Online Appendix to "Simultaneous vs Sequential: Optimal Assortment Recommendation in Multi-Store Retailing"

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Here we only provide proofs for main theorems and propositions. Results marked with "($\star\star\star$)" can be found in the full version of the paper via https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4592172.

Appendix A: Proofs for Section 4

We restate store 1's optimization problem discussed in Section 4 here for clarity.

$$\max_{S_1 \subseteq \mathcal{N}} \quad R(S_1, \boldsymbol{c}) \triangleq \frac{\sum_{k\in S_1} r_k v_k + \sum_{k\in S_2\setminus S_1} r_k v_k \cdot \exp(-c^k)}{1+\sum_{k\in S_1} v_k + \sum_{k\in S_2\setminus S_1} v_k \cdot \exp(-c^k)}. \tag{A.1}$$

Given $\boldsymbol{c}$, we denote the optimal solution to (A.1) by $S_M^*(\boldsymbol{c})$ and the optimal objective value of (A.1) by $R_M^*(\boldsymbol{c})$, respectively. When there are multiple optimal assortments, to simplify the analysis, we focus on the optimal assortment that is revenue-ordered with the least elements (the existence of such an assortment will be confirmed by Proposition A.1). For instance, if $\{1,\dots,k\}$ and $\{1,\dots,k+1\}$ are both optimal to problem (A.1), then we let $S_M^*(\boldsymbol{c}) = \{1,\dots,k\}$.

A.1. Proof of Theorem 1

To prove Theorem 1, we need the following two propositions.

PROPOSITION A.1. *For any given $\boldsymbol{c}$, there must exist a revenue-ordered assortment that is optimal to (A.1). Moreover, we have $r_i > R_M^*(\boldsymbol{c})$ if and only if $i \in S_M^*(\boldsymbol{c})$.*

The proof of Proposition A.1 is in Appendix A.2. We then study the effect of the utility discount factor. Define $\boldsymbol{c} \geq \tilde{\boldsymbol{c}}$ if $c^j \geq \tilde{c}^j$ for any $j \in [n]$ and there exists at least one i such that $c^i > \tilde{c}^i$. We have the following proposition.

PROPOSITION A.2. *If $\boldsymbol{c} \geq \tilde{\boldsymbol{c}}$, we have $R_M^*(\boldsymbol{c}) \geq R_M^*(\tilde{\boldsymbol{c}})$.*

The proof of Proposition A.2 is in Appendix A.3. We then offer a formal proof of Theorem 1.

Proof of Theorem 1: For any assortment set $\boldsymbol{S} = (S_1, S_2, \dots, S_m)$, we denote $S_{\bar{2}} = S_2 \cup S_3 \cup \dots \cup S_m$ and $h^k = \min_{j\in\{l|k\in S_l^*, l=2,\dots,m\}} c_{1j}^k$. We then consider the following assortment optimization problem under $S_{\bar{2}}$ and $\boldsymbol{h} = (h^1, h^2, \dots, h^n)$:

$$\max_{S_1 \subseteq \mathcal{N}} \quad R(S_1, \boldsymbol{h}) \triangleq \frac{\sum_{k\in S_1} r_k v_k + \sum_{k\in S_{\bar{2}}\setminus S_1} r_k v_k \cdot \exp(-h^k)}{1+\sum_{k\in S_1} v_k + \sum_{k\in S_{\bar{2}}\setminus S_1} v_k \cdot \exp(-h^k)}. \tag{A.2}$$

Notice that S_1 is only a feasible solution to problem (A.2). Thus, we must have $R(S_1, \boldsymbol{h}) \leq R_M^*(\boldsymbol{h})$ where $R_M^*(\boldsymbol{h})$ denotes the optimal objective value of (A.2). Also, due to Proposition A.2, we must have $R_M^*(\boldsymbol{h}) \leq R_M^*(\tilde{\boldsymbol{h}})$ where $\tilde{\boldsymbol{h}} = (\infty, \dots, \infty)$. Therefore, we have $R(S_1, \boldsymbol{h}) \leq R_M^*(\tilde{\boldsymbol{h}})$. We notice that $R(S_1, \boldsymbol{h})$ is the expected revenue of store 1 under the solution $\boldsymbol{S}$. We also notice that the solution $(\tilde{S}, \tilde{S}, \dots, \tilde{S})$ has its expected revenue for each store equal to $R_M^*(\tilde{\boldsymbol{h}})$. Thus, the revenue obtained in store 1 under $\boldsymbol{S}$ must be less than that under $(\tilde{S}, \dots, \tilde{S})$. Repeat this argument for all stores, we are able to prove Theorem 1. ■

A.2. Proof of Proposition A.1 ($\star\star\star$)

A.3. Proof of Proposition A.2 ($\star\star\star$)

A.4. Proof of Proposition 2

Proof: The proof is quite similar to that of Theorem 1. We first denote the optimal solution to the following assortment optimization problem as $\hat{S}$:

$$\max_{S\subseteq\mathcal{N}} \quad \frac{\sum_{k\in S} r_k v_k + \sum_{k\in\mathcal{N}\setminus S} r_k v_k \cdot \exp\left(-c^k\right)}{1+\sum_{k\in S} v_k + \sum_{k\in\mathcal{N}\setminus S} v_k \cdot \exp\left(-c^k\right)}. \tag{A.3}$$

Due to Proposition A.1, we know that $\hat{S}$ is revenue-ordered. For any assortment set $\boldsymbol{S}=(S_1,S_2,\ldots,S_{m-1},\mathcal{N})$, we denote $h^k=\min_{j\in\{l|k\in S_l, l=2,\ldots,m\}} c^k_{1j}$ and consider the following problem:

$$\max_{S_1\subseteq\mathcal{N}} \quad R(S_1,\boldsymbol{h}) \triangleq \frac{\sum_{k\in S_1} r_k v_k + \sum_{k\in\mathcal{N}\setminus S_1} r_k v_k \cdot \exp\left(-h^k\right)}{1+\sum_{k\in S_1} v_k + \sum_{k\in\mathcal{N}\setminus S_1} v_k \cdot \exp\left(-h^k\right)}. \tag{A.4}$$

Clearly, we have $R(S_1,\boldsymbol{h})\le R^*_M(\boldsymbol{h})$ where $R^*_M(\boldsymbol{h})$ denotes the optimal objective value of (A.4). Also, due to Proposition B.2, we must have $R^*_M(\boldsymbol{h})\le R^*_M(\tilde{\boldsymbol{h}})$ where $\tilde{\boldsymbol{h}}=(c^1,c^2,\ldots,c^k)$. Therefore, we have $R(S_1,\boldsymbol{h})\le R^*_M(\tilde{\boldsymbol{h}})$. We again notice that $R(S_1,\boldsymbol{h})$ is the expected revenue of store 1 under the solution $\boldsymbol{S}$. We also notice that the solution $(\hat{S},\hat{S},\ldots,\hat{S},\mathcal{N})$ has its expected revenue for each store equal to $R^*_M(\tilde{\boldsymbol{h}})$. Thus, the revenue obtained in store 1 under $\boldsymbol{S}$ must be less than that under $(\hat{S},\hat{S},\ldots,\hat{S},\mathcal{N})$. Repeat this argument for all stores, we have proved that the optimal solution should be $(\hat{S},\hat{S},\ldots,\hat{S},\mathcal{N})$. Moreover, we know that $R^*_M(\tilde{\boldsymbol{h}})\le R^*_M((\infty,\ldots,\infty))$, which is the optimal expected revenue of the single-store problem. Then, based on Proposition A.1, we must have that $|\hat{S}|\ge|\tilde{S}|$. ■

Appendix B: Proofs for Section 5

B.1. Proof of Theorem 2

Proof: Theorem 2 can be viewed as a special case of Theorem 4, thus the proof is omitted here. ■

B.2. Proof of Theorem 3

Proof: Theorem 3 can also be viewed as a special case of Theorem 4. The only thing we need to show is that when $c_1=c_2=\cdots=c_m=c$, we must have $S^*_1=S^*_2=\cdots=S^*_{m-1}$. To see this, we first define $\beta=\exp\left(-c\right)$ with $\beta\in(0,1)$. By Theorem 4, we know that there exists an optimal solution $(S^*_1,S^*_2,\ldots,S^*_m)$ such that $S^*_i\subseteq S^*_m$ for all $i=1,2,\ldots,m-1$. We can then express the total expected revenue from all m stores under this solution as:

$$\begin{aligned}
R(S^*_1,\ldots,S^*_m) &= \sum_{i=1}^{m-1}\lambda_i\left(\frac{W(S^*_i)}{1+V(S^*_i)}+\frac{W(S^*_m\setminus S^*_i)\cdot\beta}{(1+V(S^*_i))(1+V(S^*_i)+V(S^*_m\setminus S^*_i)\cdot\beta)}\right)+\lambda_m\frac{W(S^*_m)}{1+V(S^*_m)}\\
&= \sum_{i=1}^{m-1}\lambda_i\left(\frac{W(S^*_i)}{1+V(S^*_i)}+\frac{W(S^*_m\setminus S^*_i)\cdot\beta}{(1+V(S^*_i))(1+V(S^*_i)+V(S^*_m\setminus S^*_i)\cdot\beta)}+\frac{\lambda_m}{1-\lambda_m}\frac{W(S^*_m)}{1+V(S^*_m)}\right).
\end{aligned}$$

