Online Appendix to: Managing Payment Flexibility in Rent-to-Own Contracts for Off-Grid Energy Products

This online appendix presents auxiliary results that are useful for proving the results in the paper, as well as the proofs for the results in Section F.4.

G. Auxiliary Results for Proofs of Section 3

The following lemmas are useful in the proof of unimodality of the value function in Lemma 1.

Lemma 3 (Auxiliary Results)**.** *The following results hold,*

(i) Order-up-to 1 is the optimal policy for state (0,1), hence $\mathbb{E}_B[V(0,1,B)] = \frac{1-q}{1-\beta q}\left(\frac{v}{1-\beta} - 1\right)$.

(ii) $\mathbb{E}_B[V(0,2,B)] \geq \frac{1-q}{1-\beta q}\left(v - 1 + \beta\frac{1-q}{1-\beta q}\left(\frac{v}{1-\beta}-1\right)\right)$.

(iii) $\mathbb{E}_B[V(a,1,B)] \geq qv\left(\frac{1-(\beta q)^a}{1-\beta q}\right) + \left(\frac{v}{1-\beta}-1\right)\left(\frac{1-q}{1-\beta q}\right)$.

(iv) Assume $\beta \leq 1/(1+q)$*, then* $\frac{\beta(1-q)}{(1-\beta q)^2} \leq 1$ *for all* $q \in [0,1]$.

Proof. (*i*) Since $v > 1$ and $a = 0$, then in state $(0,1)$ it is optimal to make a payment as soon as the budget allows, hence

$$\mathbb{E}_B[V(0,1,B)] = q\beta\mathbb{E}_B[V(0,1,B)] + (1-q)\left(v - 1 + \beta\frac{v}{1-\beta}\right) \iff \mathbb{E}_B[V(0,1,B)] = \frac{1-q}{1-\beta q}\left(\frac{v}{1-\beta}-1\right),$$

where the first equality follows the Bellman equation (3) and (4), and the equivalence follows by simplifying the expression, completing the proof of part (*i*).

(*ii*) We lower bound $\mathbb{E}_B[V(0,2,B)]$ using a feasible order-up-to 1 policy, denoted by Π^1,

$$\mathbb{E}_B[V(0,2,B)] \geq \mathbb{E}_B^{\Pi^1}[V(0,2,B)] = q\beta\mathbb{E}_B^{\Pi^1}[V(0,2,B)] + (1-q)(v - 1 + \beta\mathbb{E}_B[V(0,1,B)])$$
$$\iff \mathbb{E}_B[V(0,2,B)] \geq \mathbb{E}_B^{\Pi^1}[V(0,2,B)] = \frac{1-q}{1-\beta q}\left(v - 1 + \beta\frac{1-q}{1-\beta q}\left(\frac{v}{1-\beta}-1\right)\right),$$

where the first inequality follows since order-up-to 1 is feasible for the state $(0,2)$, and the first equality follows since, from (*i*), order-up-to 1 is optimal for the state $(0,1)$. The equivalence follows from simplifying the expression and replacing part (*i*), completing the proof of part (*ii*).

(*iii*) The proof is by induction on a. For $a = 0$, the result follows from part (*i*). Assume that the statement is true for $a \geq 0$, i.e,

$$\mathbb{E}_B[V(a,1,B)] \geq qv\left(\frac{1-(\beta q)^a}{1-\beta q}\right) + \left(\frac{v}{1-\beta}-1\right)\left(\frac{1-q}{1-\beta q}\right), \tag{EC.24}$$

then we show that the statement also holds for $a+1$. We lower bound $\mathbb{E}_B[V(a+1,1,B)]$ using a feasible policy, denoted by Π, which makes a payment in the current period if the budget allows and follows the optimal policy in future periods. Then,

$$\begin{aligned}\mathbb{E}_B[V(a+1,1,B)] \geq& \mathbb{E}_B^{\Pi}[V(a+1,1,B)] = qv + \beta q\mathbb{E}_B[V(a,1,B)] + (1-q)\left(\frac{v}{1-\beta}-1\right)\\ \geq& qv + \beta q\left(qv\left(\frac{1-(\beta q)^a}{1-\beta q}\right) + \left(\frac{v}{1-\beta}-1\right)\left(\frac{1-q}{1-\beta q}\right)\right) + (1-q)\left(\frac{v}{1-\beta}-1\right)\\ =& qv\left(\frac{1-(\beta q)^{a+1}}{1-\beta q}\right) + \left(\frac{v}{1-\beta}-1\right)\left(\frac{1-q}{1-\beta q}\right),\end{aligned}$$

where the first inequality follows since Π is a feasible policy for the state (a+1,1), the first equality follows by the definition of Π, the second inequality follows from the induction assumption (EC.24), and the second equality follows from simplifying the expression, completing the proof of part (*iii*).

(*iv*) $\beta(1-q)/(1-\beta q)^2 \leq 1$ and $\beta \leq 1/(1+q)$ is equivalent to $\beta^2 q^2 - \beta q + 1 - \beta \geq 0$, which we show now.

First, note that $\beta \leq 1/(1+q)$ implies that $\beta^2 q^2 - \beta q + 1 - \beta$ is decreasing. Indeed,

$$\frac{\partial(\beta^2 q^2 - \beta q + 1 - \beta)}{\partial\beta} = 2\beta q^2 - q - 1 < 0 \iff \beta < \frac{1+q}{2q^2} \tag{EC.25}$$

where the inequality holds for all $\beta \le 1/(1+q)$ and $q \in [0,1]$, since then $\beta \le 1/(1+q) < (1+q)/2q^2$. Indeed, the latter inequality is equivalent to $q^2 < 2q+1$ which holds for all $q \in [0,1]$.

Therefore, it is enough to verify $\beta^2 q^2 - \beta q + 1 - \beta \ge 0$ at the upper bound $\beta = 1/(1+q)$, which evaluates to

$$\beta^2 q^2 - \beta q + 1 - \beta\big|_{\beta=1/(1+q)} = \frac{q^2}{(1+q)^2} - \frac{q}{1+q} + 1 - \frac{1}{1+q} = \frac{q^2}{(1+q)^2} \ge 0, \tag{EC.26}$$

where the second equality follows since the last three terms on the left hand side add up to 0, and the inequality holds for all $q \in [0,1]$, concluding the proof of part (iv). □

Lemma 4 (Concavitity of the Value Function in the Number of Installments)**.** *Assume $\beta \le 1/(1+q)$, then $\mathbb{E}_B[V(0,o,B)] - \mathbb{E}_B[V(0,o+1,B)]$ is decreasing in o for all $o \in \{0,\dots,\}$.*

Proof. Assuming $\beta \le 1/(1+q)$, we show the following equivalent statement to the lemma,

$$\mathbb{E}_B[V(0,o-1,B)] - \mathbb{E}_B[V(0,o,B)] \ge \mathbb{E}_B[V(0,o,B)] - \mathbb{E}_B[V(0,o+1,B)]. \tag{EC.27}$$

for all $o \in \{1,\dots\}$. We prove (EC.27) by modifying the optimal policy starting from state $(0,o)$, which we denote by $\Pi^*(o)$, to construct feasible policies starting from states $(0,o-1)$ and $(0,o+1)$, which we denote by $\hat{\Pi}(o-1)$ and $\hat{\Pi}(o+1)$, respectively, and showing that

$$\mathbb{E}_B[V^{\hat{\Pi}(o-1)}(0,o-1,B)] - \mathbb{E}_B[V(0,o,B)] \ge \mathbb{E}_B[V(0,o,B)] - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(0,o+1,B)]. \tag{EC.28}$$

Indeed, we can then prove (EC.27) from the following chain of inequalities,

$$\begin{aligned}&\mathbb{E}_B[V(0,o-1,B)] - \mathbb{E}_B[V(0,o,B)] \ge \mathbb{E}_B[V^{\hat{\Pi}(o-1)}(0,o-1,B)] - \mathbb{E}_B[V(0,o,B)]\\ \ge&\mathbb{E}_B[V(0,o,B)] - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(0,o+1,B)] \ge \mathbb{E}_B[V(0,o,B)] - \mathbb{E}_B[V(0,o+1,B)],\end{aligned} \tag{EC.29}$$

where the first and last inequalities follow from the feasibility of the policies $\hat{\Pi}(o-1)$ and $\hat{\Pi}(o+1)$, respectively. While the second inequality is given by (EC.28). We show both of these now.

