Ride-Hailing Networks with Strategic Drivers: The Impact of Platform Control Capabilities on Performance.

E-Companion (Proofs)

Proof of Lemma 1. For regime X, let $(s_X(n_X), r_X(n_X), q_X(n_X))$ denote an optimal solution of (14) at the equilibrium participating capacity n_X obtained from (15).

Clearly, $(s_X(n_X), r_X(n_X), q_X(n_X), n_X))$ is a feasible solution of (13), with objective value $\Pi_X(n_X)$. Note that better solutions to (13), if any, can only be achieved at $n > n_X$ since $\Pi_X(n)$ increases in n (as will be shown later in each regime X). However, any such solution $(s, r, q, n) \in \mathcal{C}_X$ at $n > n_X$ cannot satisfy constraint (12e) because

$$\pi(s, r, n) \leq \pi_X(n) \leq \pi_X(n_X) = F^{-1}\left(\frac{n_X}{N}\right) < F^{-1}\left(\frac{n}{N}\right),$$

where the first inequality follows from (16), the second inequality is due to $\pi_X(n)$ decreasing in n obtained in Step 2 (as will be shown later in each regime X), and the last inequality follows from the assumption that $F(\cdot)$ is continuously increasing on $[0, \infty)$. Therefore, $\{s(n_X), r(n_X), q(n_X), n_X\}$ is also the optimal solution to (13) and we have $n_X^* = n_X$ and $\Pi_X^* = \Pi_X(n_X)$. □

Proof of Proposition 1. We start with the following observations about the optimal solution.

(i) *Allocating all capacity n towards serving riders (i.e., $r = 0, q = 0$) is feasible (hence optimal) if and only if $n \leq n_1^C$.* To see this, let

$$S_1^C = (S_{11}, S_{12}, S_{12}\frac{t_{21}}{t_{12}}, S_{22}). \tag{A.1}$$

Then $n_1^C = S_1^C \cdot 1 = \overline{S} - (\Lambda_{21} - \Lambda_{12})t_{21}$ is the maximum service capacity without repositioning ($r = 0$). Therefore with $r = 0$ and $q = 0$, $\overline{s} \leq n_1^C \Leftrightarrow n \leq n_1^C$ by (12b); if $n > n_1^C$, then $r = 0, q = 0$ is not feasible.

(ii) *Allowing for repositioning capacity $r_{12} \geq 0$, the maximum service capacity achievable (assuming n is sufficiently large) is*

$$\begin{cases} n_1^C + r_{12}\frac{t_{21}}{t_{12}}, \text{ with } r_{21} = 0 & \text{if } r_{12} \in [0, n_2^C - \overline{S}] \\ \overline{S}, \text{ with } r_{21} = (S_{12} + r_{12})\frac{t_{21}}{t_{12}} - S_{21} > 0 & \text{if } r_{12} > n_2^C - \overline{S} \end{cases}.$$

To see this, from (12a) we have $s_{21} = (s_{12} + r_{12})\frac{t_{21}}{t_{12}} - r_{21}$, hence for $r_{12} \in [0, n_2^C - \overline{S}]$,

$$\max_{0 \leq s \leq S, r_{21} \geq 0} \overline{s} = \max_{0 \leq s \leq S, r_{21} \geq 0} s_{11} + s_{12} + (s_{12} + r_{12})\frac{t_{21}}{t_{12}} - r_{21} + s_{22} = n_1^C + r_{12}\frac{t_{21}}{t_{12}},$$

where the maximum is achieved at $s_{11} = S_{11}, s_{12} = S_{12}, s_{22} = S_{22}, r_{21} = 0$. Note that the service capacity reaches its upper bound $\overline{S}$ when r_{12} reaches $n_2^C - \overline{S}$, i.e., $n_1^C + (n_2^C - \overline{S})\frac{t_{21}}{t_{12}} = \overline{S}$. For $r_{12} > n_2^C - \overline{S}$, the maximum service capacity *stays* at $\overline{S}$ with $s = S$, but $r_{21} = (S_{12} + r_{12})\frac{t_{21}}{t_{12}} - S_{21} > 0$ by (12a).

With these observations, we can derive the optimal structure given by the Proposition. Zone (1) follows directly from observation (i). In zone (2) and (3) where $n > n_1^C$, by observation (ii), the optimization problem with least repositioning travel cost (i.e., avoiding unnecessary repositioning capacity) can be simplified as

$$\max_{r_{12}} \left\{ n_1^C + r_{12}\frac{t_{21}}{t_{12}} : \; n_1^C + r_{12}\frac{t_{21}}{t_{12}} + r_{12} \le n, \; r_{12} \in [0, n_2^C - \overline{S}], \; r_{21} = 0 \right\}.$$

When $n_1^C < n \le n_2^C$ (zone (2)), the inequality constraint is binding and the optimal solution is

$$r_{12} = (n - n_1^C)\frac{t_{12}}{t_{12} + t_{21}}, \; r_{21} = 0, \quad s = S_1^C + \left(0, 0, (n - n_1^C)\frac{t_{21}}{t_{12} + t_{21}}, 0\right), \quad q = 0.$$

For $n > n_2^C$ (zone (3)), the inequality constraint is not binding. With all the demand served ($s = S$), the extra capacity waits in queues and the optimal solution is

$$r_{12} = n_2^C - \overline{S}, \; r_{21} = 0, \quad s = S, \quad q \in \left\{(q_1, q_2) : q_1 + q_2 = n - n_2^C\right\}.$$

□

Proof of Corollary 1. (i) It follow from (18) that $\pi_C(n)$ is continuously decreasing in n with zero limit value as $n \to \infty$. This directly implies the existence and uniqueness of the participation equilibrium.

(ii) Condition (16) requires that the per-driver profit is maximized subject to (12a)–(12c) at any $n > 0$ under the platform's optimal capacity allocation prescribed by Proposition 1. Note that

$$\pi(s, r, n) = \frac{(\bar{\gamma}p - c)\overline{s} - c\overline{r}}{n}.$$

By Proposition 1, $\overline{s} = n, r = 0$ for n in the scarce capacity zone $(0, n_1^C]$, clearly $\pi(s, r, n)$ is maximized; for n in the ample capacity zone (n_2^C, ∞), $\overline{s} = \overline{S}$ and $r = (n_2^C - \overline{S}, 0)$ involves the minimum repositioning capacity $\overline{r} = n_2^C - \overline{S}$, hence $\pi(s, r, n)$ is also maximized.

For fixed n in the moderate capacity zone $(n_1^C, n_2^C]$, further increasing $\pi(s, r, n)$ requires

$$(\bar{\gamma}p - c)\Delta\overline{s} - c\Delta\overline{r} > 0 \quad \Rightarrow \quad \Delta\overline{r} < \frac{\bar{\gamma}p - c}{c}\Delta\overline{s}.$$

Since $\overline{s}$ is maximized under the platform's optimal capacity allocation, it can only be decreased or remain unchanged, i.e., $\Delta\overline{s} \le 0$. Hence

$$\Delta\overline{r} < \frac{\bar{\gamma}p - c}{c}\Delta\overline{s} \le \frac{t_{12}}{t_{21}}\Delta\overline{s} \le 0, \tag{A.2}$$

where the second inequality follows from Assumption 2. We next show that (A.2) cannot hold due to the platform's optimal capacity allocation and the flow balance constraint:

- By Proposition 1, the platform's optimal capacity allocation in the moderate capacity zone $(n_1^C, n_2^C]$ has $s_{11} = S_{11}, s_{12} = S_{12}, s_{22} = S_{22}$ and $r_{12} > 0, r_{21} = 0$, hence s_{11}, s_{12}, s_{22} cannot be increased while r_{21} cannot be reduced. It then follows from $\Delta r_{12} + \Delta r_{21} = \Delta\overline{r} < 0$ by (A.2) that r_{12} must be reduced. To conclude, any change of these capacity variables must satisfy

$$\Delta s_{11}, \Delta s_{12}, \Delta s_{22} \le 0, \quad \Delta r_{12} < 0, \; \Delta r_{21} \ge 0. \tag{A.3}$$

- By flow balance constraint (12a), i.e., $(s_{12} + r_{12})/t_{12} = (s_{21} + r_{21})/t_{21}$, its change satisfies

$$\frac{\Delta s_{12} + \Delta r_{12}}{t_{12}} = \frac{\Delta s_{21} + \Delta r_{21}}{t_{21}},$$

which implies

$$\Delta s_{21} = \frac{t_{21}}{t_{12}}(\Delta s_{12} + \Delta r_{12}) - \Delta r_{21} \le \frac{t_{21}}{t_{12}}(\Delta r_{12} + \Delta r_{21}) - \left(1 + \frac{t_{21}}{t_{12}}\right)\Delta r_{21} \le \frac{t_{21}}{t_{12}}\Delta\overline{r}, \tag{A.4}$$

where the two inequalities follow from $\varDelta s_{12} \le 0$ and $\varDelta r_{21} \ge 0$ in (A.3), respectively.