We denote the optimal solution and the optimal objective value of the following problem as $(S^{'},S^{''})$ and R^* respectively. We know that $S^{'}$ and $S^{''}$ are both revenue-ordered and $S^{'}\subseteq S^{''}$.

$$\begin{aligned}
\max_{S_1,S_2} \quad & (1-\lambda_m)\frac{W(S_1)}{1+V(S_1)}+(1-\lambda_m)\frac{\beta W(S_2\setminus S_1)}{(1+V(S_1))(1+V(S_1)+\beta V(S_2\setminus S_1))}\\
&+\lambda_m\frac{W(S_2)}{1+V(S_2)}+\lambda_m\frac{\beta W(S_1\setminus S_2)}{(1+V(S_2))(1+V(S_2)+\beta V(S_1\setminus S_2))}.
\end{aligned}$$

Then, we have $R(S_1^*,\dots,S_m^*) \le R^*$ and the inequality holds as equality when we choose the solution as $(S^{'},S^{'},\dots,S^{'},S^{''})$. Therefore, all of the optimal solutions of the SQR problem will be dominated by a solution of $(S^{'},S^{'},\dots,S^{'},S^{''})$, which leads to the conclusion that $(S^{'},S^{'},\dots,S^{'},S^{''})$ is indeed an optimal solution. ■

B.3. Proof of Theorem 4

The proof of Theorem 4 consists of three steps. In this section, we list the results for each step and prove them in the following subsections.

Step 1:

In step 1, we first prove the following proposition.

PROPOSITION B.1. *Suppose the utility discounts are store-wise homogeneous. Assume* $\lambda_m = \min\{\lambda_1,\dots,\lambda_m\}$ *and* $c_m = \max\{c_1,\dots,c_m\}$*, then there exists an optimal solution* $(S_1^*,S_2^*,\dots,S_m^*)$ *to the SQR problem such that* $S_i^* \subseteq S_m^*$ *for all* $i = 1,2,\dots,m-1$.

Based on Proposition B.1, we prove the following lemma.

LEMMA B.1. *Suppose the utility discounts are store-wise homogeneous. Assume* $\lambda_m = \min\{\lambda_1,\dots,\lambda_m\}$ *and* $c_1 \le c_2 \le \cdots \le c_m$*, then there exists an optimal solution* $(S_1^*,S_2^*,\dots,S_m^*)$ *to the SQR problem such that* $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_m^*$.

Step 2:

In step 2, we first prove the following lemma.

LEMMA B.2. *There exists a revenue-ordered (within the products in* S_2*) assortment* S_1^* *such that it is optimal to problem* (8).

Using the result of Lemma B.2, we prove the following proposition.

PROPOSITION B.2. *Suppose the utility discounts are store-wise homogeneous. Assume* $\lambda_m = \min\{\lambda_1,\dots,\lambda_m\}$ *and* $c_1 \le c_2 \le \cdots \le c_m$*, then there exists an optimal solution* $(S_1^*,S_2^*,\dots,S_m^*)$ *to the SQR problem such that* $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_m^*$ *and* $S_1^*,S_2^*,\dots,S_m^*$ *are all revenue-ordered.*

Step 3:

Based on the results of the first two steps, we prove Theorem 4.

Restate of Theorem 4: *Suppose the utility discounts are store-wise homogeneous. Assume* $\lambda_m = \min\{\lambda_1,\dots,\lambda_m\}$ *and* $c_1 \le c_2 \le \cdots \le c_m$*, then there exists an optimal solution* $(S_1^*,S_2^*,\dots,S_m^*)$ *for the SQR problem such that* $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq \tilde{S} \subseteq S_m^*$ *and* $S_1^*,S_2^*,\dots,S_m^*$ *are all revenue-ordered.*

B.4. Proofs in Step 1

In this section, we prove Proposition B.1 and Lemma B.1 in Step 1. To prove Proposition B.1, we need the following lemma.

LEMMA B.3. ($\star\star\star$) *Consider an instance of the SQR problem, where there are three stores ($m=3$) and four products ($n=4$). The revenues and the attraction values of the products are denoted as (R_1,R_2,R_3,R_4) and (V_1,V_2,V_3,V_4) respectively. We set $\lambda_1=\lambda_2=1/2$ and $\lambda_3=0$. We assume the utility discounts are store-wise homogeneous and use $c_1,c_2,c_3>0$ to denote them. We also assume $R_2\geq R_3$ and $c_1\leq c_2$. Under this problem instance, we denote three recommendation strategies as $\boldsymbol{S}=\{\{1,2\},\{1,3\},\{4\}\}$, $\boldsymbol{S_1}=\{\{1\},\{1,2,3\},\{4\}\}$ and $\boldsymbol{S_2}=\{\{1,2\},\{1,2,3\},\{4\}\}$. Then, we must have $R(\boldsymbol{S})\leq\max(R(\boldsymbol{S_1}),R(\boldsymbol{S_2}))$.*

Utilizing the result of Lemma B.3, we prove Proposition B.1 as follows.

Proof of Proposition B.1: We prove this argument by contradiction. Without loss of generality, we suppose that $S_1^*\setminus S_m^*\neq\emptyset$. The total expected revenue generated by store 1 and store m can be written as:

$$\begin{aligned}R(S_1^*,S_m^*)=\lambda_1&\left(\frac{W(S_1^*)}{1+V(S_1^*)}+\frac{W(\bar{S}\setminus S_1^*)\cdot\beta_1}{(1+V(S_1^*))(1+V(S_1^*)+V(\bar{S}\setminus S_1^*)\cdot\beta_1)}\right)\\&+\lambda_m\left(\frac{W(S_m^*)}{1+V(S_m^*)}+\frac{W(\bar{S}\setminus S_m^*)\cdot\beta_m}{(1+V(S_m^*))(1+V(S_m^*)+V(\bar{S}\setminus S_m^*)\cdot\beta_m)}\right),\end{aligned}$$

where we denote $\bar{S}=S_1^*\cup S_2^*\cup\cdots\cup S_m^*$. We then define the following quantities:

$$\begin{aligned}&R_1=\frac{W(S_1^*\cap S_m^*)}{V(S_1^*\cap S_m^*)},R_2=\frac{W(S_1^*\setminus S_m^*)}{V(S_1^*\setminus S_m^*)},R_3=\frac{W(S_m^*\setminus S_1^*)}{V(S_m^*\setminus S_1^*)},R_4=\frac{W(\bar{S}\setminus(S_1^*\cup S_m^*))}{V(\bar{S}\setminus(S_1^*\cup S_m^*))},\\&V_1=V(S_1^*\cap S_m^*),V_2=V(S_1^*\setminus S_m^*),V_3=V(S_m^*\setminus S_1^*),V_4=V(\bar{S}\setminus(S_1^*\cup S_m^*)).\end{aligned}$$

We can then reformulate the expression of $R(S_1^*,S_m^*)$ as follows:

$$\begin{aligned}R(S_1^*,S_m^*)=\lambda_1&\frac{R_1V_1+R_2V_2}{1+V_1+V_2}+\lambda_1\frac{\beta_1(R_3V_3+R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}\\&+\lambda_m\frac{R_1V_1+R_3V_3}{1+V_1+V_3}+\lambda_m\frac{\beta_m(R_2V_2+R_4V_4)}{(1+V_1+V_3)(1+V_1+\beta_mV_2+V_3+\beta_mV_4)}=\lambda_1\pi_1+\lambda_m\pi_m,\end{aligned}$$

where we denote

$$\begin{aligned}\pi_1&=\frac{R_1V_1+R_2V_2}{1+V_1+V_2}+\frac{\beta_1(R_3V_3+R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}.\\\pi_m&=\frac{R_1V_1+R_3V_3}{1+V_1+V_3}+\frac{\beta_m(R_2V_2+R_4V_4)}{(1+V_1+V_3)(1+V_1+\beta_mV_2+V_3+\beta_mV_4)}.\end{aligned}$$

We then consider the following two cases:

Case 1: $R_2\geq R_3$

We denote $\boldsymbol{S_1}=(S_1^*\cap S_m^*,S_1^*\cup S_m^*)$ and express the total expected revenue generated by store 1 and store m under $\boldsymbol{S_1}$ as follows:

$$\begin{aligned}R(\boldsymbol{S_1})=\lambda_1&\frac{R_1V_1}{1+V_1}+\lambda_1\frac{\beta_1(R_2V_2+R_3V_3+R_4V_4)}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)}\\&+\lambda_m\frac{R_1V_1+R_2V_2+R_3V_3}{1+V_1+V_2+V_3}+\lambda_m\frac{\beta_mR_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)}=\lambda_1\pi_1^1+\lambda_m\pi_m^1,\end{aligned}$$

where we denote

$$\begin{aligned}\pi_1^1&=\frac{R_1V_1}{1+V_1}+\frac{\beta_1(R_2V_2+R_3V_3+R_4V_4)}{(1+V_1)(1+V_1+\beta_1V_2+\beta_1V_3+\beta_1V_4)}.\\\pi_m^1&=\frac{R_1V_1+R_2V_2+R_3V_3}{1+V_1+V_2+V_3}+\frac{\beta_mR_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)}.\end{aligned}$$

