E-Companion to: Managing Payment Flexibility in Rent-to-Own Contracts for Off-Grid Energy Products

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A. Notations

We present the table below summarizing the notations used throughout the paper.

Notation	Description
$v > 1$	per-period usage value
$c = 1$	payment per installment
$\beta \in [0,1]$	consumer discount factor
$B_t \sim Geo(q)$	random effective budget in period t
$b_t \in \mathbb{N}_0$	realized effective budget in period t
$q \in [0,1]$	geometric budget distribution parameter
$I \in \mathbb{N}$	repayment term
$o_t \in \{0,1,\ldots,I\}$	number of outstanding payments in period t
$a_t \in \{0,1,\ldots,I\}$	number of advanced payments in period t
$x_t \in \{0,1,\ldots,\min(b_t,o_t)\}$	number of installments paid in period t
$X_t^* \in \{0,1,\ldots\}$	consumer optimal repayment policy in period t
$V(a,o,b)$	consumer optimal value-to-go
$\bar{a} \in \{a,\ldots,a+o\}$	consumer optimal order-up-to level
$\mathbb{E}[\tau]$	consumer expected time to ownership
$\mathbb{E}[\Pi]$	firm expected profit
$\alpha \in [\beta,1]$	firm discount factor
$\delta \in [0,1]$	firm repossession parameter
$R \geq I$	repossession term
$G = R - I$	grace period
$F \in \{1,2,4\}$	repayment frequency

B. Proofs of Results from Section 3

Proof of Lemma 1: First, note that the the reformulated value function (5) is clearly increasing at $y=0$, i.e., $\beta\mathbb{E}_B[V(0,o,B)] \leq v-1+\beta\mathbb{E}_B[V(0,o-1,B)]$ for all $o \in \{1,\ldots\}$. Indeed, the inequality follows from $v>1$ and $\mathbb{E}_B[V(0,o,B)] \leq \mathbb{E}_B^{\Pi}[V(0,o-1,B)] \leq \mathbb{E}_B[V(0,o-1,B)]$, where Π is the feasible policy for state $(0,o-1)$ that mimics the optimal policy for state $(0,o)$ except that it ignores its last payment.

We now show that if $\beta \leq \frac{1}{1+q}$, then $v-y+\beta\mathbb{E}_B[V(y-1,a+o-y,B)]$ is unimodal in $y \in \{\max(a,1),\ldots,a+o\}$. For ease of notation, let $f(y)=\mathbb{E}_B[V(y-1,a+o-y,B)]$ in the rest of this proof. Let $\bar{y} \in \{1,\ldots,a+o-1\}$ be the smallest local maximum of $v-y+\beta f(y)$. If such $\bar{y}$ does not exist, then there is no local maximum before ownership (given by $\bar{y}=a+o$), thus implying that order-up-to ownership is the only local and thus global maximum, completing the proof of unimodality in this case. If $\bar{y}$ exists, then we now show that $v-y+\beta f(y)$ is decreasing for all $y \geq \bar{y}$, thus implying that $\bar{y}$ is only local and global maximum, completing the proof of unimodality.

We proceed by induction. The base case, $v-y+\beta f(y)$ being decreasing at $y=\bar{y} \in \{1,\ldots,a+o-1\}$ holds by definition of $\bar{y}$. For the induction step, assume $v-y+\beta f(y)$ is decreasing at some value $\hat{y} \geq \bar{y} \geq 1$, then we show that $v-y+\beta f(y)$ is also decreasing at $\hat{y}+1 \leq a+o-1$. Namely, assuming

$$v-\hat{y}+\beta f(\hat{y}) \geq v-\hat{y}-1+\beta f(\hat{y}+1) \iff \beta f(\hat{y})-\beta f(\hat{y}+1)+1 \geq 0. \tag{EC.1}$$

we show

$$v-\hat{y}-1+\beta f(\hat{y}+1) \geq v-\hat{y}-2+\beta f(\hat{y}+2) \iff \beta f(\hat{y}+1)-\beta f(\hat{y}+2)+1 \geq 0. \tag{EC.2}$$

We first lower bound $f(\hat{y}+1)=\mathbb{E}_B[V(\hat{y},a+o-\hat{y}-1,B)]$ as follows,

$$\begin{aligned} f(\hat{y}+1) &\geq q\left(v+\beta\mathbb{E}_B[V(\hat{y}-1,a+o-\hat{y}-1,B)]\right)+(1-q)\left(\mathbb{E}_B[V(\hat{y}+1,a+o-\hat{y}-2,B)]-1\right) \\ &\geq q\left(v+\beta f(\hat{y})\right)+(1-q)\left(f(\hat{y}+2)-1\right), \end{aligned} \tag{EC.3}$$

where the first inequality follows by, starting from state $(\hat{y},a+o-\hat{y}-1)$, following an order-at-least 1 policy in the first period and then following the optimal policy afterwards, and recalling that the budgets are geometrically distributed, hence memoryless. The second inequality follows from

$\mathbb{E}_B[V(\hat{y}-1,a+o-\hat{y}-1,B)] \geq \mathbb{E}_B^{\Pi}[V(\hat{y}-1,a+o-\hat{y},B)] \geq f(\hat{y}) = \mathbb{E}_B[V(\hat{y}-1,a+o-\hat{y},B)]$, where Π is the feasible policy for state $(\hat{y}-1,a+o-\hat{y}-1)$ that mimics the optimal policy for state $(\hat{y}-1,a+o-\hat{y})$ except that it ignores its last payment.

To conclude, we now prove (EC.2),

$$\begin{aligned}
\beta f(\hat{y}+1)-\beta f(\hat{y}+2)+1 \geq& \beta qv+\beta^2 qf(\hat{y})-\beta qf(\hat{y}+2)-\beta(1-q)+1\\
\geq& \beta qv+\beta^2 q(f(\hat{y}+1)-1/\beta)-\beta qf(\hat{y}+2)-\beta(1-q)+1\\
=& \beta qv+\beta q\left(\beta f(\hat{y}+1)-f(\hat{y}+2)\right)+1-\beta\\
\geq& \beta qv+\beta q\left(\beta f(a+o-1)-f(a+o)\right)+1-\beta\\
\geq& \beta qv+\beta q\left(\beta\left[qv\left(\frac{1-(\beta q)^{a+o-2}}{1-\beta q}\right)+\left(\frac{v}{1-\beta}-1\right)\left(\frac{1-q}{1-\beta q}\right)\right]-\frac{v}{1-\beta}\right)+1-\beta\\
=& \frac{-(\beta q)^{a+o}v-\beta^2 q(1-q)+(1-\beta)(1-\beta q)}{1-\beta q}\\
\geq& \frac{-(\beta q)^2-\beta^2 q(1-q)+(1-\beta)(1-\beta q)}{1-\beta q}=\frac{1-\beta-\beta q}{1-\beta q}\geq 0,
\end{aligned} \tag{EC.4}$$

where the first inequality follows from multiplying (EC.41) by β, adding 1 to both sides, and rearranging terms. The second inequality follows by replacing the induction hypothesis (EC.1). The first equality results from simplifying and rearranging terms. The third inequality follows from Lemma 5, which shows that $\beta f(y+1)-f(y+2)$ is minimized at $y=a+o-2$. The fourth inequality follows from replacing $f(a+o-1)=\mathbb{E}_B[V(a+o-2,1,B)] \geq qv\left(\frac{1-(\beta q)^{a+o-2}}{1-\beta q}\right)+\left(\frac{v}{1-\beta}-1\right)\left(\frac{1-q}{1-\beta q}\right)$ from Lemma 3(*iii*) and $f(a+o)=v/(1-\beta)$ from (4). The second equality follows from simplifying the expression. The fifth inequality follows since $(\beta q)^{a+o-2}v \leq (\beta q)^{\hat{y}}v \leq \beta(f(\hat{y}+1)-f(\hat{y})) \leq 1$, where the first inequality follows from $\hat{y}+1 \leq a+o-1$, the second inequality follows since $f(\hat{y}+1)=\mathbb{E}_B[V(\hat{y},a+o-\hat{y}-1,B)]$ attains an additional discounted value of $\beta^{\hat{y}-1}v$ compared to $f(\hat{y})=\mathbb{E}_B[V(\hat{y}-1,a+o-\hat{y},B)]$ in the sample path where zero budget is attained for the first $\hat{y}$ consecutive periods, which occurs with probability $q^{\hat{y}}$, and the last inequality is the induction hypothesis (EC.1). The third equality follows from simplifying the expression. The last inequality in (EC.4) follows from the numerator being non-negative for all $\beta \leq \frac{1}{1+q}$, concluding the proof. □

Proof of Theorem 1: Lemma 1 showed that the reformulated Bellman equation (5) is unimodal in y. Therefore, there exists a $\bar{y} \in \{a,\ldots,a+o\}$ such that $v1_{\{y\geq 1\}}-y+\beta\mathbb{E}_B[V((y-1)^+,a+o-y,B)] \leq v1_{\{y+1\geq 1\}}-y-1+\beta\mathbb{E}_B[V(y,a+o-y-1,B)]$ for all $y \in \{a,\ldots,\bar{y}-1\}$, and $v1_{\{y\geq 1\}}-y+\beta\mathbb{E}_B[V((y-1)^+,a+o-y,B)] \geq v1_{\{y+1\geq 1\}}-y-1+\beta\mathbb{E}_B[V(y,a+o-y-1,B)]$ for all $y \in \{\bar{y},\ldots,a+o-1\}$. Thus, we let $\bar{y}=\bar{a}(a+o)=\arg\max_{y\in\{a,\ldots,a+o\}}\{v1_{\{y\geq 1\}}-y+\beta\mathbb{E}_B[V((y-1)^+,a+o-y,B)]\}$, completing the proof of the theorem. □

Proof of Theorem 2: (A) We show that the order-up-to 1 policy from part (A) is optimal if and only if $\beta q\mathbb{E}_B[V(1,0,B)] \leq 1$. First, we show that if the policy in part (A) is optimal, then $\beta q\mathbb{E}_B[V(1,0,B)] \leq 1$. From Lemma 6(I), since the policy in part (A) is optimal then $\beta\left(\mathbb{E}_B[V(1,l-2,B)]-\mathbb{E}_B[V(0,l-1,B)]\right) \leq 1$ for all $l \in \{2,\ldots,I\}$. In particular, for $l=2$,

$$\begin{aligned}
\beta\left(\mathbb{E}_B[V(1,0,B)]-\mathbb{E}_B[V(0,1,B)]\right) \leq 1 \iff& \beta q\mathbb{E}_B[V(1,0,B)]-1-\beta\left(\frac{1-q}{1-\beta q}\right)\left(\beta q\mathbb{E}_B[V(1,0,B)]-1\right) \leq 0\\
\iff& \left(\beta q\mathbb{E}_B[V(1,0,B)]-1\right)\left(1-\frac{\beta(1-q)}{1-\beta q}\right) \leq 0,
\end{aligned} \tag{EC.5}$$

where the first equivalence follows by splitting $\beta\mathbb{E}_B[V(1,0,B)]$ into $\beta q\mathbb{E}_B[V(1,0,B)]$ and $\beta(1-q)\mathbb{E}_B[V(1,0,B)]$ and replacing $\mathbb{E}_B[V(0,1,B)]=\left(\frac{1-q}{1-\beta q}\right)\left(\frac{v}{1-\beta}-1\right)=\left(\frac{1-q}{1-\beta q}\right)\left(\mathbb{E}_B[V(1,0,B)]-1\right)$ from Lemma 3(i). The final equivalence follows from simplifying. Finally, the second term in (EC.5) is non-negative for all $\beta,q \in [0,1]$, therefore, we conclude that $\beta\left(\mathbb{E}_B[V(1,0,B)]-\mathbb{E}_B[V(0,1,B)]\right) \leq 1$ if and only if $\beta q\mathbb{E}_B[V(1,0,B)] \leq 1$, as required.

