How Advance Sales can Reduce Profits: When to Buy, When to Sell, and What Price to Charge - Online appendix

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1 Proof of Theorem 14

Proof. Since subtracting p from v is just moving to the left of v, we will ignore p. The statement in Theorem 14 becomes

- $(0,0)$ if (v,k) is in Region 1.
- $\Big(0\ , v_1+k+R(v_1+k)\Big)$ if (v,k) is in Region 2.
- $\Big(v_1+R(v_1)\ , W[A]-\Big(1+\frac{k}{v_1}\Big)R(v_1)\Big)$ if (v,k) is in Region 3.
- $\Big(v_1+R(v_1)\ , 0\Big)$ if (v,k) is in Region 4.
- The first statement follows immediately from the definition of Region 1 (see Table 1 in the main text).
- We wish to prove that the equilibrium value of λ_2 in Region 2 equals $v_1+k+R(v_1+k)$. Recall that in Region 2,

  $$v_1+k=\frac{\lambda_2 e^{\lambda_1}}{1-e^{-\lambda_2}}. \tag{1}$$

  Since in Region 2 $\lambda_1=0$, (1) becomes

  $$v_1+k=\frac{\lambda_2}{1-e^{-\lambda_2}}. \tag{2}$$

  First, we show that $\lambda_2=v_1+k+R(v_1+k)$ indeed solves (2).

  Substituting $\lambda_2=v_1+k+R(v_1+k)$ in the right-hand side of (2) gives

  $$\frac{v_1+k+R(v_1+k)}{1-e^{-(v_1+k)}e^{-R(v_1+k)}}. \tag{3}$$

  By Property W5

  $$e^{-R(v_1+k)}=\frac{R(v_1+k)}{-(v_1+k)e^{-(v_1+k)}}. \tag{4}$$

Substituting this in (3) gives

$$\frac{v_1 + k + R(v_1 + k)}{1 + \frac{R(v_1+k)}{v_1+k}} = v_1 + k,$$

which is the left-hand side of (2).

Now, because the right-hand side of (2) is strictly monotonic in λ_2, then for any given pair (v_1, k) the value $\lambda_2 = v_1 + k + R(v_1 + k)$ *uniquely* solves (2).

- To find λ_1 in Regions 3 and 4 recall that $u_1 = 0$ there. Thus,

  $$v_1 = \frac{\lambda_1}{1 - e^{-\lambda_1}}. \tag{5}$$

  Note that substituting $k = 0$ in (2) and changing λ_2 to λ_1 yields (5). Hence the proof given earlier for λ_2 in Region 2 applies for λ_1 in Regions 3 and 4, and gives $\lambda_1 = v_1 + R(v_1)$.

- Lastly, we have yet to prove that the equilibrium value of λ_2 in Region 3 is $W[A] - \left(1 + \frac{k}{v_1}\right) R(v_1)$.

  Recall that we consider c as the unit value. Recall also that $v_1 = v - p - k$, and that $v_2 = v - p$. Note that given v and k, the values v_1 and v_2 are functions of p. We find that most of the expressions appearing in our analysis are functions of $v - p - k$. Thus we focus on v_1. The four regions in terms of v_1 are

  - Region 1 is reached iff $-k \leq v_1 < 1 - k$.
  - Region 2 is reached iff $1 - k \leq v_1 \leq 1$.
  - Region 3 is reached iff $1 < v_1 < v(k) - k$.
  - Region 4 is reached iff $v(k) - k \leq v_1$,

  where $v(k)$ was defined as the point separating Regions 3 and 4 (when the regions were expressed in terms of v).

  What is left to prove is that the equilibrium value of λ_2 in Region 3 is $W[A] - \left(1 + \frac{k}{v_1}\right) R(v_1)$.

  Recall that $A = -(v_1 + k)e^{-v_1 + \frac{k}{v_1} R(v_1)}$.

  Using the two central equations $u_1 = 0$ and $u_2 = 0$, that hold in Region 3 (see (4) and (5) in the main paper) we have

  $$v_1 = \frac{\lambda_1}{1 - e^{-\lambda_1}}, \tag{6}$$

  and

  $$v_2 = \frac{\lambda_2 e^{\lambda_1}}{1 - e^{-\lambda_2}}. \tag{7}$$

  We already proved that in Regions 3 and 4 the equilibrium value of λ_1 is $v_1 + R(v_1)$, and that it uniquely solves (6). Substituting this in (7) gives

$$v_2 = \frac{\lambda_2 e^{v_1 + R(v_1)}}{1 - e^{-\lambda_2}},$$

which is the same as

$$v_2 e^{-v_1 - R(v_1)} = \frac{\lambda_2}{1 - e^{-\lambda_2}}. \tag{8}$$

Since the right-hand side of (8) is monotonic in λ_2, for any given pair v_1, v_2 at most one value of λ_2 satisfies (8). We now show that the proposed solution $\lambda_2 = W[A] - \frac{v_2 R(v_1)}{v_1}$ satisfies (8). Substituting this λ_2 in the right-hand side of (8) gives

$$\frac{W[A] - \frac{v_2 R(v_1)}{v_1}}{1 - e^{\frac{v_2 R(v_1)}{v_1}} e^{-W[A]}}.$$

