf{x}_l,1))\cdot\Delta_{\mathbf{x}_l}(1)-c\geq 0$. By definition again, $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\leq 0$ which results in $(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0=0=S(\mathbf{x}_l;\Theta,N)$. So we have verified that $S(\mathbf{x}_l;\Theta,N)=(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ under *Case B*.

Case C that $1\leq S(\mathbf{x}_l;\Theta,N)\leq T-1$ which implies that the following two conditions hold by the characterization of personalized thresholds in (11):

$$h+p((\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)),(\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)+1))\cdot\Delta_{\mathbf{x}_l}(S(\mathbf{x}_l;\Theta,N)+1)-c\geq 0, \tag{35}$$

$$h+p((\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)-1),(\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)))\cdot c-c<0, \tag{36}$$

where the second inequality originates from the fact that $\Delta_{\mathbf{x}_l}(S(\mathbf{x}_l;\Theta,N))=c$ by the single dimensional MDP (10) and (35).

To show $S(\mathbf{x}_l;\Theta,N)=(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ under *Case C*, it suffices to show that $S(\mathbf{x}_l;\Theta,N)\leq(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ and $S(\mathbf{x}_l;\Theta,N)\geq(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ hold simultaneously as follows.

First, we will show that $S(\mathbf{x}_l;\Theta,N)\leq(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ which is equivalent to $S(\mathbf{x}_l;\Theta,N)\leq\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\vee 0$ since $S(\mathbf{x}_l;\Theta,N)\leq T$. Prove by contradiction. Assuming that $S(\mathbf{x}_l;\Theta,N)>\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\vee 0$ which is equivalent to $S(\mathbf{x}_l;\Theta,N)>\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil$ due to $S(\mathbf{x}_l;\Theta,N)>0$ under *Case C*. Since $S(\mathbf{x}_l;\Theta,N)$ is an integer, we must have $S(\mathbf{x}_l;\Theta,N)-1\geq\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\geq f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$. By the definition of $f_{nl}(\cdot)$ and monotonicity of $\delta(w)$ (increasing in w under **Case (18a)**), $\delta(S(\mathbf{x}_l;\Theta,N)-1)=h+p((\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)-1),(\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)))\cdot c-c\geq 0$ which is contradict with (36). So, we have shown that $S(\mathbf{x}_l;\Theta,N)\leq(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$.

Second, we will show that $S(\mathbf{x}_l;\Theta,N)\geq(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0$ which is equivalent to $S(\mathbf{x}_l;\Theta,N)\geq\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T$ since $S(\mathbf{x}_l;\Theta,N)\geq 0$. Prove by contradiction again. Assuming that $S(\mathbf{x}_l;\Theta,N)<\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T$ which is equivalent to $S(\mathbf{x}_l;\Theta,N)<\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil$ since $S(\mathbf{x}_l;\Theta,N)<T$ under *Case C*. Since $S(\mathbf{x}_l;\Theta,N)$ is an integer, we must have $S(\mathbf{x}_l;\Theta,N)<f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$. By the definition of $f_{nl}(\cdot)$ and monotonicity of $\delta(w)$ (increasing in w under **Case (18a)**) again, $0>\delta(S(\mathbf{x}_l;\Theta,N))=h+p((\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)),(\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)+$

$1)) \cdot c - c \geq h + p((\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)),(\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)+1)) \cdot \Delta_{\mathbf{x}_l}(S(\mathbf{x}_l;\Theta,N)+1) - c$ which is contradict with (35), where the last inequality comes from (34). So, we have shown that $S(\mathbf{x}_l;\Theta,N) \geq (\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil \wedge T) \vee 0$. Combine the two steps above together, we have shown that $S(\mathbf{x}_l;\Theta,N) = (\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil \wedge T) \vee 0$ under *Case C*.

Combine *Case A*, *B* and *C* together, we have shown that $S(\mathbf{x}_l;\Theta,N) = (\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil \wedge T) \vee 0$.

Step 2. To characterize the close form of the personalized threshold $S(\mathbf{x}_l;\Theta,N) = (\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil \wedge T) \vee 0$ under **Case (18a)**, it remains to characterize the close form of the nonlinear function $f_{nl}(\cdot)$.

Recall the reformulation of $\delta(w)$ in (33), we have

$$\begin{aligned}
\delta(w) \geq 0 &\Leftrightarrow h\left(1 - \frac{1}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) + \frac{1}{1-e^a}\cdot e^{aw}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\right) - c\cdot e^{aw}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \geq 0 \\
&\Leftrightarrow h\left(1 - \frac{1}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\right) + \left(\frac{h}{1-e^a} - c\right)\cdot e^{aw}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \geq 0 \\
&\Leftrightarrow w \geq \frac{1}{a}\left\{\ln(h) - \ln\left(c - \frac{h}{1-e^a}\right)\right\} - \frac{1}{a}\ln(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)) + \frac{1}{a}\ln\left(1 - \frac{1}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\right) \\
f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)) &= \frac{1}{a}\left\{\ln(h) - \ln\left(c - \frac{h}{1-e^a}\right)\right\} - \frac{1}{a}\ln(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)) + \frac{1}{a}\ln\left(1 - \frac{1}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\right)
\end{aligned}$$

where the last equivalence comes from the fact that $\frac{h}{1-e^a} - c < 0$ under **Case (18a)** and the last equality comes from the definition of $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)) := \min\{w \in \mathbb{R} : \delta(w) \geq 0\}$. So, we have characterize the close of the nonlinear function $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)$, so does the personalized threshold $S(\mathbf{x}_l;\Theta,N)$ under **Case (18a)**.

Next, we turn to **Case (18b)**.

Case (18b) is with conditions that $h < c\cdot(1-e^a)$ and $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \geq 1-e^a$.

In this case we wish to verify that $S(\mathbf{x}_l;\Theta,N) = T$. Actually, $\delta(0) = h - \bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\cdot c \leq h - (1-e^a)\cdot c < 0$ where the two inequalities originate from conditions in **Case (18b)**. It is easy to check that $\delta(w)$ is decreasing in w for $w \in [0, T-1]$ by the reformulation of $\delta(w)$ in (33) when noting that $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \geq 1-e^a$ under **Case (18b)**. So, $\delta(w) \leq \delta(0) < 0$ for $w \in \langle T-1\rangle$, which results in

$$h + p((\mathbf{x}_l,w),(\mathbf{x}_l,w+1))\cdot\Delta_{\mathbf{x}_l}(w+1) - c \leq h + p((\mathbf{x}_l,w),(\mathbf{x}_l,w+1))\cdot c - c = \delta(w) < 0, \forall w \in \langle T-1\rangle,$$

where the first inequality comes from the fact that $\Delta_{\mathbf{x}_l}(w) \leq c$ for $w \in \langle T\rangle$ in (34). The conditions above actually imply that $S(\mathbf{x}_l;\Theta,N) = T$ by its characterization in (11). So, we have shown that $S(\mathbf{x}_l;\Theta,N) = T$ under **Case (18b)**.

Now we begin to consider the remaining cases when $h \geq c\cdot(1-e^a)$.

Case (19a) is with conditions that $h \geq c\cdot(1-e^a)$ and $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \leq 1-e^a$.

We wish to verify that $S(\mathbf{x}_l;\Theta,N) = 0$ in this case. By the reformulation (33) and the fact that $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \leq 1-e^a$, $\delta(w)$ is increasing in w for $w \in [0,T-1]$ which implies that $\delta(w) \geq \delta(0) \geq 0$ for $w \in [0,T-1]$, where the last inequality comes from the fact that $\delta(0) = h - \bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\cdot c \geq h - (1-e^a)\cdot c \geq 0$ under **Case (19a)**. Prove by contradiction to show that $S(\mathbf{x}_l;\Theta,N) = 0$ in this case. If $S(\mathbf{x}_l;\Theta,N) > 0$, we must have $\delta(S(\mathbf{x}_l;\Theta,N)-1) = h + p((\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)-1),(\mathbf{x}_l,S(\mathbf{x}_l;\Theta,N)))\cdot c - c < 0$ by (36). Here comes the contradiction since we have shown that $\delta(w) \geq 0$ for $w \in [0,T-1]$. So, we have verified that $S(\mathbf{x}_l;\Theta,N) = 0$ under **Case (19a)**.

Case (19b) is with conditions that $h \geq c\cdot(1-e^a)$ and $1-e^a < \bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \leq h/c$.