Next, we show that if $\beta q\mathbb{E}_B[V(1,0,B)] \leq 1$ then the policy in part (A) is optimal. From (EC.5), $\beta q\mathbb{E}_B[V(1,0,B)] \leq 1$ is equivalent to $\beta\left(\mathbb{E}_B[V(1,0,B)]-\mathbb{E}_B[V(0,1,B)]\right) \leq 1$. Lemma 7 then implies $\beta\left(\mathbb{E}_B[V(1,l-2,B)]-\mathbb{E}_B[V(0,l-1,B)]\right) \leq 1$ for all $l \in \{2,\ldots,I\}$, which, from Lemma 6(I), is equivalent to the policy in part (A) being optimal, concluding the proof of part (A).

(B) We show that the order-up-to policy in part (B) is optimal if and only if (6) and (7) hold. The only if direction follows from Lemma 6(II), i.e., if the policy in part (B) is optimal, then (6) and (7) hold since they correspond to (EC.41) when $l=i$ and (EC.44) when $l=i+1$, respectively.

Next, we show that if (6) and (7) hold then the policy in part (B) is optimal. To do so, we show that (i) (7) implies (EC.44) and (EC.42), and (ii) (6) implies (EC.41), (EC.43) and (EC.40). Namely, (6) and (7) imply all the inequalities in Lemma 6(II), which are equivalent to the policy from part (B) being optimal.

(i) Indeed, Lemma 7 shows that (7), i.e., $\beta\left(\mathbb{E}_B[V(k-1,i-k+1,B)-\mathbb{E}_B[V(k-2,i-k+2,B)]\right)1_{\{i<I\}}<1$ implies (EC.44), i.e., $\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)1_{\{l<I\}}<1$ for all $l\in\{i+1,\dots,I\}$.

Further, we now show that (EC.44), i.e., $\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)1_{\{l<I\}}<1$ for all $l\in\{i+1,\dots,I\}$, implies (EC.42), i.e., $\beta\left(\mathbb{E}_B[V(k,l-k-1,B)-\mathbb{E}_B[V(k-1,l-k,B)]\right)1_{\{l<I\}}<1$ for all $l\in\{k+1,\dots,i\}$. Note that,

$$\begin{aligned}1>&\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)\ge\beta\left(\mathbb{E}_B[V(k-1,I-1-k,B)-\mathbb{E}_B[V(k-2,I-k,B)]\right)\\ \ge&\beta\left(\mathbb{E}_B[V(k,0,B)-\mathbb{E}_B[V(k-1,1,B)]\right)\ge\beta\left(\mathbb{E}_B[V(k,l-k-1,B)-\mathbb{E}_B[V(k-1,l-k,B)]\right),\end{aligned}\tag{EC.6}$$

where the first inequality is (EC.44). The second inequality follows from Lemma 7 and increasing l to $I-1$. The third inequality follows from combining Propositions 5 and 6. The last inequality follows from Lemma 7 and is valid for all $l\in\{k+1,\dots,I\}$. Hence, (EC.6) shows that (EC.44) implies (EC.42), completing the proof of (i).

(ii) We now show that (6), i.e., $\beta\left(\mathbb{E}_B[V(k-1,i-k,B)-\mathbb{E}_B[V(k-2,i-k+1,B)]\right)\ge1$ implies (EC.41), i.e., $\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)\ge1$ for all $l\in\{k+1,\dots,i\}$ by contradiction. Assume (EC.41) does not hold for some $\bar{l}\in\{k+1,\dots,i\}$, i.e., $\beta\left(\mathbb{E}_B[V(k-1,\bar{l}-k,B)-\mathbb{E}_B[V(k-2,\bar{l}-k+1,B)]\right)<1$. Then, from Lemma 7 we must have $\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)<1$ for all $l\in\{\bar{l},\dots,i\}$, which includes $l=i$ and thus contradicts (6). Hence, (6) implies (EC.41).

Further, we now show that (EC.41), i.e., $\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)\ge1$ for all $l\in\{k+1,\dots,i\}$, implies (EC.43), i.e., $\beta\left(\mathbb{E}_B[V(k-2,l-k+1,B)-\mathbb{E}_B[V(k-3,l-k+2,B)]\right)\ge1$ for all $l\in\{i+1,\dots,I\}$ by contradiction. Assume (EC.43) does not hold for some $\bar{l}\in\{i+1,\dots,I\}$, i.e., $\beta\left(\mathbb{E}_B[V(k-2,\bar{l}-k+1,B)-\mathbb{E}_B[V(k-3,\bar{l}-k+2,B)]\right)<1$, then we show that $\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)<1$ for all $l\in\{k+1,\dots,i\}$, a contradiction with (EC.41). Following the same steps as in (EC.6), we get

$$1>\beta\left(\mathbb{E}_B[V(k-2,\bar{l}-k+1,B)-\mathbb{E}_B[V(k-3,\bar{l}-k+2,B)]\right)\ge\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right),$$

for all $l\in\{k+1,\dots,I\}$, contradicting (EC.41). Hence, (EC.41) implies (EC.43).

To complete the proof of part (ii), we now show that (EC.41), i.e., $\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)\ge1$ for all $l\in\{k+1,\dots,i\}$, implies (EC.40), i.e., $\beta\left(\mathbb{E}_B[V(l-1,0,B)-\mathbb{E}_B[V(l-2,1,B)]\right)\ge1$ for all $l\in\{2,\dots,k\}$ by contradiction. As before, assume (EC.40) does not hold for some $\bar{l}\in\{2,\dots,k\}$, i.e., $\beta\left(\mathbb{E}_B[V(\bar{l}-1,0,B)-\mathbb{E}_B[V(\bar{l}-2,1,B)]\right)<1$, then

$$\begin{aligned}1>&\beta\left(\mathbb{E}_B[V(\bar{l}-1,0,B)-\mathbb{E}_B[V(\bar{l}-2,1,B)]\right)\ge\beta\left(\mathbb{E}_B[V(\bar{l}-1,k+1-\bar{l},B)-\mathbb{E}_B[V(\bar{l}-2,k+2-\bar{l},B)]\right)\\ \ge&\beta\left(\mathbb{E}_B[V(\bar{l},0,B)-\mathbb{E}_B[V(\bar{l},1,B)]\right)\ge\cdots\ge\beta\left(\mathbb{E}_B[V(k-1,1,B)-\mathbb{E}_B[V(k-2,2,B)]\right),\end{aligned}\tag{EC.7}$$

where the first inequality is the contradiction assumption. The second inequality follows from Lemma 7. The third inequality follows from combining Propositions 5 and 6. The last inequality follows by iterating the previous two arguments until $\bar{l}=k$. The last inequality contradicts with (EC.41) when $l=k+1$. Hence, (EC.41) implies (EC.40), concluding the proof of part (ii), thus part (B) and the theorem. □

Proof of Theorem 3. We show that the result in the theorem holds for each sample path of budget realizations; thus, it holds in expectation. Specifically, for any sequence of budgets realizations $\{b_t\}_{t=0}^{\infty}$, let $\{x_{it}^*\}_{t=0}^{\infty}$, $i\in\{1,2\}$ be the realized consumer's optimal payments (cf., Theorem 1), and $\{a_{it}^*,o_{it}^*\}_{t=0}^{\infty}$, $i\in\{1,2\}$ be the realized advanced and outstanding payment state variables.

We then show that if the consumer's optimal order-up-to policy satisfies Properties 1 and 2, then, without loss of generality (w.l.o.g.), we have

$$a^*_{1t} \geq a^*_{2t} \text{ for all } t \in \{0, 1, \ldots\}, \tag{EC.8}$$

$$a^*_{1t} + o^*_{1t} \leq a^*_{2t} + o^*_{2t} \text{ for all } t \in \{0, 1, \ldots\}. \tag{EC.9}$$

Indeed, equations (EC.8) and (EC.9) imply $o^*_{1t} \leq o^*_{2t}$ for all $t \in \{0, 1, \ldots\}$, hence $\tau_1 \leq \tau_2$ by definition. Further, note, by iterating equation (1) until reaching the initial condition $(a_0, o_0) = (0, I)$, that $o^*_{it} = I - \sum_{l=0}^{t-1} x^*_{il}$, $i \in \{1, 2\}$. Therefore, $o^*_{1t} \leq o^*_{2t}$ for all $t \in \{0, 1, \ldots\}$ is equivalent to $\sum_{k=0}^{t} x^*_{1k} \geq \sum_{i=0}^{t} x^*_{2k}$ for all $t \in \{0, 1, \ldots\}$, which Lemma 8 shows that holds if and only if $\Pi_1 = \sum_{t=0}^{\infty} \alpha^t x^*_{1t} > \sum_{t=0}^{\infty} \alpha^t x^*_{2t} = \Pi_2$, proving the theorem.

We now turn to show equations (EC.8) and (EC.9) to complete the proof. If $x^*_{1t} = x^*_{2t}$ for all $t \in \{0, 1, \ldots\}$ then $a^*_{1t} = a^*_{2t}$ and $o^*_{1t} = o^*_{2t}$ for all $t \in \{0, 1, \ldots\}$ and we are done. Hence, assume w.l.o.g. that there exists a period $\tau \in \{1, \ldots\}$ such that $x^*_{1\tau} > x^*_{2\tau}$ and $x^*_{1k} = x^*_{2k}$ for all $k \in \{0, \ldots, \tau - 1\}$. To simplify the notation, assume w.l.o.g. that $\tau = 1$. Note that then, $a^*_{10} = a^*_{20}$ and $o^*_{10} = o^*_{20}$.

We prove equations (EC.8) and (EC.9) by induction on t.