Because of Property W5 this expression equals

$$\frac{W[A] - \frac{v_2 R(v_1)}{v_1}}{1 - \frac{e^{\frac{v_2 R(v_1)}{v_1}} W[A]}{A}},$$

which equals

$$\frac{\left(W[A] - \frac{v_2 R(v_1)}{v_1}\right) A}{A - e^{\frac{v_2 R(v_1)}{v_1}} W[A]}.$$

Since $v_2 = v_1 + k$ the above equals

$$\frac{\left(\frac{v_2 R(v_1)}{v_1} - W[A]\right) v_2 e^{-v_1 + \frac{k R(v_1)}{v_1}}}{-v_2 e^{-v_1 + \frac{k R(v_1)}{v_1}} - e^{\frac{v_2 R(v_1)}{v_1}} W[A]}$$

$$= \frac{\left(\frac{v_2 R(v_1)}{v_1} - W[A]\right) e^{R(v_1) + \frac{k R(v_1)}{v_1}}}{-v_2 e^{-v_1 + \frac{k R(v_1)}{v_1}} - e^{\frac{v_2 R(v_1)}{v_1}} W[A]} \cdot v_2 e^{-v_1 - R(v_1)}$$

Hence we need to prove that the quotient above equals 1, namely that

$$\left(\frac{v_2 R(v_1)}{v_1} - W[A]\right) e^{R(v_1) + \frac{k R(v_1)}{v_1}} = -v_2 e^{-v_1 + \frac{k R(v_1)}{v_1}} - e^{\frac{v_2 R(v_1)}{v_1}} W[A]. \tag{9}$$

Note that the expression $e^{R(v_1) + \frac{k R(v_1)}{v_1}}$ appearing on the left-hand side satisfies

$$e^{R(v_1) + \frac{k R(v_1)}{v_1}} = e^{\left(\frac{v_1 + k}{v_1}\right) R(v_1)} = e^{\frac{v_2 R(v_1)}{v_1}}.$$

Substituting this in the left-hand side of (9) gives

$$\frac{v_2 R(v_1)}{v_1} e^{\frac{v_2 R(v_1)}{v_1}} - e^{\frac{v_2 R(v_1)}{v_1}} W[A].$$

Hence we only need to prove that

$$\frac{v_2 R(v_1)}{v_1} e^{\frac{v_2 R(v_1)}{v_1}} = -v_2 e^{-v_1 + \frac{k R(v_1)}{v_1}}.$$

This is equivalent to proving that

$$\frac{R(v_1)}{v_1} e^{\frac{v_2 R(v_1)}{v_1} + v_1 - \frac{k R(v_1)}{v_1}} = -1.$$

Note that the left-hand side of the above equation equals

$$\frac{R(v_1)}{v_1} e^{\frac{(v_2 - k) R(v_1)}{v_1} + v_1} = \frac{R(v_1)}{v_1} e^{R(v_1) + v_1}.$$

By (4) the above equals

$$\frac{R(v_1)}{v_1} \frac{(-v_1 e^{-v_1}) e^{v_1}}{R(v_1)} = -1.$$

□

To prove Theorem 1 in the main paper, we first need to establish several results, using the Lambert function $W[x]$. The following lemma proves essential properties of the function $R(v_1)$.

Lemma 1.

- *R1.* $R(v_1)$ *is negative and increasing for all* $v_1 > 1$.
- *R2.* $R(1) = -1$.
- *R3.* $R(v(k) - k) = \frac{k}{v(k)} - 1$.
- *R4.* $R'(v_1) = \frac{-R(v_1)}{R(v_1)+1} \cdot \frac{v_1 - 1}{v_1}$.

Proof.

1. Since $-v_1 < -1 < 0$, then $-v_1 e^{-v_1} < 0$. Hence by Properties W1 (that W is increasing) and W4 (that $W[0] = 0$), we get $R(v_1) = W[-v_1 e^{-v_1}] < 0$. Now $(-v_1 e^{-v_1})' = e^{-v_1}(v_1 - 1) > 0$ (since $1 < v_1$). The Lambert function is increasing and so $W[-v_1 e^{-v_1}]$ increases with v_1, proving R1.