First, we will verify that the personalized threshold $S(\mathbf{x}_l;\Theta,N)$ is either 0 or T in this case, and then characterize the conditions under which it will be one of them. Prove by contradiction to show that $S(\mathbf{x}_l;\Theta,N)$ is either 0 or T. Assuming that $1 \leq S(\mathbf{x}_l;\Theta,N) \leq T-1$ under **Case (19b)**. By the reformulation (33) and the fact that $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) > 1-e^a$, $\delta(w)$ is decreasing in w for $w \in [0,T-1]$ under **Case (19b)**. To derive the contradiction, define $w_1 := \lceil\max\{w \in \langle T-1\rangle : \delta(w) \geq 0\}\rceil$ ($w_1 \geq 0$

since $\delta(0) = h - \bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\cdot c \geq 0$ under **Case (19b)**). We will derive the contradiction of $1 \leq S(\mathbf{x}_l;\Theta,N) \leq T-1$ by considering different cases depending on $w_1 = T-1$ (*Case A*) or $w_1 < T-1$ (*Case B*) as follows.

Case A that $w_1 = T-1$ which implies that $\delta(w) \geq 0$ for $w \in \langle T-2\rangle$ by the definition of w_1 and decreasing of $\delta(w)$. It follows that $\delta(S(\mathbf{x}_l;\Theta,N)-1) = h + p((\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)-1),(\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)))\cdot c - c \geq 0$ since $S(\mathbf{x}_l;\Theta,N)-1 \in \langle T-2\rangle$ (since $1 \leq S(\mathbf{x}_l;\Theta,N) \leq T-1$ by assumption), which contradicts (36).

Case B that $w_1 < T-1$. We consider two additional cases as follows to derive contradiction of $1 \leq S(\mathbf{x}_l;\Theta,N) \leq T-1$ depending on $S(\mathbf{x}_l;\Theta,N) > w_1$ (*Case B.1*) or $S(\mathbf{x}_l;\Theta,N) \leq w_1$ (*Case B.2*).

Case B.1 that $S(\mathbf{x}_l;\Theta,N) > w_1$. By definition of w_1, we must have $0 > \delta(S(\mathbf{x}_l;\Theta,N)) \geq h + p((\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)),(\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)+1))\cdot \Delta_{\mathbf{x}_l}(S(\mathbf{x}_l;\Theta,N)+1) - c$ which contradicts (35), where the last inequality comes from (34).

Case B.2 that $S(\mathbf{x}_l;\Theta,N) \leq w_1$. By the definition of w_1, we must have $0 \leq \delta(S(\mathbf{x}_l;\Theta,N)-1) = h + p((\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)-1),(\mathbf{x}_l, S(\mathbf{x}_l;\Theta,N)))\cdot c - c$ which contradicts (36).

So, we have verified that the personalized threshold $S(\mathbf{x}_l;\Theta,N)$ is either 0 or T under **Case (19b)**. It remains to characterize the necessary and sufficient conditions under which $S(\mathbf{x}_l;\Theta,N) = T$ which are exactly the conditions stated in (19b) by the characterization of personalized thresholds in (11).

Case (19c) is with conditions that $h \geq c\cdot(1-e^a)$ and $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) > h/c$.

We wish to verify $S(\mathbf{x}_l;\Theta,N) = T$ in this case. By the reformulation (33) and the fact that $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) > 1-e^a$, $\delta(w)$ is decreasing in w for $w \in [0,T-1]$ which implies that $\delta(w) \leq \delta(0) < 0$ for $w \in [0,T-1]$, where the last inequality comes from the fact that $\delta(0) = h - \bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\cdot c < 0$ under **Case (19c)**. It follows that for any $w \in \langle T-1\rangle$

$$h + p((\mathbf{x}_l, w),(\mathbf{x}_l, w+1))\cdot\Delta_{\mathbf{x}_l}(w+1) - c \leq h + p((\mathbf{x}_l, w),(\mathbf{x}_l, w+1))\cdot c - c = \delta(w) < 0,$$

which implies that $S(\mathbf{x}_l;\Theta,N) = T$ by its characterization in (11), where the first inequality is due to $\Delta_{\mathbf{x}_l}(w) \leq c$ for $w \in \langle T\rangle$ by (34). So, we have verified that $S(\mathbf{x}_l;\Theta,N) = T$ under **Case (19c)**.

That completes the proof of Theorem 2. □

In the remaining part, we will first present Proposition 4 and its proof, based on which Theorem 3 can be proved.

PROPOSITION 4 (PIECEWISE LINEAR APPROXIMATION OF PERSONALIZED THRESHOLDS). *Under the Markovian logit model shown in (13) with network N and parameters $\Theta = (\boldsymbol{\eta}, \boldsymbol{\beta}, a, b)$, denote $X(\epsilon)$ as the region of features $\mathbf{x}_l$ such that their corresponding probability of making another consecutive orders is bounded by ϵ, that is $X(\epsilon) = \{\mathbf{x}_l \in X \mid \sum_{\tau\in\langle T-1\rangle} \bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N) \leq \epsilon\}$. If in addition $h < c\cdot(1-e^a)$ and $\epsilon < 1-e^{aT}$, then for any $\mathbf{x}_l \in X(\epsilon)$ the personalized threshold $S(\mathbf{x}_l;\Theta,N)$ can be bounded by a piecewise-linear function as:*

$$0 \leq S(\mathbf{x}_l;\Theta,N) - (\lceil f_l(\mathbf{x}_l;\Theta)\rceil \wedge T)\vee 0 \leq \left\lfloor 1 + \frac{1}{a}\ln\left(1 + \frac{\epsilon\wedge\epsilon_U}{1-\epsilon\wedge\epsilon_U}\right) + \frac{1}{a}\ln\left(1 - \frac{\epsilon\wedge\epsilon_U}{1-e^{aT}}\right)\right\rfloor, \tag{37}$$

where the linear function $f_l(\cdot)$ and scalar ϵ_U are defined in (20) and (21).

Proof of Proposition 4. Firstly, we will verify that the personalized threshold $S(\mathbf{x}_l;\Theta,N)$ shares the close form in (18a) with conditions of Proposition 4. For any $\mathbf{x}_l \in X(\epsilon)$, we have $\sum_{\tau\in\langle T-1\rangle} \bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N) = \frac{1-e^{aT}}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \leq \epsilon < 1-e^{aT}$ which implies that $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) < 1-e^a$, where the equality is obtained by substituting (29). Together with the assumption that $h < c\cdot(1-e^a)$, the personalized threshold share the close form $S(\mathbf{x}_l;\Theta,N) = (\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil \wedge T)\vee 0$ for any $\mathbf{x}_l \in X(\epsilon)$ by Theorem 2. So, to approximate the personalized threshold $S(\mathbf{x}_l;\Theta,N)$, it suffices to approximate the nonlinear function $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$. We will firstly bound the nonlinear function $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$ with a linear counterpart $f_l(\mathbf{x}_l;\Theta)$ (in **Step 1**) and then show that this will result in the piecewise-linear approximation (37) of the personalized threshold (in **Step 2**).

Step 1. Bound $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$ with $f_l(\mathbf{x}_l;\Theta)$.