Base Case: By definition of τ (assumed w.l.o.g. to be 1), we have $a^*_{10} = a^*_{20}$ and $o^*_{10} = o^*_{20}$. Hence, equations (EC.8) and (EC.9) hold when $t = 0$.

Induction Step: Assume equations (EC.8) and (EC.9) hold for some $t \in \{0, 1, \ldots, \}$. We now show that then they also hold for $t + 1$. Indeed,

$$\begin{aligned} a^*_{1(t+1)} &= (a^*_{1t} + x^*_{1t} - 1)^+ = (a^*_{1t} + \min(\bar{a}_1(a^*_{1t} + o^*_{1t}) - a^*_{1t}, b_t) - 1)^+ \\ &= (\min(\bar{a}_1(a^*_{1t} + o^*_{1t}), b_t + a^*_{1t}) - 1)^+ \geq (\min(\bar{a}_2(a^*_{2t} + o^*_{2t}), b_t + a^*_{2t}) - 1)^+ = a^*_{2(t+1)}, \end{aligned}$$

where the first three equalities follow by definition, see equation (2) and Theorem 1, and straightforward manipulations. The inequality holds since both terms inside the min() on the left-hand side dominate the ones on the right-hand side. Specifically, $\bar{a}_1(a^*_{1t} + o^*_{1t}) \geq \bar{a}_2(a^*_{2t} + o^*_{2t})$ holds from the following chain of inequalities,

$$\bar{a}_1(a^*_{1t} + o^*_{1t}) \geq \bar{a}_1(a^*_{2t} + o^*_{2t}) \geq \bar{a}_2(a^*_{2t} + o^*_{2t}), \tag{EC.10}$$

where the first inequality in (EC.10) follows from Property 2 and the induction hypothesis for equation (EC.9), and the second inequality in (EC.10) follows from Property 1. While $b_t + a^*_{1t} \geq b_t + a^*_{2t}$ holds from the induction hypothesis for equation (EC.8). The last equality is analogous to the first three equalities combined, completing the induction step for equation (EC.8).

We now show the induction step for equation (EC.9). Indeed,

$$\begin{aligned} a^*_{1(t+1)} + o^*_{1(t+1)} &= (a^*_{1t} + x^*_{1t} - 1)^+ + o^*_{1t} - x^*_{1t} = a^*_{1t} + o^*_{1t} - 1_{\{a^*_{1t} + x^*_{1t} \geq 1\}} \\ &\leq a^*_{2t} + o^*_{2t} - 1_{\{a^*_{1t} + x^*_{1t} \geq 1\}} \leq a^*_{2t} + o^*_{2t} - 1_{\{a^*_{2t} + x^*_{2t} \geq 1\}} = a^*_{2(t+1)} + o^*_{2(t+1)}, \end{aligned}$$

where the first equality follows by definition, see equations (1) and (2). The second equality can be verified in two exhaustive cases: if $a^*_{1t} + x^*_{1t} \geq 1$ then both sides of the equality evaluate to $a^*_{1t} + o^*_{1t} - 1$, and if $a^*_{1t} + x^*_{1t} < 1$ then $a^*_{1t} = x^*_{1t} = 0$ (since both terms are non-negative integers), thus both sides of the equality evaluate to o^*_{1t}. The first inequality follows from the induction hypothesis for equation (EC.9). The second inequality holds in all cases except when $1_{\{a^*_{2t} + x^*_{2t} \geq 1\}} > 1_{\{a^*_{1t} + x^*_{1t} \geq 1\}}$, which leads to a contradiction with the induction hypothesis for equation (EC.8). Indeed, $1_{\{a^*_{2t} + x^*_{2t} \geq 1\}} > 1_{\{a^*_{1t} + x^*_{1t} \geq 1\}}$ if and only if $a^*_{2t} + x^*_{2t} \geq 1$ and $a^*_{1t} = x^*_{1t} = 0$ (since both terms are non-negative integers). Since by definition $x^*_{1t} = \min(\bar{a}_1(a^*_{1t} + o^*_{1t}) - a^*_{1t}, b_t)$ (cf. Theorem 1) and $\bar{a}_1(a^*_{1t} + o^*_{1t}) \geq 1$ (cf. Theorem 2), then $a^*_{1t} = x^*_{1t} = 0$ implies $b_t = 0$, thus $x^*_{2t} = 0$. Therefore, $a^*_{2t} + x^*_{2t} \geq 1$ implies $a^*_{2t} \geq 1 > 0 = a^*_{1t}$, a contradiction with the induction hypothesis for equation (EC.8). The last equality is analogous to the first two equalities combined, completing the induction step for equation (EC.9).

We conclude by proving the last statement in the theorem. Let $\mathcal{Z}$ be the feasible set of actions the firm can take to change the model parameters and, for any $z \in \mathcal{Z}$, let us abuse the notation and denote by $\mathbb{E}_B[\Pi(z)]$ and $\mathbb{E}_B[\tau(z)]$ the induced firm's expected profits and the consumer's expected time to ownership, respectively. Then, by definition of optimality, $z^* \in \arg\max_{z\in\mathcal{Z}} \mathbb{E}_B[\Pi(z)]$ if and only if $\mathbb{E}_B[\Pi(z^*)] \geq \mathbb{E}_B[\Pi(z)]$ for all $z \in \mathcal{Z}$. In this proof, we previously showed that the latter holds if and only if $\mathbb{E}_B[\tau(z^*)] \leq \mathbb{E}_B[\tau(z)]$ for all $z \in \mathcal{Z}$. And again by definition of optimality, the latter holds if and only iff $z^* \in \arg\min_{z\in\mathcal{Z}} \mathbb{E}_B[\tau(z)]$, completing the proof of the theorem. □

Proof of Proposition 1: (*A*) The proof of this part directly follows when the optimal policy is paying at most one installment per period. Under this policy, the payments made by the consumer, denoted by P_t, are i.i.d. random variables and $\mathbb{E}[P_t] = (1-q)$. Thus, the expected time to ownership is the sum of I independent Geometric random variables, i.e., $\mathbb{E}\left[\sum_{t=1}^{\overline{\tau}} P_t\right] = \mathbb{E}[\tau^*]\mathbb{E}[P_t] = I \iff \mathbb{E}[\tau^*] = \frac{I}{\mathbb{E}[P_t]} = \frac{I}{1-q}$, concluding the proof of part (*A*).

(*B*) The first case is when the optimal policy is order-up-to 2 for $a+o \in \{2,\ldots,i\}$ and order-up-to-1 for $a+o \in \{i+1,\ldots,I\}$. For the portion of order-up-to-1, we have the expected time to ownership from part (*A*), i.e., $(I-i)/(1-q)$. Next, we derive the expected time to ownership for the portion of order-up-to 2. Let τ_i denote the random time to ownership when $a+o=i$. Since we start with $a=0$ and $o=i$, then we get

$$\mathbb{E}[\tau_i] = q(\mathbb{E}[\tau_i]+1) + q(1-q)(\mathbb{E}[\tau_{i-1}]+1) + (1-q)^2(\mathbb{E}[\tau_{i-2}]+1), \tag{EC.11}$$

where the equality follows from the order-up-to 2 policy and the probability of having a budget of 0, 1, and 2. Now let $\hat{\tau}_i = (1-q)\tau$. Then, (EC.11) turns into $\mathbb{E}[\hat{\tau}_i] = 1 + q\mathbb{E}[\hat{\tau_{i-1}}] + (1-q)\mathbb{E}[\hat{\tau_{i-2}}]$, which implies

$$\mathbb{E}[\hat{\tau}_i] - \mathbb{E}[\hat{\tau_{i-1}}] = 1 - (1-q)\left(\mathbb{E}[\hat{\tau_{i-1}}] - \mathbb{E}[\hat{\tau_{i-2}}]\right). \tag{EC.12}$$

By replacing the recursive equations in (EC.12) we get

$$\mathbb{E}[\hat{\tau}_i] - \mathbb{E}[\hat{\tau_{i-1}}] = 1-(1-q)+\cdots+(1-q)^{i-1}\left(\mathbb{E}[\hat{\tau}_1]-\mathbb{E}[\hat{\tau}_0]\right) = \frac{1-(-(1-q))^i}{2-q} \iff \mathbb{E}[\tau_i]-\mathbb{E}[\tau_{i-1}] = \frac{1-(-(1-q))^i}{(2-q)(1-q)}, \tag{EC.13}$$

where the first equality follows from replacing the recursive equation (EC.12) for all $i \leq 2$. The second equality follows from the fact that $\mathbb{E}[\hat{\tau}_1] = (1-q)$, $\mathbb{E}[\hat{\tau}_0] = 0$ and the sum of finite geometric series with factor $-(1-q) \in [-1, 0]$. The equivalence follows from replacing $\hat{\tau}_i = (1-q)\tau$.

To complete the proof of the first case, we follow the recursive equations in (EC.13) until $\mathbb{E}[\tau_0] = 0$ to get

$$\mathbb{E}[\tau^*] = \frac{I-i}{1-q} + \frac{\sum_{j=1}^{i} \frac{1-(-(1-q))^j}{2-q}}{1-q},$$

concluding the proof of the first case in part (*B*).

To prove the second case in part (*B*), we consider an absorbing Markov chain formulated in a 1-dimensional setting where payments are independent of a, and only depend on o since the optimal policy is order-up-to budget. Therefore, we reformulate the absorbing Markov chain into a 1-dimensional format where states are characterized only with o. Let Q be a $(I-1)\times(I-1)$ matrix obtained from removing the row and column associated with state $o=0$ in the transition matrix. Also, let $B_{i,j}$ be the (i,j) term in the $(I_Q - Q)^{-1}$ matrix. We are interested about $B_{1,j}$ which is associated with state $o=I$, and thus $\mathbb{E}[\tau] = \sum_{j=1}^{I-1} B_{1,j}$. Let k be the largest number of bundle payments made by the order-up-to ownership policy. Then, we show that the following recursive formula holds, that is

$$B_{1,1} = \frac{1}{1-q}, \tag{EC.14}$$

$$B_{1,I} = \sum_{i=1}^{k-1} q(1-q)^{i-1} B_{1,I-i} + (1-q)^{k-1} B_{1,I-k}. \tag{EC.15}$$

Equation (EC.14) shows the number of times the Markov chain remains in the initial state, that is the expected time until a success in the geometric distribution. Equation (EC.15) is also the sum of expected time for transition from state $(1, I-i)$ to $(1, I)$ for $i \in \{1,..,k\}$, i.e., the sum of expected time to make a payment equal to i in state $(I-i)$.

We next show that given equations (EC.14) and (EC.15), then $B_{1,I} = q/(1-q)$ for all $I \geq 2$ when $k = I$. We use induction as follows.