We first define the feasible policies $\hat{\Pi}(o-1)$ and $\hat{\Pi}(o+1)$ as follows. Both $\hat{\Pi}(o-1)$ and $\hat{\Pi}(o+1)$ mimic $\Pi^*(o)$ except $\hat{\Pi}(o-1)$ does not make the last payment made by $\Pi^*(o)$, while $\hat{\Pi}(o+1)$ makes one additional payment optimally after $\Pi^*(o)$ attains ownership.

From their definition, $\hat{\Pi}(o-1)$ and $\hat{\Pi}(o+1)$ are feasible and (EC.28) is reduced as follows. Let $\hat{t}$ be the period where $\hat{\Pi}(o-1)$ reaches ownership, and $a_{\hat{t}}$ be the advanced payments induced by $\Pi^*(o)$. Then, (EC.28) reduces to

$$\begin{aligned}&\mathbb{E}_B[V^{\hat{\Pi}(o-1)}(a_{\hat{t}},0,B)] - \mathbb{E}_B[V(a_{\hat{t}},1,B)] \ge \mathbb{E}_B[V(a_{\hat{t}},1,B)] - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(a_{\hat{t}},2,B)]\\ \iff& \frac{v}{1-\beta} - \mathbb{E}_B[V(a_{\hat{t}},1,B)] \ge \mathbb{E}_B[V(a_{\hat{t}},1,B)] - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(a_{\hat{t}},2,B)].\end{aligned} \tag{EC.30}$$

We now show (EC.30) by considering two possible cases: (1) Let Ω_1 denote all the sample paths where there exists a first time after $\hat{t}$ such that the advanced payments induced by $\Pi^*(o)$ are fully consumed before its last payment, and (2) Let Ω_1^c denote its complement.

For case (1), we can, without loss of generality, focus on the first time after $\hat{t}$ such that the advanced payments induced by $\Pi^*(o)$ are fully consumed before its last payment. Then, (EC.30) reduces to

$$\begin{aligned}&\frac{v}{1-\beta} - \mathbb{E}_B[V(a_{\hat{t}},1,B) \mid \Omega_1] \ge \mathbb{E}_B[V(a_{\hat{t}},1,B) \mid \Omega_1] - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(a_{\hat{t}},2,B) \mid \Omega_1]\\ \iff& \frac{v}{1-\beta} - \mathbb{E}_B[V(0,1,B) \mid \Omega_1] \ge \mathbb{E}_B[V(0,1,B) \mid \Omega_1] - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(0,2,B) \mid \Omega_1]\\ \iff& \frac{v}{1-\beta} - \frac{1-q}{1-\beta q}\left(\frac{v}{1-\beta} - 1\right) \ge \frac{1-q}{1-\beta q}\left(\frac{v}{1-\beta} - 1\right) - \frac{1-q}{1-\beta q}\left(v - 1 + \beta\frac{1-q}{1-\beta q}\left(\frac{v}{1-\beta} - 1\right)\right)\\ \iff& qv\left(1 - \frac{\beta(1-q)}{(1-\beta q)^2}\right) + \frac{(1-\beta)(1-q)}{(1-\beta q)^2} \ge 0,\end{aligned}$$

where the first equivalence follows from the definition of Ω_1, i.e., since we focus on the first time after $\hat{t}$ such that the advanced payments induced by $\Pi^*(o)$ are fully consumed before its last payment, then no payments have been made since $\hat{t}$. The second equivalence follows from Lemma 3(i) and because $\hat{\Pi}(o+1)$ follows the policy in the lower bound in Lemma 3(ii). The third equivalence follows from simplifying the expression. Finally, the last inequality holds for any $\beta \leq 1/(1+q)$ from Lemma 3(iv), completing the proof in case (1).

For case (2), let $\tilde{t}$ be the period when $\Pi^*(o)$ makes its last payment. Then, (EC.30) reduces to

$$\frac{v}{1-\beta} - \mathbb{E}_B[V(a_{\hat{t}},1,B) \mid \Omega_1^c] \geq \mathbb{E}_B[V(a_{\hat{t}},1,B) \mid \Omega_1^c] - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(a_{\hat{t}},2,B) \mid \Omega_1^c]$$
$$\iff 1 \geq \frac{v}{1-\beta} - \mathbb{E}_B[V^{\hat{\Pi}(o+1)}(a_{\tilde{t}},1,B) \mid \Omega_1^c],$$

where the equivalence follows from the definition of Ω_1^c, i.e., after $\hat{t}$ the advanced payments induced by $\Pi^*(o)$ are not fully consumed before its last payment, then no payments have been made from $\hat{t}$ to $\tilde{t}$ and one payment is made at $\tilde{t}$. Further, the latter payment achieves ownership for $\Pi^*(o)$. Finally, the inequality holds again from the definition of Ω_1^c, and the observation that the additional payment made optimally by $\hat{\Pi}(o+1)$ is made at a period weakly after $\tilde{t}$, completing the proof in case (2) and thus the proof of the lemma. □

Lemma 5 (Structural Property of the Value Function)**.** $\beta\mathbb{E}_B[V((y-1)^+,a+o-y,B)]-\mathbb{E}_B[V(y,a+o-y-1,B)]$ *is decreasing in* y *for all* $y \in \{a,\ldots,a+o-2\}$, $a \geq 0$, $o \geq 2$, $a+o \leq I$.

Proof. We show the following equivalent statement to the lemma,

$$\begin{aligned}&\mathbb{E}_B[V(y+1,a+o-y-2,B)] - \mathbb{E}_B[V(y,a+o-y-1,B)]\\&\geq \beta(\mathbb{E}_B[V(y,a+o-y-1,B)] - \mathbb{E}_B[V((y-1)^+,a+o-y,B)]),\end{aligned} \tag{EC.31}$$

for all $y \in \{a,\ldots,a+o-2\}$. We prove (EC.31) by modifying the optimal policy starting from state $(y,a+o-y-1)$, which we denote by $\Pi^*(y)$, to construct feasible policies starting from states $(y+1,a+o-y-2)$ and $((y-1)^+,a+o-y)$, which we denote by $\hat{\Pi}(y+1)$ and $\hat{\Pi}(y-1)$, respectively, and showing that

$$\begin{aligned}&\mathbb{E}_B[V^{\hat{\Pi}(y+1)}(y+1,a+o-y-2,B)] - \mathbb{E}_B[V(y,a+o-y-1,B)]\\&\geq \beta(\mathbb{E}_B[V(y,a+o-y-1,B)] - \mathbb{E}_B[V^{\hat{\Pi}(y-1)}((y-1)^+,a+o-y,B)]).\end{aligned} \tag{EC.32}$$

Indeed, we can then prove (EC.31) analogously to (EC.29) in Lemma 4.

We first define the feasible policies $\hat{\Pi}(y+1)$ and $\hat{\Pi}(y-1)$ in tandem as follows. Both $\hat{\Pi}(y+1)$ and $\hat{\Pi}(y-1)$ mimic $\Pi^*(y)$ except for at most one deviation each. Let a_t^{y-1} denote the advanced payments induced by $\hat{\Pi}(y-1)$ in period t, and x_t^* denote $\Pi^*(y)$'s optimal payment in period t. On the one hand, (1) $\hat{\Pi}(y-1)$ attempts to makes an additional payment with respect to $\Pi^*(y)$ in the earliest period t' such that $a_{t'}^{y-1}=0$ and $x_{t'}^*=0$; if the budget allows such a payment $\hat{\Pi}(y-1)$ mimics $\Pi^*(y)$ forever after, if not then $\hat{\Pi}(y-1)$ acts optimally forever after. Alternatively, if no such period t' exists (i.e., $x_t^*>0$ in each period t such that $a_t^{y-1}=0$, if any), then (2) $\hat{\Pi}(y-1)$ follows the optimal policy to make the last payment starting from the period and state when $\Pi^*(y)$ attains ownership. On the other hand, (1) $\hat{\Pi}(y+1)$ makes one less payment than $\Pi^*(y)$ in period $t'+1$ if $x_{t'+1}^*>0$ and mimics $\Pi^*(y)$ forever after, if $x_{t'+1}^*=0$ then $\hat{\Pi}(y+1)$ acts optimally forever after. Alternatively, if no such period t' exists, then (2) $\hat{\Pi}(y-1)$ does not make the last payment made by $\Pi^*(y)$.

Since $\hat{\Pi}(y+1)$ and $\hat{\Pi}(y-1)$ are feasible by definition, we now conclude the proof by showing (EC.32) in the two possible cases discussed in their definition: (1) Let denote all the sample paths such that there a first period t' such that $a_{t'}^{y-1}=0$ and $x_{t'}^*=0$, and (2) Let Ω_1^c denote its complement.

