Electronic Companion of "Online Demand Fulfillment in Highly Asymmetric Markets"

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EC.1. Proof of Theorem 1

Before delving into the proof of Theorem 1, we first provide proofs of Lemmas 1–3, and then present Proposition EC.1 as a variation of Theorem 1 in Xu et al. (2020).

Proof of Lemma 1: Let $H(t) = C_i(t) - \epsilon_m \hat{c}_i^m t$, for any given $1 \leq m \leq M$ and $i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}$. Since $|H(t)| = [H(t)]^+ + [H(t)]^-$, we will bound $\mathbb{E}[H(t)]^+$ and $\mathbb{E}[H(t)]^-$. Note that, $C_i(t+1) = C_i(t)$ if the $(t+1)$st arrival is not from $\mathcal{J}_{Lm}$, which is with probability $1 - \epsilon_m$. By part 1(a) of DLDP, there are two random variables at any t,

$$W(t) = H(t) - H(t-1) = \begin{cases} -\epsilon_m \hat{c}_i^m, & \text{with probability } 1-\epsilon_m, \\ (1-\epsilon_m)\hat{c}_i^m - \frac{1}{2}\min\left\{\hat{c}_i^m, \frac{\hat{c}_i^m}{\hat{c}^m}\eta^{Lm}\right\}, & \text{with probability } \epsilon_m, \end{cases}$$

corresponding to the scenario where $H(t) \geq 0$ and

$$W'(t) = H(t) - H(t-1) = \begin{cases} -\epsilon_m \hat{c}_i^m, & \text{with probability } 1-\epsilon_m, \\ (1-\epsilon_m)\hat{c}_i^m + \frac{1}{2}\min\left\{\hat{c}_i^m, \frac{\hat{c}_i^m}{\hat{c}^m}\eta^{Lm}\right\}, & \text{with probability } \epsilon_m, \end{cases}$$

corresponding to the scenario where $H(t) < 0$. Furthermore, $\{W(t) : t \geq 1\}$ and $\{W'(t) : t \geq 1\}$ are both iid random variables.

- Bound $\mathbb{E}[H(t)]^+$. Define a Lindley process $\hat{H}(t) = \left[\hat{H}(t-1) + W(t)\right]^+$ with $\hat{H}(0) = H(0)$. As a Lindley process, $\mathbb{E}[\hat{H}(t)]$ must be increasing in t and converges to $\mathbb{E}[\hat{H}(\infty)]$ where $\hat{H}(\infty) = \max_{n\geq 1}\left\{\sum_{t=1}^{n} W(t)\right\}$. Furthermore, $\mathbb{E}[\hat{H}(\infty)] = \sum_{n=1}^{\infty} \frac{\mathbb{E}\left[\sum_{t=1}^{n} W(t)\right]^+}{n}$ by Kingman's formula in Kingman (1962) and hence $\mathbb{E}[\hat{H}(\infty)]$ is bounded as $\epsilon \to 0$. That is, for any $\epsilon > 0$, since both $-\mathbb{E}[W(t)]$ and $\mathrm{Var}(W(t))$ are in the order of ϵ, by the Central Limit Theorem and the tail behavior of normal distribution,

$$\frac{1}{n}\mathbb{E}\left[\sum_{t=1}^{n} W(t)\right]^+ \sim \frac{\mathbb{E}\left[-n\epsilon + \sqrt{n\epsilon}Z\right]^+}{n} \leq e^{-\sqrt{n\epsilon}}\sqrt{\frac{\epsilon}{n}},$$

where Z is the standard normal random variable. As $\epsilon \to 0$, $\sum_{n=1}^{\infty} e^{-\sqrt{n\epsilon}}\sqrt{\frac{\epsilon}{n}} \le \int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx < \infty$, hence $\sup_{\epsilon\to 0}\mathbb{E}[\hat{H}(\infty)] < \infty$.