<u>Base case:</u> $B_{1,2} = qB_{1,1} = q/(1-q)$

Induction step: Assume $B_{1,j} = q/(1-q)$ for all $j \in \{3,..,I-1\}$ with $k = j$. Then, from (EC.14) and (EC.15) we have

$$B_{1,I} = \frac{q^2}{1-q}\sum_{i=1}^{I-2}(1-q)^{i-1} + \frac{1}{1-q}q(1-q)^{I-2} = \frac{q}{1-q}, \tag{EC.16}$$

where the first equality follows from replacing $B_{1,j} = q/(1-q)$ for all $j \in \{3,..,I-1\}$. The second equality follows from the sum of geometric series with ratio $(1-q)$, completing the proof of induction. Therefore, given (EC.16) we derive the expected time of ownership, i.e., $\mathbb{E}[\tau^*] = \sum_{j=1}^{I-1} B_{1,j} = \dfrac{1}{1-q} + (I-1)\dfrac{q}{1-q} = \dfrac{qI}{1-q} + 1$, concluding the proof of part (B). □

Proof of Lemma 2. Let (A_t^*, O_t^*) be the sequence of states induced by the consumer's optimal repayment policy X_t^*. Recall, from Theorem 1 that $X_t^*(A_t^*, O_t^*, B_t) = \min(\bar{a}(A_t^* + O_t^*) - A_t^*, B_t)$. We show that $A_t^* \le \bar{a}(A_t^* + O_t^*) - 1$ almost surely for all $t \in \{0, 1, \ldots\}$. Hence, $X_t^* \ge 1$ whenever the budget allows, i.e., $\mathbb{P}(X_t^* = 0) = \mathbb{P}(B_t = 0) = q$.

We show that $A_t^* \le \bar{a}(A_t^* + O_t^*) - 1$ holds for all $t \in \{0, 1, \ldots\}$ for any sample path of budget realizations $\{b_t\}_{t=0}^{\infty}$. Specifically, let $\{x_t^*\}_{t=0}^{\infty}$ be the realized consumer's optimal payments (cf., Theorem 1), and $\{a_t^*, o_t^*\}_{t=0}^{\infty}$ be the realized advanced and outstanding payment state variables.

First, consider $t = 0$. Recall that the initial state is $(a_0^*, o_0^*) = (0, I)$, i.e., consumers start with no advanced payments and I outstanding payments to own the product. From Theorem 2, $\bar{a}(a_0^* + o_0^*) = \bar{a}(I) \in \{1, \ldots, I\}$. Hence, $a_0^* = 0 \le \bar{a}(I) - 1 \in \{0, \ldots, I-1\}$, completing the proof when $t = 0$.

Now consider any $t \in \{0, 1, \ldots\}$, then

$$a_{t+1}^* = (a_t^* + x_t^* - 1)^+ = (a_t^* + \min(\bar{a}(a_t^* + o_t^*) - a_t^*, b_t) - 1)^+ \le \bar{a}(a_t^* + o_t^*) - 1 \le \bar{a}(a_{t+1}^* + o_{t+1}^*) - 1,$$

where the first equality follows from equation (2), the second equality follows from Theorem 1, the first inequality follows since from Theorem 2 we have $\bar{a}(a_t^* + o_t^*) \in \{1, \ldots, a_t^* + o_t^*\}$, and the second inequality follows from Property 2 and the observation that

$$a_t^* + o_t^* \ge a_t^* + o_t^* - 1_{\{a_t^* + x_t^* \ge 1\}} = (a_t^* + x_t^* - 1)^+ + o_t^* - x_t^* = a_{t+1}^* + o_{t+1}^*,$$

where the first equality can be verified in two exhaustive cases: if $a_t^* + x_t^* \ge 1$ then both sides of the equality evaluate to $a_t^* + o_t^* - 1$, and if $a_t^* + x_t^* < 1$ then $a_t^* = x_t^* = 0$ (since both terms are non-negative integers), thus both sides of the equality evaluate to o_{1t}^*. The second equality follows from equations (1) and (2), completing the proof of the lemma. □

C. Proofs of Results from Section 4

Proof of Proposition 2: First, we show as $q \to 0$, then $\mathbb{E}[\tau^4] < \mathbb{E}[\tau^2] < \mathbb{E}[\tau^1]$. Let $q = 0$. Since there is no future income uncertainty, paying order-up-to F gains higher consumer's value than ordering more. The latter follows from condition $\beta q \mathbb{E}_B[V(1,0,B)] < 1$ in Theorem 2(A) for optimal order-up-to 1 policy under $F = 1$. We can extend this condition for $F = 2$ and $F = 4$, i.e., $\beta(1-(1-q)^2)\mathbb{E}_B[V(3,0,B)] < 2$ and $\beta(1-(1-q)^4)\mathbb{E}_B[V(7,0,B)] < 4$, respectively. Indeed, all the above inequalities hold when $q = 0$. Thus, we use the closed-forms derived in Proposition 7, which considers order-up-to F policy.

We now derive and compare the the closed-forms of expected time to ownership under each design F. When $F = 1$, the optimal policy is order-up-to 1 policy, therefore, the expected time to ownership follows Proposition 1(A), i.e., $\mathbb{E}[\tau^1] = I/(1-q)$. When $F = 2$, the optimal policy is order-up-to 2, thus, the expected time to ownership follows Proposition 7(i), i.e., $\mathbb{E}[\tau^2] = I/(2(1-q)^2) + (I/2 - 2)(1-q)^2$. When $F = 4$, the optimal policy is order-up-to 4 policy, thus, the expected time to ownership follows Proposition 7(ii), i.e., $\mathbb{E}[\tau^4] = I/(4(1-q)^4)$. Thus, when $q = 0$ we get $\mathbb{E}[\tau^4] = I/4 < \mathbb{E}[\tau^2] = I - 2 < \mathbb{E}[\tau^1] = I$, where the first inequality follows from $I \ge 4$ since monthly payments must be feasible.

Next, we show that as $q \to 1$, then $\mathbb{E}[\tau^4] > \mathbb{E}[\tau^2] > \mathbb{E}[\tau^1]$. Let $q = 1$. Then, the optimal policy is order-up-to ownership, which follows from the condition $1 \le \beta^{I-1} q^{I-1} \mathbb{E}_B[V(I-1,0,B)]$ in Theorem 2(B). We use the expected time to ownership derived in Proposition 7 given $q \to 1$. Thus, we have $\mathbb{E}[\tau^4] = I/(4(1-q)^4) > \mathbb{E}[\tau^2] = I/(2(1-q)^2) > \mathbb{E}[\tau^1] = I/((1-q))$, where the inequalities follow from $q \le 1$ and $q \to 1$, thus $4(1-q)^4 < 2(1-q)^2 < (1-q)$, concluding the proof. □

Proof of Proposition 3: The result in part (i) follows from Proposition 8 where $\bar{q} = q'$ and $\bar{\bar{q}}_i = \hat{q}$. The reason is that $\frac{1-\beta}{\beta v} \geq 1 \geq q$ implies $\frac{\beta q v}{1-\beta} < 1$, where the latter inequality implies no bundled payments under all F (see Lemma 9), which is the case in Proposition 8. Therefore, to complete the proof we show part (ii), i.e., if $2I(1-\frac{1-\beta}{\beta v})^2 + (2I-8)(1-\frac{1-\beta}{\beta v})^6 - I < 0$ then there exists a $\bar{\bar{q}}_{ii} \in [\bar{q}_0, \bar{\bar{q}}_i]$ such that $\mathbb{E}[\tau^2] \leq \min\{\mathbb{E}[\tau^1], \mathbb{E}[\tau^4]\}$ for all $q \in [\bar{q}, \bar{\bar{q}}_{ii}]$.

If $\frac{1-\beta}{\beta v} < 1$, then we show that

$$2I(1-\frac{1-\beta}{\beta v})^2 + (2I-8)(1-\frac{1-\beta}{\beta v})^6 - I < 0 \tag{EC.17}$$

is a sufficient condition such that no bundled payments happen for all $q \in [0, \bar{q}]$ where $\bar{q}$ satisfies $2I(1-\bar{q})^2 + (2I-8)(1-\bar{q})^6 - I = 0$. If condition (EC.17) holds, then given the definition of $\bar{q}$ together with the $2I(1-q)^2 + (2I-8)(1-q)^6 - I$ being decreasing in q, we must have $\bar{q} < \frac{1-\beta}{\beta v}$. From Lemma 9, the latter inequality implies that no bundled payments happen for all $q \in [0, \bar{q}]$, which is the case in Proposition 8. Thus, (EC.17) implies that $\bar{q} = q'$.

To complete the proof, we compare the expected time to ownership under $F = 2$ and $F = 1$. We use the expected times to ownership derived in Proposition 7(i) since at $q = \bar{q}$ no bundled payments happen. We now show that the expected time to ownership under $F = 2$ (EC.53) is lower than $F = 1$ (EC.52) for all $q \in [\bar{q}, \bar{\bar{q}}_{ii}]$. For all $q > \bar{q}$ we have $2I(1-q)^2 + (2I-8)(1-q)^6 - I < 0$, which follows from the definition of $\bar{q}$. This inequality implies $I + (I-4)(1-q)^4 - I/(2(1-q)^2) < 0$ since $2I(1-q)^2 \geq 0$, which then implies $I + (I-4)(1-q)^4 - 2I(1-q) < 0$. The latter inequality follows from the reverse of (EC.55) since $\mathbb{E}[\tau^4] < \mathbb{E}[\tau^1]$ at $q = \bar{q}$. The inequality $I + (I-4)(1-q)^4 - 2I(1-q) < 0$ is the reverse of (EC.57), thus implying that $\mathbb{E}[\tau^2] \leq \mathbb{E}[\tau^1]$. Finally, we define $\bar{\bar{q}}_{ii}$ as the intersection of expected times to ownership under $F = 2$ and $F = 1$ (can include bundled payments). Thus, $\bar{\bar{q}}_{ii} \leq \bar{\bar{q}}_i$, concluding the proof. □

D. Numerical results for Non-unimodality of the Value Function

In Figure EC.1, we illustrate a numerical example where the value function is non-unimodal when condition $\beta \leq \frac{1}{1+q}$ in Lemma 1 does not hold. We observe that there are two local maximums (the red (left) and the green (right) points) of the consumer's utility. Thus, we show that the utility function is not necessarily unimodal in y. Intuitively, when β and q are both very large (i,e., the consumer highly values the future product consumption at the same time that her future income is very uncertain), the benefit of fully paying the outstanding payments ($x = o$) is very high such that the consumer distorts from the structural ordering policy and makes full payment. Thus, we observe that the green (right) local maximum point distorts the unimodality of the value function due to the high value gained by attaining ownership.