For Case (1), let a_t^y and o_t^y denote the advanced and outstanding payments induced by $\Pi^*(y)$ in period t, respectively. Note that the conditions of case (1) imply that $\hat{\Pi}(y-1)$ and $\hat{\Pi}(y+1)$ have not deviated from $\Pi^*(y)$ up to periods $\hat{t}-1$ and $\hat{t}$, respectively (hence $a_{t'}^y=1$ and $o_{t'}^y \geq 1$). Then,

$$\begin{aligned}&\mathbb{E}_B[V^{\hat{\Pi}(y+1)}(y+1,a+o-y-2,B) \mid \Omega_1] - \mathbb{E}_B[V(y,a+o-y-1,B) \mid \Omega_1]\\&\geq \beta(\mathbb{E}_B[V(y,a+o-y-1,B) \mid \Omega_1] - \mathbb{E}_B[V^{\hat{\Pi}(y-1)}((y-1)^+,a+o-y,B) \mid \Omega_1])\\\iff &\beta^{t'}((1-q_o)1 + q(v+\beta\mathbb{E}_B[V^{\hat{\Pi}(y+1)}(0,o_{t'}^y-1,B) \mid \Omega_1] - \beta\mathbb{E}_B[V(0,o_{t'}^y,B) \mid \Omega_1]))\\&\geq \beta(\beta^{t'-1}((1-q)1 + q_o(v+\beta\mathbb{E}_B[V(0,o_{t'}^y,B) \mid \Omega_1] - \beta\mathbb{E}_B[V^{\hat{\Pi}(y-1)}(0,o_{t'}^y+1,B)) \mid \Omega_1]))\\\iff &\mathbb{E}_B[V^{\hat{\Pi}(y+1)}(0,o_{t'}^y-1,B) \mid \Omega_1] - \mathbb{E}_B[V(0,o_{t'}^y,B) \mid \Omega_1] \geq \mathbb{E}_B[V(0,o_{t'}^y,B) \mid \Omega_1] - \mathbb{E}_B[V^{\hat{\Pi}(y-1)}(0,o_{t'}^y+1,B)) \mid \Omega_1],\\\iff &\mathbb{E}_B[V(0,o_{t'}^y-1,B) \mid \Omega_1] - \mathbb{E}_B[V(0,o_{t'}^y,B) \mid \Omega_1] \geq \mathbb{E}_B[V(0,o_{t'}^y,B) \mid \Omega_1] - \mathbb{E}_B[V(0,o_{t'}^y+1,B)) \mid \Omega_1],\end{aligned}$$

where the first equivalence follows since, for the right hand side of the inequality, by definition $\hat{\Pi}(y-1)$ makes a payment in period t' with probability $1-q$ incurring in an additional cost of 1 with respect to $\Pi^*(y)$ and mimics the latter attaining the same outcomes forever after. While with probability q no budget is available for $\hat{\Pi}(y-1)$ to make a payment in period t', hence $\hat{\Pi}(y-1)$ loses a value v with respect to $\Pi^*(y)$ in period t', and from period $t'+1$ on acts optimally. Analogously, for the left hand side, since by assumption $a^y_{t'}=1$ and $x^*_{t'}=0$, then $a^y_{t'+1}=0$ and $\Pi^*(y)$ makes at least one payment in period $t'+1$, if the budget allows. Hence, by definition, $\hat{\Pi}(y+1)$ makes one less payment in period t' with probability $1-q$ incurring in a saving of 1 with respect to $\Pi^*(y)$ and mimics the latter attaining the same outcomes forever after. While with probability q no budget is available for $\Pi^*(y)$ to make a payment in period $t'+1$, hence $\hat{\Pi}(y+1)$ wins a value v with respect to $\Pi^*(y)$ in period t', and from period $t'+1$ on acts optimally. The second equivalence follows by simplifying the inequality. The third equivalence follows since, by definition, both $\hat{\Pi}(y-1)$ and $\hat{\Pi}(y+1)$ act optimally in these continuation paths. Finally, the last inequality holds from the concavity of the value function in the number of installments, shown in Lemma 4.

Case (2) is analogous to Case (2) in Lemma 4, and thus omitted, completing the proof. □

The following proposition and Lemma are useful in the proof of Theorem 2.

Proposition 5 (Ordered Thresholds). *$\forall k\in\{2,...,I-1\}$ and $\beta\le\frac{1}{1+q}$, if $\beta\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)\le 1$ then*

$$1\ge\beta\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)\ge\beta^k q^k\mathbb{E}\left[V(k,0,B)\right].$$

Proof. We show that the optimal objective value of the minimization problem below is non-negative to conclude the proof. The inequality in the statement then follows by multiplying the optimal objective value by $\beta^{k-1}q^{k-1}\ge 0$.

$$\min_{q,\beta}\quad\frac{\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)}{\beta^{k-2}q^{k-1}}-\beta q\mathbb{E}\left[V(k,0,B)\right]\tag{EC.33}$$

$$\text{s.t.}\quad\beta\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)\le 1\tag{EC.34}$$

$$0\le q_o\le 1\tag{EC.35}$$

$$0\le\beta\le 1.\tag{EC.36}$$

First, we show that (i) at optimality (EC.34) is tight if and only if (EC.35) is tight. Next, we show that (ii) (EC.33) is decreasing in q, which combined with (i) implies constraints (EC.34) and (EC.35) must be tight at optimality. Finally we conclude the proof by showing that (iii) the optimal value of the objective function (EC.33) is non-negative when $\beta\le\frac{1}{1+q}$.

(i) First, we show that if $\beta\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)=1$ then $q=1$ at optimality. If $\beta\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)=1$ then the objective function in (EC.33) becomes $Obj_1=1/(\beta^{k-1}q^{k-1})-\beta qv/(1-\beta)$, where the first term follows from the assumption and the second term follows since $\mathbb{E}\left[V(k,0,B)\right]=v/(1-\beta)$, which the latter follows from (4). We now take the first order derivative of Obj_1 w.r.t q to get

$$\frac{\partial Obj_1}{\partial q}=\frac{-(k-1)}{\beta^{k-1}q^k}-\frac{\beta v}{1-\beta}<0,$$

where the inequality follows since $q,\beta\in[0,1]$ and $k\in\{2,...,I-1\}$. Therefore, the objective function in (EC.33) is decreasing in q when $\beta\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)=1$, thus at optimality we must have $q=1$. Namely, at optimality if (EC.34) is tight then (EC.35) must be tight. To complete the proof of part (i), we show that if $q=1$ then $\beta\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)=1$ at optimality. If $q=1$, then the consumer's future income will be zero and she only gets the value for her advanced payments. Therefore, for all $\beta\in[0,1)$ we get

$$\lim_{I\to\infty}\mathbb{E}\left[V(k-1,I-k,B)\right]]=\sum_{i=0}^{k-2}\beta^i v=\frac{1-\beta^{k-1}}{1-\beta}v,\tag{EC.37}$$

Thus, for all $\beta\in[0,1)$, we have $\lim_{I\to\infty}\left(\mathbb{E}\left[V(k-1,I-k,B)\right]-\mathbb{E}\left[V(k-2,I-k+1,B)\right]\right)=v(1-\beta^{k-1})/(1-\beta)-v(1-\beta^{k-2})(1-\beta)=v\beta^{k-2}$ and the objective function in (EC.33) becomes

$Obj_2 = (1 - \beta/(1-\beta))v$. We take the first order derivative of Obj_2 w.r.t β to get $\frac{\partial Obj_2}{\partial \beta} = -\frac{1-\beta+\beta}{(1-\beta)^2} v < 0$. For $\beta = 1$,

$$\lim_{I\to\infty} (\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)]) = \sum_{i=0}^{k-2} \beta^i v - \sum_{i=0}^{k-3} \beta^i v = v, \tag{EC.38}$$

where the first equality is given by $q = 1$ (see the first equality in EC.37), the second equality follows from $\beta = 1$. Thus, the objective function in (EC.33) becomes $Obj_3 = (1/\beta^{k-2} - \beta/(1-\beta))v$ when $\beta = 1$ and this function is decreasing in β since $\partial Obj_3/\partial\beta = -((k-2)/\beta^{k-1} + 1/(1-\beta)^2)v < 0$.

Therefore, when $q = 1$ the objective function in (EC.33) is decreasing in β, thus at optimality we must have that at least one of (EC.34) or (EC.36) is tight. If (EC.34) is tight that would suffice. Therefore, assume that (EC.36) is tight, i.e., $\beta = 1$, then we show that constraint (EC.34) is violated, a contradiction. Specifically, from (EC.38) we have $\lim_{I\to\infty}(\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)]) = v > 1$, where the inequality follows from $v > 1$. The former implies that constraint (EC.34) is violated when (EC.36) is tight. Therefore, at optimality if (EC.35) is tight then (EC.34) must be tight, completing the proof of (i).