We now show that, for any given t,

$$\mathbb{E}[H(t)]^+ \le \mathbb{E}[\hat{H}(t)] + 2\hat{c}_i^m, \tag{EC.1}$$

by establishing that $H(t) \le \hat{H}(t) + 2\hat{c}_i^m$ along any sample path. Hence $\mathbb{E}[H(t)]^+$ is bounded from above by a constant independent of ϵ and t. Suppose that there exists a sample path with $H(t) > \hat{H}(t) + 2\hat{c}_i^m$ for some t. Since $\hat{H}(t) \ge 0$, it follows that $H(t) > 2\hat{c}_i^m$, and hence $H(t-1) > 0$ (the adjustment from $H(t-1)$ to $H(t)$ cannot exceed $2\hat{c}_i^m$), which implies that

$$\begin{aligned} H(t-1) &= H(t) - W(t) \\ &> \hat{H}(t) + 2\hat{c}_i^m - W(t) = \left[\hat{H}(t) + W(t)\right]^+ + 2\hat{c}_i^m - W(t) > \hat{H}(t-1) + 2\hat{c}_i^m. \end{aligned}$$

- Bound $\mathbb{E}[H(t)]^-$. Following a similar argument, we can show that $\mathbb{E}[H(t)]^- \le \mathbb{E}[\hat{H}'(t)] + 2\hat{c}_i^m$, where $\hat{H}'(t) = \left[\hat{H}'(t-1) - W'(t)\right]^+$, and $\sup_{\epsilon\to 0}\mathbb{E}[\hat{H}'(\infty)] < \infty$. Thus, $\mathbb{E}[H(t)]^-$ is also bounded from above by a constant independent of ϵ and t.

□

Proof of Lemma 2: The proof follows the same arguments in the proof of Lemma 1 utilizing the mean-reverting nature of the process $\{C_i(t) - (1-\epsilon)\hat{c}_i^H t\}$ for $i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, \cdots, i_M a\}$, so it is omitted. □

Proof of Lemma 3: Similar to the proof of Lemma 1, we will bound $\mathbb{E}[Z_m(t)]^+$ and $\mathbb{E}[Z_m(t)]^-$. Recall that $\hat{c}^H = 1 - \sum_{i\in\{i_1 a,\cdots,i_M a\}} \hat{c}_i^H$. For $i = i_m a$, note that $C_i(t+1) = C_i(t)$ if the $(t+1)$st arrival is not from $\mathcal{J}_H$, which is with probability ϵ. By part 2(b) of DLDP, there are two random variables at any t,

$$W(t) = Z_m(t) - Z_m(t-1) = \begin{cases} \epsilon\frac{\hat{c}_{i_m a}^H}{1-\hat{c}^H} - \epsilon_m - \frac{\eta^{L_m}}{3(M-\gamma_t)}, & \text{with probability } 1-\epsilon, \\ 1 - \epsilon_m - (1-\epsilon)\frac{\hat{c}_{i_m a}^H}{1-\hat{c}^H}, & \text{with probability } \epsilon_m, \\ -\epsilon_m - (1-\epsilon)\frac{\hat{c}_{i_m a}^H}{1-\hat{c}^H}, & \text{with probability } \epsilon - \epsilon_m, \end{cases}$$

corresponding to the scenario where $Z_m(t) \geq 0$ and

$$W'(t) = Z_m(t) - Z_m(t-1) = \begin{cases} \epsilon \frac{\hat{c}^H_{i_m a}}{1-\hat{c}^H} - \epsilon_m + \frac{\eta^{L_m}}{3(M-\gamma_t)}, & \text{with probability } 1-\epsilon, \\ 1 - \epsilon_m - (1-\epsilon)\frac{\hat{c}^H_{i_m a}}{1-\hat{c}^H}, & \text{with probability } \epsilon_m, \\ -\epsilon_m - (1-\epsilon)\frac{\hat{c}^H_{i_m a}}{1-\hat{c}^H}, & \text{with probability } \epsilon - \epsilon_m, \end{cases}$$

corresponding to the scenario where $Z_m(t) < 0$. Furthermore, $\{W(t) : t \geq 1\}$ and $\{W'(t) : t \geq 1\}$ are both iid random variables.