2. To prove R2, note that by Property W2 $R(1) = W[-e^{-1}] = -1$.

3. We wish to prove that $R(v(k) - k) = \frac{k}{v(k)} - 1$. Recall that $v(k)$ satisfies $k = v(k) - \frac{\ln v(k)}{1 - \frac{1}{v(k)}}$. Hence $(k - v(k))(1 - \frac{1}{v(k)}) = -\ln v(k)$, and so $k - v(k) = -\ln v(k) + (\frac{k}{v(k)} - 1)$. Adding

$\ln(k - v(k))$ to both sides of the equation gives

$$(k - v(k)) + \ln(k - v(k)) = \ln\Big(\frac{k}{v(k)} - 1\Big) + \Big(\frac{k}{v(k)} - 1\Big).$$

Thus

$$(k - v(k))e^{k - v(k)} = \Big(\frac{k}{v(k)} - 1\Big)e^{\frac{k}{v(k)} - 1}.$$

Applying the Lambert function to both sides of the equation gives

$$W[(k - v(k))e^{k - v(k)}] = W\Big[\Big(\frac{k}{v(k)} - 1\Big)e^{\frac{k}{v(k)} - 1}\Big]. \tag{10}$$

The left-hand side of (10) is $R(v(k) - k)$. Now by W6, since $\frac{k}{v(k)} - 1 \geq -1$ then

$$W\Big[\Big(\frac{k}{v(k)} - 1\Big)e^{\frac{k}{v(k)} - 1}\Big] = \frac{k}{v(k)} - 1,$$

proving R3.

4.

$$R'(v_1) = W'[-v_1 e^{-v_1}]e^{-v_1}(v_1 - 1). \tag{11}$$

Hence by Property W3,

$$R'(v_1) = \frac{R(v_1)e^{-v_1}(v_1 - 1)}{-v_1 e^{-v_1}(R(v_1) + 1)} = \frac{-R(v_1)}{R(v_1) + 1} \cdot \frac{v_1 - 1}{v_1},$$

proving R4.

$\square$

For the next result we need the following lemma.

Lemma 2. *The expression* $v_1 + (v_1 + k)R(v_1)$, *is negative in Region 3.*

Proof. Substituting $v_1 = v(k) - k$, which is the right end of Region 3, in $v_1 + (v_1 + k)R(v_1)$, and recalling that by R3 $R(v(k) - k) = \frac{k}{v(k)} - 1$, gives 0. Additionally, by R4, the derivative $1 + R(v_1) + (v_1 + k)R'(v_1)$ of this expression is

$$1 + R(v_1) + \frac{-R(v_1)(v_1 + k)(v_1 - 1)}{v_1(R(v_1) + 1)}.$$

By R1 and R2, in Region 3 $1 + R(v_1) > 0$, and $-R(v_1) > 0$. Hence the derivative is positive and so the expression $v_1 + (v_1 + k)R(v_1)$ is negative in Region 3. $\square$

Recall that $A = A(v_1) = -(v_1 + k)e^{-v_1 + \frac{kR(v_1)}{v_1}}$,

Lemma 3.

1. *$W[A(v(k)-k)]=-1$.*

2. *$A(v_1)$ decreases with v_1 for $1<v_1<v(k)-k$.*

Proof.

1. Since $W[\cdot]$ is strictly monotonic and by Property W2 $W[-e^{-1}]=-1$, we must prove that $A(v(k)-k)=-e^{-1}$. Now‘

$$A(v(k)-k)=-v(k)e^{k-v(k)+\frac{kR(v(k)-k)}{v(k)-k}}. \tag{12}$$

From R3, $R(v(k)-k)=\frac{k}{v(k)}-1$. Substituting this in the right-hand side of (12) gives

$$-v(k)e^{k-v(k)+\frac{k(k/v(k)-1)}{v(k)-k}}=-v(k)e^{k-v(k)-\frac{k}{v(k)}}=-e^{k-v(k)-\frac{k}{v(k)}+\ln v(k)},$$

Substituting $\ln v(k)=-(k-v(k))\Big(1-\frac{1}{v(k)}\Big)$ gives

$$-e^{k-v(k)-\frac{k}{v(k)}-(k-v(k))\left(1-\frac{1}{v(k)}\right)}=-e^{-1}.$$

2. To prove that $A(v_1)$ is decreasing note that

$$A'(v_1)=\frac{e^{-v_1+\frac{kR}{v_1}}\Big(v_1(v_1+k-1)+kR\Big)\big(v_1+(v_1+k)R\big)}{v_1^2(R+1)}, \tag{13}$$

where $R=R(v_1)$. By properties R1 and R2, the denominator is positive. Thus we need to prove that the numerator is negative. In Region 3, $v_1>1$. Thus by properties R1 and R2, $v_1+R>1+R>0$. Hence the expression in the first parentheses of (13), $v_1(v_1+k-1)+kR$ which equals $k(v_1+R)+v_1(v_1-1)$, is positive. The expression in the second parentheses is negative by Lemma 2. Thus, $A'(v_1)$ is negative implying that $A(v_1)$ is indeed decreasing there.

