Incorporating Income Disparity and Utility Heterogeneity in Resource Allocation: Online Appendices

## A. Optimizing the Income-Dependent Menu: Compact Formulation

In this appendix, we drive the simplified constraints in (13) and (14).

**Lemma A.1** *In the problem* (9)-(11)*, the set of individual rationality constraints in* (10) *can be reduced to:*

$$G(s_{i1}, g_1) + F(f_i + m_{i1}) \geq F(f_i), \qquad \forall i \in \mathcal{I}. \tag{27}$$

Lemma A.1 states that we only need to ensure the individual rationality constraint for the consumers with $j = 1$, the lowest resource utility parameter. The proof shows that the individual rationality constraints for $j > 1$ are implied by the first individual rationality constraint and the incentive compatibility constraints.

**Proof of Lemma A.1**: First, consider the incentive compatibility constraint (11) for any $i$ and for $j > k = 1$.

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq G(s_{i1}, g_j) + F(f_i + m_{i1}).$$

Because $G(s, g)$ is increasing in $g$, we have $G(s_{i1}, g_j) + F(f_i + m_{i1}) \geq G(s_{i1}, g_1) + F(f_i + m_{i1})$.

Therefore, if the individual rationality constraint for group $j = 1$ in (27) is satisfied, the above two inequalities lead to $G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq F(f_i)$ for $j > 1$. In other words, the individual rationality constraint for $j > 1$ is implied by the incentive compatibility constraints and the individual rationality constraint for $j = 1$. □

**Lemma A.2** *In the problem* (9)-(11)*, the set of incentive compatibility constraints in* (11) *can be reduced to:*

$$G(s_{i,j+1}, g_j) - G(s_{ij}, g_j) \leq F(f_i + m_{ij}) - F(f_i + m_{i,j+1}) \leq G(s_{i,j+1}, g_{j+1}) - G(s_{ij}, g_{j+1}), \forall i, \forall j < J.$$

Lemma A.2 states that we only need to ensure the incentive compatibility constraints that prevent adjacent deviations, i.e., a consumer in group $j$ has no incentive to mimic a consumer in group $j + 1$ and vice versa.

**Proof of Lemma A.2**: Consider indices $j$, $x$, and $k$, such that $1 \leq j < x < k \leq J$. The incentive-compatibility constraints ensure that consumers within the same income group $i$, but with different resource utility indices, $j$, $x$, and $k$, have no incentive to mimic each other. In what follows, we prove that the incentive-compatibility constraints between groups $(i, j)$ and $(i, x)$, together with those between groups $(i, x)$ and $(i, k)$, imply the incentive-compatibility constraints between groups $(i, j)$ and $(i, k)$.

First, if consumer $(i, j)$ will not mimic being in group $(i, x)$ and consumer $(i, x)$ will not mimic being in group $(i, k)$, we must have

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq G(s_{ix}, g_j) + F(f_i + m_{ix}), \tag{28}$$

$$G(s_{ix}, g_x) + F(f_i + m_{ix}) \geq G(s_{ik}, g_x) + F(f_i + m_{ik}). \tag{29}$$

Second, using the same logic as in the proof of Proposition 1, we can show that the incentive-compatibility constraints between groups $(i, x)$ and $(i, k)$ along with the supermodularity of $G(s, g)$ imply that $s_{ix} \leq s_{ik}$. Then, the supermodularity of $G(s, g)$ and $g_j < g_x$ imply that

$$G(s_{ik}, g_x) - G(s_{ix}, g_x) \geq G(s_{ik}, g_j) - G(s_{ix}, g_j). \tag{30}$$

Summing up the three inequalities (28)-(30), we have

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq G(s_{ik}, g_j) + F(f_i + m_{ik}).$$

That is, consumer $(i, j)$ will not mimic being in group $(i, k)$.

Similarly, we can show that if consumer $(i, k)$ will not mimic being in group $(i, x)$ and consumer $(i, x)$ will not mimic being in group $(i, j)$, then consumer $(i, k)$ will not mimic being in group $(i, j)$.