By similar arguments as in the proof of Lemma 1, we can obtain $\mathbb{E}[Z_m(t)]^+ \leq \mathbb{E}\hat{Z}_m(t) + 2$, where $\hat{Z}_m(\cdot)$ is a Lindley process recursively defined as $\hat{Z}_m(t+1) = \left[\hat{Z}_m(t) + W(t)\right]^+$ with $\hat{Z}_m(0) = 0$. It is straightforward to verify that $-\mathbb{E}[W(t)]$ is in the order of $(1-\epsilon)$ and $\mathrm{Var}(W(t))$ is in the order of 1. By the Central Limit Theorem and the tail property of a normal distribution, we have that $\frac{1}{n}\mathbb{E}\left[\sum_{t=1}^{n} W(t)\right]^+ \sim \frac{1}{n}\mathbb{E}\left[-n(1-\epsilon) + C_1\sqrt{n}Z\right]^+ \leq \frac{e^{-\sqrt{n}(1-\epsilon)}}{\sqrt{n}}$ for some constant C_1, where Z follows the standard normal distribution. Using Kingman's formula, we have $\mathbb{E}[\hat{Z}_m(\infty)] = \sum_{n=1}^{\infty} \frac{\mathbb{E}\left[\sum_{t=1}^{n} W(t)\right]^+}{n}$ which is bounded by some constant independent of ϵ and t. It follows that $\mathbb{E}[Z_m(t)]^+$ is uniformly bounded, and so is $\mathbb{E}[Z_m(t)]^-$ by similar arguments. □

PROPOSITION EC.1. *For a given set of non-negative functions $C_i(t)$, $1 \leq t \leq K$, such that $\sum_{i=1}^{I} C_i(t) = t$ and the systems $(\mathscr{A}, \{C_i(t) - C_i(t-1)\}_{i=1,\dots I}, \mathbf{p})$ have positive GCGs, denoted as $\tilde{\eta}^t$, we have*

$$\mathbb{E}\left[\sum_{i=1}^{I} |L_i(K) - C_i(K)|\right] \leq \ln 64 \times \max\left\{\frac{1}{C_{\min}}, \frac{I}{\tilde{\eta}_{\min}}\right\},$$

where $C_{\min} = \min\{C_i(t) - C_i(t-1) : 1 \leq i \leq I,\ 1 \leq t \leq K\}$ and $\tilde{\eta}_{\min} = \min\{\tilde{\eta}^t : 1 \leq t \leq K\}$.

Proof: The proof is a straightforward extension of Theorem 1 in Xu et al. (2020) and hence is omitted. □

Proof of Theorem 1: Under the DLDP, the total expected number of lost sales is bounded from above by

$$\sum_{i \in \mathcal{I}} \mathbb{E}|L_i(T) - c_i T| = \sum_{i \in \{i_1, i_2, \cdots, i_M\}} \mathbb{E}|L_i(T) - c_i T| + \sum_{i \notin \{i_1, i_2, \cdots, i_M\}} \mathbb{E}|L_i(T) - c_i T|$$

$$= \sum_{m=1}^{M} \mathbb{E}|L_{i_m}(T) - c_{i_m}T| + \sum_{m=1}^{M} \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} \mathbb{E}|L_i(T) - c_i T|$$
$$+ \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|L_i(T) - c_i T|. \tag{EC.2}$$

We will bound each of the three terms in (EC.2) as follows:

- Noting that the total inventory at resource i_m is $c_{i_m} = (1-\epsilon)\hat{c}^H_{i_m a} + \epsilon_m \hat{c}^{Lm}_{i_m b}$, we have

$$\begin{aligned}
\sum_{m=1}^{M} \mathbb{E}|L_{i_m}(T) - c_{i_m}T| &= \sum_{m=1}^{M} \mathbb{E}|L_{i_m a}(T) + L_{i_m b}(T) - \epsilon_m \hat{c}^{Lm}_{i_m b} T - (1-\epsilon)\hat{c}^H_{i_m a} T| \\
&\le \sum_{m=1}^{M} \mathbb{E}|L_{i_m a}(T) - C_{i_m a}(T)| + \sum_{m=1}^{M} \mathbb{E}|L_{i_m b}(T) - C_{i_m b}(T)| \\
&\quad + \sum_{m=1}^{M} \mathbb{E}|C_{i_m a}(T) + C_{i_m b}(T) - \epsilon_m \hat{c}^{Lm}_{i_m b} T - (1-\epsilon)\hat{c}^H_{i_m a} T|. \qquad \text{(EC.3)}
\end{aligned}$$