□

Recall that $R=R(v_1)$, and that $v(k)$ is defined by the equation $k=v(k)-\frac{\ln v(k)}{1-\frac{1}{v(k)}}$. We examine $W[A]$ as a function of k. Given v_1 denote by k_0 the value of k that satisfies $v_1=v(k_0)-k_0$. Denote $u_0=v(k_0)$. Also denote by A_0 the value of A that corresponds to k_0, namely $A_0=-(v_1+k_0)e^{-v_1+\frac{k_0R}{v_1}}$.

Lemma 4. *Given v_1 :*

1. *In Region 3 $W[A]$ increases with k.*

2. *In Region 3 $\frac{d}{dk}W[A]$ decreases with k.*

3. $\frac{d}{dk}W[A]\Big|_{k=k_0} = \frac{1}{u_0}$.

Proof.

1. To prove that $W[A]$ increases with k in Region 3 note that

$$\frac{dA}{dk} = -\frac{e^{-v_1+\frac{kR}{v_1}}}{v_1}(v_1 + (v_1 + k)R).$$

By Lemma 2, $v_1 + (v_1 + k)R < 0$ in Region 3. Thus $\frac{dA}{dk} > 0$ and A increase with k in Region 3. Since W is an increasing function, $W[A]$ also increases with k in Region 3.

2. Using Property W3 yields

$$\frac{d}{dk}W[A] = \frac{(v_1 + (v_1 + k)R)W[A]}{v_1(v_1 + k)(W[A] + 1)}. \tag{14}$$

The above equals

$$\frac{v_1 + (v_1 + k)R}{v_1(v_1 + k)} - \frac{(v_1 + (v_1 + k)R)}{v_1(v_1 + k)(W[A] + 1)}.$$

Differentiating the above expression with respect to k gives

$$\frac{d^2}{dk^2}W[A] = \frac{-v_1^2}{(v_1(v_1 + k))^2} + \frac{v_1(v_1 + k)\frac{d}{dk}W[A](v_1 + (v_1 + k)R)}{(v_1(v_1 + k)(W[A] + 1))^2}. \tag{15}$$

The first term in (15) is negative. The second term is also negative since we proved in Part 1 that $\frac{d}{dk}W[A]$ is positive, and by Lemma 2 that $v_1 + (v_1 + k)R$ is negative.

3. By (14)

$$\frac{d}{dk}W[A] = \frac{(v_1 + (v_1 + k)R)W[A]}{v_1(v_1 + k)(W[A] + 1)}.$$

Note that for $k = k_0$ the numerator vanishes according to Lemma 2, and the denominator vanishes according to Part 1 of Lemma 3.

By L'Hôpital's rule

$$\frac{d}{dk}W[A]\Big|_{k=k_0} = \frac{\frac{d}{dk}\Big((v_1 + (v_1 + k)R)W[A]\Big)\Big|_{k=k_0}}{\frac{d}{dk}\Big(v_1(v_1 + k)(W[A] + 1)\Big)\Big|_{k=k_0}}.$$

Hence

$$\frac{d}{dk}W[A]\Big|_{k=k_0} = \frac{RW[A_0] + (v_1 + (v_1 + k_0)R)\frac{d}{dk}W[A]\Big|_{k=k_0}}{v_1(W[A_0] + 1) + v_1(v_1 + k_0)\frac{d}{dk}W[A]\Big|_{k=k_0}}. \tag{16}$$

Denote $s = \frac{d}{dk} W[A]\Big|_{k=k_0}$. Then (16) becomes

$$s = \frac{RW[A_0] + (v_1 + (v_1 + k_0)R)s}{v_1(W[A_0] + 1) + v_1(v_1 + k_0)s}.$$

Recalling that $v_1 + (v_1 + k_0)R$ appearing in the numerator equals 0, and that $W[A_0] = -1$ yields

$$s = \frac{-R}{v_1(v_1 + k_0)s}. \tag{17}$$

Note that $s = 0$ does not solve (17); hence the above expression is well defined. From (17) we get

$$s^2 = \frac{-R}{v_1(v_1 + k_0)}. \tag{18}$$

By R3, $R = R(v_1) = R(u_0 - k_0) = \frac{k_0}{u_0} - 1$. Substituting this and also $v_1 = u_0 - k_0$ in (18) gives

$$s^2 = \frac{1 - \frac{k_0}{u_0}}{(u_0 - k_0)u_0} = \frac{u_0 - k_0}{(u_0 - k_0)u_0^2} = \frac{1}{u_0^2}. \tag{19}$$

Now, from the proof of the first part of Lemma 4 (that in Region 3 $W[A]$ increases with k), in Region 3 $\frac{d}{dk}W[A]$ is non-negative. In particular, $s \geq 0$. Combining this with the fact that $s \neq 0$, which we established earlier, gives $s > 0$. This, together with (19), implies that $s = \frac{1}{u_0}$.

□

We can now prove Theorem 1 (see proof in the main paper).