Recall the reformulation of $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)$ in (32) and the definition of $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$ in (17), we can reformulate the nonlinear function as

$$\begin{aligned}
f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)) =& \frac{1}{a}\left\{\ln(h)-\ln\left(c-\frac{h}{1-e^a}\right)\right\}+\frac{1}{a}\cdot\ln\left(1-\frac{1}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\right)\\
&-\frac{1}{a}\cdot\left\{\boldsymbol{\eta}\mathbf{x}_l+b-\ln\left(\sum_{\tau=0}^{T-1}\exp(\beta_\tau+a\tau)+\frac{1-e^{aT}}{1-e^a}\cdot\exp(\boldsymbol{\eta}'\mathbf{x}_l+b)\right)\right\}\\
=&\frac{1}{a}\left\{\ln(h)-\ln\left(c-\frac{h}{1-e^a}\right)\right\}+\frac{1}{a}\cdot\ln\left(1-\frac{1}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\right)\\
&-\frac{1}{a}\cdot\left\{\boldsymbol{\eta}\mathbf{x}_l+b-\ln\left(\sum_{\tau=0}^{T-1}\exp(\beta_\tau+a\tau)\right)-\ln\left(1+\frac{\frac{1-e^{aT}}{1-e^a}\cdot\exp(\boldsymbol{\eta}\mathbf{x}_l+b)}{\sum_{\tau=0}^{T-1}\exp(\beta_\tau+a\tau)}\right)\right\}\\
=&f_l(\mathbf{x}_l;\Theta)+\frac{1}{a}\ln\left(1+\frac{\frac{1-e^{aT}}{1-e^a}\cdot\exp(\boldsymbol{\eta}\mathbf{x}_l+b)}{\sum_{\tau=0}^{T-1}\exp(\beta_\tau+a\tau)}\right)+\frac{1}{a}\cdot\ln\left(1-\frac{1}{1-e^a}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\right)\\
=&f_l(\mathbf{x}_l;\Theta)+\frac{1}{a}\ln\left(1+\frac{\epsilon'(\mathbf{x}_l;\Theta)}{1-\epsilon'(\mathbf{x}_l;\Theta)}\right)+\frac{1}{a}\ln\left(1-\frac{1}{1-e^{aT}}\epsilon'(\mathbf{x}_l;\Theta)\right)
\end{aligned}$$

where the first equality is obtained by substituting the reformulation of $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)$ in (32) into the definition of $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$ in (17), the second and third equality are by simple calculation, and the finally equality originates from the the definition of $\epsilon'(\mathbf{x}_l;\Theta):=\frac{1-e^{aT}}{1-e^a}\cdot\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)\le\epsilon$.

Define the function $g(\epsilon'):=\frac{1}{a}\ln\left(1+\frac{\epsilon'}{1-\epsilon'}\right)+\frac{1}{a}\ln\left(1-\frac{1}{1-e^{aT}}\epsilon'\right)$ for $\epsilon'\in[0,\epsilon]$, then $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))=f_l(\mathbf{x}_l;\Theta)+g(\epsilon'(\mathbf{x}_l;\Theta))$ by the reformulation above. The function can be reformulated as $g(\epsilon')=\frac{1}{a}\ln\left(\frac{1}{1-e^{aT}}\cdot\left(1-\frac{e^{aT}}{1-\epsilon'}\right)\right)$ which implies that it is increasing in ϵ'. This will result in the bound $0=g(0)\le g(\epsilon'(\mathbf{x}_l;\Theta))\le g(\epsilon)$ which implies the bound for the nonlinear function $f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))$ for any $\mathbf{x}_l\in X(\epsilon)$:

$$0\le f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))-f_l(\mathbf{x}_l;\Theta)\le g(\epsilon)=\frac{1}{a}\ln\left(1+\frac{\epsilon}{1-\epsilon}\right)+\frac{1}{a}\ln\left(1-\frac{1}{1-e^{aT}}\epsilon\right). \tag{38}$$

With the bound in (38) between $f_{nl}(\cdot)$ and $f_l(\cdot)$, we are ready to characterize the piecewise-linear approximation (37) of the personalized threshold in **Step 2**.

Step 2. Piecewise linear approximation for the personalized threshold $S(\mathbf{x}_l;\Theta,N)$.

Divide the set $X(\epsilon)$ into two disjoint subsets: $X(\epsilon)=X_1(\epsilon)\cup X_2(\epsilon)$, where

$$\begin{aligned}
X_1(\epsilon)&:=\left\{x\in X(\epsilon)\,\middle|\,\sum_{\tau=0}^{T-1}\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)>\epsilon_U\right\},\\
X_2(\epsilon)&:=\left\{x\in X(\epsilon)\,\middle|\,\sum_{\tau=0}^{T-1}\bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N)\le\epsilon_U\right\}.
\end{aligned}$$

To show the bound (37) holds for any $\mathbf{x}_l\in X(\epsilon)$, it suffices to show the bound holds for any $\mathbf{x}_l\in X_1(\epsilon)$ and $\mathbf{x}_l\in X_2(\epsilon)$ respectively.

For any $\mathbf{x}_l\in X_2(\epsilon)$, we must have

$$\begin{aligned}
0\le&S(\mathbf{x}_l;\Theta,N)-(\lceil f_l(\mathbf{x}_l;\Theta)\rceil\wedge T)\vee 0\\
=&(\lceil f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))\rceil\wedge T)\vee 0-(\lceil f_l(\mathbf{x}_l;\Theta)\rceil\wedge T)\vee 0\\
\le&\lfloor 1+f_{nl}(\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N))-f_l(\mathbf{x}_l;\Theta)\rfloor\\
\le&\left\lfloor 1+\frac{1}{a}\ln\left(1+\frac{\epsilon\wedge\epsilon_U}{1-\epsilon\wedge\epsilon_U}\right)+\frac{1}{a}\ln\left(1-\frac{\epsilon\wedge\epsilon_U}{1-e^{aT}}\right)\right\rfloor
\end{aligned}$$

which is exactly the bound (37), where the second inequality comes from Lemma 3 and the last inequality originates form the bound (38) when noting that $X_2(\epsilon)\subseteq X(\epsilon\wedge\epsilon_U)$.

It remains to verify the bound for any $\mathbf{x}_l \in X_1(\epsilon)$. Denote the deterministic utility of purchasing another order as $u(\mathbf{x}_l) = \boldsymbol{\eta}'\mathbf{x}_l + b$, then both $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)$ and $f_l(\mathbf{x}_l;\Theta)$ can be written as functions of $u(\mathbf{x}_l)$ by their definitions in (32) and (20) as follows:

$$\bar{q}'_{0,M}(u(\mathbf{x}_l);\Theta,N) = \frac{\exp(u(\mathbf{x}_l))}{\sum_{\tau=0}^{T-1}\exp\left(\beta_\tau + a\tau\right) + \frac{1-e^{aT}}{1-e^{a}}\exp(u(\mathbf{x}_l))}, \tag{39}$$

$$f'_l(u(\mathbf{x}_l);\Theta) := -\frac{1}{a}u(\mathbf{x}_l) + C_0, \tag{40}$$

both of which are strictly increasing in $u(\mathbf{x}_l)$. Then, for any $\mathbf{x}_l \in X_1(\epsilon)$, we must have $\frac{1-e^{aT}}{1-e^{a}} \cdot \bar{q}'_{0,M}(u(\mathbf{x}_l);\Theta,N) > \epsilon_U = \frac{1-e^{aT}}{1-e^{a}} \cdot \bar{q}'_{0,M}(-a(T-1-C_0);\Theta,N)$ (the "=" here is by definition of ϵ_U) which implies that $u(\mathbf{x}_l) > -a(T-1-C_0)$ by the increasing of $\bar{q}'_{0,M}(u;\Theta,N)$ in u. This will lead to $f'_l(u(\mathbf{x}_l);\Theta) > f'_l(-a(T-1-C_0);\Theta) = T-1$ since $f'_l(u;\Theta)$ is also strictly increasing in u. It follows that $0 \le S(\mathbf{x}_l;\Theta,N) - (\lceil f_l(\mathbf{x}_l;\Theta)\rceil \wedge T)\vee 0 = S(\mathbf{x}_l;\Theta,N) - (\lceil f'_l(u(\mathbf{x}_l);\Theta)\rceil \wedge T)\vee 0 = S(\mathbf{x}_l;\Theta,N) - T$ (the "$\le$" comes from (38)), which implies that $S(\mathbf{x}_l;\Theta,N) = T$ and $S(\mathbf{x}_l;\Theta,N) - (\lceil f_l(\mathbf{x}_l;\Theta)\rceil \wedge T)\vee 0 = 0$. So we have verified that the bound (37) also holds for any $\mathbf{x}_l \in X_1(\epsilon)$, which completes the proof of Proposition 4. □

Proof of Theorem 3. By Proposition 4, the bound in Theorem 3 holds for any features $\mathbf{x}_l \in X(\epsilon)$ with any $\epsilon < 1-e^{aT}$. It remains to show it still holds for any $\mathbf{x}_l \in X^c$, where the set X^c is defined as

$$\begin{aligned} X^c &= \left\{ \mathbf{x}_l \in X \,\middle|\, \sum_{\tau\in\langle T-1\rangle} \bar{q}_{\tau,M}(\mathbf{x}_l;\Theta,N) \ge 1-e^{aT} \right\} \\ &= \left\{ \mathbf{x}_l \in X \,\middle|\, \frac{1-e^{aT}}{1-e^{a}}\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \ge 1-e^{aT} \right\} \end{aligned}$$

where the second equality is obtained by substituting (29). Then, we have $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N) \ge 1-e^{a}$ for any $\mathbf{x}_l \in X^c$ by definition and $h < c\cdot(1-e^a)$ by the assumption that $h < c\cdot(1-e^a)/(1+e^a)$, which implies that $S(\mathbf{x}_l;\Theta,N) = T$ by (18b) in Theorem 2. To verify the bound in Theorem 3, it suffices to show that the piecewise-linear function $f_{pl}(\mathbf{x}_l) = (\lceil f_l(\mathbf{x}_l;\Theta)\rceil \wedge T)\vee 0$ equals T for any $\mathbf{x}_l \in X^c$ as follows.