By (A.3) and (A.4), we have

$$\varDelta \bar{s} = \sum \varDelta s_{ij} \le \varDelta s_{21} \le \frac{t_{21}}{t_{12}} \varDelta \bar{r},$$

which is a clear contradiction with $\varDelta \bar{s} > \frac{t_{21}}{t_{12}} \varDelta \bar{r}$ from (A.2). Therefore $\pi(s, r, n)$ is indeed maximized subject to (12a)–(12c) and the optimality condition holds. □

Proof of Lemma 2. If $\bar{s}_1 = s_{11} + s_{12} = 0$ and $\widetilde{\eta}_1 < 1$, then any driver who chooses not to not reposition at location 1 will stay there forever. In steady state, all drivers queue at location 1, $W_1 = \infty$, $T^q(\widetilde{\eta}_1; s, q) = \infty$, and hence $\widetilde{\pi}(\widetilde{\eta}_1; s, q) = 0$. Otherwise, we are not in this degenerate case and by Remark 4, $s_{21} > 0$, so that the two locations are commuting and drivers' expected steady-state profit rate follows from the Renewal Reward Theory. Without loss of generality, we calculate the time functions over cycles starting and ending at the low-demand location (1). Let $p_{lk} = \lambda_{lk}/(\lambda_{l1} + \lambda_{l2})$ denote the probability of serving a lk-ride at location l. The expected service, repositioning and queueing time functions are as follows:

- The expected service time in a cycle is given by
$$T^s(\widetilde{\eta}_1; s) = (1 - \widetilde{\eta}_1) \left[p_{11} t_{11} + p_{12} \left(t_{12} + \frac{1 - p_{21}}{p_{21}} t_{22} + t_{21} \right) \right] + \widetilde{\eta}_1 \left(\frac{1 - p_{21}}{p_{21}} t_{22} + t_{21} \right),$$
where $\frac{1-p_{21}}{p_{21}} t_{22}$ gives the expected time serving local demand at location 2. This follows from the fact that the number of local rides at location 2 a driver serves ("failures") before picking a ride back to location 1 ("success") follows a geometric distribution of "success" probability p_{21}.

- The expected repositioning time in a cycle is simply $T^r(\widetilde{\eta}_1) = \widetilde{\eta}_1 t_{12}$.

- The expected queueing delay in a cycle is given by
$$T^q(\widetilde{\eta}_1; s, q) = (1 - \widetilde{\eta}_1) \left(W_1 + p_{12} \frac{1}{p_{21}} W_2 \right) + \widetilde{\eta}_1 \frac{1}{p_{21}} W_2,$$
where $\frac{1}{p_{21}} W_2$ gives the expected queueing time at location 2. This follows from the fact that the number of queueing delays ("trials") at location 2 a driver encounters before picking a ride back to location 1 ("success") follows a geometric distribution of "success" probability p_{21}. $W_l = Q_l/(\lambda_{l1} + \lambda_{l2})$ is the queueing time at location l due to Little's Law.

Substituting the above time functions in (19) we have:

$$\widetilde{\pi}(\widetilde{\eta}_1; s, q) = \frac{(\bar{\gamma} p - c) \left[(1 - \widetilde{\eta}_1) s_{21} \frac{t_{12}}{t_{21}} (s_{11} + s_{12}) + (\widetilde{\eta}_1 s_{11} \frac{t_{12}}{t_{11}} + s_{12})(s_{21} + s_{22}) \right] - c \widetilde{\eta}_1 s_{21} \frac{t_{12}}{t_{21}} (s_{11} \frac{t_{12}}{t_{11}} + s_{12})}{(1 - \widetilde{\eta}_1) s_{21} \frac{t_{12}}{t_{21}} (s_{11} + s_{12} + q_1) + (\widetilde{\eta}_1 s_{11} \frac{t_{12}}{t_{11}} + s_{12})(s_{21} + s_{22} + q_2) + \widetilde{\eta}_1 s_{21} \frac{t_{12}}{t_{21}} (s_{11} \frac{t_{12}}{t_{11}} + s_{12})}. \tag{A.5}$$

□

Proof of Proposition 2. First, note that with $\eta_2(s, r) = 0$, $r_{21} = 0$ by (20), and $r_{12}(s)$ follows from the flow balance constraint (12a). The driver-incentive compatible capacity allocation must satisfy (21). Differentiating $\widetilde{\pi}(\widetilde{\eta}_1; s, q)$ given in (A.5) wrt $\widetilde{\eta}_1$, we get

$$\frac{\partial \widetilde{\pi}}{\partial \widetilde{\eta}_1} = \frac{s_{21} \frac{t_{12}}{t_{21}} (s_{11} \frac{t_{12}}{t_{11}} + s_{12}) \left[(s_{21} + s_{22}) \bar{\gamma} p - \left(s_{21} + s_{22} + s_{21} \frac{t_{12}}{t_{21}} \right) c \right]}{\left[(1 - \widetilde{\eta}_1) s_{21} \frac{t_{12}}{t_{21}} (s_{11} + s_{12} + q_1) + (\widetilde{\eta}_1 s_{11} \frac{t_{12}}{t_{11}} + s_{12})(s_{21} + s_{22} + q_2) + \widetilde{\eta}_1 s_{21} \frac{t_{12}}{t_{21}} (s_{11} \frac{t_{12}}{t_{11}} + s_{12}) \right]^2}$$

$$\times \left[q_1 - (q_1^*(s) + k(s)q_2)\right],$$

where

$$q_1^*(s) = \frac{(s_{11}+s_{12})s_{21}\frac{t_{12}}{t_{21}} + (s_{21}+s_{22})s_{12}}{(s_{21}+s_{22}) - \left(s_{21}+s_{22}+s_{21}\frac{t_{12}}{t_{21}}\right)\frac{c}{\widetilde{\gamma}p}}, \quad k(s) = \frac{(s_{11}+s_{12}) - s_{11}\frac{c}{\widetilde{\gamma}p}}{(s_{21}+s_{22}) - \left(s_{21}+s_{22}+s_{21}\frac{t_{12}}{t_{21}}\right)\frac{c}{\widetilde{\gamma}p}}. \tag{A.6}$$

The sign of $\partial\widetilde{\pi}/\partial\widetilde{\eta}_1$ only depends on the sign of $q_1 - (q_1^*(s) + k(s)q_2)$. Note that $\eta_1(s,r) = r_{12}/(s_{11}\frac{t_{12}}{t_{11}} + s_{12} + r_{12})$ from (20), so by (21):

(i) When $\bar{s}_1 > 0 = r_{12}$, $\eta_1(s,r) = 0$, which requires $\partial\widetilde{\pi}/\partial\widetilde{\eta}_1 \le 0$, hence $q_1 \le q_1^*(s) + k(s)q_2$;

(ii) When $\bar{s}_1, r_{12} > 0$, $\eta_1(s,r) \in (0,1)$, which requires $\partial\widetilde{\pi}/\partial\widetilde{\eta}_1 = 0$ (equivalently, $\widetilde{\pi}(0;s,q) = \widetilde{\pi}(1;s,q)$), hence $q_1 = q_1^*(s) + k(s)q_2$ with $q_1^*(s), k(s) > 0$;

(iii) When $\bar{s}_1 = 0 < r_{12}$, $\eta_1(s,r) = 1$ and $q_1^*(s) = k(s) = 0$, all drivers reposition at location 1 without waiting in a queue ($q_1 = 0$ and $\widetilde{\pi}(0;s,q) = 0 < \widetilde{\pi}(1;s,q)$).

It follows that $q \in \mathcal{D}(s)$ as defined in (22). □

Proof of Proposition 3. We want to show there is a unique feasible capacity utilization satisfying (12a)–(12c), (23)–(25) and (22), i.e., $(s,r,q,n) \in \mathcal{M}$ given by (26). Note that $r_{21} = 0$ by (22). By (12a) and (23) we can express service capacities in terms of s_{12} and r_{12}:

$$s_{11} = s_{12}\frac{S_{11}}{S_{12}}, \quad s_{21} = (s_{12}+r_{12})\frac{t_{21}}{t_{12}}, \quad s_{22} = (s_{12}+r_{12})\frac{t_{21}}{t_{12}}\frac{S_{22}}{S_{21}}. \tag{A.7}$$

We will focus on s_{12}, r_{12}, q, and recover the remaining quantities using (A.7). The other constraints, (12b), (22), (24), (25) and (12c), are rewritten below.

$$\frac{n_1^M}{S_{12}}s_{12} + \left[\frac{t_{21}}{t_{12}}\left(1 + \frac{S_{22}}{S_{21}}\right) + 1\right]r_{12} + q_1 + q_2 = n, \tag{A.8}$$

$$q_1 \begin{cases} \le q_1^*(s) + k(s)q_2 & \text{if } r_{12} = 0 \\ = q_1^*(s) + k(s)q_2 & \text{if } r_{12} > 0 \end{cases}, \tag{A.9}$$

$$(S_{12} - s_{12})r_{12} = 0, \tag{A.10}$$

$$(S_{12} - s_{12})q_1 = 0, \quad \left(S_{21} - (s_{12}+r_{12})\frac{t_{21}}{t_{12}}\right)q_2 = 0, \tag{A.11}$$

$$0 \le s_{12} \le S_{12}, \quad 0 \le \frac{t_{21}}{t_{12}}(s_{12}+r_{12}) \le S_{21}, \quad r_{12} \ge 0, \quad q \ge 0. \tag{A.12}$$

We make the following observations about the feasible solution.