1.1 Profit maximization - elaborating on the results in the main paper Section 5.2

The first step is to find an explicit expression for the local maximum of π in each region. However, these local maxima may appear outside the region in which they are valid; in that case they are irrelevant. In other words, the prices p that are relevant candidates for the global maximum π are the maxima for which $(v - p, k)$ still lies in the original region of (v, k). For instance, if (v, k) lies in Region 4, then the local maximum for Region 4 may have $(v - p, k)$ lie in Region 3. That local maximum is not relevant, since in Region 3 there is a different expression for the local maximum of π. For a region with no relevant local maximum we need to check its end points. Among these candidates, the p that attains the maximum value for $\pi(p)$ is the price that maximizes the seller's expected profit.

1.1.1 Proof of Theorem 4

Recall that the right end of Region 3 was denoted as p_3. Then $p_3 = v - k - 1$. Recall that p_2 and p_4 denote the local maxima of Regions 2 and 4 respectively. Then p_2, p_3, and p_4 are the only candidates for the global maximum p^* of the expected profit π. Recall that if p_2 does not belong to Region 2 it is irrelevant. Similarly for p_4 and Region 4. In contrast, $p_3 = v - k - 1$, which is the point separating between Regions 2 and 3, always exists (when Region 3 exists). In order to prove Theorem 4, we wish to find explicit expressions for p_i and π_i in each region. Denote $\pi_i = \pi(p_i) fori = 2, 3, 4$.

Definition 5.

$$W = W[-(k+1)e^{-(k+1)}].$$

Proposition 6. *The candidates p_2, p_4 and $p_,$ for the global maxima p^* of the expected profit π and their associated expected profits π_i are*

1.

$$p_2 = v - \frac{\ln(v)}{1 - \frac{1}{v}}, \quad \pi_2 = v - 1 - \ln(v). \tag{20}$$

2.

$$p_3 = v - k - 1, \quad \pi_3 = (v - k - 1)\left(1 - e^{-(W+k+1)}\right). \tag{21}$$

3.

$$p_4 = v - k - \frac{\ln(v-k)}{1 - \frac{1}{v-k}}, \quad \pi_4 = v - k - 1 - \ln(v-k), \tag{22}$$

Proof.

1. Recall that in Region 2, $\lambda_1 = 0$. Hence $(v - p)\frac{1-e^{-\lambda_2}}{\lambda_2} = 1$, and so in particular,

$$p_2 = v - \frac{\lambda_2}{1 - e^{-\lambda_2}}. \tag{23}$$

Substituting this in $\pi = p\left(1 - e^{-(\lambda_1+\lambda_2)}\right)$, gives

$$\pi_2 = \left(v - \frac{\lambda_2}{1 - e^{-\lambda_2}}\right)\left(1 - e^{-\lambda_2}\right) = v(1 - e^{-\lambda_2}) - \lambda_2. \tag{24}$$

To find the local maximum in Region 2 compute

$$\frac{d}{d\lambda_2}\pi = ve^{-\lambda_2} - 1. \tag{25}$$

The unique solution for $\frac{d}{d\lambda_2}\pi = 0$ is $\lambda_2^* = \ln v$. Substituting this in (23) and (24) gives p_2 and π_2, as stated in (20).

2. Because $p_3 = v - k - 1$ is on the border between Regions 2 and 3, $\lambda_1 = 0$, and so

$$\pi_3 = (v - k - 1)\left(1 - e^{-\lambda_2}\right). \tag{26}$$

Now, $p = v - k - 1$ implies that $v_1 = 1$. Hence, by Theorem 14, in Region 3

$$\lambda_2 = W[A] - (k+1)R(1).$$

By Property W2, $R(1) = W[-e^{-1}] = -1$. Note also that by Definition, for $v_1 = 1$,

$$A = -(k+1)e^{-(k+1)}.$$

Hence

$$\lambda_2 = W[-(k+1)e^{-(k+1)}] - (k+1)(-1) = W + k + 1.$$

Substituting $\lambda_2 = W + k + 1$ in (26) gives (21).

3. Recall that in Region 4, $\lambda_2 = 0$. Hence,

$$p_4 = v - k - \frac{\lambda_1}{1 - e^{-\lambda_1}}. \tag{27}$$

So,

$$\pi_4 = (v-k)(1 - e^{-\lambda_1}) - \lambda_1. \tag{28}$$

Now,

$$\frac{d}{d\lambda_1}\pi = (v-k)e^{-\lambda_1} - 1, \tag{29}$$

giving the profit-maximizing value $\lambda_1^* = \ln(v-k)$. Substituting this in (27) and (28), gives p_4 and π_4, as stated in (22).

□

Proposition 7. *For all $k > 0$, and $v > 1$,*

1. $\pi_4 < \pi_2$

2. $\pi_3 < \pi_2$,

Proposition 7 is important since it implies that whenever $p_2 \in$ *Region 2* (namely, the local maximum of Region 2 still lies in Region 2) then the global maximum π^* is attained at p_2, and so $\pi^* = \pi_2$.

Proof.