We denote $\boldsymbol{S_2} = (S_1^*, S_1^* \cup S_m^*)$ and express the total expected revenue generated by store 1 and store m under $\boldsymbol{S_2}$ as follows:

$$\begin{aligned} R(\boldsymbol{S_2}) = &\lambda_1 \frac{R_1V_1 + R_2V_2}{1+V_1+V_2} + \lambda_1 \frac{\beta_1(R_3V_3 + R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)} \\ &+ \lambda_m \frac{R_1V_1 + R_2V_2 + R_3V_3}{1+V_1+V_2+V_3} + \lambda_m \frac{\beta_m R_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)} = \lambda_1\pi_1^2 + \lambda_m\pi_m^2, \end{aligned}$$

where we denote

$$\begin{aligned} \pi_1^2 &= \frac{R_1V_1 + R_2V_2}{1+V_1+V_2} + \frac{\beta_1(R_3V_3 + R_4V_4)}{(1+V_1+V_2)(1+V_1+V_2+\beta_1V_3+\beta_1V_4)}. \\ \pi_m^2 &= \frac{R_1V_1 + R_2V_2 + R_3V_3}{1+V_1+V_2+V_3} + \frac{\beta_m R_4V_4}{(1+V_1+V_2+V_3)(1+V_1+V_2+V_3+\beta_mV_4)}. \end{aligned}$$

Based on the result of Lemma B.3, we must have:

$$\max(\pi_1^1 + \pi_m^1, \pi_1^2 + \pi_m^2) \geq \pi_1 + \pi_m.$$

We first notice that we must have $\pi_m^1 = \pi_m^2 \leq \pi_m$. Otherwise, we should have $R(S_1^*, S_m^*) < R(S_1^*, S_1^* \cup S_m^*)$ and note that by replacing the assortment in store m with $S_1^* \cup S_m^*$, the total expected revenue generated by the stores other than 1 and m will not change since the utility discounts are store-wise homogeneous. Then, the total expected revenue generated by all stores will be strictly higher if we make the replacement, which is contradictory to the optimality of $(S_1^*, \ldots, S_m^*)$. Then, if $\pi_1^1 + \pi_m^1 \geq \pi_1 + \pi_m$, we must have $\pi_1^1 \geq \pi_1$. Notice that $\lambda_1(\pi_1^1 - \pi_1) \geq \lambda_m(\pi_1^1 - \pi_1) \geq \lambda_m(\pi_m - \pi_m^1)$ and this implies $R(\boldsymbol{S_1}) \geq R(\boldsymbol{S})$. By the same reasoning, we can also imply that $R(\boldsymbol{S_2}) \geq R(\boldsymbol{S})$ if $\pi_1^2 + \pi_m^2 \geq \pi_1 + \pi_m$. To conclude, we have $\max(R(\boldsymbol{S_1}), R(\boldsymbol{S_2})) \geq R(\boldsymbol{S})$ in this case. Note that under $\boldsymbol{S}$, $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$, the total expected revenue generated by the stores other than 1 and m are the same. Therefore, the optimal solution $(S_1^*, \ldots, S_m^*)$ is dominated by one of the replacements (replace $\boldsymbol{S}$ with $\boldsymbol{S_1}$ or replace $\boldsymbol{S}$ with $\boldsymbol{S_2}$), which is enough to prove the desired result.

Case 2: $R_2 \leq R_3$

The proof for this case is largely the same as the proof for the first case. The only difference is that we now should let $\boldsymbol{S_2} = (S_m^*, S_1^* \cup S_m^*)$ in this case, and the result follows. ■

Based on Proposition B.1, we prove Lemma B.1 as follows.

Proof of Lemma B.1: We first define $\beta_i = \exp(-c_i)$ with $\beta_i \in (0,1)$. By Proposition B.1, we know that there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$ such that $S_i^* \subseteq S_m^*$ for all $i = 1, 2, \ldots, m-1$. We can then express the total expected revenue from all m stores under this solution as:

$$\begin{aligned} R(S_1^*, \ldots, S_m^*) &= \sum_{i=1}^{m-1} \lambda_i \left(\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{W(S_m^* \setminus S_i^*) \cdot \beta_i}{(1+V(S_i^*))(1+V(S_i^*) + V(S_m^* \setminus S_i^*) \cdot \beta_i)} \right) + \lambda_m \frac{W(S_m^*)}{1+V(S_m^*)} \\ &= \sum_{i=1}^{m-1} \lambda_i \left(\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{W(S_m^* \setminus S_i^*) \cdot \beta_i}{(1+V(S_i^*))(1+V(S_i^*) + V(S_m^* \setminus S_i^*) \cdot \beta_i)} + \frac{\lambda_m}{1-\lambda_m} \frac{W(S_m^*)}{1+V(S_m^*)} \right). \end{aligned}$$

Suppose S_m^* is fixed, we know that S_i^* must be the optimal solution to the following problem:

$$\max_{S_i : S_i \subseteq S_m^*} \quad \frac{W(S_i)}{1+V(S_i)} + \frac{\beta_i W(S_m^* \setminus S_i)}{(1+V(S_i))(1+V(S_i) + \beta_i V(S_m^* \setminus S_i))}. \tag{B.1}$$

We then prove the following claim:

Claim: Suppose the optimal solution to problem (B.1) is unique, then $|S_i^*|$ must be weakly decreasing in β_i.

Proof of the claim: We prove this claim by contradiction. For $0 \le \beta'' \le \beta'$, denote the optimal solution to problem (B.1) when $\beta_i = \beta'$ and $\beta_i = \beta''$ as S' and S'', respectively. We assume $|S'| > |S''|$. Due to the optimality and uniqueness of S' and S'', we must have:

$$\frac{W(S')}{1+V(S')} + \frac{\beta' W(S_2 \setminus S')}{(1+V(S'))(1+V(S')+\beta' V(S_2 \setminus S'))} > \frac{W(S'')}{1+V(S'')} + \frac{\beta' W(S_2 \setminus S'')}{(1+V(S''))(1+V(S'')+\beta' V(S_2 \setminus S''))},$$

and

$$\frac{W(S'')}{1+V(S'')} + \frac{\beta'' W(S_2 \setminus S'')}{(1+V(S''))(1+V(S'')+\beta'' V(S_2 \setminus S''))} > \frac{W(S')}{1+V(S')} + \frac{\beta'' W(S_2 \setminus S')}{(1+V(S'))(1+V(S')+\beta'' V(S_2 \setminus S'))},$$

which then leads to the following inequality:

$$\begin{aligned} &\frac{W(S_2 \setminus S')}{1+V(S')}\left(\frac{\beta'}{1+V(S')+\beta' V(S_2 \setminus S')} - \frac{\beta''}{1+V(S')+\beta'' V(S_2 \setminus S')}\right) \\ &\quad > \frac{W(S_2 \setminus S'')}{1+V(S'')}\left(\frac{\beta'}{1+V(S'')+\beta' V(S_2 \setminus S'')} - \frac{\beta''}{1+V(S'')+\beta'' V(S_2 \setminus S'')}\right). \end{aligned} \tag{B.2}$$

We then consider the following set function in S:

$$\begin{aligned} f(S) &= \frac{W(S_2 \setminus S)}{1+V(S)}\left(\frac{\beta'}{1+V(S)+\beta' V(S_2 \setminus S)} - \frac{\beta''}{1+V(S)+\beta'' V(S_2 \setminus S)}\right) \\ &= \frac{(\beta' - \beta'')W(S_2 \setminus S)}{(1+V(S)+\beta' V(S_2 \setminus S))(1+V(S)+\beta'' V(S_2 \setminus S))}. \end{aligned}$$

It is then easy to see that when $S'' \subseteq S'$, we must have $f(S') \le f(S'')$. According to Lemma B.2, we know that S' and S'' are both revenue-ordered. Additionally, we have $|S'| > |S''|$ and thus we have $S'' \subseteq S'$ indeed. However, (B.2) implies that $f(S') > f(S'')$, which causes a contradiction. The claim has then been proved.

Then, since $c_1 \le c_2 \le \ldots \le c_m$, we must then have $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq S_m^*$ and the result in Lemma B.1 has been proved. ■

B.5. Proofs in Step 2

In this section, we prove Lemma B.2 and Proposition B.2 in Step 2. We first prove Lemma B.2 as follows.