(ii) We now show that the objective function in (EC.33) is decreasing in q. Indeed, both terms in (EC.33) are decreasing in q.

Note that the second term in the objective function is equal to $-\beta q \frac{v}{1-\beta}$ following from (4), which is decreasing in q. For the first term, note that

$$\begin{aligned}
&\frac{\partial}{\partial q} \frac{\lim_{I\to\infty} (\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)])}{\beta^{k-2} q^{k-1}} \\
&= \frac{\frac{\partial \lim_{I\to\infty} (\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)])}{\partial q}}{\beta^{k-2} q^{k-1}} \\
&\quad - \frac{\lim_{I\to\infty} (\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)]) \frac{\partial \beta^{k-2} q^{k-1}}{\partial q}}{(\beta^{k-2} q^{k-1})^2} \\
&\leq \frac{\frac{\partial \beta^{k-2} q^{k-1} v}{\partial q}}{\beta^{k-2} q^{k-1}} - \frac{\beta^{k-2} q^{k-1} v \frac{\partial \beta^{k-2} q^{k-1}}{\partial q}}{(\beta^{k-2} q^{k-1})^2} = \frac{(k-1)v}{q} - \frac{(k-1)v}{q} = 0,
\end{aligned}$$

where all the equalities follow from differentiation rules and elemental algebra. The inequality follows from two related observations:

(1) $\lim_{I\to\infty} (\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)]) \geq \beta^{k-2} q^{k-1} v$ since $\beta^{k-2} v$ is the value of the left hand side of the inequality along the coupled sample path where no positive budget is realized in the first $k-1$ periods, which occurs with probability q^{k-1}, and

(2) $\dfrac{\partial \lim_{I\to\infty} (\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)])}{\partial q} \leq \dfrac{\partial \beta^{k-2} q^{k-1} v}{\partial q}$ since this coupled sample path attains the largest product of the realized difference in the left hand side of the inequality times the marginal effect of q on the probability the sample path. Indeed, from the optimality of an order-up-to $\bar{a}(a+o)$ policy in Theorem 1, the same order-up-to policy is followed in both states $(k-1, I-k)$ and $(k-2, I-k+1)$ since both satisfy $a + o = k-1$. It follows that the only other possible realized difference in the left hand side of the inequality corresponds to saving a payment instead, which, since the budget has a geometric distribution, occurs with probability proportional to $\mathbb{P}(B > b) = (1-q)^b$ for some budget value b. The latter is *decreasing* in q, thus the product of the realized difference in the left hand side of the inequality times the marginal effect of q on the probability the sample path is upper bounded by $\partial \beta^{k-2} q^{k-1} v/\partial q \geq 0$ in this case as well, completing the proof of (ii).

(iii) To complete the proof, we show that if $\beta \leq \frac{1}{1+q}$ then the optimal value of the objective function (EC.33) (Obj^*) is non-negative. Specifically, from (i), (ii) and (EC.37) it follows that we can replace $q^* = 1$ and $\beta^* \lim_{I\to\infty} (\mathbb{E}[V(k-1, I-k, B)] - \mathbb{E}[V(k-2, I-k+1, B)]) = {\beta^*}^{k-1} v = 1$ in the objective function (EC.33) to get

$$Obj^* = \frac{1}{{\beta^*}^{k-1}} - \frac{\beta^*}{1-\beta^*} \frac{1}{{\beta^*}^{k-1}} = \frac{1 - 2\beta^*}{(1-\beta^*){\beta^*}^{k-1}},$$

where the firs equality follows by replacing $v = 1/{\beta^*}^{k-1}$ and the second equality follows from simplifying the expression.

Furthermore, since $\beta^* \in [0,1]$ then $Obj^* \geq 0$ if and only if $1-2\beta^* \geq 0$. The latter inequality follows from the unimodality condition $\beta \leq \frac{1}{1+q} = \frac{1}{2}$ when $q=1$. Thus, we conclude that $1-2\beta^* \geq 0$ for all $k \in \{2,...,I-1\}$, or equivalently the optimal value of the objective function $Obj^* \geq 0$, completing the proof. ☐

Lemma 6 (Necessary and Sufficient Conditions for Optimality)**.** *For all $a+o=l \in \{2,\dots,I\}$,*

(I) The policy from Theorem 2(A) is optimal if and only if $\forall l \in \{2,\dots,I\}$

$$\beta\left(\mathbb{E}_B[V(1,l-2,B)-\mathbb{E}_B[V(0,l-1,B)]\right) \leq 1, \tag{EC.39}$$

(II) The policy from Theorem 2(B) is optimal if and only if

(i) $\forall l \in \{2,\dots,k\}$

$$1 \leq \beta\left(\mathbb{E}_B[V(l-1,0,B)-\mathbb{E}_B[V(l-2,1,B)]\right) \tag{EC.40}$$

(ii) $\forall l \in \{k+1,\dots,i\}$

$$1 \leq \beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right) \quad \textit{and} \tag{EC.41}$$

$$\beta\left(\mathbb{E}_B[V(k,l-k-1,B)-\mathbb{E}_B[V(k-1,l-k,B)]\right)1_{\{l<I\}} < 1, \tag{EC.42}$$

(iii) $\forall l \in \{i+1,\dots,I\}$

$$1 \leq \beta\left(\mathbb{E}_B[V(k-2,l-k+1,B)-\mathbb{E}_B[V(k-3,l-k+2,B)]\right) \quad \textit{and} \tag{EC.43}$$

$$\beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)1_{\{l<I\}} < 1. \tag{EC.44}$$

Proof. We prove (I), $(II).(i)$, $(II).(ii)$ and $(II).(iii)$ separately. In each case, we use the unimodality of the reformulated value function (5) in y (see Lemma 1) and the optimal order-up-to policy as a function of $a+o$ only (see Theorem 1) to derive the equivalences in the lemma.

(I) For all $l \in \{2,\dots,I\}$, we show that the optimal order-up-to policy is $\bar{a}=1$ if and only if (EC.39) holds. Given the unimodality of the value function (5) (see Lemma 1), it is necessary and sufficient to compare order-up-to 1 with no-ordering and order-up-to 2. The former always holds as in the first part of the proof of Lemma 1. Thus, it is necessary and sufficient to consider the latter. Given the optimal order-up-to policy as a function of $a+o$ (see Theorem 1), then, without loss of generality, it is sufficient to show that order-up-to $\bar{a}=1$ is optimal at state $a=0$ and $o=l$, to conclude about all other states where $a+o=l$. Hence, order-up-to 1 is optimal if and only if

$$v-2+\beta\mathbb{E}_B[V(1,l-2,B)] \leq v-1+\beta\mathbb{E}_B[V(0,l-1,B)] \iff \beta\left(\mathbb{E}_B[V(1,l-2,B)-\mathbb{E}_B[V(0,l-1,B)]\right) \leq 1, \tag{EC.45}$$

concluding the proof of part (I).

(II) (i) For all $l \in \{2,\dots,k\}$, we show that the optimal order-up-to policy is $\bar{a}=\min(l,k)=l$ if and only if (EC.40) holds. The unimodality of the value function (5) from Lemma 1 implies it is necessary and sufficient to compare order-up-to l with order-up-to $l-1$ and $l+1$ to conclude about its optimality. Since $a+o=l \leq k$, then no more than l payments are left and the comparison with order-up-to $l+1$ is redundant in this case. Furthermore, since from Theorem 1 the optimal order-up-to policy is a function of $a+o$, it is sufficient to show that order-up-to $\bar{a}=l$ policy is optimal at state $a=0$ and $o=l$ to conclude about all other states where $a+o=l$.

Hence, order-up-to $\bar{a}=l$ is optimal if and only if

$$v-(l-1)+\beta\mathbb{E}_B[V(l-2,1,B)] \leq v-l+\beta\mathbb{E}_B[V(l-1,0,B)] \iff 1 \leq \beta\left(\mathbb{E}_B[V(l-1,0,B)-\mathbb{E}_B[V(l-2,1,B)]\right), \tag{EC.46}$$

concluding the proof of part $(II).(i)$.

(ii) For all $l \in \{k+1,\dots,i\}$, we show that the optimal order-up-to policy is $\bar{a}=\min(l,k)=k$ if and only if the conditions in part (ii) are met. Analogous to the argument in part $(II).(i)$, given Lemma 1 and Theorem 1, it is sufficient to compare order-up-to k with order-up-to $k-1$ and $k+1$ at state $a=0$ and $o=l$ to conclude about all other states where $a+o=l$.