1. By (22), $\pi_4 = v - k - 1 - \ln(v-k)$, and by (20), $\pi_2 = v - 1 - \ln(v)$. Note that for $y \geq 1$, the value of $y - 1 - \ln y$ is increasing. This result and the inequalities $1 \leq v - k < v$ imply that $\pi_4 < \pi_2$.

2. To prove that $\pi_3 < \pi_2$ for $k > 0$ we must prove that

$$(v-k-1)\left(1 - e^{-(W+k+1)}\right) \leq v - 1 - \ln v,$$

which is equivalent to proving that

$$(v-k-1)\left(1-e^{-(W+k+1)}\right)-v+\ln v \leq -1. \tag{30}$$

We will find the maximum value of the left-hand side of (30) and show that it equals -1. To find the maximum we solve

$$\left((v-k-1)\left(1-e^{-(W+k+1)}\right)-v+\ln v\right)' = 1-e^{-W-k-1}-1+\frac{1}{v}=0.$$

The solution is $v=e^{W+k+1}$. The second derivative of the left-hand side of (30) is $(1-e^{-W-k-1}-1+\frac{1}{v})' = -\frac{1}{v^2}$. Hence e^{W+k+1} is a local maximum. Note that

$$e^{W+k+1}=e^{W}e^{k+1}=\frac{-(k+1)e^{-k-1}e^{k+1}}{W}=\frac{k+1}{-W}. \tag{31}$$

We now show that for $v=e^{W+k+1}$ the left-hand side of (30) is -1. By Property W5 we obtain

$$=\left(-\frac{k+1}{W}-k-1\right)\left(1+\frac{We^{-(k+1)}}{(k+1)e^{-(k+1)}}\right)+\frac{k+1}{W}+\ln\left(\frac{k+1}{-W}\right)$$

$$=\left(-\frac{k+1}{W}-k-1\right)\left(1+\frac{W}{k+1}\right)+\frac{k+1}{W}+\ln\left(\frac{k+1}{-W}\right),$$

which gives

$$-k-2-W+\ln\left(-\frac{k+1}{W}\right). \tag{32}$$

Since by (31) $\frac{k+1}{-W}=e^{W+k+1}$

$$\ln\left(\frac{k+1}{-W}\right)=\ln e^{(W+k+1)}=W+k+1.$$

Substituting this in (32) gives

$$-k-2-W+\ln\left(\frac{k+1}{-W}\right)=-k-2-W+W+k+1=-1.$$

$\square$

When does p_2 belong to Region 2?

As explained earlier, Proposition 7 implies that whenever $p_2 \in$ *Region 2* (namely, the local maximum of Region 2 still lies in Region 2), then the global maximum π^* is attained at p_2. We now find the conditions guaranteeing that p_2 lies in Region 2. Recall that $W=W[-(k+1)e^{-(k+1)}]$.

Lemma 8. *If* $1 \leq v \leq e^{W+k+1}$, *then* p_2 *lies in Region 2.*

Proof. We will first prove that p_2 lies in Region 2, iff $\frac{\ln v}{1-\frac{1}{v}} \leq k+1$. Then, we will prove that $\frac{\ln v}{1-\frac{1}{v}} \leq k+1$ iff $v \leq e^{W+k+1}$. Recall that Region 2 refers to all p satisfying $v-k-1 \leq p \leq v-1$. First, by L'Hôpital's rule

$$\lim_{v\to 1} \frac{\ln v}{1-\frac{1}{v}} = \frac{\lim_{v\to 1}\frac{1}{v}}{\lim_{v\to 1}\frac{1}{v^2}} = \frac{1}{1} = 1.$$

Because $\frac{\ln v}{1-\frac{1}{v}}$ is increasing for all $v \geq 1$,

$$\frac{\ln v}{1-\frac{1}{v}} \geq 1,$$

and so

$$v - \frac{\ln v}{1-\frac{1}{v}} \leq v-1,$$

proving that $p_2 \leq v-1$. We now prove that $p_2 \geq v-k-1$, iff $\frac{\ln v}{1-\frac{1}{v}} \leq k+1$. The latter is equivalent to

$$v - \frac{\ln v}{1-\frac{1}{v}} \geq v-k-1.$$

Hence in this case $p_2 \geq v-k-1$, and so p_2 lies in Region 2. We will show that the unique solution for

$$\frac{\ln v}{1-\frac{1}{v}} = k+1 \tag{33}$$

is e^{W+k+1}. By (31)

$$e^{W+k+1} = -\frac{k+1}{W}.$$

Hence

$$We^{W+k+1} = -(k+1),$$

implying that

$$(W+k+1)e^{W+k+1} - (k+1)e^{W+k+1} = -(k+1),$$

and so

$$(W+k+1)e^{W+k+1} = (k+1)\left(e^{W+k+1}-1\right).$$

This is equivalent to

$$\frac{(W+k+1)e^{W+k+1}}{e^{W+k+1}-1} = k+1,$$

which implies

$$\frac{W+k+1}{1-\frac{1}{e^{W+k+1}}} = k+1.$$

Hence e^{W+k+1} solves (33). Because the left-hand side of (33) is strictly increasing, e^{W+k+1} uniquely solves (33). ☐

Corollary 9. *If* $1 \leq v \leq e^{W+k+1}$, *then* $p^* = p_2 = v - \frac{\ln(v)}{1-\frac{1}{v}}$, *and* $\pi^* = \pi_2 = v - 1 - \ln(v)$.