Proof of Lemma B.2: Denote the optimal solution to problem (8) as S_1^*. Since we have $S_1^* \subseteq S_2$, S_2 is partitioned into S_1^* and $S_2 \setminus S_1^*$. If S_1^* is not revenue-ordered, then we must have product j and k with $r_j > r_k$ such that $j \in S_2 \setminus S_1^*$ and $k \in S_1^*$. Based on the optimality of S_1^*, we know that moving product j to S_1^* will at least not increase the total expected revenue, which can be written as:

$$\begin{aligned} &\frac{W(S_1^*)+r_j v_j}{1+V(S_1^*)+v_j} + \frac{\beta(W(S_2 \setminus S_1^*) - r_j v_j)}{(1+V(S_1^*)+v_j)(1+V(S_1^*)+v_j+\beta(V(S_2 \setminus S_1^*)-v_j))} \\ &\le \frac{W(S_1^*)}{1+V(S_1^*)} + \frac{\beta W(S_2 \setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2 \setminus S_1^*))}. \end{aligned} \tag{B.3}$$

Note that (B.3) can be further transformed to

$$\begin{aligned} &\frac{r_j v_j}{1+V(S_1^*)+v_j}\left(1 - \frac{\beta}{1+V(S_1^*)+\beta V(S_2 \setminus S_1^*)+(1-\beta)v_j}\right) \le W(S_1^*)\left(\frac{1}{1+V(S_1^*)} - \frac{1}{1+V(S_1^*)+v_j}\right) \\ &\quad + \beta W(S_2 \setminus S_1^*)\left(\frac{1}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2 \setminus S_1^*))} - \frac{1}{(1+V(S_1^*)+v_j)(1+V(S_1^*)+\beta V(S_2 \setminus S_1^*)+(1-\beta)v_j))}\right). \end{aligned}$$

We also note that

$$\frac{1}{1+V(S_1^*)} - \frac{1}{1+V(S_1^*)+v_j} = \frac{v_j}{(1+V(S_1^*))(1+V(S_1^*)+v_j)},$$

and

$$\frac{1}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}-\frac{1}{(1+V(S_1^*)+v_j)(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j))}$$
$$=\frac{(1+V(S_1^*))\cdot(1-\beta)v_j+(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j)v_j}{(1+V(S_1^*))(1+V(S_1^*)+v_j)(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j))}.$$

Therefore, we can finally transform (B.3) into the following inequality:

$$r_j\le\frac{\frac{W(S_1^*)}{1+V(S_1^*)}+\frac{\beta W(S_2\setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}+\frac{\beta(1-\beta)W(S_2\setminus S_1^*)}{(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j)}}{1-\frac{\beta}{1+V(S_1^*)+\beta V(S_2\setminus S_1^*)+(1-\beta)v_j}}=U_j.$$

Based on the optimality of S_1^*, we also know that moving product k out of S_1^* will at least not increase the total expected revenue, which can be written as:

$$\frac{W(S_1^*)-r_kv_k}{1+V(S_1^*)-v_k}+\frac{\beta(W(S_2\setminus S_1^*)+r_kv_k)}{(1+V(S_1^*)-v_k)(1+V(S_1^*)-v_k+\beta(V(S_2\setminus S_1^*)+v_k))}$$
$$\le\frac{W(S_1^*)}{1+V(S_1^*)}+\frac{\beta W(S_2\setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}.$$

With some algebraic calculations, the above inequality can be rewritten as:

$$r_k\ge\frac{\frac{W(S_1^*)}{1+V(S_1^*)}+\frac{\beta W(S_2\setminus S_1^*)}{(1+V(S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))}+\frac{\beta(1-\beta)W(S_2\setminus S_1^*)}{(1+V(S_1^*)+\beta V(S_2\setminus S_1^*))(1+V(S_1^*)+\beta V(S_2\setminus S_1^*)-(1-\beta)v_k)}}{1-\frac{\beta}{1+V(S_1^*)+\beta V(S_2\setminus S_1^*)-(1-\beta)v_k}}=L_k.$$

Therefore, we have the following relationship:

$$L_k\le r_k<r_j\le U_j.$$

However, checking the expressions of U_j and L_k, we find that $U_j\le L_k$, which comes to a contradiction. ■

Using the result of Lemma B.2, we prove Proposition B.2. We need the following lemma.

LEMMA B.4. $(\star\star\star)$ *For $x\ge 0$, we define a function $f(x)$ as follows:*

$$f(x)=\sum_{i=1}^{m}\frac{a_i+b_ix}{d_i+x}.$$

If $a_i,b_i>0$ for all $i\in[m]$, $\frac{a_1}{b_1}\ge\cdots\ge\frac{a_m}{b_m}$ and $0<d_1\le\cdots\le d_m$, then $f(x)$ is quasi-convex in x for $x\ge 0$.

We then prove Proposition B.2 as follows.

Proof of Proposition B.2: Under the given conditions, denote the optimal solution to the SQR problem as $\boldsymbol{S^*}=(S_1^*,S_2^*,\ldots,S_m^*)$. By Lemma B.1, we must have $S_1^*\subseteq S_2^*\subseteq\cdots\subseteq S_m^*$. If S_m^* is revenue-ordered, then $S_1^*,S_2^*,\ldots,S_{m-1}^*$ are all revenue-ordered by Lemma B.2 and the proof is completed. Therefore, we only need to show that S_m^* is indeed revenue-ordered. We prove it by contradiction. Suppose S_m^* is not revenue-ordered, we then let k be the smallest index such that $k\notin S_m^*$. We denote $S_{m,l}^*=\{j|j<k,j\in S_m^*\}$ and $S_{m,r}^*=\{j|j>k,j\in S_m^*\}$. We consider the following two cases:

Case 1: $S_{m,l}^*\subseteq S_{m-1}^*$

Under the optimal solution $\boldsymbol{S^*}=(S_1^*,S_2^*,\ldots,S_m^*)$, the total expected revenue is written as follows:

$$R(\boldsymbol{S^*})=\sum_{i=1}^{m-1}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_iW(S_{m-1}^*\setminus S_i^*)+\beta_iW(S_m^*\setminus S_{m-1}^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_iV(S_{m-1}^*\setminus S_i^*)+\beta_iV(S_m^*\setminus S_{m-1}^*))}\right]$$
$$+\lambda_m\frac{W(S_{m-1}^*)+W(S_m^*\setminus S_{m-1}^*)}{1+V(S_{m-1}^*)+V(S_m^*\setminus S_{m-1}^*)}.$$

We consider a feasible solution $\boldsymbol{S_1} = (S_1^*, \ldots, S_{m-1}^*, S_m^* \cup \{k\})$ with its total expected revenue given as follows:

$$R(\boldsymbol{S_1}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{\beta_i W(S_{m-1}^* \setminus S_i^*) + \beta_i W(S_m^* \setminus S_{m-1}^*) + \beta_i r_k v_k}{(1+V(S_i^*))(1+V(S_i^*) + \beta_i V(S_{m-1}^* \setminus S_i^*) + \beta_i V(S_m^* \setminus S_{m-1}^*) + \beta_i v_k)} \right] + \lambda_m \frac{W(S_{m-1}^*) + W(S_m^* \setminus S_{m-1}^*) + r_k v_k}{1 + V(S_{m-1}^*) + V(S_m^* \setminus S_{m-1}^*) + v_k}.$$

We consider another feasible solution $\boldsymbol{S_2} = (S_1^*, \ldots, S_{m-1}^*, S_{m-1}^*)$ with its total expected revenue given as follows:

$$R(\boldsymbol{S_2}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{\beta_i W(S_{m-1}^* \setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*) + \beta_i V(S_{m-1}^* \setminus S_i^*))} \right] + \lambda_m \frac{W(S_{m-1}^*)}{1+V(S_{m-1}^*)}.$$

We aim to show the following relationship:

$$R(\boldsymbol{S^*}) \le \max\{R(\boldsymbol{S_1}), R(\boldsymbol{S_2})\}. \tag{B.4}$$

To prove (B.4), we first denote $Z^* = R(\boldsymbol{S^*}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S_i^*)}{1+V(S_i^*)}$, $Z_1 = R(\boldsymbol{S_1}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S_i^*)}{1+V(S_i^*)}$ and $Z_2 = R(\boldsymbol{S_2}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S_i^*)}{1+V(S_i^*)}$. Then, it is equivalent to prove $Z^* \le \max\{Z_1, Z_2\}$. For notation brevity, we define $R_i = \frac{W(S_{m-1}^* \setminus S_i^*)}{V(S_{m-1}^* \setminus S_i^*)}$ and $V_i = V(S_{m-1}^* \setminus S_i^*)$ for $i = 1, \ldots, m-1$ if $S_{m-1}^* \setminus S_i^* \ne \emptyset$, while define $R_i = 0$ and $V_i = 0$ otherwise. We also define $r = \frac{W(S_m^* \setminus S_{m-1}^*)}{V(S_m^* \setminus S_{m-1}^*)}$ and $v = V(S_m^* \setminus S_{m-1}^*)$ if $S_m^* \setminus S_{m-1}^* \ne \emptyset$, while define $r = 0$ and $v = 0$ otherwise. Since $S_{m,l}^* \subseteq S_{m-1}^*$, we must then have $r \le r_k$. Note that we can express Z^*, Z_1 and Z_2 as

$$Z^* = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v}{(1+V(S_i^*))(1+V(S_i^*) + \beta_i V_i + \beta_i v)} + \lambda_m \frac{W(S_{m-1}^*) + rv}{1+V(S_{m-1}^*) + v}.$$

$$Z_1 = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v + \beta_i r_k v_k}{(1+V(S_i^*))(1+V(S_i^*) + \beta_i V_i + \beta_i v + \beta_i v_k)} + \lambda_m \frac{W(S_{m-1}^*) + rv + r_k v_k}{1+V(S_{m-1}^*) + v + v_k}.$$

$$Z_2 = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i}{(1+V(S_i^*))(1+V(S_i^*) + \beta_i V_i)} + \lambda_m \frac{W(S_{m-1}^*)}{1+V(S_{m-1}^*)}.$$