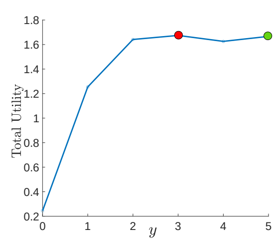
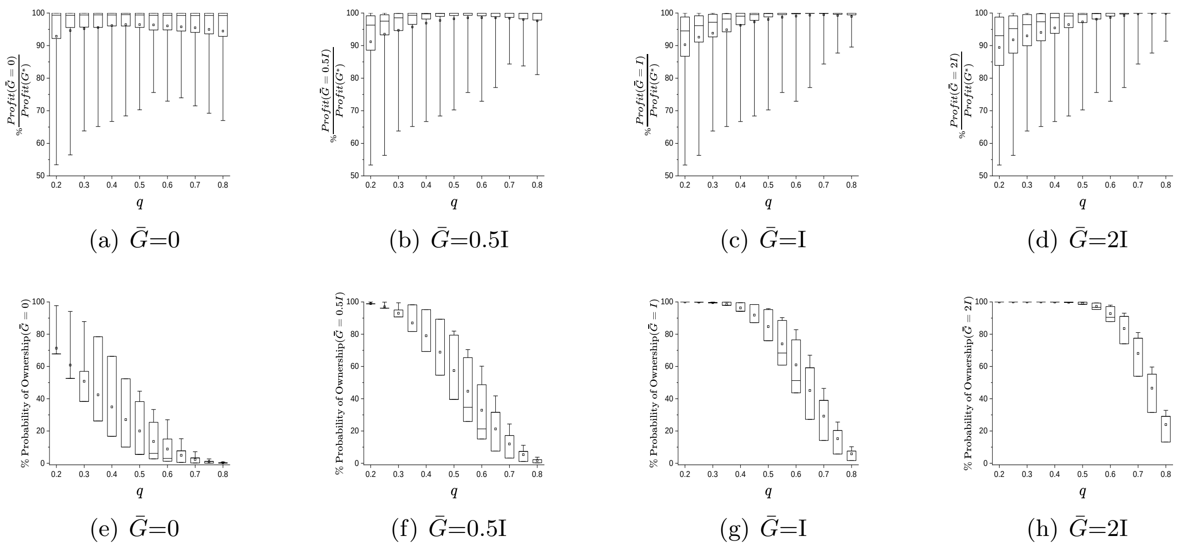


Figure EC.1 Example of non-unimodal value function (5) **(**$q = 0.9$, $\beta = 0.7$, $v = 2$, $I = 5$, $a = 0$**).**

E. Numerical Results for Finite Grace Period

Figure EC.2 presents the box plots of the firm's profit ratio and the consumer's probability of ownership for Figure 8 for each uniform grace period $\bar{G}$.

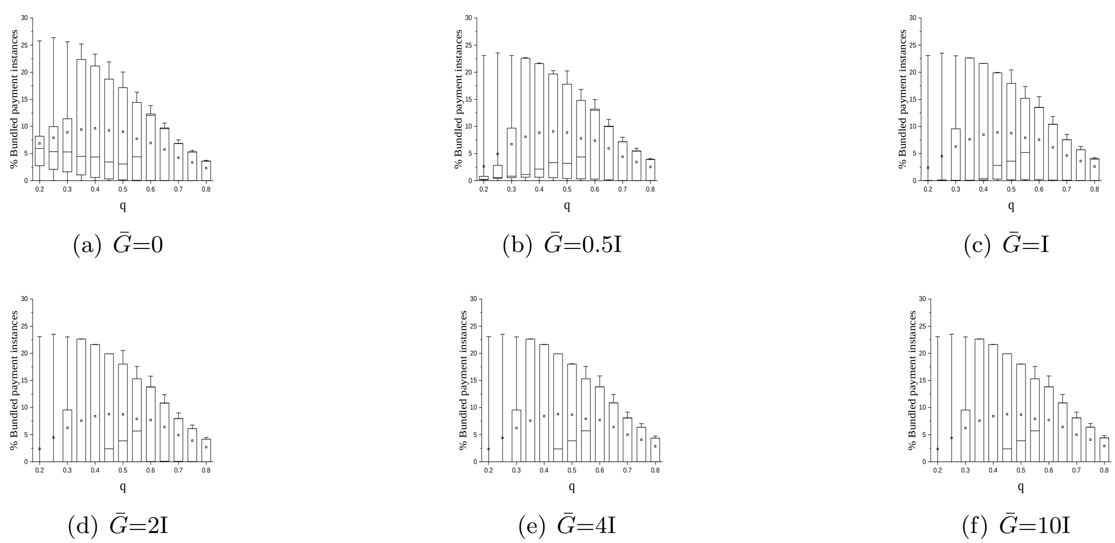


(a) $\bar{G}$=0 (b) $\bar{G}$=0.5I (c) $\bar{G}$=I (d) $\bar{G}$=2I

(e) $\bar{G}$=0 (f) $\bar{G}$=0.5I (g) $\bar{G}$=I (h) $\bar{G}$=2I

Figure EC.2 Firm's profit ratio w.r.t. firm's optimal profit when (a) $\bar{G}$=0, (b) $\bar{G}$=0.5I, (c) $\bar{G}$=I, and (d) $\bar{G}$=2I. Consumer's probability of ownership when (e) $\bar{G}$=0, (f) $\bar{G}$=0.5I, (g) $\bar{G}$=I, and (h) $\bar{G}$=2I. Data points correspond to the problem instances $(v, \beta, \alpha, \delta)$ summarized in Table 2. Each box plot consists of 1575 instances.

Figure EC.3 presents the average percentage of bundled payments across 1000 random budget instances for various grace periods. We generate 1000 random budget samples, each for 200 periods. The budget in each period is randomly drawn from a geometric distribution with parameter q. Then, for each uniform grace period $\bar{G}$, we use the same 1000 budget instances for $\bar{G}$ periods ($\bar{G} \leq 200$). For each budget instance, we determine the consumer's optimal repayment policy based on the realized budget in each period. We then calculate the percentage of periods in which bundled payments occur (i.e., when $x > 1$) until ownership is achieved. Averaging this value across all 1000 budget instances yields the average percentage of bundled payments, which corresponds to a single point in the bar plots. To generate multiple points for each q, we vary the parameters v and β as specified in Table 1. Figure EC.3(f) represents our base model with an infinite grace period, where ownership is guaranteed with probability 1. Notably, this figure illustrates that bundled payments are as prevalent as that observed in the base model with an infinite grace period.



(a) $\bar{G}$=0 (b) $\bar{G}$=0.5I (c) $\bar{G}$=I

(d) $\bar{G}$=2I (e) $\bar{G}$=4I (f) $\bar{G}$=10I

Figure EC.3 Average percentage of bundled payments over 1000 random budget instances for data points that correspond to the problem instances (v, β) summarized in Table 1, for $I = 8$. Each box plot consists of 210 points where each point is the average of 1000 instances.

F. Details on Model Extensions from Section 5

F.1. Allowing Consumer Savings

We consider a model where, in each period, the consumer can save any remaining budget after fulfilling her immediate consumption needs. The rate of immediate consumption is exogenous. When the immediate consumption per period is very high (e.g., an upper bound on the budget), the consumer cannot save for future periods, and our base model in (3) is retrieved. However, when the immediate consumption need per period is moderate (e.g., the expected budget), the consumer saves any remaining budget after fulfilling their immediate consumption needs in a given period.

We assume that the consumer derives less value from immediate per-period consumption than from product usage value v, consistent with the definition of the budget B, and we normalize the value of immediate consumption to zero. Then, the Bellman equation for this model is given by

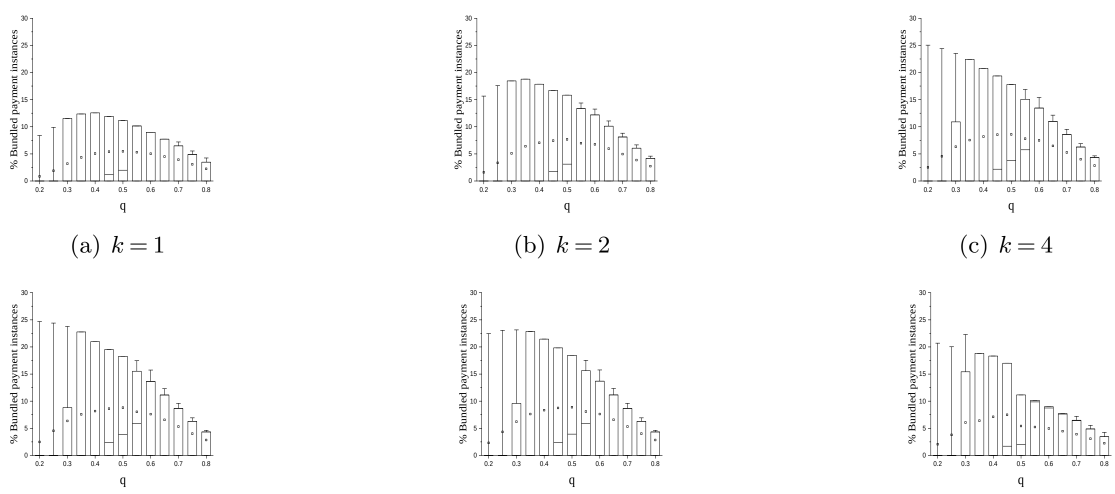
$$V(a,o,b,s) = \max_{x\in\{0,\ldots,\min(o,b+s)\}} \left\{ v\mathbb{1}_{\{a+x\geq 1\}} - x + \beta \mathbb{E}_B[V((a+x-1)^+, o-x, B, (s+b-x-k)^+)] \right\}, \qquad \text{(EC.18)}$$

with a terminal value $V(0,0,b,s) = \frac{v}{1-\beta}$, where the state variable $s \in \{0,1,...\}$ is the total savings at the end of the period. The exogenous immediate consumption per-period is given by $k \in \{0,1,...,\max(B)\}$, which is independent of q. In addition, we examine the case where consumption equals the mean of the budget ($k = \mathbb{E}(B)$), which varies with q. The inability to save on average is consistent with findings in the literature on low-income individuals in LMICs, such as Deaton (1989). In particular, note that for $k = \max(B)$, the consumer has no savings since any leftover budget is used for immediate consumption, leading to our base model in (3).

In the following subsections, we focus on evaluating the extent to which the three main results of this paper hold for the model with savings in (EC.18). Specifically, we confirm the persistence of (1) the prevalence of bundled payments even in the absence of transaction costs, (2) the dominance of moderate repayment frequency when income uncertainty is moderate, and (3) the effectiveness of a moderate grace period in balancing the firm's profit with the consumer's access to the product.

Due to the analytical intractability of the model with savings (particularly regarding the consumer's optimal repayment policy), we rely on extensive numerical analysis, using the instances detailed in Tables 1 and 2.

F.1.1. Bundled Payments. We first numerically show that bundled payments occur in the model with savings for non-zero consumption $k > 0$. Figure EC.4 illustrates the percentage of payments that are bundled across 1000 random budget instances for different consumption levels. We use the same 1000 budget samples as in Figure EC.3. Each budget sample consists of 200 periods, where 200 periods is enough to attain ownership in all 1000 instances.