(i) *The allocation where all capacity serves rider demand, i.e., $r_{12} = 0, q = 0$, $s_{12} = \frac{n}{n_1^M}S_{12}$, is feasible if and only if $n \le n_1^M$.* (a) "⇒": this is immediate; this also implies that if $n > n_1^M$, then $r_{12} > 0$ or $q > 0$. (b) "⇐": given $n \le n_1^M$, suppose first that $r_{12} > 0$, then (A.10) implies that $s_{12} = S_{12}$, thus $\bar{s} > n_1^M$ and $n > n_1^M$, a contradiction; second, that $q_1 > 0$, then (A.11) implies that $s_{12} = S_{12}$, thus $\bar{s} \ge n_1^M$ and $n > n_1^M$, a contradiction; or third, that $q_2 > 0$, then (A.11) implies that $s_{21} = S_{21}, r_{12} > 0$, thus $n > n_1^M$ is still a contradiction. Hence $n \le n_1^M \Rightarrow r_{12} = 0, q = 0, s_{12} = \frac{n}{n_1^M}S_{12}$, which satisfies (A.8)–(A.12) and is thus feasible.

(ii) *When $n > n_1^M$, any feasible solution must serve all demand at location 1, i.e., $s_{12} = S_{12}$, and moreover, $q_1^*(s) \equiv q_1^*(S)$.* By (i), $n > n_1^M$ implies that $r_{12} > 0$ and/or $q \ne 0$, and either of these assertions implies that $s_{12} = S_{12}$ by (A.10) and (A.11). To show that $q_1^*(s) \equiv q_1^*(S)$, we substitute s_{11}, s_{21}, s_{22} in $q_1^*(s)$ in (A.6) by (A.7),

$$q_1^*(s) = \frac{(\frac{S_{11}}{S_{12}} + 1)\frac{t_{12}}{t_{21}} + (1 + \frac{S_{22}}{S_{21}})}{(1 + \frac{S_{22}}{S_{21}}) - (1 + \frac{S_{22}}{S_{21}} + \frac{t_{12}}{t_{21}})\frac{c}{\bar{\gamma}p}} s_{12},$$

which is equal to a constant multiplying s_{12}. For $s_{12} = S_{12}$, we have that $q_1^*(s) \equiv q_1^*(S)$.

With these two observations, we derive the feasible solution given by the Proposition.

Zone (1): the result follows directly from observation (i).

Zone (2): for $n_1^M < n \le n_2^M := n_1^M + q_1^*(S)$, observation (ii) gives $s_{12} = S_{12}$ and $q_1^*(s) = q_1^*(S)$, so (A.8), (A.9) and (A.12) immediately imply that $r_{12} = 0$. Putting $s_{12} = S_{12}$ and $r_{12} = 0$ into (A.11) we get $q_2 = 0$. It also follows from (A.8) that $q_1 = n - n_1^M$. In this zone q_1 increases with n while $r_{12} = q_2 = 0$. The feasible solution is summarized as

$$r = 0, \quad s = \left(S_{11}, S_{12}, S_{21}\frac{\Lambda_{12}}{\Lambda_{21}}, S_{22}\frac{\Lambda_{12}}{\Lambda_{21}}\right), \quad q = (n - n_1^M, 0).$$

Zone (3): for $n_2^M < n \le n_3^M := n_2^C + q_1^*(S)$, if $s_{12} = S_{12}$ and $r_{12} = 0$, then $q_2 = 0$ by (A.11), which implies that $q_1 \le q_1^*(S)$ by (A.9). By (A.8), this contradicts the fact that $n > n_2^M$. It follows that $r_{12} > 0$, and (A.9) yields $q_1 = q_1^*(S) + k(s)q_2$. Together with $s_{12} = S_{12}$, it is then easy to verify that (A.8), (A.11) and $n \le n_3^M$ imply that $q_2 = 0$. In this zone r_{12} increases with n while $q_1 = q_1^*(S), q_2 = 0$ and the feasible solution is as follows:

$$r_{12} > 0, \quad s = \left(S_{11}, S_{12}, (S_{12} + r_{12})\frac{t_{21}}{t_{12}}, (S_{12} + r_{12})\frac{t_{21}}{t_{12}}\frac{S_{22}}{S_{21}}\right), \quad q = (q_1^*(S), 0).$$

Zone (4): for $n > n_3^M$, the above argument still implies that $r_{12} > 0$ and $q_1 = q_1^*(S) + k(s)q_2$. And, by (A.8) and (A.12) we get that $q_2 > 0$. It then follows from (A.11) that $r_{12} = S_{21}\frac{t_{12}}{t_{21}} - S_{12} = n_2^C - \overline{S}$ and $s = S$. In this zone q_1 and q_2 increase with n while s and r stay constant. The feasible solution is given by

$$r = (n_2^C - \overline{S}, 0), \quad s = S, \quad q = (q_1^*(S) + k(S)q_2, q_2).$$

This completes the proof. □

Proof of Corollary 2. (i) Substituting $\overline{s}$ and $\overline{r}$ from Proposition 3 into (12d) yields

$$\pi_M(n) = \frac{(\bar{\gamma}p - c)\overline{s} - c\overline{r}}{n} = \begin{cases} \bar{\gamma}p - c & \text{zone (1) } (n \le n_1^M), \\ \frac{n_1^M}{n}(\bar{\gamma}p - c) & \text{zone (2) } (n_1^M < n \le n_2^M), \\ \frac{S_{21}+S_{22}}{S_{21}+S_{22}+S_{21}\frac{t_{12}}{t_{21}}}\bar{\gamma}p - c & \text{zone (3) } (n_2^M < n \le n_3^M), \\ \frac{1}{n}(\bar{\gamma}p\overline{S} - cn_2^C) & \text{zone (4) } (n > n_3^M). \end{cases} \tag{A.13}$$

It is easy to see that $\pi_M(n)$ is continuously decreasing in n and that $\lim_{n\to\infty} \pi_M(n) = 0$. Therefore, the participation equilibrium condition (12e), $n = NF(\pi_M(n))$, has a unique solution n_M.

(ii) By Proposition 3, at any $n > 0$ there is a *unique* feasible driver capacity allocation under the constraints in (26), hence the per-driver profit is naturally maximized. □

Proof of Proposition 4. We have the following observations about the optimal solution.

(i) *Allocating all capacity n towards serving riders (i.e., $r_{12} = 0, q = 0$) is feasible (hence optimal) if and only if $n \le n_1^A := n_1^C$.* This is the same as observation (i) in the Proof of Proposition 1. Also notice that constraint (22) is satisfied.

(ii) *If for some capacity level n_1 the service capacity $\overline{s} > n_1^A$, then for all capacity levels $n_2 \geq n_1$ the optimal solution involves repositioning.* First, note that by the definition of n_1^A, $\overline{s} > n_1^A$ implies that $r_{12} > 0$ (which holds at n_1), and hence we only need to show that the optimal service capacity at n_2 is higher than n_1^A. It suffices to find one feasible solution at n_2 that has the same service capacity as at n_1, which is higher than n_1^A. To achieve this, let the service capacity vector s and the repositioning capacity $r_{12} > 0$ at n_2 be the same as those at n_1, respectively, and put the extra capacity $n_2 - n_1$ into q satisfying $q_1 = q_1^*(s) + k(s)q_2$. In this way all constraints are still satisfied while the service capacity $\overline{s} > n_1^A$ remains unchanged.

(iii) $r_{12} > 0$ *for all* $n > n_1^A + q_1^*(S_1^A)$, *where* $S_1^A = S_1^C$ *defined in* (A.1) *such that* $S_1^A \cdot 1 = n_1^A$. It suffices to show that at capacity levels in the right neighborhood of $n_1^A + q_1^*(S_1^A)$, the optimal service capacity is higher than n_1^A, which then, by observation (ii), will prove the result. To show this, for an arbitrarily small $\epsilon > 0$, let n_ϵ be the minimum feasible total capacity to provide service vector $S_1^A + (0, 0, \epsilon, 0)$, and hence service capacity $n_1^A + \epsilon > n_1^A$. Following constraints (12a)–(12c) and (22), we have

$$n_\epsilon = n_1^A + \epsilon + \frac{t_{12}}{t_{21}}\epsilon + q_1^*(S_1^A + (0, 0, \epsilon, 0)),$$

and $n_0 = n_1^A + q_1^*(S_1^A)$ for $\epsilon = 0$. It is easy to see that n_ϵ increases in ϵ, since by definition (A.6),

$$\frac{\partial q_1^*(s)}{\partial s_{21}} = \frac{((s_{11} + s_{12})\bar{\gamma}p - s_{11}c)s_{22}\frac{t_{12}}{t_{21}}}{\left[(s_{21} + s_{22})\bar{\gamma}p - (s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}})c\right]^2} > 0, \quad \forall s_{22} > 0, s_{11} + s_{12} > 0, \tag{A.14}$$

i.e., $q_1^*(s)$ increases wrt s_{21} when $s_{22}, s_{11} + s_{12} > 0$. Therefore, the optimal service capacity must be higher than n_1^A at capacity levels in the right neighborhood of $n_1^A + q_1^*(S_1^A)$.