Since $r \le r_k$, we have the following:

$$Z_1 \ge \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r(v + v_k)}{(1+V(S_i^*))(1+V(S_i^*) + \beta_i V_i + \beta_i (v + v_k))} + \lambda_m \frac{W(S_{m-1}^*) + r(v+v_k)}{1+V(S_{m-1}^*) + (v+v_k)}.$$

We then define a function $f(x)$ for $x \ge 0$ as follows:

$$f(x) = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r x}{(1+V(S_i^*))(1+V(S_i^*) + \beta_i V_i + \beta_i x)} + \lambda_m \frac{W(S_{m-1}^*) + rx}{1+V(S_{m-1}^*) + x}.$$

We can now apply the result of Lemma B.4. Specifically, we let $a_i = \frac{\lambda_i R_i V_i}{1+V(S_i^*)}$, $b_i = \frac{\lambda_i r}{1+V(S_i^*)}$ and $d_i = \frac{1+V(S_i^*)+\beta_i V_i}{\beta_i}$ for $i = 1, \ldots, m-1$. We also let $a_m = \lambda_m W(S_{m-1}^*)$, $b_m = \lambda_m r$ and $d_m = 1 + V(S_{m-1}^*)$. Note that $\frac{a_i}{b_i} = \frac{R_i V_i}{r} = \frac{W(S_{m-1}^* \setminus S_i^*)}{r}$ for $i = 1, \ldots, m-1$ and $\frac{a_m}{b_m} = \frac{W(S_{m-1}^*)}{r}$. Thus, we have $\frac{a_m}{b_m} \ge \frac{a_1}{b_1} \ge \frac{a_2}{b_2} \ge \cdots \ge \frac{a_{m-1}}{b_{m-1}}$. Also, note that $d_i = \frac{1-\beta_i}{\beta_i}(1+V(S_i^*)) + 1 + V(S_{m-1}^*)$ for $i = 1, \ldots, m-1$. Thus, we also have $d_m \le d_1 \le d_2 \le \cdots \le d_{m-1}$. Based on these two conditions, we know that the function $f(x)$ must be quasi-convex for $x \ge 0$ and thus we have the following:

$$\max\{Z_1, Z_2\} \ge \max\{f(v+v_k), f(0)\} \ge f(v) = Z^*.$$

Thus, inequality (B.4) is proved. □

Case 2: $S^*_{m-1} \subseteq S^*_{m,l}$

Under the optimal solution $\boldsymbol{S^*} = (S^*_1, S^*_2, \dots, S^*_m)$, the total expected revenue is written as follows:

$$R(\boldsymbol{S^*}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S^*_i)}{1+V(S^*_i)} + \frac{\beta_i W(S^*_{m,l} \setminus S^*_i) + \beta_i W(S^*_{m,r})}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V(S^*_{m,l} \setminus S^*_i) + \beta_i V(S^*_{m,r}))}\right] + \lambda_m \frac{W(S^*_{m,l}) + W(S^*_{m,r})}{1+V(S^*_{m,l}) + V(S^*_{m,r})}.$$

We consider a feasible solution $\boldsymbol{S_1} = (S^*_1, \dots, S^*_{m-1}, S^*_m \cup \{k\})$ with its total expected revenue given as follows:

$$\begin{aligned} R(\boldsymbol{S_1}) = &\sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S^*_i)}{1+V(S^*_i)} + \frac{\beta_i W(S^*_{m,l} \setminus S^*_i) + \beta_i W(S^*_{m,r}) + \beta_i r_k v_k}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V(S^*_{m,l} \setminus S^*_i) + \beta_i V(S^*_{m,r}) + \beta_i v_k)}\right] \\ &+ \lambda_m \frac{W(S^*_{m,l}) + W(S^*_{m,r}) + r_k v_k}{1+V(S^*_{m,l}) + V(S^*_{m,r}) + v_k}. \end{aligned}$$

We consider another feasible solution $\boldsymbol{S_2} = (S^*_1, \dots, S^*_{m-1}, S^*_{m,l})$ with its total expected revenue given as follows:

$$R(\boldsymbol{S_2}) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S^*_i)}{1+V(S^*_i)} + \frac{\beta_i W(S^*_{m,l} \setminus S^*_i)}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V(S^*_{m,l} \setminus S^*_i))}\right] + \lambda_m \frac{W(S^*_{m,l})}{1+V(S^*_{m,l})}.$$

We aim to show the following relationship:

$$R(\boldsymbol{S^*}) \le \max\{R(\boldsymbol{S_1}), R(\boldsymbol{S_2})\}. \tag{B.5}$$

To prove (B.5), we first denote $Z^* = R(\boldsymbol{S^*}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S^*_i)}{1+V(S^*_i)}$, $Z_1 = R(\boldsymbol{S_1}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S^*_i)}{1+V(S^*_i)}$ and $Z_2 = R(\boldsymbol{S_2}) - \sum_{i=1}^{m-1} \lambda_i \frac{W(S^*_i)}{1+V(S^*_i)}$. Then, it is equivalent to prove $Z^* \le \max\{Z_1, Z_2\}$. For notation brevity, we define $R_i = \frac{W(S^*_{m,l} \setminus S^*_i)}{V(S^*_{m,l} \setminus S^*_i)}$ and $V_i = V(S^*_{m,l} \setminus S^*_i)$ for $i = 1, \dots, m-1$ if $S^*_{m,l} \setminus S^*_i \ne \emptyset$, while define $R_i = 0$ and $V_i = 0$ otherwise. We also define $r = \frac{W(S^*_{m,r})}{V(S^*_{m,r})}$ and $v = V(S^*_{m,r})$ if $S^*_{m,r} \ne \emptyset$, while define $r = 0$ and $v = 0$ otherwise. Based on the definition, we must then have $r \le r_k$. Note that we can express Z^*, Z_1 and Z_2 as

$$\begin{aligned} Z^* &= \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i v)} + \lambda_m \frac{W(S^*_{m,l}) + rv}{1+V(S^*_{m,l}) + v}. \\ Z_1 &= \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r v + \beta_i r_k v_k}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i v + \beta_i v_k)} + \lambda_m \frac{W(S^*_{m,l}) + rv + r_k v_k}{1+V(S^*_{m,l}) + v + v_k}. \\ Z_2 &= \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i)} + \lambda_m \frac{W(S^*_{m,l})}{1+V(S^*_{m,l})}. \end{aligned}$$

Since $r \le r_k$, we have the following:

$$Z_1 \ge \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r(v + v_k)}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i (v + v_k))} + \lambda_m \frac{W(S^*_{m,l}) + r(v+v_k)}{1+V(S^*_{m,l}) + (v + v_k)}.$$

We then define a function $f(x)$ for $x \ge 0$ as follows:

$$f(x) = \sum_{i=1}^{m-1} \lambda_i \frac{\beta_i R_i V_i + \beta_i r x}{(1+V(S^*_i))(1+V(S^*_i)+\beta_i V_i + \beta_i x)} + \lambda_m \frac{W(S^*_{m,l}) + rx}{1+V(S^*_{m,l}) + x}.$$

We can now apply the result of Lemma B.4. Specifically, we let $a_i = \frac{\lambda_i R_i V_i}{1+V(S^*_i)}$, $b_i = \frac{\lambda_i r}{1+V(S^*_i)}$ and $d_i = \frac{1+V(S^*_i)+\beta_i V_i}{\beta_i}$ for $i = 1, \dots, m-1$. We also let $a_m = \lambda_m W(S^*_{m,l})$, $b_m = \lambda_m r$ and $d_m = 1 + V(S^*_{m,l})$. Note that $\frac{a_i}{b_i} = \frac{R_i V_i}{r} = \frac{W(S^*_{m,l} \setminus S^*_i)}{r}$ for $i = 1, \dots, m-1$ and $\frac{a_m}{b_m} = \frac{W(S^*_{m,l})}{r}$. Thus, we have $\frac{a_m}{b_m} \ge \frac{a_1}{b_1} \ge \frac{a_2}{b_2} \ge \dots \ge \frac{a_{m-1}}{b_{m-1}}$. Also, note that $d_i = \frac{1-\beta_i}{\beta_i}(1+V(S^*_i)) + 1 + V(S^*_{m,l})$ for $i = 1, \dots, m-1$. Thus, we also have $d_m \le d_1 \le d_2 \le \dots \le d_{m-1}$.

Based on these two conditions, we know that the function $f(x)$ must be quasi-convex for $x \geq 0$ and thus we have the following:

$$\max\{Z_1, Z_2\} \geq \max\{f(v+v_k), f(0)\} \geq f(v) = Z^*.$$

Thus, inequality (B.5) is proved. □

For the optimal solution $\boldsymbol{S^*} = (S_1^*, S_2^*, \ldots, S_m^*)$, suppose the second case happens, i.e., $S_{m-1}^* \subseteq S_{m,l}^*$, we then have either $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_m^* \cup \{k\})$ or $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_{m,l}^*)$. If the latter case happens, the proof is completed since $S_{m,l}^*$ is already revenue-ordered. If the former case happens, the proof is also completed if $S_m^* \cup \{k\}$ is revenue-ordered. Otherwise, we let $S_m^* \leftarrow S_m^* \cup \{k\}$ and follow the previous argument to analyze the current optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$, which will finally lead to a conclusion that there exists a revenue-ordered assortment S_m^* such that it is optimal.