Therefore, the incentive-compatibility constraints between adjacent groups $(i, j)$ and $(i, j + 1)$ for all $j < J$ imply the full set of incentive-compatibility constraints in (11). □

## B. Proofs

**Proof of Lemma 1**: We first prove that the first-best monetary allocation $m_{ij}^{\mathsf{FB}}$ (weakly) decreases in the income index $i$ for any given utility index $j$. For an arbitrary feasible solution to (3)-(7) that does not satisfy this property, i.e., one where $m_{ij} < m_{kj}$ with $i < k$, for some $i, k \in \mathcal{I}$ and some $j \in \mathcal{J}$, we will prove that this solution can be strictly improved.

Consider a consumer in group $(i, j)$ and another consumer in group $(k, j)$. Because $f_i < f_k$ and $m_{ij} < m_{kj}$, we have $f_i + m_{ij} < f_k + m_{kj}$. We propose a new allocation for these two consumers. (Similarly, we can propose new allocations for all consumers in these two groups, and the remainder of the proof will only require slight modifications to account for $\theta_{ij}$ and $\theta_{kj}$.) We consider two cases.

<u>*Case 1*</u>: Non-binding individual rationality constraint for $(k, j)$: $G(s_{kj}, g_j) + F(f_k + m_{kj}) > F(f_k)$.

We propose a new monetary allocation, $\widehat{m}_{ij} = m_{ij} + \epsilon_1$ and $\widehat{m}_{kj} = m_{kj} - \epsilon_1$, where $\epsilon_1 > 0$ is small enough so that consumer $(k, j)$'s individual rationality constraint remains non-binding and that $f_i + m_{ij} < f_i + \widehat{m}_{ij} < f_k + \widehat{m}_{kj} < f_k + m_{kj}$. Then, the strict concavity of $F(\cdot)$ implies that

$$F(f_i + \widehat{m}_{ij}) + F(f_k + \widehat{m}_{kj}) > F(f_i + m_{ij}) + F(f_k + m_{kj}),$$

which strictly improves the sum of the utilities of these two consumers.

Case 2: Binding individual rationality constraint for $(k, j)$: $G(s_{kj}, g_j) + F(f_k + m_{kj}) = F(f_k)$.

This binding constraint implies $m_{ij} < m_{kj} \leq 0$. Furthermore, we have $s_{ij} > s_{kj}$ because

$$G(s_{ij}, g_j) \geq F(f_i) - F(f_i + m_{ij}) > F(f_k) - F(f_k + m_{ij}) > F(f_k) - F(f_k + m_{kj}) = G(s_{kj}, g_j),$$

where the first inequality is the individual rationality constraint for $(i, j)$, the second inequality follows from the strict concavity of $F(\cdot)$, and the third inequality follows from $m_{ij} < m_{kj}$.

Consider a new monetary allocation, $\widehat{m}_{ij} = m_{ij} + \epsilon_2$ and $\widehat{m}_{kj} = m_{kj} - \epsilon_2$, and a new resource allocation, $\widehat{s}_{ij} = s_{ij} - \epsilon_3$ and $\widehat{s}_{kj} = s_{kj} + \epsilon_3$, satisfying the following properties: consumer $(k, j)$'s individual rationality constraint remains binding, $G(\widehat{s}_{kj}, g_j) + F(f_k + \widehat{m}_{kj}) = F(f_k)$, and $\epsilon_2 > 0$ and $\epsilon_3 > 0$ are small enough such that $f_i + m_{ij} < f_i + \widehat{m}_{ij} < f_k + \widehat{m}_{kj} < f_k + m_{kj}$ and $s_{kj} < \widehat{s}_{kj} < \widehat{s}_{ij} < s_{ij}$. Then, the strict concavity of $G(\cdot, g_j)$ and $F(\cdot)$ implies that the new allocation changes consumer $(i, j)$'s resource and financial utilities respectively by:

$$G(\widehat{s}_{ij}, g_j) - G(s_{ij}, g_j) > G(s_{kj}, g_j) - G(\widehat{s}_{kj}, g_j),$$
$$F(f_i + \widehat{m}_{ij}) - F(f_i + m_{ij}) > F(f_k + m_{kj}) - F(f_k + \widehat{m}_{kj}).$$