Recall that both $\bar{q}_{0,M}(\mathbf{x}_l;\Theta,N)$ and $f_l(\mathbf{x}_l;\Theta)$ can be equivalently written as $\bar{q}'_{0,M}(\mathbf{x}_l;\Theta,N)$ (in (39)) and $f'_l(\mathbf{x}_l;\Theta)$ (in (40)) respectively, which are strictly increasing functions of $u(\mathbf{x}_l) = \boldsymbol{\eta}'x + b$. By definition of the set $\mathbf{x}_l \in X^c$, it holds that $\frac{1-e^{aT}}{1-e^{a}}\bar{q}'_{0,M}(u(\mathbf{x}_l);\Theta,N) \ge 1-e^{aT}$. By simple calculation, it is easy to verify that $1-e^{aT} > \epsilon_U$ under the imposed assumption that $h < c\cdot(1-e^a)/(1+e^a)$, which implies that $\frac{1-e^{aT}}{1-e^{a}}\bar{q}'_{0,M}(u(\mathbf{x}_l);\Theta,N) > \epsilon_U = \frac{1-e^{aT}}{1-e^{a}} \cdot \bar{q}'_{0,M}(-a(T-1-C_0);\Theta,N)$ (the "=" here is by definition of ϵ_U). This will result in $u(\mathbf{x}_l) > -a(T-1-C_0)$ by increasing of the function $\bar{q}'_{0,M}(u;\Theta,N)$ in u. It follows that $f'_l(u(\mathbf{x}_l);\Theta) > f'_l(-a(T-1-C_0);\Theta) = T-1$ by increasing of the function $f'_l(u;\Theta)$ in u, which implies that $f_{pl}(\mathbf{x}_l) = (\lceil f_l(\mathbf{x}_l;\Theta)\rceil \wedge T)\vee 0 = (\lceil f'_l(u(\mathbf{x}_l);\Theta)\rceil \wedge T)\vee 0 = T = S(\mathbf{x}_l;\Theta,N)$. So we have completed the proof of Theorem 3. □

LEMMA 3. *For any scalars A and B with $A \ge B$, it holds that* $0 \le (\lceil A\rceil \wedge T)\vee 0 - (\lceil B\rceil \wedge T)\vee 0 \le \lfloor A-B+1\rfloor$.

Proof of Lemma 3. To prove Lemma 3, it suffices to prove the bound that $(\lceil A\rceil \wedge T)\vee 0 - (\lceil B\rceil \wedge T)\vee 0 \le A-B+1$ when noting that both $(\lceil A\rceil \wedge T)\vee 0$ and $(\lceil B\rceil \wedge T)\vee 0$ are integers. The bound will be verified by considering different cases depending on $A \le 0$ (**Case 1**), $A \ge T$ (**Case B**), or $0 < A < T$ (**Case C**).

Case A that $A \le 0$ which implies that $B \le A \le 0$. This will lead to $(\lceil A\rceil \wedge T)\vee 0 - (\lceil B\rceil \wedge T)\vee 0 = 0 - 0 \le A-B+1$.

Case B that $A \ge T$ which implies that

$$(\lceil A\rceil \wedge T)\vee 0 - (\lceil B\rceil \wedge T)\vee 0 = T - (\lceil B\rceil \wedge T)\vee 0 = \begin{cases} 0 \le A-B+1, \text{ if } B > T-1, \\ T - \lceil B\rceil \vee 0 \le A-B+1, \textbf{ if } B \le T-1. \end{cases} \tag{41}$$

Case C that $0 < A < T$ which implies that $B \le A < T$. This will lead to

$$(\lceil A \rceil \wedge T) \vee 0 - (\lceil B \rceil \wedge T) \vee 0 = \lceil A \rceil - \lceil B \rceil \vee 0 = \begin{cases} \lceil A \rceil - \lceil B \rceil \le A - B + 1, & \text{if } B > 0, \\ \lceil A \rceil \le A + 1 \le A - B + 1, & \text{if } B \le 0. \end{cases} \quad (42)$$

This completes the proof of Lemma 3. ☐

Proof of Proposition 3. To prove Proposition 3, it suffices to show that each term $\ln(\Pr(\mathbf{x}_l^k, DT_l^k; \Theta))$ in (23) is concave in parameters $\Theta = (\boldsymbol{\eta}, \boldsymbol{\beta}, a, b)$ for any $k \in [K]$ and $l \in [L_k]$. By (13), we have

$$\begin{aligned} \Pr(\mathbf{x}_l^k, DT_l^k; \Theta) &= \prod_{w=0}^{DT_l^k - 2} \bar{p}_{w,w+1}(\mathbf{x}_l^k; \Theta, N) \cdot \bar{p}_{DT_l^k - 1, y_l^k}(\mathbf{x}_l^k; \Theta, N) \\ &= \prod_{w=0}^{DT_l^k - 2} \frac{\exp\left(a + g_{w+1}(\mathbf{x}_l^k; \Theta, N)\right)}{\exp(g_w(\mathbf{x}^k; \Theta, N))} \cdot \frac{\mathcal{I}(y_l^k = \triangle) \cdot \exp(\beta_{DT_l^k - 1}) + \mathcal{I}(y_l^k = M) \cdot \exp(\boldsymbol{\eta}' \mathbf{x}_l^k + b)}{\exp(g_{DT_l^k - 1}(\mathbf{x}_l^k; \Theta, N))} \\ &= \frac{\mathcal{I}(y_l^k = \triangle) \cdot \exp(\beta_{DT_l^k - 1} + a(DT_l^k - 1)) + \mathcal{I}(y_l^k = M) \cdot \exp(\boldsymbol{\eta}' \mathbf{x}_l^k + b + a(DT_l^k - 1))}{\exp(g_0(\mathbf{x}_l^k; \Theta, N))}. \end{aligned} \quad (43)$$

It follows that $\ln(\Pr(\mathbf{x}_l^k, DT_l^k; \Theta)) = \mathcal{I}(y_l^k = \triangle) \cdot (\beta_{DT_l^k - 1} + a(DT_l^k - 1)) + \mathcal{I}(y_l^k = M) \cdot (\boldsymbol{\eta}' \mathbf{x}_l^k + b + a(DT_l^k - 1)) - g_0(\mathbf{x}_l^k; \Theta, N)$, where the first term is linear in Θ. Hence, it suffices to show that $g_0(\mathbf{x}_l^k; \Theta, N)$ is convex in Θ. By Lemma 5, $g_0(\mathbf{x}_l^k; \Theta, N)$ can be reformulated as $g_0(\mathbf{x}; \Theta, N) = \ln\left(\sum_{\tau=0}^{T-1} \exp\left(\beta_\tau + a\tau\right) + \exp(\boldsymbol{\eta}\mathbf{x} + b + a\tau)\right)$, whose convexity in Θ is confirmed by the following Lemma 4. This completes the proof. ☐

Lemma 4 (Convexity of log exponential functions). *The function* $g(\mathbf{x}) = \ln(\sum_{i=1}^n p_i \exp(\mathbf{b}_i' \mathbf{x} + c_i))$ *with* $p_i \ge 0$ *for* $i = 1, 2, \ldots, n$ *is a convex function in* $\mathbf{x}$.