Now, suppose the first case happens, i.e., $S_{m,l}^* \subseteq S_{m-1}^*$, we then have either $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_m^* \cup \{k\})$ or $R(\boldsymbol{S^*}) \leq R(S_1^*, \ldots, S_{m-1}^*, S_{m-1}^*)$. If the former case happens, the proof is completed if $S_m^* \cup \{k\}$ is revenue-ordered. Otherwise, we let $S_m^* \leftarrow S_m^* \cup \{k\}$. We then fix S_m^* and update S_i^* for $i \in \{1, \ldots, m-1\}$ by solving problem (8) with $S_2 = S_m^*$ and $\beta = \beta_i$. Note that the total expected revenue will not decrease after this operation and the updated $S_1^*, \ldots, S_{m-1}^*$ should all be revenue-ordered within S_m^*. We can then follow the previous argument to analyze the updated solution. If the latter case happens, we let $S_m^* \leftarrow S_{m-1}^*$. We can also follow the previous argument to analyze the updated solution. To be specific, we now consider two cases: $S_{m,l}^* \subseteq S_{m-2}^*$ and $S_{m-2}^* \subseteq S_{m,l}^*$. If the case $S_{m,l}^* \subseteq S_{m-2}^*$ happens, for instance, we then have either $R(S_1^*, S_2^*, \ldots, S_{m-1}^*, S_m^*) \leq R(S_1^*, S_2^*, \ldots, S_{m-2}^*, S_{m-1}^* \cup \{k\}, S_m^* \cup \{k\})$ or $R(S_1^*, S_2^*, \ldots, S_{m-1}^*, S_m^*) \leq R(S_1^*, S_2^*, \ldots, S_{m-2}^*, S_{m-2}^*, S_{m-2}^*)$. If we continue the analysis, one can show that it will finally lead to a conclusion that either 1) there exists a revenue-ordered assortment S_m^* such that it is optimal or 2) the initial optimal solution $\boldsymbol{S^*}$ is dominated by a solution $(S', S', \ldots, S')$ where S' is not revenue-ordered. However, in the latter case, such a solution is further dominated by the solution $(\tilde{S}, \tilde{S}, \ldots, \tilde{S})$ where $\tilde{S}$ is the single-store optimal assortment, which is revenue-ordered. Therefore, the proof is completed. ■

B.6. Proof in Step 3

Finally, we give a proof to Theorem 4 as follows.

Proof: Based on Proposition B.2, we have shown that under the given conditions, there exists an optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$ to the SQR problem such that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_m^*$ and $S_1^*, S_2^*, \ldots, S_m^*$ are all revenue-ordered. Therefore, to prove Theorem 4, we only need to show that $S_1^* \subseteq S_2^* \subseteq \cdots \subseteq S_{m-1}^* \subseteq \tilde{S} \subseteq S_m^*$. We consider the following two cases:

Case 1: $S_1^* \subseteq \cdots \subseteq S_k^* \subseteq \tilde{S} \subset S_{k+1}^* \subseteq \cdots \subseteq S_m^*$ for $k \in \{0, \ldots, m-2\}$

Under the optimal solution $(S_1^*, S_2^*, \ldots, S_m^*)$, the total expected revenue is written as follows:

$$R(S_1^*, S_2^*, \ldots, S_m^*) = \sum_{i=1}^{m-1} \lambda_i \left[\frac{W(S_i^*)}{1+V(S_i^*)} + \frac{\beta_i W(S_m^* \setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^* \setminus S_i^*))} \right] + \lambda_m \frac{W(S_m^*)}{1+V(S_m^*)}.$$

We then consider a feasible solution $(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*)$ with its total expected revenue given as follows:

$$\begin{aligned} R(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*) = &\sum_{i=1}^{k}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_i W(S_m^*\setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*))}\right] \\ &+\sum_{i=k+1}^{m-1}\lambda_i\left[\frac{W(\tilde{S})}{1+V(\tilde{S})}+\frac{\beta_i W(S_m^*\setminus \tilde{S})}{(1+V(\tilde{S}))(1+V(\tilde{S})+\beta_i V(S_m^*\setminus \tilde{S}))}\right]+\lambda_m\frac{W(S_m^*)}{1+V(S_m^*)}. \end{aligned}$$

Due to the fact that $\tilde{S}$ is the optimal assortment in the single-store problem, we must have:

$$\frac{W(S_i^*)}{1+V(S_i^*)}\le\frac{W(\tilde{S})}{1+V(\tilde{S})}. \tag{B.6}$$

Due to the fact that $|\tilde{S}|<|S_i^*|\le|S_m^*|$ for $i=k+1,\dots,m-1$, we must have:

$$V(S_i^*)\ge V(\tilde{S}) \quad\text{and}\quad \beta_i W(S_m^*\setminus S_i^*)\le\beta_i W(S_m^*\setminus\tilde{S}). \tag{B.7}$$

Additionally, notice that the following relationship holds: $1+V(S_i^*)+V(S_m^*\setminus S_i^*)=1+V(\tilde{S})+V(S_m^*\setminus\tilde{S})$. Since we also have $(1-\beta_i)V(S_m^*\setminus S_i^*)\le(1-\beta_i)V(S_m^*\setminus\tilde{S})$ for $i=k+1,\dots,m-1$, we then have:

$$1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*)\ge 1+V(\tilde{S})+\beta_i V(S_m^*\setminus\tilde{S}). \tag{B.8}$$

Therefore, combining (B.6), (B.7) and (B.8), we can show that $R(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*)\ge R(S_1^*,S_2^*,\dots,S_m^*)$, which implies that $(S_1^*,\dots,S_k^*,\tilde{S},\dots,\tilde{S},S_m^*)$ should also be an optimal solution.

Case 2: $S_1^*\subseteq S_2^*\subseteq\cdots\subseteq S_{m-1}^*\subseteq S_m^*\subseteq\tilde{S}$

Under the optimal solution $(S_1^*,S_2^*,\dots,S_m^*)$, the total expected revenue is written as follows:

$$R(S_1^*,S_2^*,\dots,S_m^*)=\sum_{i=1}^{m-1}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_i W(S_m^*\setminus S_i^*)}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*))}\right]+\lambda_m\frac{W(S_m^*)}{1+V(S_m^*)}.$$

We denote $j=|S_m^*|+1$. Since $|\tilde{S}|>|S_m^*|$, we must have $j\in\tilde{S}$. We then consider a feasible solution $(S_1^*,\dots,S_{m-1}^*,S_m^*\cup\{j\})$ with its total expected revenue expressed as follows:

$$R(S_1^*,\dots,S_{m-1}^*,S_m^*\cup\{j\})=\sum_{i=1}^{m-1}\lambda_i\left[\frac{W(S_i^*)}{1+V(S_i^*)}+\frac{\beta_i W(S_m^*\setminus S_i^*)+\beta_i r_j v_j}{(1+V(S_i^*))(1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*)+\beta_i v_j)}\right]+\lambda_m\frac{W(S_m^*)+r_j v_j}{1+V(S_m^*)+v_j}.$$

Then, since $\tilde{S}$ is the optimal assortment of the single-store problem and $j\in\tilde{S}$, we must have:

$$r_j\ge\frac{W(\tilde{S})}{1+V(\tilde{S})}\ge\frac{W(S_m^*)}{1+V(S_m^*)},$$

which implies the following inequality:

$$\frac{W(S_m^*)+r_j v_j}{1+V(S_m^*)+v_j}\ge\frac{W(S_m^*)}{1+V(S_m^*)}. \tag{B.9}$$

Notice that we also have the following relationship:

$$r_j\ge\frac{W(\tilde{S})}{1+V(\tilde{S})}\ge\frac{W(S_m^*\setminus S_i^*)}{1+V(S_m^*\setminus S_1^*)}\ge\frac{W(S_m^*\setminus S_i^*)}{\frac{1}{\beta_i}\cdot(1+V(S_i^*))+V(S_m^*\setminus S_i^*)}=\frac{\beta_i W(S_m^*\setminus S_i^*)}{1+V(S_i^*)+\beta_i V(S_m^*\setminus S_i^*)}. \tag{B.10}$$

Therefore, combining (B.9) and (B.10), we can show that $R(S_1^*,\dots,S_{m-1}^*,S_m^*\cup\{j\})\ge R(S_1^*,S_2^*,\dots,S_m^*)$, which implies that $(S_1^*,S_2^*,\dots,S_m^*)$ should also be an optimal solution. We can then incrementally add products to S_m^* until $S_m^*=\tilde{S}$ and conclude that $(S_1^*,\dots,S_{m-1}^*,\tilde{S})$ should also be an optimal solution. ■

B.7. Proof of Proposition 3

Proof: Our proof is mainly adapted from Rusmevichientong et al. (2014). We show that any instance of the partition problem can be reduced to an instance of the assortment feasibility problem constructed below. We first formally define the partition problem as follows.