Summing the above two inequalities and noting that the sum on the right-side vanishes to zero because the individual rationality constraint for $(k, j)$ remains binding under the new allocation, we have $G(\widehat{s}_{ij}, g_j) + F(f_i + \widehat{m}_{ij}) - G(s_{ij}, g_j) - F(f_i + m_{ij}) > 0$, i.e., the utility of consumer $(i, j)$ is strictly higher under the new allocation, while the utility of consumer $(k, j)$ does not change.

In both cases, we have found feasible allocations that strictly improve the total utility. Therefore, the first-best monetary allocation $m_{ij}^{\mathsf{FB}}$ must (weakly) decrease in the income index $i$.

Next, we prove that $m_{ij}^{\mathsf{FB}}$ (weakly) decreases in the resource utility index $j$ for any income index $i$. For an arbitrary feasible solution to (3)-(7) that does not satisfy this property, i.e., one where $m_{ij} < m_{ik}$ with $j < k$, for some $i \in \mathcal{I}$ and $j, k \in \mathcal{J}$, we will prove that this solution can be strictly improved.

Consider a consumer in group $(i, j)$ and another consumer in group $(i, k)$. We propose a new allocation for these two consumers.

<u>*Case 1*</u>: Non-binding individual rationality constraint for $(i, k)$: $G(s_{ik}, g_k) + F(f_i + m_{ik}) > F(f_i)$.

We propose a new monetary allocation, $\widehat{m}_{ij} = m_{ij} + \epsilon_1$ and $\widehat{m}_{ik} = m_{ik} - \epsilon_1$, where $\epsilon_1 > 0$ is small enough so that consumer $(i, k)$'s individual rationality constraint remains non-binding and that $m_{ij} < \widehat{m}_{ij} < \widehat{m}_{ik} < m_{ik}$. Then, the strict concavity of $F(\cdot)$ implies that

$$F(f_i + \widehat{m}_{ij}) + F(f_i + \widehat{m}_{ik}) > F(f_i + m_{ij}) + F(f_i + m_{ik}),$$

which strictly improves the original allocation.

<u>*Case 2*</u>: Binding individual rationality constraint for $(i, k)$: $G(s_{ik}, g_k) + F(f_i + m_{ik}) = F(f_i)$.

This binding constraint implies $m_{ij} < m_{ik} \leq 0$. Furthermore, we have $s_{ij} > s_{ik}$ because

$$G(s_{ij}, g_j) \geq F(f_i) - F(f_i + m_{ij}) > F(f_i) - F(f_i + m_{ik}) = G(s_{ik}, g_k) > G(s_{ik}, g_j),$$

where the first inequality is the individual rationality constraint for $(i, j)$, the second inequality follows from $m_{ij} < m_{ik}$, and the last inequality is because $G(s, g)$ strictly increases in $g$ and $g_j < g_k$.

Consider a new monetary allocation, $\widehat{m}_{ij} = m_{ij} + \epsilon_2$ and $\widehat{m}_{ik} = m_{ik} - \epsilon_2$, and a new resource allocation, $\widehat{s}_{ij} = s_{ij} - \epsilon_3$ and $\widehat{s}_{ik} = s_{ik} + \epsilon_3$, satisfying the following properties: consumer $(i, k)$'s individual rationality constraint remains binding, $G(\widehat{s}_{ik}, g_k) + F(f_i + \widehat{m}_{ik}) = F(f_i)$, and $\epsilon_2 > 0$ and $\epsilon_3 > 0$ are small enough such that $m_{ij} < \widehat{m}_{ij} < \widehat{m}_{ik} < m_{ik}$ and $s_{ik} < \widehat{s}_{ik} < \widehat{s}_{ij} < s_{ij}$. Then, the new allocation changes consumer $(i, j)$'s resource and financial utilities respectively by:

$$G(\widehat{s}_{ij}, g_j) - G(s_{ij}, g_j) > G(s_{ik}, g_j) - G(\widehat{s}_{ik}, g_j) > G(s_{ik}, g_k) - G(\widehat{s}_{ik}, g_k),$$
$$F(f_i + \widehat{m}_{ij}) - F(f_i + m_{ij}) > F(f_i + m_{ik}) - F(f_i + \widehat{m}_{ik}),$$

where the first and last inequalities are due to the strict concavity of $G(\cdot, g_j)$ and $F(\cdot)$, and the second inequality follows from the supermodularity of $G(s, g)$.

Summing the above inequalities and noting that the sum on the right-side vanishes to zero because the individual rationality constraint for $(i, k)$ remains binding under the new allocation, we have $G(\widehat{s}_{ij}, g_j) + F(f_i + \widehat{m}_{ij}) - G(s_{ij}, g_j) - F(f_i + m_{ij}) > 0$, i.e., the utility of consumer $(i, j)$ is strictly higher under the new allocation, while the utility of consumer $(i, k)$ does not change.

In both cases, we have found feasible allocations that strictly improve the total utility. Therefore, the first-best monetary allocation $m_{ij}^{\mathsf{FB}}$ must (weakly) decrease in $j$. $\square$

**Proof of Lemma 2**: (i) We prove the first statement of the lemma by contradiction. Suppose the first-best solution to (3)-(7) satisfies the following: for some $i \in \mathcal{I}$ and $j < k$,

$$G(s_{ij}^{\mathsf{FB}}, g_j) + F(f_i + m_{ij}^{\mathsf{FB}}) > F(f_i) \quad \text{and} \quad G(s_{ik}^{\mathsf{FB}}, g_k) + F(f_i + m_{ik}^{\mathsf{FB}}) = F(f_i). \tag{31}$$

We will prove that such a solution can be strictly improved, contradicting to the fact that it is the first-best.

Consider a consumer in group $(i, j)$ and another consumer in group $(i, k)$. We propose a new allocation for these two consumers. Since $j < k$, Lemma 1 leads to $m_{ij}^{\mathsf{FB}} \geq m_{ik}^{\mathsf{FB}}$. Thus, we consider the following two cases.

<u>*Case 1*</u>: $m_{ij}^{\mathsf{FB}} = m_{ik}^{\mathsf{FB}}$. In this case, we must have $G(s_{ij}^{\mathsf{FB}}, g_j) > G(s_{ik}^{\mathsf{FB}}, g_k)$ in view of (31). This implies $s_{ij}^{\mathsf{FB}} > s_{ik}^{\mathsf{FB}}$ because $g_j < g_k$. Therefore,

$$G_j'(s_{ij}^{\mathsf{FB}}) < G_j'(s_{ik}^{\mathsf{FB}}) < G_k'(s_{ik}^{\mathsf{FB}}) \tag{32}$$

where the first inequality is due to the strict concavity of $G_j(s) \equiv G(s, g_j)$ and the second inequality follows from the supermodularity of $G(s, g)$.

We propose a new resource allocation, $\widehat{s}_{ij} = s_{ij}^{\mathsf{FB}} - \epsilon$ and $\widehat{s}_{ik} = s_{ik}^{\mathsf{FB}} + \epsilon$, where $\epsilon$ is small so that consumer $(i, j)$'s individual rationality constraint remains non-binding and the order of marginal resource utilities in (32) remains unchanged. This new allocation shifts a small amount of resource from consumer $(i, j)$ with a lower marginal utility to consumer $(i, k)$ with a higher marginal utility, thereby strictly improving the original allocation.