(a) $k=1$ (b) $k=2$ (c) $k=4$

(d) $k=6$ (e) $k=8=\max(B)$ (f) $k=E(B)$

Figure EC.4 Average percentage of bundled payments over 1000 random budget instances for data points that correspond to the problem instances (v,β) summarized in Table 1, for $I=8$.

Figures EC.4(a)–(e) illustrate the average percentage of bundled payments for $k \in \{1, 2, 4, 6, 8\}$, where $k = 8 = \max(B)$ corresponds to our base model in (3). Figure EC.4(f) illustrates the average percentage of bundled payments when k is set to the rounded up mean budget, which depends on q, i.e., $k = \mathbb{E}(B)$. In this case, the consumer is expected to consume her average budget per period. We observe that bundled payments occur in the model with savings when consumption is nonzero. In particular, for moderate to high values of k, such as $k \geq 4$, bundled payments occur frequently, similar to the base model (i.e., $k = 8$). Moreover, in the case where the consumer spends her average budget in each periodwe still observe that bundled payments remain as prevalent as in the base model. As k decreases to smaller values, such as $k = 2$ and $k = 1$, we observe a reduction in bundled payments, which eventually disappear entirely when $k = 0$. This outcome is intuitive since, when $k = 0$, the consumer can fully save all her remaining budget without any costs and thus only pays for her current-period product usage if she has sufficient funds. However, the case of $k = 0$ is not relevant in the context that we study, where consumers face challenges to fully save (see Dupas and Robinson (2013a) and Dupas et al. (2018)). Therefore, the numerical results in Figure EC.4 suggest that the first main result of this paper, that bundled payments occur in the absence of transaction costs, qualitatively holds in the model with savings for sufficiently high values of k.

F.1.2. Moderate Flexibility in Repayment Frequency/Installment Amount. We next numerically illustrate that the bi-weekly design outperforms the weekly and monthly designs in terms of expected time to ownership when q is moderate in the model with savings. Table EC.1 presents the range of q values where the bi-weekly design yields the lowest expected time to ownership compared to the weekly and monthly designs for different exogenous consumption levels k. The reported intervals represent the average range over 210 instances, with model parameters (v, β) varying as specified in Table 1.

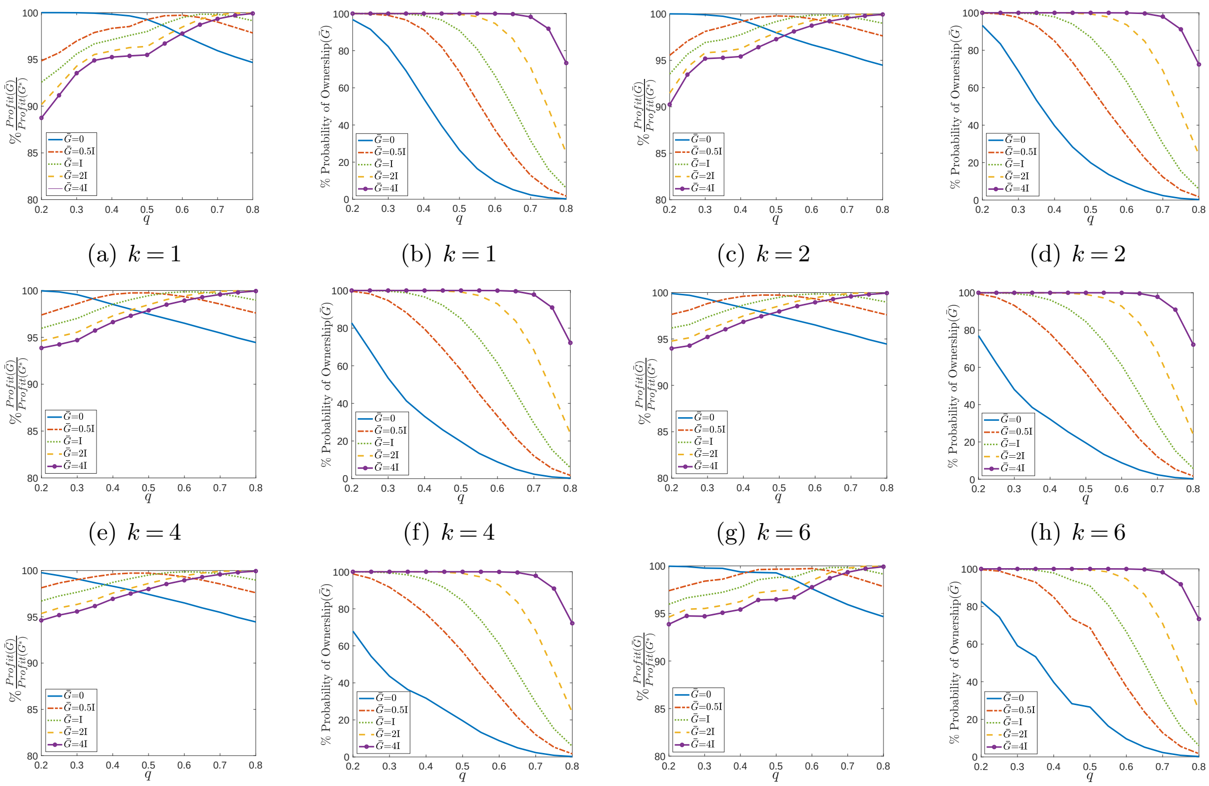
As before, $k = 8$ corresponds to the base model, for which in Section 4.1 we analytically demonstrated that the bi-weekly design dominates the other two designs when q is moderate. Interestingly, this result remains valid with minimal changes as k decreases to 4. This minimal change is analogous to our observations on the percentage of bundled payments in Figure EC.4 for $k = 4$ and $k = 6$. However, as k drops below 4, we observe a significant reduction in the length of the dominance interval, eventually disappearing entirely at $k = 0$, where the bi-weekly design no longer holds the lowest expected time to ownership. In this case, the monthly design always generates the lowest expected time to ownership. However, as mentioned earlier, the case of $k = 0$ is not particularly relevant to our context of study. Therefore, the overall numerical results in Table EC.1 illustrate that the second main result of this paper, that a moderate installment amount is preferable when income uncertainty is moderate, qualitatively holds in the model with savings for sufficiently high values of k.

	$k=1$	$k=2$	$k=4$	$k=6$	$k=8=max(B)$	$k=\mathbb{E}(B)$
Average Interval	[0.3869, 0.4343]	[0.325, 0.4228]	[0.3145, 0.4297]	[0.3215, 0.4318]	[0.3213, 0.4368]	[0.3160, 0.4228]
Length Mean	0.0474	0.0978	0.1152	0.1103	0.1155	0.1068
Length Std	0.0248	0.0367	0.0394	0.0343	0.0382	0.0375
Length CI	[0.0440, 0.0507]	[0.0929, 0.1028]	[0.1099, 0.1206]	[0.1056, 0.1149]	[0.1103, 0.1206]	[0.1017, 0.1188]

Table EC.1 Average interval of q and it's statistical properties where the bi-weekly design is dominant. The average is over data points that correspond to the problem instance (v, β) summarized in Table 1, for $I = 8$. We calculate the 95% confidence interval (CI).

F.1.3. Moderate Flexibility in Grace Period. We next numerically show that the observation that a moderate grace period effectively balances consumer welfare and the firm's profit also holds in the model with savings. In Figure EC.5, for different consumption levels k, we plot the firm's average percentage expected profit ratio (relative to its profit under the optimal grace period G^*) and the average consumer's probability of ownership, given a uniform grace period, $\bar{G}$. The reported average is over instances with $G^* \geq 0$ and with model parameters $(v, \beta, \alpha, \delta)$ summarized in Table 2. The firm's expected profit function is given in (12).

As before, $k = 8$ corresponds to the base model, thus the plots are identical to the numerical results in Figure 8. We observe that our main result, which supports an intermediate grace period of $\bar{G} = I$ or $\bar{G} = 2I$, holds in the model with savings for nonzero k. For instance, when $k = 4$ and $q = 0.3$, increasing the grace period from $\bar{G} = 0$ to $\bar{G} = I$ significantly enhances consumer access to the product by more than 40%, while reducing the firm's expected profit by less than 3%. Therefore, our third main result, that a moderate grace period benefits both consumers and firms operating as social enterprises, qualitatively holds in the model with savings for sufficiently high values of k.



(a) $k = 1$ (b) $k = 1$ (c) $k = 2$ (d) $k = 2$

(e) $k = 4$ (f) $k = 4$ (g) $k = 6$ (h) $k = 6$

(i) $k = 8 = \max(B)$ (j) $k = 8 = \max(B)$ (k) $k = \mathbb{E}(B)$ (l) $k = \mathbb{E}(B)$

Figure EC.5 Impact of uniform grace periods on firm's average expected profit ratio w.r.t. firm's optimal profit and consumer's average probability of ownership. These results are across problem instances $(v, \beta, \alpha, \delta)$ in Table 2, for $I = 8$.

F.2. Finite Product Lifetime

We consider a product with a finite lifetime, i.e., the product only lasts for a finite number of usage periods. For a finite lifetime $L \geq I$, the Bellman equation is the same as in our base model (equation (3)) but with a different terminal value depending on the product lifetime:

$$V(0,0,b) = \frac{v(1-\beta^{L-I})}{1-\beta}. \tag{EC.19}$$

As the difference between the product lifetime and the repayment term, $L - I$, approaches infinity, as assumed in the base model, the terminal value in (EC.19) converges to the terminal value of the base model in equation (4) in the paper, i.e., $\lim_{(L-I)\to\infty} \frac{v(1-\beta^{L-I})}{1-\beta} = \frac{v}{1-\beta}$. The role of the terminal value in this model setup is to reflect that once I payments have been made, ownership is acquired, and after the product has been used for those I paid periods (i.e., $a = o = 0$), the product's remaining lifetime is reduced to $L - I$ periods. Note that the terminal value in (EC.19) corresponds to a product whose value is $v(1 - \beta^{L-I})$ in our base model, while all other aspects of the model align with our base framework. Therefore, the qualitative insights derived from our base model, such as bundled payment behavior and the role of moderate flexibility in repayment frequency and grace period, continue to hold in this extended setting.