PARTITION PROBLEM:
INPUTS: Set of items indexed by $1, 2, \dots, n$ and the size $w_i \in \mathbb{Z}_+$ associated with each item i.
QUESTION: Is there a subset $S \subseteq \{1, \dots, n\}$ such $\sum_{i\in S} w_i = \sum_{i\in\{1,\dots,n\}\setminus S} w_i$?

We construct an instance of the SQR problem as follows. Let $T = \frac{1}{2}\sum_{i=1}^n w_i$. There are three stores with $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1+4T}{1+8T}, \frac{4T}{1+8T}, 0)$. That is, store 3 does not contribute to the total revenue but stores 1 and 2 can view products from store 3. We consider $n+2$ products and label them as $\{1, \dots, n, n+1, n+2\}$. The revenues of these products are set as:

$$r_i = \begin{cases} (1+8T)(3+4T) & \text{if } i = 1, \dots, n \\ 4(1+8T)(3+4T) & \text{if } i = n+1 \\ 100(1+8T)(3+4T) & \text{if } i = n+2. \end{cases}$$

We first set $c_{12}^k = \infty$ and $c_{21}^k = \infty$ for $k = 1, \dots, n+2$, i.e., store 1(2) can not see any product from store 2(1). For product $n+2$, we let $v_{n+2} = 1$ and $c_{13}^{n+2} = c_{23}^{n+2} = \infty$. For product $n+1$, we let $v_{n+1} = \frac{1}{2}$ and $c_{13}^{n+1} = 0$ and $\exp(-c_{23}^{n+1}) = \frac{2}{7}$. For product $k = 1, \dots, n$, we let $v_k = 4w_i$ and $\exp(-c_{13}^k) = \frac{1}{2}$ and $\exp(-c_{23}^k) = \frac{1+4T}{7T} \in (0, 1)$.

Denote the optimal solution to the above instance as (S_1^*, S_2^*, S_3^*) and $r = (1+8T)(3+4T)$ for convenience. We first argue that $\{n+2\} \in S_1^*$. If not, since $c_{13}^{n+2} = \infty$, adding product $n+2$ to S_1^* must increase the revenue of store 1 and does not affect the other stores' revenue. We then argue that $S_1^* = \{n+2\}$, i.e., all other products will not be included in S_1^*. Clearly, when $S_1^* = \{n+2\}$, the revenue of store 1 is at least $100r/(1+1) = 50r$. If other products are added, the revenue of store 1 is at most $(100r + 4r \cdot 1/2)/(1+1+1/2) + 4r < 50r$. We also note that the products chosen in S_1^* will not affect the revenue of store 2 since $c_{21}^k = \infty$ for $k = 1, \dots, n+2$. Therefore, we have shown that $S_1^* = \{n+2\}$. Due to the same reason, we have that $S_2^* = \{n+2\}$. We then argue that $\{n+1\} \in S_3^*$ since adding it will always increase the second-stage revenue of store 1 and store 2. Therefore, the problem now becomes to decide a set $S \subseteq \{1, \dots, n\}$ to offer in S_3^*. The problem can now be equivalently represented as follows:

$$\max_{S\subseteq\{1,\dots,n\}} \quad \frac{1+4T}{1+8T} \cdot \frac{2r + \sum_{i\in S} 2w_i r}{1 + \frac{1}{2} + \sum_{i\in S} 2w_i} + \frac{4T}{1+8T} \cdot \frac{4r \cdot \frac{1}{7} + \sum_{i\in S} \frac{4(1+4T)w_i}{7T} r}{1 + \frac{1}{7} + \sum_{i\in S} \frac{4(1+4T)w_i}{7T}}. \tag{B.11}$$

One can verify that based on our construction, (B.11) is equivalent to the formulation of the assortment feasibility problem in Rusmevichientong et al. (2014). Thus, the SQR problem under construction is NP-hard. We should comment that although the above construction requires that the revenues of products $1, \dots, n$ are the same, it is possible to perturb those revenues a little bit but still guarantee that the optimal solution to problem (B.11) is attained by $\sum_{i\in S} w_i = T$ since we assume $w_i \in \mathbb{Z}_+$. ■

B.8. Total revenue decreases when the utility discount increases

Claim: $R_Q^*(c)$ is monotonically decreasing in c.

Proof: From the formulation of problem (6), it is easy to see that when c decreases, $\beta = \exp(-c)$ increases, and thus the optimal objective value increases. ■

B.9. Proof of Proposition 5

Proof: We first prove Proposition 5(a). It is easy to see that $R_Q^*(\lambda)$ is a symmetric function with respect to $\lambda = 0.5$. The fact that it is piecewise linear follows from the observation that for each feasible assortment (S_1, S_2), the total expected revenue $R(S_1, S_2)$ can be viewed as a linear function in λ. Then, $R_Q^*(\lambda)$ is the maximization of a finite number of linear functions, which is convex by definition. Due to symmetry, $R_Q^*(\lambda)$ must be decreasing when $\lambda \in [0, 0.5]$ and increasing when $\lambda \in [0.5, 1]$.

We then prove Proposition 5(b). Due to symmetry, it suffices to prove that $|S_1^*|$ is weakly decreasing in λ when $\lambda \in [0, 0.5)$. We prove this claim by contradiction. For $0 \le \lambda_l < \lambda_r \le 0.5$, we denote (S_1^l, S_2^l) as the optimal solution to problem (6) when $\lambda = \lambda_l$ and (S_1^r, S_2^r) as the optimal solution to problem (6) when $\lambda = \lambda_r$. Suppose we now have $|S_1^l| < |S_1^r|$. Due to Theorem 2, we know that S_1^l, S_2^l, S_1^r and S_2^r should be revenue-ordered. Also, we must have $S_2^l \subseteq \tilde{S} \subseteq S_1^l$ and $S_2^r \subseteq \tilde{S} \subseteq S_1^r$, where $\tilde{S}$ is the optimal assortment in the single-store problem. We then express the total expected revenue when $\lambda = \lambda_l$ as follows:

$$\begin{aligned}R(S_1^l, S_2^l) &= \lambda_l \frac{W(S_1^l)}{1+V(S_1^l)} + (1-\lambda_l)\frac{W(S_2^l)}{1+V(S_2^l)} + (1-\lambda_l)\frac{\beta W(S_1^l \setminus S_2^l)}{(1+V(S_2^l))(1+V(S_2^l)+\beta V(S_1^l \setminus S_2^l))} \\ &= \lambda_l R_1(S_1^l, S_2^l) + (1-\lambda_l) R_2(S_1^l, S_2^l) = R_2(S_1^l, S_2^l) + \lambda_l (R_1(S_1^l, S_2^l) - R_2(S_1^l, S_2^l)),\end{aligned}$$

where we denote $R_1(S_1^l, S_2^l)$ ($R_2(S_1^l, S_2^l)$, resp.) as the expected revenue of store 1 (store 2, resp.) when the offered assortments are (S_1^l, S_2^l) (assuming arrival rate is 1). Similarly, we can express the total expected revenue when $\lambda = \lambda_r$ as follows:

$$R(S_1^r, S_2^r) = R_2(S_1^r, S_2^r) + \lambda_r (R_1(S_1^r, S_2^r) - R_2(S_1^r, S_2^r)).$$

Since $\tilde{S} \subseteq S_1^l$, $\tilde{S} \subseteq S_1^r$ and $|S_1^l| < |S_1^r|$, we have $R_1(S_1^l, S_2^l) \ge R_1(S_1^r, S_2^r)$. Then, we must have $R_2(S_1^l, S_2^l) \le R_2(S_1^r, S_2^r)$, since otherwise the optimal solution (S_1^r, S_2^r) when $\lambda = \lambda_r$ is dominated by the solution (S_1^l, S_2^l). We also note that $R_1(S_1^l, S_2^l) = R_1(S_1^r, S_2^r)$ and $R_2(S_1^l, S_2^l) = R_2(S_1^r, S_2^r)$ cannot hold at the same time since then the optimal solution at λ_r would be (S_1^l, S_2^l) based on our definition since $|S_1^l| < |S_1^r|$. However, this implies that the slope of the function $R_Q^*(\lambda)$ is smaller at the point $\lambda = \lambda_r$ (which is $R_1(S_1^r, S_2^r) - R_2(S_1^r, S_2^r)$) than at the point $\lambda = \lambda_l$ (which is $(R_1(S_1^l, S_2^l) - R_2(S_1^l, S_2^l))$), which is contradictory to the first argument. This completes the proof. ■

B.10. Proof of Proposition 4

Proof: The proof is directly followed by Lemma B.2 and the proof in Lemma B.1. ■

Appendix C: Proofs for Section 6

C.1. Proof of Theorem 5

We first prove Theorem 5(a) as follows.

Proof of Theorem 5(a): Denote the optimal objective of the SMR and the SQR problem as R_M^* and R_Q^*, respectively. By Theorem 1, we have $R_M^* \le R^*$, where R^* is the optimal revenue of the single-store problem. Note that we also have $R_Q^* \ge R^*$ since we can always offer $(\tilde{S}, \tilde{S}, \ldots, \tilde{S})$ under the SQR strategy. Therefore, we have $R_Q^* \ge R_M^*$. ■

To prove Theorem 5(b), we need the following Proposition.