(iv) $r_{12} = 0$ *for* $n \in [n_1^A, n_1^A + \delta]$ *for a small* $\delta > 0$. We first prove this for the optimal solution at $n = n_1^A + \delta$ by establishing that for any feasible q_1 it must be that $q_1 < q_1^*(s)$, from which we deduce $r_{12} = 0$ from (22). Then, observation (ii) yields the same result for $n \in [n_1^A, n_1^A + \delta]$. Pick

$$\delta = \min\left\{ q_1^*(S_1^A), \frac{(S_{12})^2}{S_{12}\left(1 + \frac{t_{21}}{t_{12}}\right) + S_{21} + S_{22}} \right\},$$

so that $n_1^A + \delta \leq n_1^A + q_1^*(S_1^A)$, $\delta < S_{12}$ and

$$\delta < \frac{(S_{12})^2}{S_{12}\left(1 + \frac{t_{21}}{t_{12}}\right) + S_{21} + S_{22}} \left(1 + \frac{t_{21}}{t_{12}}\right) \Rightarrow \frac{S_{12}(S_{12} - \delta)}{S_{21} + S_{22}} \left(1 + \frac{t_{21}}{t_{12}}\right) > \delta. \tag{A.15}$$

First note that the optimal solution at $n_1^A + \delta$ must have $\overline{s} \geq n_1^A$, since a feasible solution $s = S_1^A, r_{12} = 0, q_1 = 0, q_2 = \delta$ yields $\overline{s} = n_1^A$. Therefore

$$r_{12}, q_1 \leq \delta, \quad s_{21} \geq S_{12}\frac{t_{21}}{t_{12}}, \quad s_{12} \geq S_{12} - \delta > 0, \tag{A.16}$$

where the first inequality follows from $\overline{s} \geq n_1^A$ and capacity constraint (12b), the second is by $\overline{s} \geq n_1^A$, and the third follows from the second and (12a) in that $s_{12} = s_{21}\frac{t_{12}}{t_{21}} - r_{12} \geq S_{12} - \delta$.
Then,

$$q_1^*(s) = \frac{(s_{11} + s_{12})s_{21}\frac{t_{12}}{t_{21}} + (s_{21} + s_{22})s_{12}}{(s_{21} + s_{22}) - \left(s_{21} + s_{22} + s_{21}\frac{t_{12}}{t_{21}}\right)\frac{c}{\bar{\gamma}p}} \geq \frac{s_{12}s_{21}\frac{t_{12}}{t_{21}} + s_{21}s_{12}}{S_{21} + S_{22}} \geq \frac{S_{12}(S_{12} - \delta)\frac{t_{21}}{t_{12}}\left(1 + \frac{t_{12}}{t_{21}}\right)}{S_{21} + S_{22}} > \delta,$$

where the the second inequality follows from the second and third inequalities in (A.16), and the last inequality is by (A.15). This, together with the first inequality in (A.16), yields $q_1 \le \delta < q_1^*(s)$, which implies $r_{12} = 0$ by constraint (22). Hence we have shown $r_{12} = 0$ at $n = n_1^A + \delta$. By observation (ii), the optimal solution at $n \in [n_1^A, n_1^A + \delta]$ has service capacity $\overline{s} = n_1^A$ and no repositioning.

(v) *An optimal solution can serve all demand* ($s = S$) *if and only if* $n \ge n_3^A := n_2^C + q_1^*(S)$. "$\Leftarrow$": given $n \ge n_2^C + q_1^*(S)$, it is easy to verify that the capacity allocation $s = S, r = (n_2^C - \overline{S}, 0)$ and $q_1 = q_1^*(S) + k(S)q_2$ with $q_1 + q_2 = n - n_2^C$ is feasible and serves all demand (hence optimal). "$\Rightarrow$": an optimal (hence feasible) solution that serves all demand must have $s = S, r = (n_2^C - \overline{S}, 0)$ and $q_1 = q_1^*(S) + k(S)q_2$. By (12b) this yields $n = n_2^C + q_1^*(S) + k(S)q_2 + q_2 \ge n_2^C + q_1^*(S) = n_3^A$.

With these five observations, we can derive the optimal solution given by the Proposition. Zone (1) follows directly from observation (i) and zone (4) follows from observation (v). In $(n_1^A, n_3^A]$, not all drivers are serving riders and not all riders are served. There exists a threshold n_2^A such that $n_1^A < n_2^A < n_3^A$ which separates zone (2) and (3) apart: in zone (2), $(n_1^A, n_2^A]$, optimal solution has service capacity $\overline{s} = n_1^A$, no repositioning ($r = 0$), and extra capacity queues at location 1 with $q = (n - n_1^A, 0)$; whereas in zone (3), $(n_2^A, n_3^A]$, optimal solution involves repositioning ($r_{12} > 0$), serves $\overline{s} > n_1^A$, and extra capacity queues at location 1 with $q = (q_1^*(s), 0)$.[1] Note that $n_2^A > n_1^A$ by observation (iv). $n_2^A < n_3^A$ follows from $n_2^A \le n_1^A + q_1^*(S_1^A)$ by observation (iii) and $n_1^A + q_1^*(S_1^A) < n_2^C + q_1^*(S) = n_3^A$ by property (A.14). Furthermore, the fact that optimal solution involves repositioning at any capacity level in zone (3) follows from observation (ii). □

The following proofs of Lemma 3 and Propositions 6 and 7 refer to two technical lemmas, Lemmas S-1 and S-2. The statements and proofs of these lemmas as well as the proof of Proposition 5 are relegated to the Supplemental Material S1.

Proof of Lemma 3. We first consider the platform's capacity allocation problem in an "unrestricted" setting (allowing strategic demand rejection) and a "restricted" setting (disallowing strategic demand rejection), and prove properties of the corresponding equilibria. Later we will use these properties to show the optimality conditions.

Under the "*unrestricted*" platform-optimal capacity allocation of Proposition 4 where strategic demand rejection might prevail, we show in Lemma S-2 (see Supplemental Material S1) that $\pi_A(n)$ is decreasing in n and $\lim_{n\to\infty} \pi_A(n) = 0$. Hence there is a unique equilibrium participating capacity n_A that satisfies (29).

Under the "*restricted*" optimal capacity allocation where strategic demand rejection is *disallowed*, problem (31), with platform revenue $\hat{\Pi}_A(n)$ and resulting per-driver profit $\hat{\pi}_A(n)$ given in Part 2 of the statement. We first prove two properties of $\hat{\pi}_A(n)$ as in 2(i) (equilibrium condition (32)) and 2(ii) (driver optimality (33)), and properties of the optimal quantities n_A^*, π_A^*, Π_A^* as in 2(iii). Note that 2(iv) just restates (13) and does not need a proof.

2(i) Under the restricted problem, the platform's optimal capacity allocation for $n \in (\hat{n}_2^A, n_3^A]$ follows pattern (1) in Lemma S-1 (see Supplemental Material S1): only s_{21} is increasing and $\hat{\pi}_A(n)$ is continuous. Then case (i) in the proof of Lemma S-2 shows that $\hat{\pi}_A(n)$ decreases in n (note that Lemma S-2 adopts the general notation $\pi_A(n)$ which means $\hat{\pi}_A(n)$ in this case), and hence there is a unique solution $\hat{n}_A$ that solves (32). (We show $\hat{n}_A > n_A$ below.)

[1]The steady state system flow equations do not differentiate between queueing in locations 1 and 2 in this capacity regime. A more detailed transient analysis would show that when the platform makes admission control decisions, it would choose to clear the queue in the high-demand location given that the demand exceeds the available capacity, and drivers would only queue in the low-demand location.

2(ii) The optimality condition (16) in Lemma 1 can be proven similarly as under regime C (in the proof of Corollary 1), which shows that any feasible deviation from the platform's optimal capacity allocation cannot increase the per-driver profit. First, the logic for the scarce and ample capacity zones, $(0, n_1^A]$ and (n_3^A, ∞), is identical to that in the proof of Corollary 1. Second, in the moderate capacity zone (without repositioning), $(n_1^A, \hat{n}_2^A]$, there is zero repositioning capacity, i.e., $\overline{r} = 0$, which cannot be reduced, hence the key inequality (A.2) in the proof of Corollary 1 is immediately violated, and it follows that the per-driver profit is maximized. Third, in the moderate capacity zone (with repositioning), $(\hat{n}_2^A, n_3^A]$, the argument is identical to that in zone 3 of regime C. We have thus proven that for any n, the per-driver profit is *maximized* without strategic demand rejection (which may not be the case with strategic demand rejection), i.e., (33) holds.