Then, analogous to (EC.46), order-up-to k is optimal if and only if

$$1 \leq \beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)$$
$$\text{and } \beta\left(\mathbb{E}_B[V(k,l-k-1,B)-\mathbb{E}_B[V(k-1,l-k,B)]\right) < 1,$$

concluding the proof of part $(II).(ii)$.

(iii) For all $l \in \{i+1,\ldots,I\}$, we show that the optimal order-up-to policy is $\bar{a}=k-1$ if and only if the conditions in part (iii) are met. Analogous to the proof of previous parts, order-up-to $k-1$ is optimal if and only if

$$1 \le \beta\left(\mathbb{E}_B[V(k-2,l-k+1,B)-\mathbb{E}_B[V(k-3,l-k+2,B)]\right)$$
$$\text{and } \beta\left(\mathbb{E}_B[V(k-1,l-k,B)-\mathbb{E}_B[V(k-2,l-k+1,B)]\right)<1,$$

concluding the proof of part $(II).(iii)$, and thus the lemma. □

Proposition 6 (Threshold Equivalence)**.** *Condition* $\beta\left(\mathbb{E}_B[V(k-1,0,B)-\mathbb{E}_B[V(k-2,1,B)]\right)\ge 1$ *from Theorem 2[B] is equivalent to* $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]\ge 1$ *for all* $k\in\{2,\ldots,I\}$.

Proof. We proceed by induction. Note that the equivalence holds for the base case $k=2$ as shown in the proof of Theorem 2(A), i.e., $\beta\left(\mathbb{E}_B[V(1,0,B)]-\mathbb{E}_B[V(0,1,B)]\right)\ge 1$ if and only if $\beta q\mathbb{E}_B[V(1,0,B)]\ge 1$.

For the induction step, assume that $\beta\left(\mathbb{E}_B[V(l-1,0,B)-\mathbb{E}_B[V(l-2,1,B)]\right)\ge 1$ is equivalent to $\beta^{l-1}q^{l-1}\mathbb{E}_B[V(l-1,0,B)]\ge 1$ for all $l\in\{2,\ldots,k\}$, where $k\le I-1$. Then, we show that $\beta\left(\mathbb{E}_B[V(k,0,B)-\mathbb{E}_B[V(k-1,1,B)]\right)\ge 1$ is equivalent to $\beta^k q^k\mathbb{E}_B[V(k,0,B)]\ge 1$.

We split the analysis into two cases: $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]\ge 1$ and $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]<1$. First assume $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]\ge 1$, then

$$\beta\left(\mathbb{E}_B[V(k,0,B)]-\mathbb{E}_B[V(k-1,1,B)]\right)\ge 1$$
$$\iff \beta\mathbb{E}_B[V(k,0,B)]-1-\left(\frac{1-\beta^{k-1}q^{k-1}}{1-\beta q}\right)\beta qv-\frac{\beta(1-q)}{1-\beta q}\left(\mathbb{E}_B[V(k,0,B)-1\right)\ge 0$$
$$\iff \left(\beta^k q^k\mathbb{E}_B[V(k,0,B)]-1\right)\left(1-\frac{\beta(1-q)}{1-\beta q}\right)\ge 0, \tag{EC.47}$$

where the first equivalence follows since $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]\ge 1$, $\mathbb{E}_B[V(k,0,B)]=v/(1-\beta)$ from (4), and the induction hypothesis imply $\mathbb{E}_B[V(k-1,1,B)]=\frac{1-(\beta q)^{k-1}}{1-\beta q}qv+\frac{1-q}{1-\beta q}\left(\mathbb{E}_B[V(k,0,B)]-1\right)$, i.e., the inequality in Lemma 3(iii) becomes tight. Indeed, since $\beta,q_o\in[0,1]$, then $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]\ge 1$ implies $\beta^{l-1}q^{l-1}\mathbb{E}_B[V(l-1,0,B)]\ge 1$ for all $l\in\{2,\ldots,k\}$. From the induction hypothesis, we then have $\beta\left(\mathbb{E}_B[V(l-1,0,B)-\mathbb{E}_B[V(l-2,1,B)]\right)\ge 1$ for all $l\in\{2,\ldots,k\}$, which in turns implies that making at least one payment in each period starting from state $(k-1,1)$ is optimal. As a result, if $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]\ge 1$ then the feasible policy from the proof of Lemma 3(iii) becomes optimal. The second equivalence follows since $v=\mathbb{E}[V(k-1,0,B)](1-\beta)$ from (4) and simplifying the expression. Finally, note that the second term in (EC.47) is non-negative for all $\beta,q\in[0,1]$. Hence, we conclude that if $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]\ge 1$ then $\beta\left(\mathbb{E}_B[V(k,0,B)]-\mathbb{E}_B[V(k-1,1,B)]\right)\ge 1$ is equivalent to $\beta^K q^k\mathbb{E}_B[V(k,0,B)]\ge 1$.

Now assume $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]<1$. Note that then we have,

$$\beta\left(\mathbb{E}_B[V(k,0,B)]-\mathbb{E}_B[V(k-1,1,B)]\right)-1$$
$$\le \beta\mathbb{E}_B[V(k,0,B)]-1-\left(\frac{1-\beta^{k-1}q^{k-1}}{1-\beta q}\right)\beta qv-\frac{\beta(1-q)}{1-\beta q}\left(\mathbb{E}_B[V(k,0,B)]-1\right)$$
$$=\left(\beta^k q^k\mathbb{E}_B[V(k,0,B)]-1\right)\left(1-\frac{\beta(1-q)}{1-\beta q}\right)<0, \tag{EC.48}$$

where the first inequality follows from Lemma 3(iii) and replacing $\mathbb{E}_B[V(k,0,B)]=v/(1-\beta)$. The equality follows from simplifying the expression. The second inequality follows since the second term on the left-hand side is non-negative for all $\beta,q\in[0,1]$ and $\beta^k q^k\mathbb{E}_B[V(k,0,B)]\le\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]<1$, where the first inequality holds for all $\beta,q\in[0,1]$ since $\mathbb{E}_B[V(k,0,B)]=v/(1-\beta)$ for all k. Hence, we conclude that if $\beta^{k-1}q^{k-1}\mathbb{E}_B[V(k-1,0,B)]<1$ then $\beta\left(\mathbb{E}_B[V(k,0,B)]-\mathbb{E}_B[V(k-1,1,B)]\right)<1$ and $\beta^k q^k\mathbb{E}_B[V(k,0,B)]<1$, concluding the proof. □

Lemma 7. *(Ordered Thresholds in Number of Installments) For all* $k\in\{2,\ldots,I-1\}$*, if* $\beta\left(\mathbb{E}_B[V(k-1,i-k,B)-\mathbb{E}_B[V(k-2,i-k+1,B)]\right)<1$ *then*

$$\beta\left(\mathbb{E}_B[V(k-1,i-k+1,B)-\mathbb{E}_B[V(k-2,i-k+2,B)]\right)$$
$$\le\beta\left(\mathbb{E}_B[V(k-1,i-k,B)-\mathbb{E}_B[V(k-2,i-k+1,B)]\right). \tag{EC.49}$$

Proof. Let Π^0 be a feasible policy starting from state $(k-2, i-k+2)$, which makes no payment in the current period and follows the optimal policy in future periods. Then, we get

$$\begin{aligned}
&\mathbb{E}_B[V(k-1,i-k+1,B)] - \mathbb{E}_B[V(k-2,i-k+2,B)] \\
\leq&\mathbb{E}_B[V(k-1,i-k+1,B)] - \mathbb{E}_B^{\Pi^0}[V(k-2,i-k+2,B)] \\
=&(v+\beta\mathbb{E}_B[V(k-2,i-k+1,B)]) - (v+\beta\mathbb{E}_B[V(k-3,i-k+2,B)]) \\
=&\beta(\mathbb{E}_B[V(k-2,i-k+1,B)] - \mathbb{E}_B[V(k-3,i-k+2,B)]) \\
\leq&\mathbb{E}_B[V(k-1,i-k,B)] - \mathbb{E}_B[V(k-2,i-k+1,B)],
\end{aligned} \tag{EC.50}$$

where the first inequality follows from the feasibility of Π^0, the first equality follows from the definition of Π^0 and noticing that $\beta\left(\mathbb{E}_B[V(k-1,i-k,B) - \mathbb{E}_B[V(k-2,i-k+1,B)]\right) < 1$ implies that it is optimal not to order in the first period starting from state $(k-1, i-k+1)$ since the order-up-to level $\bar{a}$ satisfies $\bar{a} \leq k-1$, see (the reverse of) (EC.41) in Lemma 6. The second equality follows from simplifying. The last inequality follows from Lemma 5, concluding the proof. ☐