PROPOSITION C.1. ($\star\star\star$) *When $\boldsymbol{S}$ is given, the consumer surplus under the SMR model follows:*

$$CS^M(\boldsymbol{S}) = \sum_{i=1}^m \lambda_i \log\left(1 + \sum_{k\in S_i} v_k + \sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)\right).$$

The consumer surplus under the SQR model follows:

$$\begin{aligned} CS^Q(\boldsymbol{S}) = &\sum_{i=1}^m \lambda_i \frac{\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}{1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)} + \sum_{i=1}^m \lambda_i \frac{V(S_i)}{1+V(S_i)} \log(1+V(S_i)) \\ &+ \sum_{i=1}^m \lambda_i \frac{\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)}{(1+V(S_i))(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))} \log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)). \end{aligned}$$

We then prove Theorem 5(b) as follows.

Proof of Theorem 5(b): The result simply follows by the inequalities below:

$$\begin{aligned} CS^Q(\boldsymbol{S}) = &\sum_{i=1}^m \lambda_i \frac{\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}{1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)} + \sum_{i=1}^m \lambda_i \frac{V(S_i)}{1+V(S_i)} \log(1+V(S_i)) \\ &+ \sum_{i=1}^m \lambda_i \frac{\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)}{(1+V(S_i))(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))} \log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)) \\ \le &\sum_{i=1}^m \lambda_i \frac{\log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))}{1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)} + \sum_{i=1}^m \lambda_i \frac{V(S_i)}{1+V(S_i)} \log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)) \\ &+ \sum_{i=1}^m \lambda_i \frac{\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)}{(1+V(S_i))(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right))} \log(1+V(S_i)+\sum_{k\in\bar{S}\setminus S_i} v_k \exp\left(-c_i^k(\boldsymbol{S})\right)) = CS^M(\boldsymbol{S}). \end{aligned}$$

Appendix D: Proofs for Section 7

D.1. Proof of Proposition 6

Proof: To facilitate our analysis, we define the seller's assortment optimization problem for the single store i under the heterogeneous case as

$$R_i^* = \max_S \quad \frac{\sum_{k\in S} r_k v_{ik}}{1+\sum_{k\in S} v_{ik}}. \tag{D.1}$$

We also denote the optimal objective value of the SQR-heter problem and the SMR-heter problem as R_Q^* and R_M^*, respectively. Suppose the optimal solution to the SMR-heter problem is $\boldsymbol{S^*} = (S_1^*, \ldots, S_m^*)$, using the same analysis in Theorem 1, we can show that the revenue for each store i with $i \in [m]$ under $\boldsymbol{S^*}$ is upper bounded by the single-store optimal revenue, namely, R_i^*. Thus, we have $R_M^* \le \sum_{i=1}^n \lambda_i R_i^*$. It is also easy to see that $R_Q^* \ge \sum_{i=1}^n \lambda_i R_i^*$ always holds, which establishes the desired result. ■

D.2. Proof of Proposition 7

Proof: To prove the approximation ratio $\frac{1}{2}$, we need the following lemma.

LEMMA D.1. *Given an MNL model, suppose the revenues and the attraction values of the products are denoted as $(r_1, \ldots, r_n)$ and $(v_1, \ldots, v_n)$. Denote the optimal objective value of the following problem as R:*

$$\max_S \quad \frac{\sum_{i\in S} r_i v_i}{1+\sum_{i\in S} v_i}.$$

Given a vector $(\epsilon_1, \dots, \epsilon_n)$ *where* $0 < \epsilon_i < 1$ *for each* $i = 1, \dots, n$*, we also denote the optimal objective value of the following problem as* R'*:*

$$\max_{S} \quad \frac{\sum_{i\in S} r_i \epsilon_i v_i}{1+\sum_{i\in S}\epsilon_i v_i}.$$

We then have $R \geq R'$*.*

Proof of Lemma D.1: It suffices to prove the result by showing that if we only increase the attraction value for product i from $\epsilon_i v_i$ to v_i and keep the attraction values of other products the same, the optimal expected revenue would at least not decrease. To that end, denote the optimal solution to the problem with attraction value $\epsilon_i v_i$ as S'. Then, if $i \notin S'$, the result has been proved since S' generates the same objective value for the problem with v_i. Otherwise, if $i \in S'$, we must have r_i greater than R'. Then, increasing the attraction value for product i from $\epsilon_i v_i$ to v_i will increase the optimal expected revenue. ■

Proof of the approximation ratio $\frac{1}{2}$: Denote the optimal solution to the SQR problem as $\boldsymbol{S}^* = (S_1^*, \dots, S_m^*)$. We also let $\bar{S}^* = S_1^* \cup S_2^* \cup \dots \cup S_m^*$. We then have

$$\begin{aligned}
R_Q^H = R(\tilde{S}_1, \dots, \tilde{S}_m) &\geq \sum_{i=1}^{m} \lambda_i \cdot \frac{\sum_{k\in\tilde{S}_i} r_k v_{ik}}{1+\sum_{k\in\tilde{S}_i} v_{ik}} \\
&\geq \sum_{i=1}^{m} \frac{\lambda_i}{2}\left[\frac{\sum_{k\in S_i^*} r_k v_{ik}}{1+\sum_{k\in S_i^*} v_{ik}} + \frac{\sum_{k\in\bar{S}^*\setminus S_i^*} r_k v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right)}{1+\sum_{k\in\bar{S}^*\setminus S_i^*} v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right)}\right] \\
&\geq \sum_{i=1}^{m} \frac{\lambda_i}{2}\left[\frac{\sum_{k\in S_i^*} r_k v_{ik}}{1+\sum_{k\in S_i^*} v_{ik}} + \frac{\sum_{k\in\bar{S}^*\setminus S_i^*} r_k v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right)}{(1+\sum_{k\in S_i^*} v_{ik})(1+\sum_{k\in S_i^*} v_{ik}+\sum_{k\in\bar{S}^*\setminus S_i^*} v_{ik}\cdot\exp\left(-c_i^k(\boldsymbol{S^*})\right))}\right] \\
&= \frac{1}{2} R_Q^*.
\end{aligned}$$

Recall that $\tilde{S}_i$ is the optimal solution to the single-store problem. Therefore, the second inequality follows by applying Lemma D.1 with $\epsilon_i = \exp(-c_i^k(\boldsymbol{S^*}))$ for $i = 1, \dots, n$. We construct the following instance to prove the tightness (similar to Gao et al. 2021, see Appendix G). Suppose there are two stores and two products are offered. The arrival rates are $\lambda_1 = 1$ and $\lambda_2 = 0$. The revenues of the two products are $1 + \frac{1}{\epsilon}$ and 1, where $\epsilon \to 0$. The attraction values of the two products are the same in the two stores with $v_1 = \epsilon$ and $v_2 = \frac{1}{\epsilon}$. We also let $c_{12}^1 = c_{12}^2 = 0$. One can verify that $\tilde{S} = \{1\}$ under this instance, and the recommendation strategy $\{\{1\},\{1\}\}$ will result in a total revenue close to 1 when $\epsilon \to 0$. However, the recommendation strategy $\{\{1\},\{2\}\}$ will result in a total revenue close to 2 when $\epsilon \to 0$. The tightness then follows from the construction. ■

D.3. Proof of Proposition 8

Proof: For any given assortment set $\boldsymbol{S}$, the expected revenue obtained by the partial recommendation (homogeneous or heterogeneous) is equal to that obtained by the full recommendation but setting some of the c_{ij}^k as infinity. Based on that, recall that in Theorem 1, we have shown that under the SMR strategy, for arbitrary c_{ij}^k (even for $c_{ij}^k = \infty$), the optimal assortment in each store is the same and should be the single-store optimal assortment $\tilde{S}$ when the valuations are homogeneous. Thus, the optimal partial recommendation strategy should be $(\tilde{S}, \tilde{S}, \dots, \tilde{S})$ for the homogeneous case. When the valuations are heterogeneous, we have shown in the proof of Proposition 6 that for arbitrary c_{ij}^k (even for $c_{ij}^k = \infty$), the optimal expected revenue has an upper bound of $\sum_{i=1}^m \lambda_i R_i^*$, where R_i^* is the single-store optimal revenue in store i. That said,

$\sum_{i=1}^{m} \lambda_i R_i^*$ also serves as an upper bound for any partial recommendation strategy. However, such an upper bound can be exactly achieved by considering a partial recommendation strategy with $(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m)$ and not recommending products from other stores (if any). Thus, the optimal partial recommendation strategy should be $(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m)$ for the heterogeneous case. ■

Appendix E: Some numerical examples ($\star\star\star$)

E.1. Example where SMR improves revenue when there is capacity constraint

E.2. Example where the optimal assortments may not be revenue-ordered for the SMR model under the omnichannel setting

E.3. Example where the optimal assortments may not be revenue-ordered for the SQR model under the omnichannel setting

E.4. Example where SQR has higher consumer surplus than SMR

E.5. Problem instances for Figure 6

Appendix F: Numerical results on the heuristic performance in Proposition 7 ($\star\star\star$)

Appendix G: The integer programming formulation for the cardinality-constrained SQR problem under universally homogeneous disutilities ($\star\star\star$)

Appendix H: Joint Assortment and Pricing Optimization ($\star\star\star$)