F.3. Allowing for Partial Payments

We explicitly model the RTO firm allowing consumers to make partial payments in any integer amount under any design. As in the base model, consumers are permitted to use the product in the current period if the sum of the number of advanced payments already paid for and the number of installments paid in period t meets or exceeds the minimum required installment amount under each design (1 for weekly, 2 for bi-weekly, and 4 for monthly). This leads to the following updated Bellman equation when the state is (a, o) and the realized budget is b:

$$V^F(a,o,b) = \max_{x \in \{0,1,2,\dots,\min(o,b)\}} \left\{ v1_{\{a+x \geq F\}} - x + \beta \mathbb{E}_B[V^F((a + x - 1_{\{a+x \geq F\}})^+, o - x, B)] \right\}, \tag{EC.20}$$

with terminal value $V^F(0,0,b) = \frac{v}{1-\beta}$, where $F = 1$, $F = 2$, and $F = 4$ denote weekly, bi-weekly and monthly designs, respectively. Equation (EC.20) makes three changes with respect to equation (9) in Section 4.1. First, the feasible payment values now allow for partial payments, which can be any integer amount and are independent of the design F. Second, the consumer can use the product only if the minimum required installment under design F is met, which is captured by replacing the term $v1_{\{a+x \geq 1\}}$ with $v1_{\{a+x \geq F\}}$ in equation (9). Third, the model dynamics in equation (2) are updated to $a_{t+1} = (a_t + x_t - 1_{\{a_t + x_t \geq F\}})^+$, which accounts for cases where a partial payment is made but it is insufficient to secure immediate product usage, thereby adjusting the number of advanced payments accordingly. Note that for $F = 1$, the Bellman equation (EC.20) is equivalent to the Bellman equation of our base model in (3). Due to the analytical intractability of deriving closed-form expressions for the expected time to ownership under bi-weekly and monthly designs for problem (EC.20), we conducted extensive numerical experiments to assess the robustness of our results when allowing for partial payments.

Figure EC.6(a) illustrates the average expected time to ownership as a function of the income uncertainty q, assuming consumers are allowed to make partial payments. This average is computed using 210 different combinations of (v, β), as specified in Table 1. We compare this figure to its counterpart, Figure 6(b), where partial payments are not allowed. As expected, the expected time to ownership under the weekly design remains unchanged, since the definition of problem (EC.20) is same as problem (3) of our base model in the paper. More importantly, we observe that the bi-weekly design continues to outperform both the weekly and monthly designs for moderate values of q, confirming the robustness of our result on the benefits of intermediate repayment frequency even when partial payments are allowed.

(a) Average expected time to ownership

(b) Dominance of designs

Figure EC.6 (a) Average expected time to ownership and (b) Dominance of weekly ($F = 1$), bi-weekly ($F = 2$) and monthly ($F = 4$) designs with the lowest average expected time to ownership, when partial payments are allowed across problem instances (v, β) in Table 1, for $I = 8$.

Additionally, we conducted numerical experiments under a finite grace period to demonstrate the robustness of the dominance of intermediate repayment frequency when consumers are allowed to make partial payments. Figure EC.6(b) presents the results, averaged across 210 combinations of (v, β). A comparison with its counterpart, Figure 9 in Section 4.2, shows that the main finding, that intermediate repayment frequency performs best in moderate q region, continues to hold qualitatively even when the grace period is finite.

F.4. Endogenous Repayment Term

We consider the repayment term as the firm's decision variable and assume that payments are made in every base period. Since the product's price is fixed and no interest rate is charged for late payments, adjusting I changes the payment per installment, c, while ensuring that cI remains constant and equals the product's total price. To ensure the discreteness of the geometric budget distribution, we assume $c \in \{1, 2, 3, \ldots\}$. For example, consider an off-grid energy product priced at \$12. The firm could offer various contract options with different repayment terms, such as a 12-week plan with \$1 weekly payments, a 6-week plan with \$2 weekly payments, a 4-week plan with \$3 weekly payments, or a 3-week plan with \$4 weekly payments. The question we address is which option leads to better outcomes for both the firm's expected profit and the consumer's expected time to ownership. Specifically, is it more beneficial to offer greater flexibility through longer repayment terms, giving consumers more time to complete payments, or the opposite?

For a repayment term I with a payment unit $c \in \{1, 2, 3, \ldots\}$ the Bellman equation is

$$V^I(a, o, b) = \max_{x \in \{0,1,\ldots,\min(o,b)\}} \left\{ v 1_{\{a+x \geq 1\}} - cx + \beta \mathbb{E}_B [V((a + x - 1)^+, o - x, B)] \right\}, \tag{EC.21}$$

with a terminal value $V^I(0, 0, b) = \frac{v}{1-\beta}$. Note that feasible values of $a \in \{0, 1, \ldots, I\}$ and $o \in \{0, 1, \ldots, I\}$ and the payment unit c depend on the repayment term I. Since consumer payments are made in discrete units of the installment cost c, then, to simplify the exposition, we represent the effective budgets as $B_t \in \{0, c, 2c, \ldots\}$. Under this representation, the probability of having an effective budget $i \in \{0, c, 2c, \ldots\}$ is given by $\mathbb{P}(B_t = i) = \sum_{j=i}^{i+c-1} q(1-q)^j$. It can easily be shown that this budget specification is equivalent to random budget ${B_t}' \in \{0, 1, 2, \ldots\}$ in terms of number of installments to pay, x_t, each with a payment unit c, where ${B_t}' \sim \text{Geo}(q' = \sum_{j=0}^{c-1} q(1-q)^j)$. Therefore, we can rewrite the Bellman equation for a repayment term I and payment unit c as

$$V^I(a, o, b) = \max_{x \in \{0,1,\ldots,\min(o,b)\}} \left\{ v 1_{\{a+x \geq 1\}} - cx + \beta \mathbb{E}_{B'} [V((a + x - 1)^+, o - x, B')] \right\}, \tag{EC.22}$$

with a terminal value $V^I(0, 0, b) = \frac{v}{1-\beta}$ and a budget of ${B_t}' \sim \text{Geo}(q')$. As long as $v > c$ and $\beta \leq \frac{1}{1+q'}$, then all the results in Section 3 of the paper on the consumer's repayment policy, expected time to ownership and alignment of firm's expected profit and consumer's expected time to ownership hold by replacing the parameters $c > 1$ and q' instead of q.

Let the highest possible repayment term be the base case of I with $c = 1$. Thus, the price of the product is I. Then, to have a fair comparison for a lower repayment term $I' < I$, we keep the price of the product the same; therefore, $c'I' = I$, which implies $c' > 1$. Since c' must be discrete, due to the geometric budget distribution, then we assume that $c' = 2$ for all $I' \in [I/2, I)$, $c' = 3$ for all $I' \in [I/3, I/2)$, $c' = 4$ for all $I' \in [I/4, I/3)$, and so on. It is straightforward to show that the lower bound on I' for every interval is the best for the consumer since it has the lowest price of the product, equivalent to the price of the base case, i.e., I. It is also straightforward to show that, since the installment cost is the same in every interval, this lower bound also leads to the lowest expected time to ownership. Thus, we use the lower bound as a candidate for every interval and compare the flexibility offered to the consumer w.r.t. the lower bounds of the repayment terms. Let $V^I(a, o, b)$ be the consumer's optimal utility when the repayment term is I. Intuitively, as I decreases between the intervals, c increases and thus consumer welfare decreases. However, decreasing I does not necessarily increase the expected time to ownership and equivalently does not necessarily reduce the firm's expected profit (following from Theorem 3 in the paper).

The following result characterizes conditions under which a moderate repayment term generates the lowest expected time to ownership compared to a very low or very high repayment term.

Proposition 4. *Let $I' = I/2$, $c' = 2$, $I'' = I/3$, $c'' = 3$, $\tilde{I} = I/4$ and $\tilde{c} = 4$. In addition, let $\mathbb{E}[\tau^I]$ be the expected time to ownership under the optimal payment policy for repayment term I. Then,*

(i) If $\frac{1-\beta}{\beta v} \geq 1$ then $\mathbb{E}[\tau^{I''}] \leq \min\{\mathbb{E}[\tau^I], \mathbb{E}[\tau^{I'}], \mathbb{E}[\tau^{\tilde{I}}]\}$ for all $q \in [\frac{1}{4}, \frac{1}{3})$, and $\mathbb{E}[\tau^{I'}] \leq \min\{\mathbb{E}[\tau^I], \mathbb{E}[\tau^{I''}], \mathbb{E}[\tau^{\tilde{I}}]\}$ for all $q \in \left[\frac{1}{3}, \frac{1}{2}\right)$.

(ii) Else

(a) *if $(1-(1-\beta)/\beta v) - \frac{3}{4} < 0$ then there exists a $\bar{q} \in [\frac{1}{4}, \frac{1}{3})$such that $\mathbb{E}[\tau^{I''}] \leq \min\{\mathbb{E}[\tau^I], \mathbb{E}[\tau^{I'}], \mathbb{E}[\tau^{\tilde{I}}]\}$ $\forall q \in [\frac{1}{4}, \bar{q}] \subset [0,1]$.*

(b) *if $(1-(1-\beta)/\beta v) - \frac{2}{3} < 0$ then there exists a $\bar{\bar{q}} \in [\frac{1}{3}, \frac{1}{2})$ such that $\mathbb{E}[\tau^{I'}] \leq \min\{\mathbb{E}[\tau^I], \mathbb{E}[\tau^{I''}], \mathbb{E}[\tau^{\tilde{I}}]\}$ $\forall q \in [\frac{1}{3}, \bar{\bar{q}}] \subset [0,1]$.*

Proposition 4 establishes sufficient conditions under which, for a moderate value of q, the intermediate repayment terms $I' = I/2$ and $I'' = I/3$ outperforms both the longer repayment term I and the shorter repayment term $\tilde{I} = I/4$. The result follows from the same two effects described in Section 4.1, i.e., the *installment effect* and the *uncertainty effect*. The moderate repayment term strikes a balance between the two effects and minimizes the expected time to ownership. This result holds particularly when income uncertainty q is moderate, ensuring that consumers have a sufficiently high probability of affording the moderate payment per installment.

To illustrate the result in Proposition 4, we present an example in Figure EC.7. We plot the expected time to ownership under the consumer's optimal repayment policy as a function of q for four different repayment terms: $I = 12$, $I' = I/2 = 6$, $I'' = I/3 = 4$ and $\tilde{I} = I/4 = 3$. The case $I = 12$, with an installment cost of $c = 1$, represents the setup studied in base model in Section 3. Figure EC.7 illustrates that the expected time to ownership is minimized under the intermediate repayment term I' and I'' for moderate income uncertainty $q \in [0.205, 0.37]$, reflecting Proposition 4 beyond its sufficient conditions.

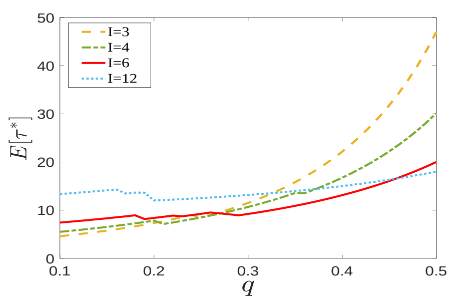


Figure EC.7 Optimal expected time to ownership under different repayment terms ($\beta = 0.5$, $v = 4$).