2(iii) $n_A \leq \hat{n}_A$ follows from the equilibrium conditions (29) and (32), together with (33) and the fact that F^{-1} is strictly increasing. To prove that $n_A^* \in [n_A, \hat{n}_A]$, first note that, by definition, n_A and $\hat{n}_A$ (and their associated optimal capacity allocations) are both feasible solutions to problem (13) with $X = A$. Next, we show that there does not exist an equilibrium participating capacity n outside the interval $[n_A, \hat{n}_A]$ that yields a higher platform revenue. On one hand, for $n < n_A$ any capacity allocation $(s, r, q) \in \mathcal{C}_A(n)$ that also satisfies the driver participation constraints (12d)–(12e) must be suboptimal: Denoting the associated platform revenue by $\Pi(s)$, we have

$$\Pi(s) \leq \Pi_A(n) \leq \Pi_A(n_A),$$

where the first inequality follows because $\Pi_A(n)$ is defined as the maximum platform revenue at n subject to $(s, r, q) \in \mathcal{C}_A(n)$ (i.e., without the driver participation constraints), and the second inequality follows from the monotonicity of $\Pi_A(\cdot)$ that is implied by Proposition 4. On the other hand, for $n > \hat{n}_A$ no solution $(s, r, q) \in \mathcal{C}_A(n)$ satisfies the driver participation constraints because

$$\pi(s, r, n) \leq \hat{\pi}_A(n) \leq \hat{\pi}_A(\hat{n}_A) = F^{-1}\left(\frac{\hat{n}_A}{N}\right) < F^{-1}\left(\frac{n}{N}\right),$$

where the first inequality follows from the optimality condition (16) in Lemma 1 that is proved in 2(ii) above, the second inequality is due to $\hat{\pi}_A(n)$ decreasing in n shown in 2(i) above, and the last inequality follows from the assumption that $F(\cdot)$ is continuously increasing on $[0, \infty)$. Therefore, the equilibrium participating capacity n_A^* must lie in the interval $[n_A, \hat{n}_A]$.

The equation for the equilibrium per-driver profit π_A^* follows from the participation equilibrium constraint (12e), and the fact that n_A^* cannot be a discontinuity point n_A of $\pi_A(\cdot)$ where $\Pi_A(n_A) = \hat{\Pi}_A(\hat{n}_A) \leq \hat{\Pi}_A(\hat{n}_A)$.

The inequalities $\pi_A(n_A^+) \leq \pi_A^* \leq \hat{\pi}_A(\hat{n}_A)$ follow because $F(\cdot)$ is increasing.

The inequality $\Pi_A^* \geq \max\{\Pi_A(n_A), \hat{\Pi}_A(\hat{n}_A\}$ follows from the fact that n_A and $\hat{n}_A$ (and their associated optimal capacity allocations) are both *feasible* solutions to problem (13) with $X = A$, which achieve objective values $\Pi_A(n_A)$ and $\hat{\Pi}_A(\hat{n}_A)$, respectively.

Now we are ready to prove the optimality conditions in the statement. Having obtained the unrestricted capacity allocation of Proposition 4, we check if $\pi_A(n_A)$ is driver-optimal and if $n_A = NF(\pi_A(n_A))$.

- If $\pi_A(n_A)$ is driver-optimal (condition (30)) and $n_A = NF(\pi_A(n_A))$: The driver-optimality of $\hat{\pi}_A(\cdot)$ proven in 2(ii) implies that $\pi_A(n_A) = \hat{\pi}_A(n_A) = F^{-1}(n_A/N)$, hence $n_A = \hat{n}_A = n^*$ by the monotonicity of $\hat{\pi}_A(\cdot)$. This proves 1(i). 1(ii) follows directly from Proposition 4 since strategic demand rejection may only be in the moderate capacity zone.

- Otherwise:
  - If $\pi_A(n_A)$ is not driver-optimal: The fact that $\hat{\pi}_A(\cdot)$ is driver-optimal and $\pi_A(n) = \hat{\pi}_A(n)$ for $n \in [0, n_1^A) \cup (n_3^A, N]$ imply that $n_A \in (n_1^A, n_3^A)$. It follows from the driver-optimality of $\hat{\pi}_A(\cdot)$ by 2(ii) that $\pi_A(n_A) < \hat{\pi}_A(n_A)$. Then by (29) we have $F^{-1}(n_A/N) \leq \pi_A(n_A) < \hat{\pi}_A(n_A)$ and hence $\hat{n}_A > n_A$ in (32) by the monotonicity of $\hat{\pi}_A(\cdot)$.
  - If $n_A \neq NF(\pi_A(n_A))$: $n_A \in (n_1^A, n_3^A)$ is the discontinuity point of $\pi_A(\cdot)$ where $\hat{\pi}_A(n_A) = \pi_A(n_A) > F^{-1}(n_A/N)$. Then the monotonicity of $\hat{\pi}_A(\cdot)$ implies $\hat{n}_A > n_A$ in (32).

To see 2(v), first note that under $\mathit{\Pi}_A(n_A) > \hat{\mathit{\Pi}}_A(\hat{n}_A)$ and $n_A \leq \hat{n}_A$ (see 2(iii)), n_A *cannot* be a discontinuity point of $\pi_A(\cdot)$ (where $\mathit{\Pi}_A(n_A) = \hat{\mathit{\Pi}}_A(n_A) \leq \hat{\mathit{\Pi}}_A(\hat{n}_A)$), hence $n_A^* \geq n_A$ is not a discontinuity point and we have $\pi_A(n_A^*) = F^{-1}(n_A^*/N)$. Next, by the driver-optimality of $\hat{\pi}_A(\hat{n}_A)$ by 2(ii), any demand rejection would strictly reduce the per-driver profit at $\hat{n}_A$ and hence cannot be an equilibrium. Thus, an optimal equilibrium *with* demand rejection must be at $n_A^* < \hat{n}_A$. By the monotonicity of $F(\cdot)$ we finally have $\pi_A(n_A^*) = F^{-1}(n_A^*/N) < F^{-1}(\hat{n}_A/N) = \hat{\pi}_A(\hat{n}_A)$.

□

Proof of Proposition 6. We first prove the case $\rho_2 = 0$ and obtain the range $(\underline{N}, \overline{N})$, and then show the existence of the threshold level $\overline{\rho}_2(N) > 0$. Note that since $S_{21} > 0$ by Assumption 1, the positivity of ρ_2 is equivalent to the positivity of S_{22}. That is, given S_{21}, there is a one-to-one mapping between ρ_2 and S_{22}, and we focus on S_{22} in what follows.

In the following three steps, under $S_{22} = 0$, we first prove several properties of the platform revenue and per-driver profit rate functions when strategic demand rejection is allowed or disallowed, then use the properties to show that strategic demand rejection is optimal when the equilibrium driver participation is in the intermediate region, and finally obtain the corresponding range of driver pool size.

(1) When $S_{22} = 0$, we have the following four properties (i)–(iv) regarding the platform revenue and per-driver profit under optimal capacity allocation allowing or disallowing strategic demand rejection:

  (i) Condition (36) in Proposition 5 holds.

  (ii) Thereshold values $n_2^A < \hat{n}_2^A < n_3^A$, as defined in (A.17).

  (iii) $\mathit{\Pi}_A(n)$ strictly increases on (n_2^A, n_3^A); $\hat{\mathit{\Pi}}_A(n)$ stays constant on $(n_2^A, \hat{n}_2^A)$ and strictly increases on $[\hat{n}_2^A, n_3^A)$; and $\mathit{\Pi}_A(n) > \hat{\mathit{\Pi}}_A(n)$ on (n_2^A, n_3^A).

  (iv) $\pi_A(n)$ remains constant on (n_2^A, n_3^A); $\hat{\pi}_A(n)$ strictly decreases on $[n_2^A, \hat{n}_2^A)$ and stays constant on $(\hat{n}_2^A, n_3^A)$; $\pi_A(n) < \hat{\pi}_A(n)$ on $(n_2^A, \hat{n}_2^A)$ and $\pi_A(n) = \hat{\pi}_A(n)$ on $[\hat{n}_2^A, n_3^A)$.

Property (i) is immediate by setting $S_{22} = 0$ in (36).