For larger v, namely $v \geq e^{W+k+1}$, we need to find where $\pi_3 \geq \pi_4$ is satisfied. In that case, $p^* = p_3$. Recall that $p_3 = v - k + 1$ is the point that separates Regions 2 and 3. Therefore it always exists and is relevant. But p_4, which is the local maximum of the expression for π in Region 4, may not belong to Region 4, and in that case it is not relevant.

Recall that $\pi_4 = v - k - 1 - \ln(v-k)$, and that $\pi_3 = (v-k-1)\left(1 - e^{-(W+k+1)}\right)$.

Let $f(v) = \pi_4 - \pi_3$. Hence,

Definition 10. $f(v) \equiv \left(1 - \frac{v}{k+1}\right) W - \ln(v-k)$.

Also, denote v_m as the minimum of f in Region 3.

Lemma 11. *Given k, $f(v)$ is a strictly convex function which has exactly two roots, the smaller of which is $k+1$.*

Proof. First, note that $f(k+1) = 0$. Now,

$$f'(v) = \left(\left(1 - \frac{v}{k+1}\right) W - \ln(v-k)\right)' = -\frac{W}{k+1} - \frac{1}{v-k}, \tag{34}$$

and we have

$$f''(v) = \frac{1}{(v-k)^2} > 0.$$

Hence f is indeed strictly concave, and so has at most two roots. To see that $k+1$ is not the only root, we need to find v_m, the minimum of f in Region 3, and show that $k+1 \neq v_m$. To solve $f'(v) = 0$, we use (34) obtaining

$$-\frac{W}{k+1} - \frac{1}{v-k} = 0.$$

This is equivalent to

$$v - k = -\frac{k+1}{W}.$$

Hence

$$v = k - \frac{k+1}{W},$$

and so $v_m = k - \frac{k+1}{W}$ is the minimum of f in Region 3. We now show that $k+1 < v_m$. Since $-(k+1)e^{-(k+1)} \geq -e^{-1}$ for all $k > 0$, and $W[\cdot]$ is an increasing function, then $W = W[-(k+1)e^{-(k+1)}] \geq W[-e^{-1}] = -1$. Thus for all $k > 0$, $W > -(k+1)$ and so

$$1 < -\frac{k+1}{W},$$

and

$$k + 1 < k - \frac{k+1}{W} = v_m.$$

Hence f has exactly two roots.

The function f is strictly convex, and $f(v_m) < 0$ (since $f(k+1) = 0$, and v_m is the minimum of f). Therefore v_m lies between the two roots. Because $k+1 < v_m$, $k+1$ is the smaller root. Denote

v_f as the larger root (a formal definition is presented after the end of the proof). Then

$$k+1 < v_m \leq v_f. \tag{35}$$

□

Definition 12. *v_f is the unique value that satisfies both $f(v) = 0$, and $v > k+1$.*

Because of Lemma 11, v_f is well defined. See Figure 1 that presents f as a function of v for $k = 1$.

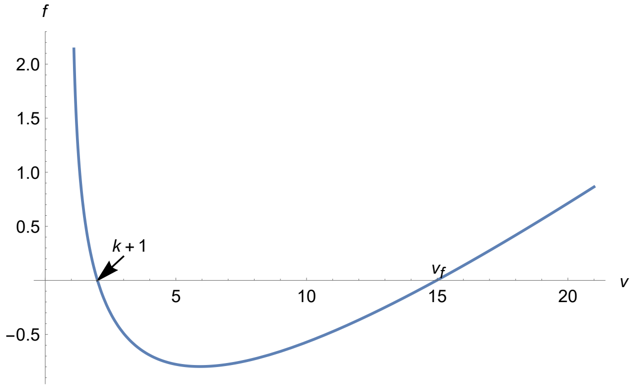

Figure 1: f as a function of v

Since $f(v) = \pi_4 - \pi_3$, it follows that for all v satisfying $k+1 < v < v_f$ $\pi_4 < \pi_3$. Recall that by Proposition 7 $\pi_3 \leq \pi_2$, and that by Lemma 8 p_2 is relevant for all v satisfying $1 \leq v \leq e^{W+k+1}$.

Corollary 13. *For every pair (v, k)*

$$e^{W+k+1} \leq v_f.$$

Proof. Recall that by (31) we have

$$e^{W+k+1} = -\frac{k+1}{W}.$$

Thus

$$e^{W+k+1} = -\frac{k+1}{W} \leq k - \frac{k+1}{W} = v_m \leq v_f,$$

where the last inequality follows from (35). □

Corollary 14.