Furthermore, since $C_{i_m b}(T) = \tau^{Lm}_T - \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} C_i(T)$ and $\epsilon_m \hat{c}^{Lm}_{i_m b} T = \epsilon_m T - \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} \epsilon_m \hat{c}^{Lm}_i T$,

$$\begin{aligned}
&\mathbb{E}|C_{i_m a}(T) + C_{i_m b}(T) - \epsilon_m \hat{c}^{Lm}_{i_m b} T - (1-\epsilon)\hat{c}^H_{i_m a} T| \\
&\le \mathbb{E}|C_{i_m a}(T) - (1-\epsilon)\hat{c}^H_{i_m a} T + \tau^{Lm}_T - \epsilon_m T| + \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} \mathbb{E}|C_i(T) - \epsilon_m \hat{c}^{Lm}_i T| \\
&= \mathbb{E}\left| Y_{i_m a}(T) + \tau^{Lm}_T - \epsilon_m T + \left[\tau^H_T - (1-\epsilon)T - \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \big(C_i(T) - (1-\epsilon)\hat{c}^H_i T \big) \right] \frac{\hat{c}^H_{i_m a}}{\sum_{\ell=1}^{M} \hat{c}^H_{i_\ell a}} \right| \\
&\quad + \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} \mathbb{E}|C_i(T) - \epsilon_m \hat{c}^{Lm}_i T| \\
&\le \mathbb{E}\left| Y_{i_m a}(T) + \tau^{Lm}_T - \epsilon_m T + [\tau^H_T - (1-\epsilon)T] \frac{\hat{c}^H_{i_m a}}{\sum_{\ell=1}^{M} \hat{c}^H_{i_\ell a}} \right| \\
&\quad + \frac{\hat{c}^H_{i_m a}}{\sum_{\ell=1}^{M} \hat{c}^H_{i_\ell a}} \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|C_i(T) - (1-\epsilon)\hat{c}^H_i T| + \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} \mathbb{E}|C_i(T) - \epsilon_m \hat{c}^{Lm}_i T| \\
&\le \mathbb{E}|Z_m(T)| + \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|C_i(T) - (1-\epsilon)\hat{c}^H_i T| + \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} \mathbb{E}|C_i(T) - \epsilon_m \hat{c}^{Lm}_i T|.
\end{aligned}$$
$$\text{(EC.4)}$$

- For any $m \in \{1, 2 \cdots, M\}$,

$$\sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_{mb}\}} \mathbb{E}|L_i(T) - c_i T| = \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_{mb}\}} \mathbb{E}|L_i(T) - \hat{c}_i^{Lm} \epsilon_m T|$$
$$\leq \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_{mb}\}} \left[\mathbb{E}|C_i(T) - \hat{c}_i^{Lm} \epsilon_m T| + \mathbb{E}|L_i^{Lm}(T) - C_i(T)| \right]. \tag{EC.5}$$

- For the third term,

$$\sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|L_i(T) - c_i T| = \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|L_i(T) - (1-\epsilon) \hat{c}_i^H T|$$
$$\leq \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|C_i(T) - (1-\epsilon) \hat{c}_i^H T| + \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|L_i(T) - C_i(T)|. \tag{EC.6}$$

By (EC.4) – (EC.6) and Proposition EC.1, we can establish that the total expected number of lost sales under the DLDP is bounded from above by

$$3 \ln 64 \max \left\{ \frac{1}{\hat{c}_{\min}}, \frac{I+M}{\eta_{\min}} \right\} + \mathbb{E}|Z_m(T)| + 2 \sum_{m=1}^{M} \sum_{i \in \mathcal{I}(\mathcal{J}_{Lm}) \setminus \{i_m b\}} \mathbb{E}|C_i(T) - \epsilon_m \hat{c}_i^{Lm} T|$$
$$+ (M+1) \sum_{i \in \mathcal{I}(\mathcal{J}_H) \setminus \{i_1 a, i_2 a, \cdots, i_M a\}} \mathbb{E}|C_i(T) - (1-\epsilon) \hat{c}_i^H T|, \tag{EC.7}$$