To obtain properties (ii) and (iii), we first show that under $S_{22} = 0$ (hence $s_{22} = 0$), we can significantly simplify the 3 patterns of optimal capacity allocation as a function of participating capacity $n \in (n_2^A, n_3^A)$ established in Lemma S-1 and its proof in the Supplemental Material S1. Setting $s_{22} = 0$ in the derivatives (S.4)–(S.6) and using $n = g(s)$ by (S.9), we get

$$\frac{\partial n}{\partial s_{11}} = \frac{\partial g(s)}{\partial s_{11}} = \left(1 + \frac{t_{12}}{t_{21}}\right) \frac{\bar{\gamma}p - c}{\bar{\gamma}p - \left(1 + \frac{t_{12}}{t_{21}}\right) c},$$

$$\frac{\partial n}{\partial s_{12}} = \frac{\partial g(s)}{\partial s_{12}} = \left(1 + \frac{t_{12}}{t_{21}}\right) \frac{\bar{\gamma} p}{\bar{\gamma} p - \left(1 + \frac{t_{12}}{t_{21}}\right) c},$$

$$\frac{\partial n}{\partial s_{21}} = \frac{\partial g(s)}{\partial s_{21}} = 1 + \frac{t_{12}}{t_{21}}.$$

Clearly $\frac{\partial n}{\partial s_{21}} < \frac{\partial n}{\partial s_{11}} < \frac{\partial n}{\partial s_{12}}$ and all are *constants* that only depend on model primitives. Noticing that $\overline{s_1}(n_3^A) = \overline{s_2}(n_3^A) = \overline{s_3}(n_3^A) = \overline{S}$ at the right end n_3^A, and the patterns specified in Lemma S-1 (i.e., with n increasing towards n_3^A: in pattern (1) only s_{21} increases towards S_{21}; in pattern (2), first s_{21} increases to S_{21} and then s_{12} increases to S_{12}; and in pattern (3), first s_{21} increases to S_{21}, then s_{11} increases to S_{11} and last s_{12} increases to S_{12}), a direct implication of the derivatives' ranking is that the left ends of each pattern (at which the service capacity is n_1^A and repositioning is about to start) are ordered as $\overline{s_3}^{-1}(n_1^A) \leq \overline{s_2}^{-1}(n_1^A) < \overline{s_1}^{-1}(n_1^A)$, and, furthermore, pattern (3) yields the largest service capacity for $n \in (\overline{s_3}^{-1}(n_1^A), n_3^A)$ and pattern (1), which *disallows* strategic demand rejection, yields the smallest service capacity for $n \in (\overline{s_1}^{-1}(n_1^A), n_3^A)$. Consequently, the platform's optimal capacity allocation follows pattern (3) when strategic demand rejection is allowed and follows pattern (1) when disallowed. Hence,

$$n_2^A := \overline{s_3}^{-1}(n_1^A) < \overline{s_1}^{-1}(n_1^A) =: \hat{n}_2^A < n_3^A, \tag{A.17}$$

which proves property (ii), and property (iii) follows immediately.

To see (iv), first note that $\pi_A(n)$ and $\hat{\pi}_A(n)$ correspond with patterns (3) and (1) established above, respectively. We can then similarly simplify the per-driver profit rate as a function of participating capacity $n \in (n_2^A, n_3^A)$ under each pattern established in the proof of Lemma S-2 in Supplemental Material S1. In specific, setting $S_{22} = 0$, we get $\pi'(n) = 0$ within each pattern given by (S.20)–(S.24). Note that under pattern (1), $\pi'(n) < 0$ for $n < \overline{s_1}^{-1}(n_1^A) = \hat{n}_2^A$. Property (iv) hence follows.

(2) Next, we show that, if $n_A^* \in (n_2^A, n_3^A)$, strategic demand rejection must be strictly optimal. We consider two cases: if $n_A^* \in (n_2^A, \hat{n}_2^A)$, then it must be that $n_2^A < n_A < \hat{n}_A < \hat{n}_2^A$, because $\pi_A(n) < \hat{\pi}_A(n)$ on $(n_2^A, \hat{n}_2^A)$ by property (1.iv) and the monotonicity assumption on $F(\cdot)$. Hence by $\mathit{\Pi}_A(n) > \mathit{\Pi}_A(n_2^A), \forall n \in (n_2^A, \hat{n}_2^A)$ and $\hat{\mathit{\Pi}}_A(n) \equiv \mathit{\Pi}_A(n_2^A), \forall n \in (n_2^A, \hat{n}_2^A)$ due to (1.iii), there must be $\hat{\mathit{\Pi}}_A(\hat{n}_A) = \mathit{\Pi}_A(n_2^A) < \mathit{\Pi}_A(n_A)$. By Lemma 3 2(v), strategic demand rejection is optimal.

If $n_A^* \in [\hat{n}_2^A, n_3^A)$, it follows from $\pi_A(n) = \hat{\pi}_A(n)$ on $[\hat{n}_2^A, n_3^A)$ by property (1.iv) that $n_A^* = n_A = \hat{n}_A$ and thus $\hat{\mathit{\Pi}}_A(n_A^*) < \mathit{\Pi}_A(n_A^*)$. In this case the optimal capacity allocation has $s_{11} < S_{11}$ or $s_{12} < S_{12}$. By Lemma 3 1(ii), strategic demand rejection is optimal.

(3) For a given driver opportunity cost distribution F, let $(\underline{N}, \overline{N})$ be the range of driver pool size such that $N \in (\underline{N}, \overline{N}) \Leftrightarrow n_A^* \in (n_2^A, n_3^A)$ under $S_{22} = 0$. This range exists and is unique because of the monotonicity assumption on $F(\cdot)$ and because $\pi_A(\cdot)$ is decreasing.

We have shown that strategic demand rejection is optimal if $S_{22} = 0$ and $N \in (\underline{N}, \overline{N})$. Now fix any $N \in (\underline{N}, \overline{N})$, we show that by continuity, the above results still hold for sufficiently small S_{22}. Specifically, we have the following properties:

(i) The platform's optimal revenues allowing or disallowing strategic demand rejection, $\mathit{\Pi}_A(n)$ and $\hat{\mathit{\Pi}}_A(n)$, are both continuous in n and S_{22}. This follows from Berge's maximum theorem, because the maximand $\mathit{\Pi}(s)$ is continuous and the feasible sets given by the constraints in $\mathit{\Pi}_A(n)$ and (31) are both continuous correspondences of (s, r, q).

(ii) The resulting per-driver profits allowing or disallowing strategic demand rejection, $\pi_A(n)$ and $\hat{\pi}_A(n)$, are both continuous in $n \in (n_2^A, n_3^A)$ and S_{22}. Given the definition of $\pi(s,r,n)$ in (12d) and the definition of $\pi_X(n)$ in Step 1 of the two-step approach described in §2.5, since the optimal solution $s(n)$ and $r(n)$ are both continuous for $n \in (n_2^A, n_3^A)$ under either pattern (3) (optimal capacity allocation allowing strategic demand rejection) or pattern (1) (optimal capacity allocation disallowing strategic demand rejection) specified in Lemma S-1, $\pi_A(n)$ and $\hat{\pi}_A(n)$ are also both continuous for $n \in (n_2^A, n_3^A)$. The continuity in S_{22} follows because the optimal solution $s(n)$ and $r(n)$ are both continuous in parameter S_{22} under either pattern (3) and pattern (1) for $n \in (n_2^A, n_3^A)$.

(iii) The drivers' opportunity cost distribution $F(\cdot)$ is continuous and strictly increasing.

(iv) Due to (ii) and (iii), the equilibrium participating capacities allowing or disallowing strategic demand rejection, n_A and $\hat{n}_A$, are both continuous in S_{22}.

Since we have shown in the first part that $\hat{\Pi}_A(\hat{n}_A) < \Pi_A(n_A)$ always holds for any $N \in (\underline{N}, \overline{N})$ at $S_{22} = 0$, it follows from (i) and (iv) that $\hat{\Pi}_A(\hat{n}_A) < \Pi_A(n_A)$ still holds for sufficiently small $S_{22} > 0$. We also know that condition (36) in Proposition 5 does not hold for sufficiently large S_{22}, hence by continuity there exists $\hat{S}_{22}(N) > 0$ such that $\hat{\Pi}_A(\hat{n}_A) < \Pi_A(n_A)$, $\forall S_{22} \in [0, \hat{S}_{22}(N)))$, i.e, strategic demand rejection is optimal in a neighborhood of $S_{22} = 0$. This translates to a threshold level $\overline{\rho}_2(N)$ on ρ_2. □

Proof of Proposition 7. We start by ranking the platform revenue and per-driver profit under the three control regimes for *fixed* participating capacity. Lemma 1 below establishes that both are higher when the platform has more control capabilities.

Lemma 1 (Ranking of equilibrium profits for fixed capacity)**.** *For fixed participating capacity n, platform controls have the following impact on profits:*
(1) More platform control increases the platform revenue: $\Pi_M(n) \le \Pi_A(n) \le \Pi_C(n)$, where

$$\Pi_M(n) < \Pi_A(n) \quad \textit{iff} \quad n \in (n_1^M, n_3^M) = (n_1^A, n_3^A) \quad \textit{and} \quad S_{22} > 0,$$
$$\Pi_A(n) < \Pi_C(n) \quad \textit{iff} \quad n \in (n_1^A, n_3^A).$$

(2) Centralized control maximizes the per-driver profit rate: $\pi_M(n) \le \pi_C(n)$ with strict inequality iff $n \in (n_1^M, n_3^M)$, $\pi_A(n) \le \hat{\pi}_A(n) \le \pi_C(n)$ with strict second inequality iff $n \in (n_1^A, n_3^A)$.
(3) Under decentralized repositioning, optimal admission control affects per-driver profit as follows:

(a) ***No change*** *($\pi_A(n) = \pi_M(n)$) under small ($n \le n_1^M$) or large ($n \ge n_3^M = n_3^A$) capacity.*

(b) ***Increase*** *($\pi_A(n) > \pi_M(n)$) under intermediate capacity ($n_1^M < n < n_3^M = n_3^A$) if (36) is not satisfied.*

(c) ***Decrease*** *($\pi_A(n) < \pi_M(n)$) under intermediate capacity ($n_2^A < n < n_2^M$) if $S_{22} = 0$.*

Proof. (1) The formulation of $\Pi_X(n)$ given by (14) has the same objective function but shrinking constraint sets $\mathcal{C}_X(n)$ (i.e., fewer constraints) for X ranging $M \to A \to C$, hence the ranking holds. The conditions for strict inequalities follow from Propositions 1, 3 and 4.