Proof of Lemma 4. The convexity of function $g(\mathbf{x})$ in $\mathbf{x}$ is equivalent to the fact that the epigraph, $\text{epi}g(\mathbf{x}) = \{(\mathbf{x}, y) | g(\mathbf{x}) \le y\}$, is a convex set. Noting that $\text{epi}g(\mathbf{x})$ can be equivalent reformulated as $\text{epi}g(\mathbf{x}) = \{(\mathbf{x}, y) | \sum_{i=1}^n p_i \exp(\mathbf{b}_i' \mathbf{x} + c_i - y) \le 0\}$ which is a convex set since $f(\mathbf{x}, y) = \sum_{i=1}^n p_i \exp(\mathbf{b}_i' \mathbf{x} + c_i - y)$ is jointly convex in $(\mathbf{x}, y)$. This completes the proof of Lemma 4. ☐

Lemma 5 (Reformulate $g_0(\mathbf{x}; \Theta)$). *The function* $g_w(\mathbf{x}; \Theta, N)$ *defined in (13) can be reformulated as*

$$g_w(\mathbf{x}; \Theta, N) = \ln\left(\sum_{\tau=w}^{T-1} \exp\left(\beta_\tau + a(\tau - w)\right) + \exp\left(\boldsymbol{\eta}\mathbf{x} + b + a(\tau - w)\right)\right), \forall w \in \langle T - 1 \rangle.$$

Proof of Lemma 5. Prove by induction on $w \in \langle T - 1 \rangle$. By (13), we have $g_{T-1}(\mathbf{x}_l; \Theta, N) = \ln(\exp(\beta_{T-1}) + \exp\left(\boldsymbol{\eta}\mathbf{x}_l + b\right))$ which is exactly the form of $g_{T-1}(\mathbf{x}_l; \Theta, N)$ we want to prove. Now assume that $g_{w+1}(\mathbf{x}_l; \Theta, N)$ shares the form in Lemma 5, it remains to show that so does $g_w(\mathbf{x}_l; \Theta, N)$. By (13) again, we have

$$\begin{aligned} g_w(\mathbf{x}_l; \Theta, N) &= \ln\left(\exp(\beta_w) + \exp(\boldsymbol{\eta}\mathbf{x}_l + b) + \exp\left(a + g_{w+1}(\mathbf{x}_l; \Theta, N)\right)\right) \\ &= \ln\left(\exp(\beta_w) + \exp(\boldsymbol{\eta}\mathbf{x}_l + b) + \exp(a) \cdot \left(\sum_{\tau=w+1}^{T-1} \exp\left(\beta_\tau + a(\tau - w - 1)\right) + \exp(\boldsymbol{\eta}\mathbf{x}_l + b + a(\tau - w - 1))\right)\right) \\ &= \ln\left(\sum_{\tau=w}^{T-1} \exp\left(\beta_\tau + a(\tau - w)\right) + \exp(\boldsymbol{\eta}\mathbf{x}_l + b + a(\tau - w))\right), \end{aligned}$$

where the second equality is obtained by substituting the induction assumption of $g_{w+1}(\mathbf{x}_l; \Theta, N)$ into the first equality. So we have completed the induction proof of Lemma 5. ☐

Lemma 6 (Equivalence of MLE and The MNL Choice Model). *There exists an MNL model with choice alternatives in the set $\mathcal{L}$ and the deterministic utility terms being linear in the order features $\mathbf{x}$, where $\mathcal{L}=\{(0,\triangle),(0,M),(1,\triangle),(1,M),\cdots,(T-1,\triangle),(T-1,M)\}$ is defined as a set that includes all the arcs to the terminal node Δ or M in network N. Under this MNL model, the choice probability of each alternative is the same as that of the corresponding arc in network N under MLM.*

Proof of Lemma 6. Recall that the MLM model with network N and parameters $\Theta=(\boldsymbol{\eta},\boldsymbol{\beta},a,b)$ assumes that the deterministic utility term is a linear function of features $\mathbf{x}_l$, and random error terms follow i.i.d. Gumbel distributions. Now we are going to construct a MNL model with alternatives in $\mathcal{L}$ such that it shares the same unconditional choice probability of each arc in $\mathcal{L}$ as the MLM model.

We construct the MNL model by further reformulating the unconditional choice probability $\bar{q}_{w,y}(\mathbf{x}_l;\Theta,N)$ of arc $(w,y)\in\mathcal{L}$ in the MLM model as follows. Firstly, by the close from of choice probabilities for the MLM model in (13), we have

$$\begin{aligned}\bar{q}_{w,y}(\mathbf{x}_l;\Theta,N)&=\prod_{\tau=0}^{w-1}\bar{p}_{\tau,\tau+1}(\mathbf{x}_l;\Theta,N)\cdot\bar{p}_{w,y}(\mathbf{x}_l;\Theta,N)\\&=\prod_{\tau=0}^{w-1}\frac{\exp\left(a+g_{\tau+1}(\mathbf{x}_l;\Theta,N)\right)}{\exp(g_\tau(\mathbf{x}_l;\Theta,N))}\cdot\frac{\mathcal{I}(y=\triangle)\cdot\exp(\beta_w)+\mathcal{I}(y=M)\cdot\exp(\boldsymbol{\eta}\mathbf{x}_l+b)}{\exp(g_w(\mathbf{x}_l;\Theta,N))}\\&=\frac{\mathcal{I}(y=\triangle)\cdot\exp(\beta_w+aw)+\mathcal{I}(y=M)\cdot\exp(\boldsymbol{\eta}\mathbf{x}_l+b+aw)}{\exp(g_0(\mathbf{x}_l;\Theta,N))}.\end{aligned}$$

By substituting the reformulation of $\exp(g_0(\mathbf{x}_l;\Theta,N))$ in Lemma 5, above expression of $\bar{q}_{w,y}(\mathbf{x}_l;\Theta,N)$ turns to

$$\bar{q}_{w,y}(\mathbf{x}_l;\Theta,N)=\frac{\mathcal{I}(y=\triangle)\cdot\exp(\beta_w+aw)+\mathcal{I}(y=M)\cdot\exp(\boldsymbol{\eta}\mathbf{x}_l+b+aw)}{\sum_{\tau=0}^{T-1}\exp\left(\beta_\tau+a\tau\right)+\exp(\boldsymbol{\eta}\mathbf{x}_l+b+a\tau)},\tag{44}$$

which is exactly the expression of choice probabilities for the MNL model that we are familiar with. The MNL model we are trying to construct is clear by setting the deterministic utility term of the alternative (w,y) in labels set $\mathcal{L}$ as the linear function $\mathcal{I}(y=\triangle)\cdot(\beta_w+aw)+\mathcal{I}(y=M)\cdot(\boldsymbol{\eta}\mathbf{x}_l+b+aw)$. The construction here confirms that the MLM and MNL model share the same choice probabilities of arcs in the label set $\mathcal{L}$, which completes the proof of Lemma 6. □

Lemma 7. *There exists a logistic regression model that is equivalent to the MLE method of MLM.*

Proof of Lemma 7. Note that the logistic regression model is an MLE method of a *multinomial logit* (MNL) choice model. Hence, it is sufficient to show the equivalence between MLM and the MNL choice model. Hence Lemma 7 is a straightforward result from Lemma 6. □

Proof of Proposition 5. Denote the denominator of (45) as $\hat{\mathbb{P}}(\mathbf{x},w)$, the estimator in (45) can be rewritten as

$$\hat{\mathbb{P}}(\tau|w;\mathbf{x})=\sum_{\mathbf{x}',w',\tau':(w',\tau')\in\mathcal{T}_{\mathbf{x}'},\mathbf{x}'\in\mathcal{X}}\frac{\mathcal{K}_1(w,w';\zeta_1)\cdot\prod_{i=3,\ldots,d_x+2}\mathcal{K}_i(x_i,x_i';\zeta_i)}{\hat{\mathbb{P}}(\mathbf{x},w)}\cdot\mathcal{K}_2(\tau,\tau';\zeta_2),$$

where the kernel function for the categorical variable is defined as

$$\mathcal{K}_2(\tau,\tau';\zeta_2)=\begin{cases}1-\zeta_2, & \text{if } \tau=\tau',\\ \zeta_2/T, & \text{if } \tau\neq\tau'.\end{cases}$$