which is independent of ϵ, where $\hat{c}_{\min} = \min\{\hat{c}_{\min}^{L1}, \ldots, \hat{c}_{\min}^{LM}\}$ with $\hat{c}_{\min}^{Lm}$ being the lowest inventory among all the dedicated resources in subnetwork $\mathscr{A}_{Lm}$, and $\eta_{\min} = \min\{\eta^{L1}, \cdots, \eta^{LM}, \eta^H\}$. The bound in (EC.7) captures three sources of lost sales: (1) $3 \ln 64 \max \left\{ \frac{1}{\hat{c}_{\min}}, \frac{I+M}{\eta_{\min}} \right\}$ is an upper bound of the lost sales due to imperfection of the DLDP measured by $\sum_i (|L_i(t) - C_i(t)|)$ for all i. (2) $\mathbb{E}|Z_m(T)|$ bounds the lost sales caused by inventory imbalance among the shared resources. (3) The rest captures lost sales attributed to inventory imbalance among the resources dedicated to the subnetworks measured by $(|C_i(t) - c_i t|)$ for $i \neq i_m$.

Lastly, Theorem 1 follows immediately from (EC.7), combined with Lemmas 1 – 3. □

EC.2. Other Proofs

Proof of Proposition 1. We first note that a long chain can be viewed as a circular chain with the resources and markets as nodes along the circle and $2J$ arcs between markets and their suppliers. If we label the market nodes from 1 to J clockwise, we can view a decision $\{q_1, q_2, \cdots, q_J\}$ as a permutation of $\{r_1, r_2, \cdots, r_J\}$, where q_j is the arranged market at position j. Without loss of generality, we fix $q_1 = r_1$. For convenience, the index $J+j$ represents the index j.

1. When $\ell = 2$, $\mathcal{J}_H$-connectivity can be achieved if and only if the two random draws from $\mathcal{J}_L$ result in the same number, which is with probability $\sum_{j=1}^{J} q_j^2$, or two consecutive market nodes on the circle, which is with probability $2\sum_{j=1}^{J} q_j q_{j+1}$. Thus, the risk probability is $1-\sum_{j=1}^{J} q_j^2 - 2\sum_{j=1}^{J} q_j q_{j+1}$. Minimizing the risk probability is equivalent to finding a permutation that maximizes $\sum_{j=1}^{J} q_j q_{j+1}$.

Now we use difference analysis to investigate the properties of the optimal permutation denoted as $\{q_j^* : j = 1, \cdots, J\}$. We first claim that, if $q_J^* \geq q_2^*$, then $q_{J-1}^* \leq q_2^*$, since otherwise we could swap q_J^* with q_1^*, resulting in the difference of the risk probability

$$q_{J-1}^* q_J^* + q_1^* q_2^* - q_{J-1}^* q_1^* - q_J^* q_2^* = (q_1^* - q_J^*)(q_2^* - q_{J-1}^*) < 0,$$

contradicting the optimality of $\{q_j^* : j = 1, \cdots, J\}$. Furthermore, it also holds that $q_{J-1}^* \geq q_3^*$, since otherwise we could swap q_J^* with q_2^*, resulting in the difference of the risk probability

$$q_{J-1}^* q_J^* + q_3^* q_2^* - q_{J-1}^* q_2^* - q_3^* q_J^* = (q_2^* - q_J^*)(q_3^* - q_{J-1}^*) < 0,$$

also a contradiction. Continuing this process, we can prove that $q_{J-j}^* \geq q_{j+2}^*$ and $q_{J-j}^* \leq q_{j+1}^*$, for $0 \leq j \leq \frac{J-2}{2}$. That is, $(q_1^*, q_J^*, q_2^*, q_{J-1}^*, \cdots,)$ must be a decreasing sequence, completing the proof.