(2) $\pi_M(n) \le \pi_C(n)$ follows directly from the specification of $\pi_C(n)$ and $\pi_M(n)$ in (A.13) and (18), respectively. $\pi_A(n) \le \hat{\pi}_A(n)$ is given by (33). To see $\hat{\pi}_A(n) \le \pi_C(n)$: for $n \le \hat{n}_2^A$ and $n \ge n_3^A$ (zone (1), (2) and (4) under regime A disallowing strategic demand rejection), $\hat{\pi}_A(n) = \pi_A(n) \le \pi_C(n)$ follows directly from the specification of $\pi_C(n)$ and $\pi_A(n)$ in (18) and (S.19), respectively; for $\hat{n}_2^A < n < n_3^A$ (zone

(3)), $\hat{\pi}_A(n) \le \pi_C(n)$ follows from Lemma S-1. The strict inequality conditions also follow from the above specification.

(3) Part (a) follows directly from the specification of $\pi_M(n)$ and $\pi_A(n)$ in (A.13) and (S.19), respectively, and the fact that $n_1^M \le n_1^A$. To see part (b), first note that for (36) to be not satisfied, there must be $\rho_2 > 0$ and equivalently $S_{22} > 0$, which implies $n_1^M < n_1^C = n_1^A$ by (27). Thus $\pi_M(n) < \pi_A(n)$ for $n_1^M < n \le n_1^A$ by the specification of $\pi_M(n)$ and $\pi_A(n)$ in (A.13) and (S.19), respectively. Then given (36) is not satisfied (no strategic demand rejection in regime A), it follows from Lemma S-1 that $\pi_M(n) < \pi_A(n)$ for $n_1^A < n < n_3^A = n_3^M$. Together we have proven part (b). For part (c), given $S_{22} = 0$, one can see that the optimal capacity allocation in the *absence* of strategic demand rejection is *equivalent* to that under regime M, i.e., $\hat{\mathit{\Pi}}_A(n) = \mathit{\Pi}_M(n), \hat{\pi}_A(n) = \pi_M(n)$ and $\hat{n}_2^A = n_2^M$, see the proof of Proposition 6. As a result, we may simply compare $\pi_A(n)$ with $\hat{\pi}_A(n)$ under $S_{22} = 0$. It then follows immediately from properties (ii) and (iv) in Part (1) of the proof of Proposition 6 that $\pi_A(n) < \hat{\pi}_A(n) = \pi_M(n)$ for $n_2^A < n < \hat{n}_2^A = n_2^M$, and $\pi_A(n) = \pi_M(n)$ for $n \le n_2^A$ or $n \ge n_2^M$. □

Remark: Part (1) of Lemma 1 also implies that riders benefit from increasing platform control capability: An important performance metric for the riders is the network-wise service level, defined as the fraction of the total rider demand that is served, i.e., $\overline{s}/\overline{S}$. Since the platform revenue rate is proportional to the total service capacity ($\mathit{\Pi}(s) = \gamma p \overline{s}$), the network-wise service level is proportional to the platform revenue rate, and therefore increases with platform controls.

Back to the proof of Proposition 7. Note the following properties of $\mathit{\Pi}_X(\cdot), \pi_X(\cdot)$ and n_X^*:

(i) $\mathit{\Pi}_X(n)$ is increasing in n for $X \in \{M, A, C\}$. This follows immediately for regime C from Proposition 1. For regimes M and A, one can verify that increasing capacity can be allocated into IC queues as in (22) without any reduction in the capacity that serves rider demand.

(ii) $\pi_M(n), \hat{\pi}_A(n), \pi_C(n)$ are continuously decreasing in n, and $\pi_A(n)$ is continuously decreasing for $n > n_2^A$. This follows from (18), (A.13), (S.19) and the proof of Lemma S-2.

(iii) For $\pi_1(n) \le \pi_2(n)$ both continuously decreasing for $n \in (A, B)$, let $n_i^* \in (A, B)$ be the solution to $n_i^* = NF(\pi_i(n_i^*)), i = 1, 2$. Then $n_1^* \le n_2^*$ and $\pi_1(n_1^*) \le \pi_2(n_2^*)$, where the inequalities are strict iff $\pi_1(n_1^*) < \pi_2(n_1^*)$ or $\pi_1(n_2^*) < \pi_2(n_2^*)$. To see $n_1^* \le n_2^*$, by definition of n_1^*, $\pi_1(n) \le \pi_2(n)$ and increasingness of $F(\cdot)$, we have

$$n_1^* = NF(\pi_1(n_1^*)) \le NF(\pi_2(n_1^*)). \tag{A.18}$$

Suppose $n_2^* < n_1^*$, then by definition of n_2^*, monotonicity of $\pi_2(\cdot)$ and $F(\cdot)$, and (A.18), we have

$$n_2^* = NF(\pi_2(n_2^*)) \ge NF(\pi_2(n_1^*)) \ge n_1^*,$$

contradicting $n_2^* < n_1^*$. Hence, it must be that $n_1^* \le n_2^*$. It then follows from $NF(\pi_i(n_i^*)) = n_i^*$ and the increasingness of $F(\cdot)$ that $\pi_1(n_1^*) \le \pi_2(n_2^*)$. For the "strict" iff condition, take $\pi_1(n_1^*) < \pi_2(n_1^*)$ and the other is similar. Obviously $\pi_1(n_1^*) < \pi_2(n_1^*) \Rightarrow n_1^* \ne n_2^*$, i.e., $n_1^* < n_2^*$, which implies $NF(\pi_1(n_1^*)) < NF(\pi_2(n_2^*))$ by definition, and hence $\pi_1(n_1^*) < \pi_2(n_2^*) \le \pi_2(n_1^*)$.

With these three properties and Lemma 1, we are ready to prove the three parts in Proposition 7.

(1) The first inequality, $\mathit{\Pi}_M^* \le \mathit{\Pi}_A^*$, can be shown by discussing whether (36) is satisfied. If (36) *is not* satisfied, strategic demand rejection is suboptimal, then $\mathit{\Pi}_M^* = \mathit{\Pi}_M(n_M^*) \le \mathit{\Pi}_A(n_M^*) \le \mathit{\Pi}_A(n_A^*) = \mathit{\Pi}_A^*$, where the first inequality follows from Lemma 1 (1), which is strict if $S_{22} \ne 0$ and $n_M^* \in (n_1^M, n_3^M)$ (equivalently $N \in (N_1^M, N_3)$), and the second inequality follows from the monotonicity of $\mathit{\Pi}_A(\cdot)$ in

Property (i) and the ranking $n_M^* \le n_A^*$ given by (a) and (b) in Part (3) of the proposition. If (36) *is* satisfied, consider the value of n_M^* (equivalently the value of N): in zone (1) and (4) ($n_M^* \le n_1^M \le n_1^A$ and $n_M^* \ge n_3^M = n_3^A$, respectively, where $\Pi_M(\cdot) = \Pi_A(\cdot)$), $\Pi_M^* = \Pi_A^*$ due to Part (3) (a); in zone (2) ($n_1^M < n_M^* \le n_2^M$), it is obvious that $n_1^M < n_A^*$ since $\pi_A(n) = \pi_M(n)$ for $n \le n_1^M$, therefore $\Pi_M^* = \Pi_M(n_1^M) = \Pi_A(n_1^M) \le \Pi_A^*$, where the inequality is strict if $S_{22} \neq 0$ (so that $n_1^M < n_1^A$); in zone (3) ($n_2^M < n_M^* < n_3^M = n_3^A$), since $\hat{\pi}_A(n) \ge \pi_A(n) \ge \pi_M(n_M^*) = \pi_M(n_3^M)$ for $n < n_3^M = n_3^A$, where the first inequality follows from (33) and the second inequality is strict if $S_{22} \neq 0$ (see proof of Proposition 6), there must be $n_M^* \le n_A \le n_A^* \le \hat{n}_A$ and hence $\Pi_M^* \le \Pi_A(n_M^*) \le \Pi_A(n_A) \le \Pi_A^*$, with first two inequalities being strict if $S_{22} \neq 0$. This completes the proof of $\Pi_M^* \le \Pi_A^*$.

The second inequality, $\Pi_A^* \le \Pi_C^*$, holds because $\Pi_A^* \le \Pi_A(n_A^*) \le \Pi_C(n_A^*) \le \Pi_C(n_C^*) = \Pi_C^*$, where the first inequality follows from the fact that $\Pi_A(n)$, as the unrestricted optimal platform revenue, is the highest at any n, including n_A^*, the second inequality follows from Lemma 1 (1), which is strict if $n_A^* \in (n_1^A, n_3^A)$ (equivalently $N \in (N_1^A, N_3)$), and the third inequality follows from the monotonicity of $\Pi_C(\cdot)$ in Property (i) and the ranking $n_A^* \le n_C^*$ given in Part (2).