With the above reformulation of the estimator $\hat{\mathbb{P}}(\tau|w;\mathbf{x})$, we have

$$\sum_{\tau=0}^{T}\hat{\mathbb{P}}(\tau|w;\mathbf{x})=\sum_{\mathbf{x}',w',\tau':(w',\tau')\in\mathcal{T}_{\mathbf{x}'},\mathbf{x}'\in\mathcal{X}}\frac{\mathcal{K}_1(w,w';\zeta_1)\cdot\prod_{i=3,\ldots,d_x+2}\mathcal{K}_i(x_i,x_i';\zeta_i)}{\hat{\mathbb{P}}(\mathbf{x},w)}\cdot\left\{\sum_{\tau=0}^{T}\mathcal{K}_2(\tau,\tau';\zeta_2)\right\}.$$

$$= \sum_{\mathbf{x}',w',\tau':(w',\tau')\in\mathcal{T}_{\mathbf{x}'},\mathbf{x}'\in\mathcal{X}} \frac{\mathcal{K}_1(w,w';\zeta_1)\cdot \prod_{i=3,\dots,d_x+2} \mathcal{K}_i(x_i,x_i';\zeta_i)}{\hat{\mathbb{P}}(\mathbf{x},w)} \cdot \left\{1-\zeta_2+\frac{\zeta_2}{T}\cdot T\right\}.$$
$$=1,$$

where the second equality follows the definition of the kernel function $\mathcal{K}_2(\cdot,\cdot;\zeta_2)$ and the last equality is by the definition of $\hat{\mathbb{P}}(\mathbf{x},w)$. The non-negativity of $\hat{\mathbb{P}}(\tau|w;\mathbf{x})$ follows from the non-negativity of kernel functions which completes the proof of Proposition 5. □

B. The KCDE and RF Methods

B.1. Using KCDE to Estimate The Transition Probabilities and The Probability of Placing Consecutive Orders

According to Section 6, we obtain a set of transaction paths from the data. However, to train the KCDE method we need a sequence of observed transitions. This requires us to first translate the transaction paths to the transitions among states with no new orders. Note that each entry in $path^k=\{(\mathbf{x}_l^k,DT_l^k)\}_{l\in[L_k]}$ corresponds to a sequence of transitions in a set of single-dimensional MDP problems. Specifically, each entry $(\mathbf{x}_l^k,DT_l^k)$, $l\in[L_k-1]$, means that upon receiving an order featured by $\mathbf{x}_l^k$, a new order arrives after DT_l^k periods. This corresponds to a set of transitions: $\{((\mathbf{x}_l^k,w),(\mathbf{x}_l^k,w+1))\}_{w\in\langle DT_l^k-2\rangle}\cup\{((\mathbf{x}_l^k,DT_l^k-1),(\mathbf{x}_{l+1}^k,0))\}$. For $l=L_k$, the consumer has made the maximum number of consecutive orders and she will not place any order within T periods. This corresponds to the transitions: $\{((\mathbf{x}_l^k,w),(\mathbf{x}_l^k,w+1))\}_{w\in\langle T-1\rangle}$. Let $\mathcal{T}_{\mathbf{x}}$ denote a set of observed transitions in the single-dimensional MDP problem defined in the cluster identified by $\mathbf{x}$. Let $\mathcal{X}$ denote a set of observed order features. The sets $\mathcal{T}_{\mathbf{x}}$, $\mathbf{x}\in\mathcal{X}$, serve as standard training data for the KCDE method.

Our goal is to estimate the transition probabilities of the MDP problem defined by (5) to (9). According to Theorem 1, it suffices to estimate $p((\mathbf{x}_l,w),(\mathbf{x}_l,w+1))$, $w\in\langle T-1\rangle$, for each single-dimensional MDP. By definition, the conditional probability $p((\mathbf{x}_l,w),(\mathbf{x}_l,w+1))$ is equal to the probability that no new order arrives for $w+1$ periods divided by the probability that no new order arrives for w periods after receiving an order featured by $\mathbf{x}_l$.

We represent the probability of each event using kernel functions. Note that for each order featured by $\mathbf{x}_l$, the number of periods without new order arrivals is a categorical variable. According to the KCDE method (Hall et al. 2004), for any two values w and τ, the probability that the categorical variable equals w and τ before and after a transition, respectively, in a single-dimensional MDP featured by $\mathbf{x}_l$ is $\frac{1}{|\mathcal{T}_{\mathbf{x}_l}|}\sum_{w',\tau':(w',\tau')\in\mathcal{T}_{\mathbf{x}_l}}\mathcal{K}_1(w,w';\zeta_1)\mathcal{K}_2(\tau,\tau';\zeta_2)$, where ζ_i represents the bandwidth for $i=1,2$. Here $|\mathcal{T}_{\mathbf{x}_l}|$ denotes the cardinality of $\mathcal{T}_{\mathbf{x}_l}$, which equals the number of joint observations of (w',τ') before and after a transition in a single-dimensional MDP featured by $\mathbf{x}_l$.

The kernel function for categorical variables with a total of $T+1$ categories is defined as $\mathcal{K}_i(w,w';\zeta_i)=(\zeta_i/T)^{\mathcal{I}(w'\neq w)}(1-\zeta_i)^{1-\mathcal{I}(w'\neq w)}$, $i=1,2$ (see Equation (4) in Hall et al. (2004)). Bowman (1980) shows that this kernel type of probability density function with bandwidths estimated from an MLE method converges (in probability) to the true density function. By definition, the conditional probability for a single-dimensional MDP to transit from w to τ can be calculated as $\sum_{w',\tau':(w',\tau')\in\mathcal{T}_{\mathbf{x}_l}}\mathcal{K}_1(w,w';\zeta_1)\cdot\mathcal{K}_2(\tau,\tau';\zeta_2)\ /\sum_{w':(w',\tau')\in\mathcal{T}_{\mathbf{x}_l}}\mathcal{K}_1(w,w';\zeta_1)$.

The data includes various single-dimensional MDPs. Thus, the observed (w',τ') may be associated with different single-dimensional MDPs. We weight each observation by the kernel density function of its associated single-dimensional MDP featured by $\mathbf{x}$. In particular, we define the conditional probability of a single-dimensional MDP featured by $\mathbf{x}$ to transit from w to τ as follows:

$$\hat{\mathbb{P}}(\tau|w;\mathbf{x})=\frac{\sum_{\mathbf{x}',w',\tau':(w',\tau')\in\mathcal{T}_{\mathbf{x}'},\mathbf{x}'\in\mathcal{X}}\mathcal{K}_1(w,w';\zeta_1)\cdot\mathcal{K}_2(\tau,\tau';\zeta_2)\cdot\prod_{i=3,\dots,d_x+2}\mathcal{K}_i(x_i,x_i';\zeta_i)}{\sum_{\mathbf{x}',w':(w',\tau')\in\mathcal{T}_{\mathbf{x}'},\mathbf{x}'\in\mathcal{X}}\mathcal{K}_1(w,w';\zeta_1)\cdot\prod_{i=3,\dots,d_x+2}\mathcal{K}_i(x_i,x_i';\zeta_i)}. \tag{45}$$

Note that the weight defined by the product of kernel functions $\prod_{i=3,\dots,d_x+2}\mathcal{K}_i(x_i,x_i';\zeta_i)$ represents the similarity of the two single-dimensional MDPs featured by $\mathbf{x}$ and $\mathbf{x}'$. We assume that the more similar the two order features, the more similar the evolutions of the corresponding single-dimensional MDPs.

We follow the convention to define a kernel function $\mathcal{K}_i(x_i,x_i';\zeta_i)$ for numerical variables as $\mathcal{K}_i(x_i,x_i';\zeta_i)=\mathbb{K}((x_i-x_i')/\zeta_i)$, where $\mathbb{K}(\cdot):\mathbb{R}\to\mathbb{R}_+$ satisfies $\int\mathbb{K}(s)ds<\infty$. The following proposition ensures that the proposed conditional probability in (45) is well defined.

Proposition 5. *The conditional probability defined in (45) satisfies* $\sum_{\tau=0,\dots,T}\hat{\mathbb{P}}(\tau|w;\mathbf{x})=1$ *and* $\hat{\mathbb{P}}(\tau|w;\mathbf{x})\geq 0$ *for* $\tau=0,\dots,T$.