2. When $\ell = 3$, we first show that the risk probability is $1-\sum_{j=1}^{J} q_j^3 - 3\sum_{j=1}^{J}\left(q_j^2 q_{j+1} + q_j q_{j+1}^2\right) - 6\sum_{j=1}^{J} q_j q_{j+1} q_{j+2}$ for a given permutation $\{q_1, q_2, \cdots, q_J\}$. That is because $\mathcal{J}_H$-connectivity can be ensured if and only if the three random draws from $\mathcal{J}_L$ result in either the same number, which is with probability $\sum_{i=1}^{J} q_i^3$, or two consecutive numbers, which is with probability $3\sum_{j=1}^{J}\left(q_j^2 q_{j+1} + q_j q_{j+1}^2\right)$, or three consecutive numbers, which is with probability $6\sum_{j=1}^{J} q_j q_{j+1} q_{j+2}$. Minimizing the risk probability is equivalent to finding a permutation that maximizes $3\sum_{j=1}^{J}\left(q_j^2 q_{j+1} + q_j q_{j+1}^2\right) + 6\sum_{j=1}^{J} q_j q_{j+1} q_{j+2}$.

We now prove that swapping any q_i with q_j, $i<j$, in the permutation $\{\cdots,q_6,q_4,q_2,q_1,q_3,q_5,q_7,\cdots\}=\{\cdots,r_6,r_4,r_2,r_1,r_3,r_5,r_7,\cdots\}$ will not decrease the risk probability. We first note that q_i-q_j, $q_{i-1}-q_{j+1}$ and $q_{i-2}q_{i-1}-q_{j+1}q_{j+2}$ have the same sign.

- $j-i=1$. In this case, the difference in the risk probability after the swapping is

$$\begin{aligned}
&3q_{i-1}^2q_i+3q_{i-1}q_i^2-\left(3q_{i-1}^2q_j+3q_{i-1}q_j^2\right)+3q_j^2q_{j+1}+3q_jq_{j+1}^2-\left(3q_i^2q_{j+1}+3q_iq_{j+1}^2\right)\\
&\quad+6q_{i-2}q_{i-1}q_i+6q_jq_{j+1}q_{j+2}-(6q_{i-2}q_{i-1}q_j+6q_iq_{j+1}q_{j+2})\\
&=3(q_i-q_j)[q_{i-1}(q_{i-1}+q_i+q_j)-q_{j+1}(q_{j+1}+q_i+q_j)]+6(q_i-q_j)(q_{i-2}q_{i-1}-q_{j+1}q_{j+2})\\
&\geq 0.
\end{aligned}$$

- $j-i=2$. In this case, the difference in the risk probability is

$$\begin{aligned}
&3q_{i-1}^2q_i+3q_{i-1}q_i^2-(3q_{i-1}^2q_j+3q_{i-1}q_j^2)+3q_j^2q_{j+1}+3q_jq_{j+1}^2-(3q_i^2q_{j+1}+3q_iq_{j+1}^2)\\
&\quad+3q_{i+1}^2q_i+3q_{i+1}q_i^2-(3q_{i+1}^2q_j+3q_{i+1}q_j^2)+3q_j^2q_{j-1}+3q_jq_{j-1}^2-(3q_i^2q_{j-1}+3q_iq_{j-1}^2)\\
&\quad+6q_{i-2}q_{i-1}q_i+6q_jq_{j+1}q_{j+2}-(6q_{i-2}q_{i-1}q_j+6q_iq_{j+1}q_{j+2})\\
&\quad+6q_{i-1}q_iq_{i+1}+6q_{j-1}q_jq_{j+1}-(6q_{i-1}q_jq_{i+1}+6q_{j-1}q_iq_{j+1})\\
&=3(q_i-q_j)[q_{i-1}(q_{i-1}+q_i+q_j)-q_{j+1}(q_{j+1}+q_i+q_j)]+6(q_i-q_j)(q_{i-2}q_{i-1}-q_{j+1}q_{j+2})\\
&\quad+6(q_i-q_j)(q_{i-1}q_{i+1}-q_{j-1}q_{j+1}),\\
&\geq 0,
\end{aligned}$$

as q_i-q_j and $q_{i-1}q_{i+1}-q_{j-1}q_{j+1}$ have the same sign.