(2) This follows from properties (ii), (iii) and Lemma 1 (2) directly. To see the iff condition for strict inequalities, note that $N \in (N_1^A, N_3) \Leftrightarrow n_A^* \in (n_1^A, n_3^A) \Leftrightarrow \pi_A(n_A^*) < \pi_C(n_A^*)$, and also $N \in (N_1^A, N_3) \Leftrightarrow n_M^* \in (n_1^M, n_3^M) \Leftrightarrow \pi_M(n_M^*) < \pi_C(n_M^*)$. It then follows from Property (iii).

(3) Cases (a) and (b) follow directly from properties (ii), (iii) and cases (a) and (b) in Lemma 1 (3). For case (c), first consider $\rho_2 = 0$ (equivalently $S_{22} = 0$), it follows from case (c) (and its proof) in Lemma 1 (3) that $\pi_A(n) \le \pi_M(n) = \hat{\pi}_A(n)$ for $n \in (n_2^A, n_3^A)$, with strictly inequality when $n \in (n_2^A, n_2^M)$. This, together with properties (ii) and (iii), implies $n_M^* > n_A^*$ and $\pi_M^* > \pi_A^*$ for $n_A^* \in (n_2^A, n_2^M)$, or equivalently $N \in (\underline{N}, \overline{N}')$, where $\underline{N}, \overline{N}'$ are values of N corresponding to $n_A^* = n_2^A$ and $n_A^* = \hat{n}_2^A = n_2^M$, respectively. Then, fixing $N \in (\underline{N}, \overline{N}')$, we can show by continuity that the above results still hold for sufficiently small $\rho_2 \in [0, \widetilde{\rho}_2(N))$ like the last part of the proof of Proposition 6.

□

Proof of Proposition 8. (1) According to Proposition 7 (and Lemma 1 in its proof), if strategic demand rejection is suboptimal ((36) not satisfied), the platform revenue rate gain from admission control is positive only when $n_M^* \in (n_1^M, n_3^M)$. In this range we have

$$\Pi_M^* \ge \Pi_M(n_1^M) = \gamma p n_1^M, \tag{A.19}$$

where the equality holds for $n_M^* \in (n_1^M, n_2^M]$. By Proposition 7 (and Lemma 1 in its proof), $n_M^* \in (n_1^M, n_3^M)$ and (36) not satisfied also imply that $\pi_M^* < \pi_A^*$, thus $n_A^* \in (n_1^M, n_3^M)$ since otherwise $n_M^* = n_A^*$ and $\pi_M^* = \pi_A^*$. This yields

$$\Pi_A^* \le \Pi_A(n_3^M) = \Pi_A(n_3^A) = \gamma p \overline{S}, \tag{A.20}$$

where the equality is approached by $n_A^* \to n_3^M = n_3^A$.

Given that $N \ge n_3^M = n_3^A$, $n_X^* = NF(\pi_X(n_X^*))$ can take on values in $[0, n_3^M]$ depending on the choice of $F(\cdot)$, for $X \in \{M, A, C\}$. Consequently, the bounds in (A.19) and (A.20) can be approached and therefore

$$\max_{F(\cdot)} \frac{\Pi_A^* - \Pi_M^*}{\Pi_M^*} \le \frac{\gamma p \overline{S} - \gamma p n_1^M}{\gamma p n_1^M} = \left(\frac{\Lambda_{21}}{\Lambda_{12}} - 1\right) \frac{1}{1 + \frac{1-\rho_2}{1-\rho_1}\frac{1}{\tau}}.$$

To approach this upper bound, we need $n_M^* \in (n_1^M, n_2^M]$ and $n_A^* \to n_3^M = n_3^A$ so that $\Pi_M^* = \gamma p n_1^M$ and $\Pi_A^* \to \gamma p \overline{S}$. (Refer to Figure 1 (a) and (c) for an illustration.) This holds for opportunity cost distributions

$F(\cdot)$ satisfying

$$F^{-1}(n_2^M/N) = \pi_M(n_2^M) \quad \text{and} \quad F^{-1}(n_3^M/N) = \pi_M(n_2^M)^+,$$

i.e., the value of F at $\pi_M(n_2^M)$ is fixed at n_2^M/N ($\Rightarrow n_M^* = n_2^M$) and F grows sufficiently fast to $n_3^M/N = n_3^A/N$ at $\pi_M(n_2^M)^+ = \pi_A(n_3^A)^+$ ($\Rightarrow n_A^* \to n_3^A = n_3^M$); in words, there is a sufficiently large mass of potential drivers with opportunity cost around $\pi_M(n_2^M)^+$.

(2) According to Proposition 7 (and Lemma 1 in its proof), the platform revenue rate gain from centralized repositioning is positive only when $n_A^* \in (n_1^A, n_3^A)$. This yields

$$\Pi_A^* \geq \Pi_A(n_1^A) = \gamma p n_1^A, \tag{A.21}$$

where the equality holds for $n_A^* \in (n_1^A, n_2^A]$. By Proposition 7 (and Lemma 1 in its proof), $n_A^* \in (n_1^A, n_3^A)$ also implies that $\pi_A^* < \pi_C^*$. Thus, $n_C^* \in (n_1^A, n_3^A)$, since otherwise $n_A^* = n_C^*$ and $\pi_A^* = \pi_C^*$. This yields

$$\Pi_C^* \leq \Pi_C(n_3^A) = \gamma p \overline{S}, \tag{A.22}$$

where the equality holds for $n_C^* \in [n_2^C, n_3^A)$.

Given that $N \geq n_3^M = n_3^A$, $n_X^* = NF(\pi_X(n_X^*))$ can take on values in $[0, n_3^A]$ depending on the choice of $F(\cdot)$, for $X \in \{M, A, C\}$. Using the bounds in (A.21) and (A.22) we have

$$\max_{F(\cdot)} \frac{\Pi_C^* - \Pi_A^*}{\Pi_A^*} \leq \frac{\gamma p \overline{S} - \gamma p n_1^A}{\gamma p n_1^A} = \left(\frac{\Lambda_{21}}{\Lambda_{12}} - 1\right) \frac{1}{1 + \frac{1}{1-\rho_1}\frac{1}{\tau} + \frac{\rho_2}{1-\rho_2}\frac{\Lambda_{21}}{\Lambda_{12}}}.$$

To achieve this upper bound, we need $n_A^* \in (n_1^A, n_2^A]$ and $n_C^* \in [n_2^C, n_3^A)$ so that $\Pi_A^* = \gamma p n_1^A$ and $\Pi_C^* = \gamma p \overline{S}$. Noticing Lemma 1 (2) and Property (ii) in the proof of Proposition 7 about $\pi_A(\cdot), \pi_C(\cdot)$, this holds for opportunity cost distributions $F(\cdot)$ satisfying

$$F(\pi_A(n_2^A)) \leq n_2^A/N \quad \text{and} \quad F(\pi_C(n_2^C)) \geq n_2^C/N \tag{A.23}$$

when $\pi_A(n_2^A) < \pi_C(n_2^C)$. When $\pi_A(n_2^A) \geq \pi_C(n_2^C)$, (A.23) cannot be satisfied by any F and hence the upper bound is not tight. □

Proof of Proposition 9. Since $\pi_M(n) = \pi_A(n) = \pi_C(n)$ for $n \leq n_1^M$ and $n \geq n_3^M$, the per-driver profit rate gain from admission control only (regime A over M) and from admission control plus centralized repositioning (regime C over M) can be positive only for $n_M^* \in (n_1^M, n_3^M)$, and $n_A^*, n_C^* \in (n_1^M, n_3^M)$ simultaneously. It follows from the (decreasing) monotonicity of $\pi_X(\cdot)$, $X \in \{M, A, C\}$ that

$$\pi_M^* \geq \pi_M(n_3^M), \quad \pi_A^* \leq \pi_A(n_1^M) = \bar{\gamma} p - c, \quad \pi_C^* \leq \pi_C(n_1^M) = \bar{\gamma} p - c. \tag{A.24}$$

Therefore,

$$\max_{F(\cdot)} \frac{\pi_A^* - \pi_M^*}{\pi_M^*} = \max_{F(\cdot)} \frac{\pi_C^* - \pi_M^*}{\pi_M^*} \leq \frac{\bar{\gamma} p - c}{\pi_M(n_3^M)} - 1 = \frac{1 - \rho_2}{\tau - (1 - \rho_2 + \tau)\kappa}.$$

To achieve this upper bound, we need $n_M^* \in [n_2^M, n_3^M)$ and $n_A^*, n_C^* \in (n_1^M, n_1^A]$ so that the equalities in (A.24) are satisfied. If $n_2^M \leq n_1^A = n_1^C$, these conditions hold for $F(\cdot)$ satisfying

$$F(\pi_M(n_2^M)) \geq n_2^M/N \quad \text{and} \quad F(\pi_A(n_1^A)) \leq n_1^A/N. \tag{A.25}$$

If $n_2^M > n_1^A = n_1^C$, then (A.25) cannot be satisfied by any F and the upper bound is not tight. □