Lemma 8. *For any $\alpha \in (0,1)$, $\epsilon > 0$, and non-negative sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ such that $x_0 \geq y_0 + \epsilon$, $\sum_{i=0}^{t} x_i \geq \sum_{i=0}^{t} y_i$ for all $t \in \{0,1,\ldots\}$ if and only if $\sum_{i=0}^{\infty} \alpha^i x_i > \sum_{i=0}^{\infty} \alpha^i y_i$.*

Proof. We show that the following infinite-dimensional linear program is infeasible.

$$\begin{aligned}
\min_{x_i, y_i} \quad & 0 \\
\text{s.t.} \quad & \sum_{i=0}^{t} x_i \geq \sum_{i=0}^{t} y_i \ \forall t \in \{0,1,\ldots\} && (\delta_t) \\
& \sum_{i=0}^{\infty} \alpha^i x_i \leq \sum_{i=0}^{\infty} \alpha^i y_i && (\gamma) \\
& x_0 \geq y_0 + \epsilon && (\eta) \\
& x_i \geq 0, \ y_i \geq 0 \ \forall t \in \{0,1,\ldots\}.
\end{aligned}$$

Indeed, its dual is

$$\begin{aligned}
\max_{\delta_t, \gamma, \eta} \quad & \epsilon\eta \\
\text{s.t.} \quad & \sum_{t\geq 0} \delta_t - \gamma + \eta = 0 \\
& \sum_{t \geq i} \delta_t - \alpha^i \gamma = 0 \ \forall i \in \{1,\ldots\} \\
& \delta_t \geq 0 \ \forall t \in \{0,1,\ldots\}, \ \gamma \geq 0, \ \eta \geq 0.
\end{aligned}$$

Further, for any $\gamma > 0$, the following is a dual feasible solution, $\delta_0 = 0$, $\delta_i = \alpha^i\gamma - \alpha^{i+1}\gamma$ for all $i \in \{1,\ldots\}$, $\eta = \gamma(1-\alpha)$, whose objective function is unbounded as γ grows without limit. Hence, by standard weak duality arguments in infinite dimensional linear optimization (Romeijn et al. 1992), the primal problem is infeasible, completing the proof. ☐

H. Auxiliary Results for Proofs of Section 4

Denote the Bellman equation when the firm sets a repayment frequency where bundled payments are restricted under each repayment frequency F as follows.

$$V'^F(a,o,b) = \max_{x \in \{0, \lfloor \frac{\min(F,o,b)}{F} \rfloor F\}} \left\{ v1_{\{a+x\geq 1\}} - x + \beta\mathbb{E}_B[V'^F((a+x-1)^+, o-x, B)] \right\}, \tag{EC.51}$$

with a terminal value of $V'^F(0,0,b) = v/(1-\beta)$. The feasibility set in (EC.51) restricts the consumer to not make a payment or make a payment of F if it is needed and the budget allows. In the next proposition, we derive the expected time to ownership for each $F \in \{2,4\}$ to be used in the proof of Propositions 2, 3 and 8. Note that the expected time to ownership for $F = 1$ follows the same expression as in Proposition 1(A).

Proposition 7. *For all $I \geq 4$, the expected time to ownership under the order-up-to F policy for problem* (EC.51) *is*

(i) $\mathbb{E}[\tau^{2*}] = \dfrac{I}{2(1-q)^2} + (\frac{I}{2} - 2)(1-q)^2$ *when* $F = 2$.

(ii) $\mathbb{E}[\tau^{4*}] = \dfrac{I}{4(1-q)^4}$ *when* $F = 4$.

Proof. We prove by using induction on I as follows.

(i) Base case: First set I=4. Then we show that $\mathbb{E}[\tau^{2*}] = 2/(1-q)^2$. Let $\tau_{(a,o)}$ be the random time to ownership at state a and o. Then, we have

$$\begin{aligned}
\mathbb{E}[\tau_{(0,4)}] =& (q + q(1-q))\left(\mathbb{E}[\tau_{(0,4)}] + 1\right) + (1-q)^2\left[\mathbb{E}[\tau_{(1,2)}] + 1\right] \\
=& 1 + (q + q(1-q))\left(\mathbb{E}[\tau_{(0,4)}]\right) + (1-q)^2\left[(q + q(1-q))(\mathbb{E}[\tau_{(0,2)}] + 1) + (1-q)^2(\mathbb{E}[\tau_{(2,0)}] + 1)\right] \\
=& 1 + (q + q(1-q))\left(\mathbb{E}[\tau_{(0,4)}]\right) + (1-q)^2\left[1 + (q + q(1-q))(\frac{1}{(1-q)^2} + \mathbb{E}[\tau_{(1,0)}]) + (1-q)^2(\mathbb{E}[\tau_{(2,0)}])\right] \\
=& 1 + (q + q(1-q))\left(\mathbb{E}[\tau_{(0,4)}]\right) + (1-q)^2 + 1 - (1-q)^2,
\end{aligned}$$

where the first equality follows from $B \sim \text{Geo}(q)$ and following the order-up-to 2 policy when $F = 2$. The second and third equality follows from expanding $\mathbb{E}[\tau_{(1,2)}]$ and $\mathbb{E}[\tau_{(0,2)}]$ given the order-up-to 2 policy, respectively, and simplifying the expression to get the fourth equality. Since $\mathbb{E}[\tau_{(a,0)}] = 0$ for all $a \geq 0$, then the last equality implies $\mathbb{E}[\tau_{(0,4)}] = 2/(1-q)^2$, concluding the proof of the base case.

Induction step: Assume $\mathbb{E}[\tau^{2*}] = I/(2(1-q)^2) + (I/2 - 2)(1-q)^2$ is true for all $I \in \{4, 6, 8, \ldots, n-2\}$, where n is an even number. Then, we show $\mathbb{E}[\tau^{2*}] = I/(2(1-q)^2) + (I/2 - 2)(1-q)^2$ holds for $I = n$, i.e., we show $\mathbb{E}[\tau_{(0,n)}] = n/(2(1-q)^2) + (n/2 - 2)(1-q)^2$. Given the order-up-to 2 optimal policy we have

$$\begin{aligned}
\mathbb{E}[\tau_{(0,n)}] =& (q + q(1-q))\left(\mathbb{E}[\tau_{(0,n)}] + 1\right) + (1-q)^2\left[\mathbb{E}[\tau_{(1,n-2)}] + 1\right] \\
=& 1 + (q + q(1-q))\left(\mathbb{E}[\tau_{(0,n)}]\right) + (1-q)^2\left[1 + (q + (1-q))(\mathbb{E}[\tau_{(0,n-2)}] + 1) + (1-q)^2(\mathbb{E}[\tau_{(2,n-4)}] + 1)\right] \\
=& 1 + (q + q(1-q))\left(\mathbb{E}[\tau_{(0,n)}]\right) + (1-q)^2\left[2 + (q + (1-q))(\mathbb{E}[\tau_{(0,n-2)}]) + (1-q)^2(\mathbb{E}[\tau_{(1,n-4)}] + 1)\right] \\
=& 1 + (q + q(1-q))\left(\mathbb{E}[\tau_{(0,n)}]\right) + (1-q)^2\left[3 + (q + (1-q))(\mathbb{E}[\tau_{(0,n-2)}]) + (1-q)^2((q + q(1-q))\mathbb{E}[\tau_{(0,n-4)}]\right. \\
& \left. + (1-q)^2(\mathbb{E}[\tau_{(1,n-6)}] + 1))\right] = \ldots = \frac{n}{2(1-q)^2} + (\frac{n}{2} - 2)(1-q)^2,
\end{aligned}$$

where the equalities follow from $B \sim \text{Geo}(q)$ and and following the order-up-to 2 policy when $F = 2$, concluding the proof part (i) of the proposition.

(ii) The proof of this part is analogous to the proof of part (i). We first prove for that the base case where $I = 4$, we have $\mathbb{E}[\tau^{4*}] = 1/(1-q)^4$. We next assume that $\mathbb{E}[\tau^{4*}] = I/(4(1-q)^4)$ holds for all $I \in \{4, 8, 16, \ldots, n-4\}$, where n is a multiple of 4. Then, we show that $\mathbb{E}[\tau^{4*}] = I/(4(1-q)^4)$ also holds for $I = n$, concluding the proof of part (ii). $\square$

The next proposition is useful for the proof of Proposition 3.