- $j-i\geq 3$. In this case, the resulting difference of the risk probability is

$$\begin{aligned}
&3q_{i-1}^2q_i+3q_{i-1}q_i^2-\left(3q_{i-1}^2q_j+3q_{i-1}q_j^2\right)+3q_j^2q_{j+1}+3q_jq_{j+1}^2-\left(3q_i^2q_{j+1}+3q_iq_{j+1}^2\right)\\
&+3q_{i+1}^2q_i+3q_{i+1}q_i^2-\left(3q_{i+1}^2q_j+3q_{i+1}q_j^2\right)+3q_j^2q_{j-1}+3q_jq_{j-1}^2-\left(3q_i^2q_{j-1}+3q_iq_{j-1}^2\right)\\
&+6q_{i-2}q_{i-1}q_i+6q_jq_{j+1}q_{j+2}-(6q_{i-2}q_{i-1}q_j+6q_iq_{j+1}q_{j+2})\\
&+6q_{i-1}q_iq_{i+1}+6q_{j-1}q_jq_{j+1}-(6q_{i-1}q_jq_{i+1}+6q_{j-1}q_iq_{j+1})\\
&+6q_iq_{i+1}q_{i+2}+6q_{j-2}q_{j-1}q_j-(6q_jq_{i+1}q_{i+2}+6q_{j-2}q_{j-1}q_i)\\
&=3(q_i-q_j)[q_{i-1}(q_{i-1}+q_i+q_j)-q_{j+1}(q_{j+1}+q_i+q_j)]+6(q_i-q_j)(q_{i-2}q_{i-1}-q_{j+1}q_{j+2})\\
&\quad+6(q_i-q_j)(q_{i-1}q_{i+1}-q_{j-1}q_{j+1})+6(q_i-q_j)(q_{i+1}q_{i+2}-q_{j-2}q_{j-1})\\
&\geq 0,
\end{aligned}$$

as q_i-q_j and $q_{i+1}q_{i+2}-q_{j-2}q_{j-1}$ have the same sign.

□

Proof of Proposition 2. Let $\{q_1, q_2, \cdots, q_J\}$ be a permutation of $\{r_1, r_2, \cdots, r_J\}$. $\mathcal{J}_H$-connectivity holds only when the two markets with low shares lie in different half circles corresponding to $\mathcal{J}'$ and $\mathcal{J} \setminus \mathcal{J}'$, respectively. The extra arc will increase the probability of $\mathcal{J}_H$-connectivity by $\left(\sum_{j\in\mathcal{J}'} q_j\right)\left(\sum_{j\in\mathcal{J}\setminus\mathcal{J}'} q_j\right) = \left(\sum_{j\in\mathcal{J}'} q_j\right)\left(1 - \sum_{j\in\mathcal{J}'} q_j\right)$, which is maximized when $\sum_{j\in\mathcal{J}'} q_j$ is as close to 0.5 as possible. □

Proof of Proposition 3. With $\ell = 2$ and a budget of one leaf node, we show that it is optimal to make resource 1 dedicated to market 1 and the resource that is linked to markets $J-1$ and J as the hub.

Suppose that the dedicated resource is i and the hub is linked to markets j_1 and j_2. The change of network structure increases the risk probability if both markets j_1 and j_2 turn out to be low share markets, which is with probability $2r_{j_1}r_{j_2}$. On the other hand, such a change will decrease the risk probability if the two neighboring markets of i in the original long chain, denoted as $i-$ and $i+$, turn out to be low share markets, which is with probability $2r_{i-}r_{i+}$, or if market i is a low share market along with a market $j \neq i-, i+$, which is with probability $2r_i(1 - r_i - r_{i-} - r_{i+})$. Therefore, creating leaf node i decreases the risk probability by $g(i) = 2r_i(1 - r_i - r_{i-} - r_{i+}) + 2r_{i-}r_{i+} - 2r_{j_1}r_{j_2}$, which is maximized when $(j_1, j_2) = (J-1, J)$, for a given i.

We can easily establish that $g(i)$ is maximized at $i = 1$ as

$$\begin{aligned} g(1) - g(2) &= 2r_1(1 - r_1 - r_2 - r_3) + 2r_2r_3 - [2r_2(1 - r_2 - r_1 - r_4) + 2r_1r_4] \\ &= 2(r_1 - r_2)(1 - r_1 - r_2 - r_3 - r_4) \geq 0, \end{aligned}$$

as $r_1 \geq r_2$ and $r_1 + r_2 + r_3 + r_4 \leq 1$. Similarly, we can show that $g(1) \geq g(2) \geq g(4) \geq \cdots$ and $g(1) \geq g(3) \geq g(5) \geq \cdots$.

□