- *For all $1 \leq v \leq e^{W+k+1}$, $\quad p^* = p_2, \quad$ and $\quad \pi^* = \pi_2$.*
- *For all $e^{W+k+1} < v \leq v_f, \quad p^* = p_3, \quad$ and $\quad \pi^* = \pi_3$.*

But what happens for $v > v_f$? By Lemma 11 $\pi_4 > \pi_3$ for $v > v_f$. But is p_4 relevant (i.e., does it belong to Region 4) for $v > v_f$? The following Lemma says Yes.

Lemma 15.

1. *If $v \geq k + v(k)$ then p_4 lies in Region 4.*
2. *For every pair (v, k) $\;v_f \geq k + v(k)$.*

Proof.

1. Recall that Region 4 refers to all p satisfying $0 \leq p \leq v - v(k)$.

   Proving that $p_4 \geq 0$ requires proving that

   $$v - k - \frac{\ln(v-k)}{1 - \frac{1}{v-k}} \geq 0.$$

   This is equivalent to

   $$(v-k)\left(1 - \frac{1}{v-k}\right) - \ln(v-k) \geq 0.$$

   Hence we need to prove that

   $$(v-k) - \ln(v-k) \geq 1. \tag{36}$$

   Note that $v - k \geq v(k) \geq 1$ (where the last inequality follows from the fact that for all $k \geq 0$ $v(k) \geq 1$.) For $v - k \geq 1$ (36) always holds since the left-hand side of (36) equals 1 for $v - k = 1$ and is increasing for $v - k \geq 1$. Hence p_4 is indeed non-negative. We now prove that $p_4 \leq v - v(k)$ if $v \geq k + v(k)$. Since $\frac{\ln y}{1-\frac{1}{y}}$ increases with y, $v - k \geq v(k)$ implies that

   $$\frac{\ln(v-k)}{1 - \frac{1}{v-k}} \geq \frac{\ln v(k)}{1 - \frac{1}{v(k)}}.$$

   Hence

   $$v - k - \frac{\ln(v-k)}{1 - \frac{1}{v-k}} \leq v - k - \frac{\ln v(k)}{1 - \frac{1}{v(k)}}.$$

   The left-hand side of the above equals p_4, so we have

   $$p_4 \leq v - k - \frac{\ln v(k)}{1 - \frac{1}{v(k)}}.$$

   Recall that $v(k) = k + \frac{\ln v(k)}{1-\frac{1}{v(k)}}$, so that $p_4 \leq v - v(k)$.

2. We will prove that $k + v(k) \leq v_m$, where v_m was defined as the minimum of f. This result and the observation that $v_m \leq v_f$ (see (35) in the proof of Lemma 11) complete the proof. Recall that

   $$v_m = k - \frac{(k+1)}{W}.$$

   Hence we need to prove that

   $$v(k) \leq -\frac{(k+1)}{W}.$$

By (31)

$$-\frac{(k+1)}{W} = e^{W+k+1},$$

so we must prove that

$$v(k) \leq e^{W+k+1}.$$

Both sides of the inequality are functions of k. For $k = 0$, we obtain equality with 1 on both sides. As seen in Figure 2 from then on $v(k) < e^{W+k+1}$.

□

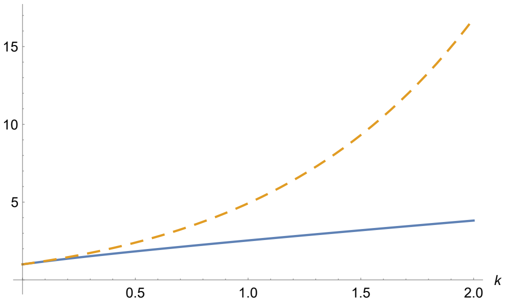

Figure 2: $v(k)$ (solid line) and e^{W+k+1} (dashed line) as functions of k

The proof of Theorem 4 follows immediately from the results in this section. Note that if k is sufficiently large then the condition $1 \leq v \leq e^{W+k+1}$ (appearing in case 2 of Theorem 4) is indeed satisfied and the upper bound $v - 1 - \ln v$ of π is thus obtained. Therefore the profit-maximizing penalty k should be large enough to satisfy the condition.

1.1.2 Proof of Corollary 5

According to Proposition 7 and Theorem 4, for all (v, k) the upper bound of π is $v - 1 - \ln v$. This upper bound is realized only when $1 \leq v \leq e^{W+k+1}$ (see case 2 in Theorem 4). In that case $\pi^* = \pi_2 = v - 1 - \ln v$, and all consumers arrivals are in period 2. If (v, k) does not satisfy this condition, then the firm's expected profit will be strictly smaller than $v - 1 - \ln v$ (since according to Proposition 7 $\pi_2 > \pi_3, \pi_4$).