We apply the likelihood cross-validation method (Li and Racine 2007) to select bandwidths for mixed data types (containing both categorical and numerical features). Specifically, the bandwidths are selected to maximize the log-likelihood function $\sum_{\mathbf{x},w,\tau:(w,\tau)\in\mathcal{T}_{\mathbf{x}},\mathbf{x}\in\mathcal{X}} z_{\mathbf{x},w,\tau}\log\left(\hat{\mathbb{P}}_{-1}(\tau|w;\mathbf{x})\right)$, where $z_{\mathbf{x},w,\tau}$ is an indicator variable that equals 1 if we observe a transition from w to τ with the order features $\mathbf{x}$ in the data, and 0 otherwise. Here, instead of using $\hat{\mathbb{P}}(\tau|w;\mathbf{x})$, we adapt the leave-one-out estimation method by Li and Racine (2007) (see page 161 in Li and Racine (2007)) to define the likelihood function. Specifically, we apply $\hat{\mathbb{P}}_{-1}(\tau|w;\mathbf{x})$ in the likelihood function, which is in the same form as (45) but with $\mathcal{T}_{\mathbf{x}}$ replaced by $\mathcal{T}_{\mathbf{x}}\setminus\{(w,\tau)\}$. That is, the kernel probability density function for a sample labeled by w,τ, and $\mathbf{x}$ should exclude the sample itself in the summation. This method is available in the class "npcdensbw" of the R package "np". By carefully setting the input parameters in "npcdensbw", we obtain the bandwidths for mixed data types and obtain the estimated transition probabilities under KCDE.

With the estimated $\hat{\mathbb{P}}(\tau|w;\mathbf{x})$, we can calculate the probability that a new order arrives in period w since receiving the latest order as $\prod_{\tau=0}^{w-1}\hat{\mathbb{P}}(\tau+1|\tau;\mathbf{x})\cdot\hat{\mathbb{P}}(0|w;\mathbf{x})$. The probability of making consecutive orders can be calculated as $\sum_{w=0}^{T-1}\prod_{\tau=0}^{w-1}\hat{\mathbb{P}}(\tau+1|\tau;\mathbf{x})\cdot\hat{\mathbb{P}}(0|w;\mathbf{x})$.

B.2. Using RF To Estimate The Transition Probabilities and The Probability of Placing Consecutive Orders

RF is an ensemble learning method that builds a set of decision trees guiding classification results, which is available in many machine learning packages. We apply the class "RandomForestClassifier" in the python package "Scikit-learn" by Pedregosa et al. (2011) to predict probabilities. Recall that $\mathcal{L}=\{(0,\triangle),(0,M),(1,\triangle),(1,M),\cdots,(T-1,\triangle),(T-1,M)\}$ is a set that includes all the arcs to the terminal node Δ or M in network N. Let $\bar{q}_{t,j}^{RF}(\mathbf{x}_l^k)$, $j\in\{\triangle,M\}$, denote the predicted probability of the arc $(t,j)\in\mathcal{L}$. Given an order $\mathbf{x}_l^k$, the predicted probability $\bar{q}_{t,j}^{RF}(\mathbf{x}_l^k)$ of each class $t\in<T-1>$ and $j\in\{\Delta,M\}$ is the average of each single tree's probability of the class in the forest. A single tree's probability of a class is the fraction of training samples of the class over all the samples in the leaf where $\mathbf{x}_l^k$ locates (cf. Pedregosa et al. (2011)).

C. Introduction to AUC, Brier Score, and Calibration Curve

AUC is a popular performance metric to assess classifiers. The standard AUC is designed only for binary classification problems. Specifically, for a given feature-response samples $\{(\mathbf{x}^k,y^k)\}_{k\in[K]}$ where $y^k\in\{-1,1\}$ is either -1 or 1, the AUC of a binary predictor $p:X\to[0,1]$ is defined as (Hand and Till 2001)

$$AUC(p(\cdot))=\frac{\sum_{(\mathbf{x},y)\in\mathcal{D}^{-1}}\sum_{(\mathbf{x}',y')\in\mathcal{D}^{1}}\mathcal{I}[p(\mathbf{x}')>p(\mathbf{x})]}{|\mathcal{D}^{-1}|\cdot|\mathcal{D}^{1}|},\tag{46}$$

where $p(\mathbf{x})$ represents the probability that the response is 1 conditioned on the feature is $\mathbf{x}$, $\mathcal{D}^i = \{(\mathbf{x}^k, y^k)|y^k = i, k \in [K]\}$ denotes the subsets of samples with their response as i for $i \in \{-1, 1\}$. From the definition, we can interpret AUC as the probability that a randomly selected positive sample shares higher predicted probability than a randomly selected negative sample. We call AUC for binary classification as AUC-binary.

Since predicting the choice probabilities in the consumer's sequential decision process is a multi-class classification problem, we need to discuss how to design AUC for a classification problem over multiple classes. We follow Hand and Till (2001) to generalize the AUC for the binary classification to the AUC for the multi-class classification. Specifically, for a I-class predictor $p(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots, p_I(\mathbf{x})) : X \to \{(p_1, p_2, \cdots, p_I)| \sum_{i \in [I]} p_i = 1\}$ where $p_i(\mathbf{x})$ denotes the probability that the response is of i-th class conditioned on the feature is $\mathbf{x}$ for any $i \in [I]$, its AUC, denoted as $AUC(p(\cdot))$, is defined as

$$AUC(p(\cdot)) = \frac{1}{I(I-1)} \sum_{i=1}^{I} \sum_{j=1, j \neq i}^{I} AUC(i, j), \tag{47}$$

$$AUC(i, j) = \frac{\sum_{(\mathbf{x}, y) \in \mathcal{D}^j} \sum_{(\mathbf{x}', y') \in \mathcal{D}^i} \mathcal{I}[p(\mathbf{x}') > p(\mathbf{x})]}{|\mathcal{D}^j| \cdot |\mathcal{D}^i|}, \tag{48}$$

where $\mathcal{D}^i = \{(\mathbf{x}^k, y^k)|y^k = i, k \in [K]\}$ denotes the subset of samples with their response as i for $i \in [I]$. We call this version of AUC for multi-class classification as AUC-Hand-Till.

The Brier Score of a multi-class predictor $p(\mathbf{x}) = (p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots, p_I(\mathbf{x})) : X \to \{(p_1, p_2, \cdots, p_I)| \sum_{i \in [I]} p_i = 1\}$ is defined as

$$\sum_{k=1}^{K} \sum_{i=1}^{I} \|p(\mathbf{x}^k) - \mathbf{e}^k\|_2^2 \tag{49}$$

where $\mathbf{e}^k = (e_1^k, \cdots, e_I^k)$ and $e_i^k = 1$ if the kth sample is of the ith class, otherwise $e_i^k = 0$ for $i = 1, \cdots, I$.

As we have mentioned in Appendix B.2, both RF and MLE are predictors of the multi-class classification problem with label set as $\mathcal{L} = \{(0, \triangle), (0, M), (1, \triangle), (1, M), \cdots, (T-1, \triangle), (T-1, M)\}$, the above AUC-Hand-Till and Brier score can be used to assess their performance.

To plot a calibration curve under MLE or RF, we first sort all the orders according to their predicted probabilities of placing consecutive orders. Then, we divide the sorted orders into five even clusters. For each cluster, we compute its orders' average predicted probability of placing consecutive orders and the empirical percentage of the orders in the cluster that have a consecutive order. Each cluster corresponds to a point with its x coordinate equals the cluster's average predicted probability of placing consecutive orders, and its y coordinate equals the empirical percentage of the orders in the cluster that have a consecutive order. The calibration curve is formed by linking the points. Note that the nearer the calibration curve is to the diagonal line, the nearer the predictions are to their empirical counterparts.

D. Comparing Out-of-Sample Costs

We further compare the out-of-sample costs of MLE, KCDE, and RF. Table 7 reports the total out-of-sample costs of MLE and KCDE over the same 20 experiments described in Section 7.1.1. Again, we randomly sample 500 transaction paths for each experiment to reduce the computation time. The table shows that MLE outperforms KCDE based on their out-of-sample costs, which is consistent with the comparison results based on other performance metrics in Section 7.1.1. Table 8 shows that the out-of-sample costs under MLE and RF are still close to each other, with the relative cost gap (defined as $\frac{C_{RF} - C_{MLE}}{C_{MLE}} \times 100\%$) ranges from -0.056% to -0.011%.