F.5. Random Product Usage Value per Period

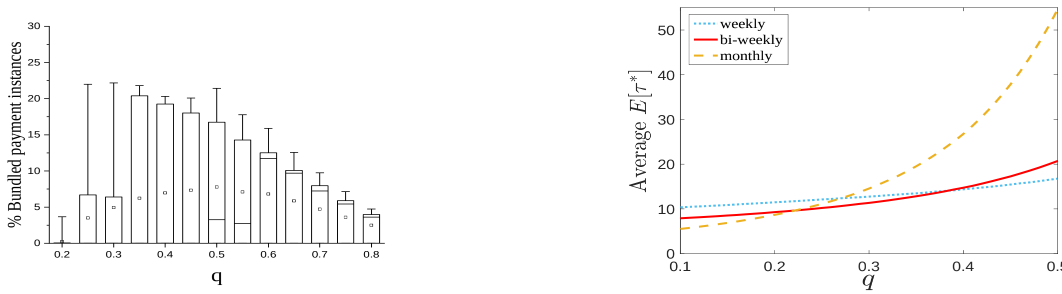
We consider a model, where the value derived in period t, denoted by W_t, is random and follows a Geometric distribution with mean $\mathbb{E}[W_t] = v > c = 1$. In each period, given the realized effective budget b_t and the realized per-period usage value w_t, the consumer chooses the number of installments x_t to pay. Then, the Bellman equation for this model is given by

$$V(a,o,b,w) = \max_{x \in \{0,\ldots,\min(o,b)\}} \left\{ w 1_{\{a+x\geq 1\}} - x + \beta \mathbb{E}_{B,W}[V((a+x-1)^+, o-x, B, W)] \right\}, \tag{EC.23}$$

with a terminal value $V(0,0,b,w) = w + \frac{\beta v}{1-\beta}$, where the state variable $w \in \{0,1,\ldots\}$ represents the realized product usage value in the current period.

Due to the analytical intractability of the model with random product usage value caused by the expansion of the state space, we rely on extensive numerical analysis to evaluate the robustness of our main findings under the model in (EC.23), where the per-period usage value is random.

F.5.1. Bundled Payments. We first numerically demonstrate that bundled payments continue to hold when the value per product usage is random across periods. Figure EC.8(a) shows the percentage of payments that are bundled across 1000 random budget instances and 1000 random per-period value instances. Each instance consists of 200 periods, which is sufficient for the consumer to attain ownership in all 1000 cases. We use the same 1000 budget instances as in Figures EC.3 and EC.4. The value in each period is drawn independently from a Geometric distribution



(a) Average percentage of bundled payments (b) Average expected time to ownership

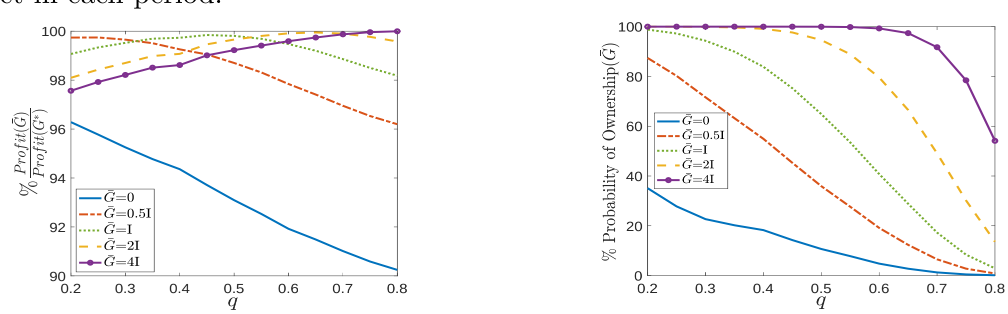
Figure EC.8 (a) Average percentage of bundled payments over 1000 random budget instances and 1000 random per-period product usage value instances and (b) Average expected time to ownership for random per-period product usage value, across problem instances (v, β) in summarized Table 1, for $I = 8$.

with parameter $\frac{1}{v+1}$, where v is selected from the values in Table 1, ensuring that the mean of random values remains $\mathbb{E}[W_t] = v$.

As shown in Figure EC.8(a), bundled payments remain prevalent even when the consumer experiences random usage value in each period. This supports our first main result: bundled payments exist even in the absence of transaction costs.

F.5.2. Moderate Flexibility in Repayment Frequency. Here, we numerically demonstrate the dominance of the bi-weekly repayment frequency in terms of the expected time to ownership when the income uncertainty, q, is moderate. Figure EC.8(b) reports the average expected time to ownership as a function of q. The reported averages range over 210 instances, with model parameters (v, β) varying as specified in Table 1, where $\mathbb{E}[W_t] = v$. The results show that bi-weekly repayment consistently outperforms both weekly and monthly designs for moderate levels of income uncertainty. This confirms that our second main result, i.e., the dominance of moderate repayment frequency under moderate uncertainty, remains valid even when the per-period product usage value is random.

F.5.3. Moderate Flexibility in Grace Period. We now verify the robustness of our third main result, that a moderate grace period balances the firm's profit and the consumer's access to the product, under the model with random per-period usage value. In Figure EC.9, we plot the firm's average expected profit ratio (relative to the profit under the optimal grace period G^*) and the average probability of ownership as functions of a uniform grace period $\bar{G}$. These averages are computed over instances where the optimal grace period $G^* \geq 0$, using model parameters $(v, \beta, \alpha, \delta)$ from Table 2. Figure EC.9 shows that an intermediate grace period, such as $\bar{G} = I$ or $\bar{G} = 2I$, significantly improves consumer access to the product—by more than 50%—while reducing the firm's expected profit by less than 3%. It even increases profit compared to offering no grace period $(\bar{G} = 0)$. Thus, this result also holds when the consumer derives a random usage value from the product in each period.



(a) Average expected profit ratio (b) Average probability of ownership

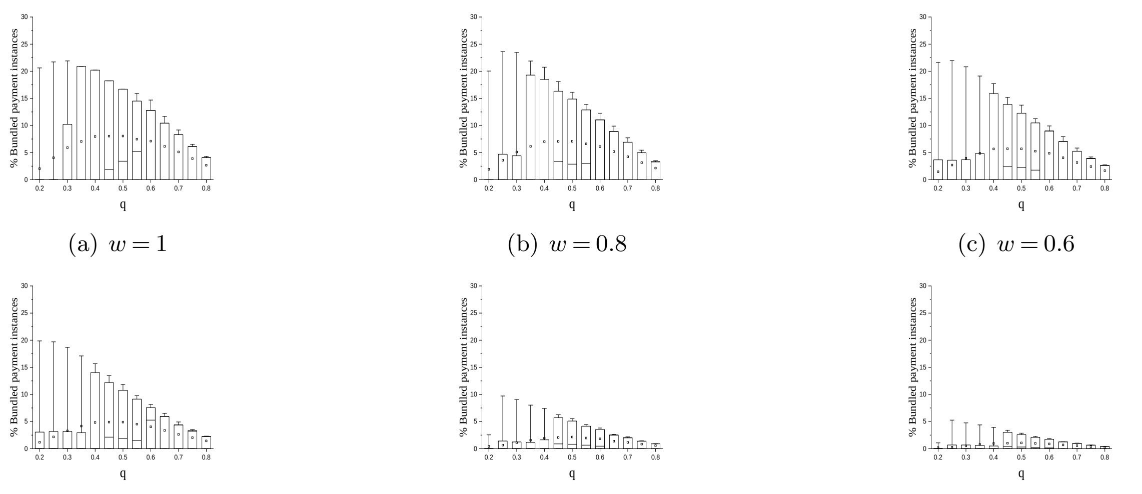
Figure EC.9 Impact of uniform grace periods on (a) firm's average expected profit ratio w.r.t. firm's optimal profit and (b) consumer's probability of ownership, for random per-period value and $I = 8$.

F.6. Allowing Negative Effective Budget

In the base model, we assume that the consumer's effective budget is nonnegative. Here, we relax this assumption to allow for negative effective budget, which may arise when total income is insufficient to satisfy more essential needs than the RTO product. In such cases, the consumer may prefer to save her budget for future essential needs rather than commit it to advance RTO payments. We consider the savings model defined in (EC.18) with a new budget distribution that allows for negative values, as described below. Recall that in this model, the consumer can save any remaining budget after meeting her immediate consumption needs.

We define the budget distribution as a transformed geometric distribution with parameter $q \in [0,1]$ and weight $w \in [0,1]$ to allow for negative effective budget realizations. Specifically, $\mathbb{P}(B_t = 0) = q$, $\mathbb{P}(B_t = i) = wq(1-q)^i$, $\mathbb{P}(B_t = -i) = (1-w)q(1-q)^i$, for each $i \in \{1, 2, \ldots\}$, where $\sum_{i\in\mathbb{Z}} \mathbb{P}(B_t = i) = 1$. Here, w is the weight assigned to the positive part of the geometric distribution. When $w = 1$, the distribution reduces to the savings model in (EC.18). As w decreases, the probability of negative effective budget increases. When $w = 0$, the distribution becomes a fully mirrored geometric distribution supported only on negative values. This transformed geometric distribution preserves a key feature of income in LMICs: the probability of observing large positive or negative budget realizations decays exponentially.

Figure EC.10 illustrates the percentage of bundled payments across 1000 randomly generated budget paths for different weights, given a per-period immediate consumption level of $k = 4$, a moderate value used in Section F.1.1. We use the same 1000 budget samples from the savings model's geometric distribution, adjusted according to the weights described above. Each sample consists of 200 periods, which is sufficient to reach ownership in all instances. Figure EC.10(a) corresponds to the savings model in (EC.18), where the budget is nonnegative. In contrast, Figures EC.10(b)-(f) consider budget distributions with support on both positive and negative values.



(a) $w = 1$ (b) $w = 0.8$ (c) $w = 0.6$

(d) $w = 0.5$ (e) $w = 0.2$ (f) $w = 0.1$

Figure EC.10 Average percentage of bundled payments over 1000 random budget instances for data points that correspond to the problem instances (v, β) summarized in Table 1, for $I = 8$ and $k = 4$.

We observe that bundled payments occur in this model when w is nonzero. In particular, for moderate to high values of w, such as $w > 0.5$, bundled payments occur frequently, similar to the savings model without negative budget mass (i.e., $w = 1$). As w decreases to smaller values, such as $w = 0.2$ and $w = 0.1$, bundled payments become less frequent and eventually disappear entirely when $w = 0$, since there is no positive budget mass. Overall, the numerical results in Figure EC.10 suggest that our main result on the occurrence of bundled payments in the absence of transaction costs continues to hold when the effective budget can be negative, as long as w is not too small, meaning that enough mass is placed on positive budget realizations.