Proposition 8. *For the setup without bundling, model EC.51, there exists three thresholds q', q'' and $\hat{q}$ such that*

(i) $\mathbb{E}[\tau^4] < \mathbb{E}[\tau^2] < \mathbb{E}[\tau^1] \quad \forall\, q \in [0, q')$,
(ii) $\mathbb{E}[\tau^2] \leq \mathbb{E}[\tau^4] < \mathbb{E}[\tau^1] \quad \forall\, q \in [q', q'')$,
(iii) $\mathbb{E}[\tau^2] \leq \mathbb{E}[\tau^1] < \mathbb{E}[\tau^4] \quad \forall\, q \in [q'', \hat{q})$,
(iv) $\mathbb{E}[\tau^1] \leq \mathbb{E}[\tau^2] < \mathbb{E}[\tau^4] \quad \forall\, q \in [\hat{q}, 1]$.

Proof. In Proposition 2 we showed that when q is sufficiently low $\mathbb{E}[\tau^4] < \mathbb{E}[\tau^2] < \mathbb{E}[\tau^1]$. Therefore, we first show that when bundled payments are restricted, i.e., model (EC.51), the expected time to ownership under $F = 4$ intersects with the expected time to ownership under $F = 2$ at a lower q than it intersects $F = 1$, i.e., $q' < q''$. We next show that each F have unique intersections, thus we have $q'' < \hat{q}$. To complete the proof we use the result shown in Proposition 2 for large enough q to show the four regions above.

We first derive the expected time to ownership under the three designs; F=1, F=2, and F=4. For $F = 1$ the expected time to ownership follows Proposition 1(A), i.e.,

$$\mathbb{E}[\tau^1] = \frac{I}{1-q}. \tag{EC.52}$$

For $F = 2$ the expected time to ownership follows Proposition 7(i) when q is low, i.e., $\mathbb{E}[\tau^2] = I/(2(1-q)^2) + (\frac{I}{2} - 2)(1-q)^2$. As q increases, bundling payments occur, which reduces the expected time to ownership (see Proposition 1). Therefore,

$$\mathbb{E}[\tau^2] \le \frac{I}{2(1-q)^2} + (\frac{I}{2} - 2)(1-q)^2. \tag{EC.53}$$

With a similar argument, for $F = 4$ the expected time to ownership follows Proposition 7(ii), and

$$\mathbb{E}[\tau^4] \le \frac{I}{4(1-q)^4}. \tag{EC.54}$$

We next show $q' < q''$ by contradiction, where q' is the intersection of expected time to ownership under $F = 4$ and $F = 2$ designs and q'' is the intersection of expected time to ownership under $F = 4$ and $F = 1$ designs, i.e.,

$$\mathbb{E}[\tau^1] = \frac{I}{1-q''} = \mathbb{E}[\tau^4] \le \frac{I}{4(1-q'')^4} \Longleftrightarrow (1-q'')^3 \le \frac{1}{4}. \tag{EC.55}$$

Assume that $q' > q''$, then we replace q'' into (EC.53) to get

$$\mathbb{E}[\tau^2] \le \frac{1}{4(1-q'')^4}\left(2I(\frac{1}{4})^{\frac{2}{3}} + 2(I-4)(\frac{1}{4})^2\right) < \frac{I}{4(1-q'')^4} = \mathbb{E}[\tau^4], \tag{EC.56}$$

where the first equality follows from replacing q'' and (EC.55) into into (EC.53). The inequality follows from the fact that $2I(\frac{1}{4})^{\frac{2}{3}} + 2(I-4)(\frac{1}{4})^2 < I$. The second inequality in (EC.56) implies that $q' < q''$, contradicting the assumption. Therefore, $q' < q''$ holds.

We conclude the proof by showing that $\hat{q} \in [0,1]$ is the unique intersection of expected time to ownership under $F = 2$ and $F = 1$, and thus given the result of Proposition 2 we have the four regions stated in the proposition. We equalize (EC.52) and (EC.53) to get $\hat{q}$. Thus,

$$\mathbb{E}[\tau^1] = \frac{I}{1-\hat{q}} = \mathbb{E}[\tau^2] \le \frac{I}{2(1-\hat{q})^2} + (\frac{I}{2} - 2)(1-\hat{q})^2 \Longleftrightarrow I + (I-4)(1-\hat{q})^4 - 2I(1-\hat{q}) \ge 0, \tag{EC.57}$$

where (EC.57) is a polynomial function of $(1-q)$ in degree 4 . Since (EC.57) is negative at $q = 0$, positive at $q = 1$, and increasing in q (or first decreasing and then increasing), then $\hat{q} \in (0,1]$ is the smallest unique root of $I + (I-4)(1-\hat{q})^4 - 2I(1-\hat{q}) = 0$, concluding the proof. □

Lemma 9. *If* $\frac{\beta q v}{1-\beta} < 1$*, then at optimality no bundled payments happen under all* $F \in \{1,2,4\}$*.*

Proof. We first show that bundled payments happen at lower values of q under $F = 1$ than $F = 2$ than $F = 4$. From Theorem 2(A), we know that bundled payments under $F = 1$ design don't happen (i.e., order-up-to policy 1 or less) if and only if $\beta q \mathbb{E}_B[V(1,0,B)] < 1$. With a similar argument as in (EC.5), we find that bundled payments under $F = 2$ and $F = 4$ designs don't happen if and only if $\beta(1-(1-q)^2)\mathbb{E}_B[V(3,0,B)] < 2$ and $\beta(1-(1-q)^4)\mathbb{E}_B[V(7,0,B)] < 4$, respectively. If $\beta q \mathbb{E}_B[V(1,0,B)] < 1$, then it implies that $\frac{\beta q v}{1-\beta}(2-q) < 2-q < 2$, which implies no bundled payment under $F = 2$. If $\beta(1-(1-q)^2)\mathbb{E}_B[V(3,0,B)] < 2$, then it implies that $\frac{\beta v}{1-\beta}(1-(1-q)^2)(1+(1-q)^2) < 2(1+(1-q)^2) < 4$, which implies no bundled payment condition under $F = 4$. Therefore, $\beta q \mathbb{E}_B[V(1,0,B)] < 1$, which is equivalent to $\frac{\beta q v}{1-\beta} < 1$ implies no bundled payments under all F, concluding the proof. □

I. Proofs for Results in Section F.4

Lemma 10. *Let* $\mathbb{E}[\tau^I]$ *be the expected time to ownership under the optimal repayment policy for the repayment term* I *for the Bellman equation* (EC.22)*. In addition, let* $I' = I/2$ *and* $c' = 2$, $I'' = I/3$ *and* $c'' = 3$*, and* $\tilde{I} = I/4$ *and* $\tilde{c} = 4$*. Then, as* $q \to 0$, $\mathbb{E}[\tau^{\tilde{I}}] < \mathbb{E}[\tau^{I''}] < \mathbb{E}[\tau^{I'}] < \mathbb{E}[\tau^I]$*. Conversely, as* $q \to 1$, $\mathbb{E}[\tau^{\tilde{I}}] > \mathbb{E}[\tau^{I''}] > \mathbb{E}[\tau^{I'}] > \mathbb{E}[\tau^I]$*.*

Proof. According to Bellman equation (EC.22), ${B_t}'$ follows a geometric distribution with parameter $\sum_{j=0}^{c-1} q(1-q)^j$. Therefore, the budget for repayment terms I, I', I'', and $\tilde{I}$ follow a geometric distribution with parameters q, $q' = \sum_{j=0}^{1} q(1-q)^j$, $q'' = \sum_{j=0}^{2} q(1-q)^j$ and $\tilde{q} = \sum_{j=0}^{3} q(1-q)^j$, respectively. Moreover, given Proposition 1 of the paper and the appropriate geometric distribution parameter, we can derive the expected time to ownership for each repayment term.

First, as $q \to 0$, then $q' \to 0$, $q'' \to 0$ and $\tilde{q} \to 0$, thus, we are in the case (A) of Proposition 1 for all repayment terms. Then, the expected time to ownerships are $\mathbb{E}[\tau^I] = I/(1-q)$, $\mathbb{E}[\tau^{I'}] = I'/(1-q')$,

$\mathbb{E}[\tau^{I''}] = I''/(1-q'')$, and $\mathbb{E}[\tau^{\tilde{I}}] = \tilde{I}/(1-\tilde{q})$ implying that $\mathbb{E}[\tau^{\tilde{I}}] = \tilde{I} = I/4 < \mathbb{E}[\tau^{I''}] = I'' = I/3 < \mathbb{E}[\tau^{I'}] = I' = I/2 < \mathbb{E}[\tau^{I}] = I$, completing the proof of the first part of the lemma.