Table 7: Out-of-sample costs of MLE and KCDE

$h(\times 10^{-3})$	Cost of KCDE	Cost of MLE	$GAP_{KCDE}(\%)$
0.262	10,174	10,164	0.098
0.861	10,169	10,155	0.138
1.775	10,145	10,117	0.277
2.788	10,109	10,065	0.437

Table 8: Out-of-sample costs of MLE and RF

$h(\times 10^{-3})$	Cost of RF	Cost of MLE	$GAP_{RF}(\%)$
0.262	298,120	298,152	-0.011
0.861	299,120	299,288	-0.056
1.775	299,674	299,786	-0.038
2.788	299,964	300,085	-0.040

E. Static Policy

The basic idea to get the optimal static threshold follows the idea of the Sample Average Approximation (SAA). Specifically, given a set of training samples $\{path^k\}_{k\in\mathcal{T}_1}$ ($\mathcal{T}_1$ is the index of training samples), the static threshold denoted by ST^* can be obtained by minimizing the empirical holding and order arrangement cost $f(ST;h,\{path^k\}_{k\in\mathcal{T}_1})$, which is defined as follows.

$$f(ST;h,\{path^k\}_{k\in\mathcal{T}_1}) = \sum_{k\in\mathcal{T}_1}\left\{\sum_{l=1}^{L_k-1}\mathcal{I}\{DT_l^k\le ST\}(DT_l^k\cdot h)+\mathcal{I}\{DT_l^k> ST\}(ST\cdot h+c)\right\} + \{ST\cdot h+c\}, \quad (50)$$

where the first term accounts for the total cost for the first L_k-1 consecutive orders whose subsequent choice is to make another order. For each order at position $l<L_k$, $DT_l^k\le ST$ indicates that a new order arrives by the holding threshold ST. According to the static policy, it only incurs a holding cost $DT_l^k\cdot h$ for the current order. In contrast, if $DT_l^k>ST$, a new order arrives after the holding threshold ST, in which case the current order has been sent to the 3PL. Therefore, not only the holding cost $ST\cdot h$ but also the order arrangement cost is incurred. The last term represents the cost incurred by the order at position L_k. By definition, no orders arrive for the maximal dwell time hence both holding and order arrangement cost is incurred.

With the optimal threshold ST^* obtained from the training samples, we can evaluate its performance on the testing samples $\{path^k\}_{k\in\mathcal{T}_2}$ where $\mathcal{T}_2$ is the indices of testing samples. The static policy is easy to implement and computationally efficient.

F. Estimating The Holding Cost Per Period h

We estimate the holding cost per period h as follows: (i) The retailer sets a target order arrangement fee reduction R. (ii) We determine the minimum threshold (number of periods) $\tau(R)$ to achieve the target. (iii) We find the maximum holding cost per period h such that the optimal static threshold is at least $\tau(R)$.

First, we can formulate a linear program to determine the threshold $\tau(R)$ to achieve the target of order arrange fee reduction.

$$\begin{aligned}\tau(R) = \min_{\tau}\ & \tau\\ s.t.\ & f(0;0,\{path^k\}_{k\in[K]}) - f(\tau;0,\{path^k\}_{k\in[K]}) \ge Rf(0;0,\{path^k\}_{k\in[K]})\\ & \tau\in\langle 30\rangle,\end{aligned} \quad (51)$$

where R denotes the target order-arrangement cost reduction level and $0\le R\le R_U = \frac{f(0;0,\{path^k\}_{k\in[K]})-f(30;0,\{path^k\}_{k\in[K]})}{f(0;0,\{path^k\}_{k\in[K]})}$. By setting h to 0, $f(\tau;0,\{path^k\}_{k\in[K]})$ represents the empirical order arrangement cost by holding the order for τ, which is decreasing in τ. Therefore, the problem can be solved via bisection search efficiently.

Given a threshold τ, we intend to find the maximal h such that the threshold is the optimal static threshold leading to the smallest total holding and order-arrangement cost, which serves as a soft constraint to ensure the timely delivery. Specifically, $h^*(\tau)$ can be estimated by solving the following linear program:

$$h^*(\tau) \in \arg\max_{h\geq 0} h \quad (52)$$
$$s.t.\ f(\tau; h, \{path^k\}_{k\in[K]}) \leq f(ST; h, \{path^k\}_{k\in[K]}), \forall ST \in \langle T\rangle, ST \neq \tau,$$

where the cost function $f(ST; h, \{path^k\}_{k\in[K]})$ defined in (50) is linear in h. We start by setting τ to $\tau(R)$. If problem (52) with $\tau = \tau(R)$ is feasible, we set the holding cost per period to $h^*(\tau(R))$ for a given target R. If problem (52) with $\tau = \tau(R)$ becomes infeasible, we set h to $h^*(\tau^*)$ where τ^* is the smallest $\tau > \tau(R)$ such that the problem (52) is feasible. It is worthwhile to point out that we can always find such a τ^* as when $\tau^* = T$, $h = 0$ is feasible. With the estimation above, the optimal static policy of the system with the estimated h as the holding cost per period can achieve the target of order-arrangement reduction level R.

For the numerical experiments in Section 7, we use transaction paths with $T = 30$ minutes. We set $R = 0.85R_U, 0.65R_U, 0.45R_U$, and $0.25R_U$, where R_U =1.72%, which results in $h = 0.262 \times 10^{-3}, 0.861 \times 10^{-3}, 1.775 \times 10^{-3}$, and 2.788×10^{-3} respectively.

G. Tables

Table 9 Description of the "SKUs" table

Field	Data type	Description	Sample value
sku_ID	string	Unique identifier of a product	b4822497a5
type	int	1P or 3P SKU	1
brand_ID	string	Brand unique identification code	c840ce7809
attribute1	int	First key attribute of the category	3
attribute2	int	Second key attribute of the category	60
activate_date	string	The date at which the SKU is first introduced	2018/3/1
deactivate_date	string	The date at which the SKU is terminated	2018/3/1

Table 10 Description of the "users" table

Field	Data type	Description	Sample value
user_ID	string	User unique identification code	000000f736
user_level	int	User level	10
first_order_month	string	First month in which the customer placed an order on JD.com (format: yyyy-mm)	2017-07
plus	int	If user is with a PLUS membership	0
gender	string	User gender (estimated)	F
age	string	User age range (estimated)	26-35
marital_status	string	User marital status (estimated)	M
education	int	User education level (estimated)	3
purchase_power	int	User purchase power (estimated)	2
city_level	int	City level of user address	1

Table 11 Description of the "orders" table

Field	Data type	Description	Sample value
order_ID	string	Order unique identification code	3b76bfcd3b
user_ID	string	User unique identification code	3cde601074
sku_ID	string	SKU unique identification code	443fd601f0
order_date	string	Order date (format: yyyy-mm-dd)	2018/3/1
order_time	string	Specific time at which the order gets placed	2018-03-01 11:10:40
quantity	int	Number of units ordered	1
type	int	1P or 3P orders	1
promise	int	Expected delivery time (in days)	2
original_unit_price	float	Original list price	99.9
final_unit_price	float	Final purchase price	53.9
direct_discount_per_unit	float	Discount due to SKU direct discount	5
quantity_discount_per_unit	float	Discount due to purchase quantity	41
bundle_discount_per_unit	float	Discount due to bundle promotion	0
coupon_discount_per_unit	float	Discount due to customer coupon	0
gift_item	int	If the SKU is with gift promotion	0
dc_ori	int	Distribution center ID where the order is shipped from	29
dc_des	int	Destination address where the order is shipped to (represented by the closest distribution center)	29

Table 12 Description of the "clicks" table

Field	Data type	Description	Sample Value
sku_ID	string	SKU unique identification code	b4822497a5
user_ID	string	User unique identification code	94ff800585
request_time	string	The time at which the customer clicks the SKU item page	2018-03-01 23:57:53
channel	string	The click channel	wechart

Table 13 Summary of consecutive orders for each user level

user level	proportion of orders for each user level	number of non-consecutive orders	number of consecutive orders	**empirical percentage of the orders that have a consecutive order**	**average predicted probability of placing consecutive orders**
1 (Individual)	28.05%	120,227	809	0.72%	0.72%
2 (Individual)	30.87%	131,948	1,273	0.90%	0.90%
3 (Individual)	21.01%	89,654	1,014	1.08%	1.07%
4 (Individual)	18.96%	79,718	2,083	2.50%	2.49%
10 (Enterprise)	1.07%	3,347	1,291	27.84%	27.80%

Table 14 Summary of consecutive orders for each position

position	1	2	3	4
empirical percentage of the orders that have a consecutive order (%)	1.16	12.29	51.89	83.40
average predicted probability of placing consecutive orders (%)	1.16	12.22	51.42	82.74