Next, as $q \to 1$, the expected time to ownership will stay same or decrease to minimum, where the minimum is when the payment policy is order-up-to ownership with expected time of $qI/(1-q)+1$ as in case (B) of Proposition 1. In any case, as $q \to 1$, then $\mathbb{E}[\tau^{\tilde{I}}] = \tilde{I}/(1-\tilde{q}) > \mathbb{E}[\tau^{I''}] = I''/(1-q'') > \mathbb{E}[\tau^{I'}] = I'/(1-q') > \mathbb{E}[\tau^{I}] = I/(1-q)$, where the inequalities follows from $4(1-\tilde{q}) = 4(1-q)^4 < 3(1-q'') = 3(1-q)^3 < 2(1-q') = 2(1-q)^2 < 1-q$ as $q \to 1$, completing the proof of the second part of the lemma and concluding the proof. □

Lemma 11. *If $\frac{\beta q v}{1-\beta} < 1$, then at optimality no bundled payments happen under repayment terms, I, $I' = I/2$, $I'' = I/3$ and $\tilde{I} = I/4$.*

Proof. We first derive the condition for no bundled payments for different repayment terms. For the base case of I and $c = 1$, the condition is exactly the same as in Theorem 2 (A), i.e., $\beta q v/(1-\beta) < 1$. For $I' = I/2$ and $c' = 2$, the updated condition is $\beta q' v/(1-\beta) < 2$ where $q' = q + q(1-q) = q(2-q)$. For $I'' = I/3$ and $c'' = 3$, the updated condition is $\beta q'' v/(1-\beta) < 3$ where $q'' = q + q(1-q) + q(1-q)^2 = q(1 + (1-q)(2-q))$. For $\tilde{I} = I/4$ and $\tilde{c} = 4$, the updated condition is $\beta \tilde{q} v/(1-\beta) < 4$ where $\tilde{q} = q + q(1-q) + q(1-q)^2 + q(1-q)^3 = q(2-q) + q(1-q)^2(2-q) = q(2-q)(1+(1-q)^2)$.

We now show that if $\beta q v/(1-\beta) < 1$ then all the above conditions hold, i.e., no bundled payments occur under repayment terms I, I', I'' and $\tilde{I}$. If $\beta q v/(1-\beta) < 1$, then we have $\beta q' v/(1-\beta) = \beta q(2-q)v/(1-\beta) < 2\beta q v/(1-\beta) < 2$, where the equality follows by the definition of q', the first inequality follows from $q(2-q) < 2q$, and the second inequality follows from $\beta q v/(1-\beta) < 1$. With a similar argument, if $\beta q v/(1-\beta) < 1$ then, we have $\beta q'' v/(1-\beta) = \beta q(1+(1-q)(2-q))v/(1-\beta) < 3\beta q v/(1-\beta) < 3$ and $\beta \tilde{q} v/(1-\beta) = \beta q(2-q)(1+(1-q)^2)v/(1-\beta) < 4\beta q v/(1-\beta) < 4$. Therefore, $\frac{\beta q v}{1-\beta} < 1$ implies no bundled payments under all repayment terms, concluding the proof. □

Proof of Proposition 4: First we show part (i). If $\frac{1-\beta}{\beta v} \geq 1$, then $\frac{\beta q v}{1-\beta} \leq \frac{\beta v}{1-\beta} \leq 1$ for all $q \in [0,1]$, where from Lemma 11 inequality $\frac{\beta q v}{1-\beta} \leq 1$ implies that no bundled payments happen under all repayment terms. Therefore, the expected time to ownership follows Proposition 1(A) of the paper, thus we have $\mathbb{E}[\tau^{I}] = \frac{I}{1-q}$, $\mathbb{E}[\tau^{I'}] = \frac{I'}{1-q'} = \frac{I}{2(1-q)^2}$, $\mathbb{E}[\tau^{I''}] = \frac{I''}{1-q''} = \frac{I}{3(1-q)^3}$, and $\mathbb{E}[\tau^{\tilde{I}}] = \frac{\tilde{I}}{1-\tilde{q}} = \frac{I}{4(1-q)^4}$. Given the ordering of expected time to ownerships in Lemma 10, it sufficient to compare the expected time to ownerships under each repayment term with its closest smaller and larger repayment terms. Thus, we have $\mathbb{E}[\tau^{I''}] \leq \mathbb{E}[\tau^{\tilde{I}}]$ and $\mathbb{E}[\tau^{I''}] < \mathbb{E}[\tau^{I'}]$ if and only if $1-q < \frac{3}{4}$ and $1-q > \frac{2}{3}$, which is equivalent to $q \in [\frac{1}{4}, \frac{1}{3})$. In addition, we have $\mathbb{E}[\tau^{I'}] \leq \mathbb{E}[\tau^{I''}]$ and $\mathbb{E}[\tau^{I'}] < \mathbb{E}[\tau^{I}]$ if and only if $1-q < \frac{2}{3}$ and $1-q > \frac{1}{2}$, which is equivalent to $q \in [\frac{1}{3}, \frac{1}{2})$, concluding the proof of part (i).

Next we show part (ii). We start with the case (a), i.e., if $1 - \frac{1-\beta}{\beta v} - \frac{3}{4} < 0$ then there exists a $\bar{q} \in [\frac{1}{4}, \frac{1}{3}]$ such that $\mathbb{E}[\tau^{I''}] \leq \min\{\mathbb{E}[\tau^{I}], \mathbb{E}[\tau^{I'}], \mathbb{E}[\tau^{\tilde{I}}]\}$ for all $q \in [\frac{1}{4}, \bar{q})$. If $1 - \frac{1-\beta}{\beta v} - \frac{3}{4} < 0$ then $\frac{1}{4} < \frac{1-\beta}{\beta v}$, which from Lemma 9 implies that no bundled payment happen for all $q \in [0, \frac{1}{4})$. Then, for all $q \geq \frac{1}{4}$ we have $1 - q - \frac{3}{4} \leq 0$, which is equivalent to $\mathbb{E}[\tau^{I''}] = \frac{I}{3(1-q)^3} \leq \mathbb{E}[\tau^{\tilde{I}}] = \frac{I}{4(1-q)^4}$. Finally, we define the intersection of $\mathbb{E}[\tau^{I''}]$ and $\min\{\mathbb{E}[\tau^{I'}], \mathbb{E}[\tau^{I}]\}$ as $\bar{q}$ (can include bundled payment since $\bar{q} \geq \frac{1}{4}$). Thus, $\mathbb{E}[\tau^{I''}] \leq \min\{\mathbb{E}[\tau^{I}], \mathbb{E}[\tau^{I'}], \mathbb{E}[\tau^{\tilde{I}}]\}$ for all $q \in [\frac{1}{4}, \bar{q})$, concluding the proof of first case of (ii).

We now show the case (b) of part (ii), i.e., if $1 - \frac{1-\beta}{\beta v} - \frac{2}{3} < 0$, then there exists a $\bar{\bar{q}} \in [\frac{1}{3}, \frac{1}{2})$ such that $\mathbb{E}[\tau^{I'}] \leq \min\{\mathbb{E}[\tau^{I}], \mathbb{E}[\tau^{I''}], \mathbb{E}[\tau^{\tilde{I}}]\}$ for all $q \in [\frac{1}{3}, \bar{\bar{q}}]$.If $1 - \frac{1-\beta}{\beta v} - \frac{2}{3} < 0$ then $\frac{1}{3} < \frac{1-\beta}{\beta v}$, which from Lemma 2 implies that no bundled payments happen for all $q \in [0, \frac{1}{3})$. Then, for all $q \geq \frac{1}{3}$ we have $1 - q - \frac{2}{3} \leq 0$, which is equivalent to $\mathbb{E}[\tau^{I'}] = \frac{I}{2(1-q)^2} \leq \mathbb{E}[\tau^{I''}] = \frac{I}{3(1-q)^3}$. Finally, we define the intersection of $\mathbb{E}[\tau^{I'}]$ and $\min\{\mathbb{E}[\tau^{I}], \mathbb{E}[\tau^{\tilde{I}}]\}$ as $\bar{\bar{q}}$ (can include bundled payment since $\bar{q} \geq \frac{1}{3}$). Thus, $\mathbb{E}[\tau^{I'}] \leq \min\{\mathbb{E}[\tau^{I}], \mathbb{E}[\tau^{I''}], \mathbb{E}[\tau^{\tilde{I}}]\}$ for all $q \in [\frac{1}{3}, \bar{\bar{q}})$, concluding the proof of the second case of (ii), thus completing the proof. □