<u>*Case 2*</u>: $m_{ij}^{\mathsf{FB}} > m_{ik}^{\mathsf{FB}}$, which leads to

$$F'(f_i + m_{ij}^{\mathsf{FB}}) < F'(f_i + m_{ik}^{\mathsf{FB}}). \tag{33}$$

In this case, the order of marginal utilities $G_j'(s_{ij}^{\mathsf{FB}})$ and $G_k'(s_{ik}^{\mathsf{FB}})$ gives three subcases:

<u>*Case 2a*</u>: $G_j'(s_{ij}^{\mathsf{FB}}) < G_k'(s_{ik}^{\mathsf{FB}})$. In this case, the same improvement strategy as in Case 1 applies.

<u>*Case 2b*</u>: $G_j'(s_{ij}^{\mathsf{FB}}) > G_k'(s_{ik}^{\mathsf{FB}})$. In this case, consumer $(i, j)$ has a higher marginal resource utility but a lower marginal financial utility than consumer $(i, k)$. Thus, we propose a new allocation, $\widehat{s}_{ij} = s_{ij}^{\mathsf{FB}} + \epsilon_1$, $\widehat{s}_{ik} = s_{ik}^{\mathsf{FB}} - \epsilon_1$, $\widehat{m}_{ij} = m_{ij}^{\mathsf{FB}} - \epsilon_2$, and $\widehat{m}_{ik} = m_{ik}^{\mathsf{FB}} + \epsilon_2$, satisfying the following properties: (31) continues to hold, and $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are small enough such that $m_{ij}^{\mathsf{FB}} > \widehat{m}_{ij} > \widehat{m}_{ik} > m_{ik}^{\mathsf{FB}}$ and $G_j'(s_{ij}^{\mathsf{FB}}) > G_j'(\widehat{s}_{ij}) > G_k'(\widehat{s}_{ik}) > G_k'(s_{ik}^{\mathsf{FB}})$. Then, the new allocation strictly improves both the sum of financial utilities and the sum of resource utilities.

<u>*Case 2c*</u>: $G_j'(s_{ij}^{\mathsf{FB}}) = G_k'(s_{ik}^{\mathsf{FB}})$. In this case, we propose the same improvement strategy as in

Case 2b, which leads to a resource utility loss of order $o(\epsilon_1)$ due to $G'_j(s^{\mathsf{FB}}_{ij}) = G'_k(s^{\mathsf{FB}}_{ik})$. Recall that $\epsilon_1$ and $\epsilon_2$ are chosen such that consumer $(i,k)$'s individual rationality constraint remains binding, implying that $\epsilon_1$ and $\epsilon_2$ are of the same order of magnitude. On the other hand, the improvement strategy improves the financial utilities by an amount of order $O(\epsilon_2)$ in view of (33), which dominates the loss of order $o(\epsilon_1)$ for small enough $\epsilon_1$ and $\epsilon_2$. Hence, there exist $\epsilon_1$ and $\epsilon_2$ such that the new allocation strictly improves the original allocation.

(ii) The Lagrangian for the first-best allocation problem in (3)-(7) is:

$$\begin{aligned}\mathcal{L} = & N \sum_{i\in\mathcal{I},j\in\mathcal{J}} \big(G_j(s_{ij}) + F(f_i + m_{ij})\big)\,\theta_{ij} + \lambda\Big(S - N\sum_{i\in\mathcal{I},j\in\mathcal{J}} s_{ij}\theta_{ij}\Big) + \sum_{i\in\mathcal{I},j\in\mathcal{J}} \mu_{ij}s_{ij} \\ & + \sum_{i\in\mathcal{I},j\in\mathcal{J}} \xi_{ij}\Big(G_j(s_{ij}) + F(f_i+m_{ij}) - F(f_i)\Big) + \eta\Big(M - N\sum_{i\in\mathcal{I},j\in\mathcal{J}} m_{ij}\theta_{ij}\Big),\end{aligned}$$

where $G_j(s) \equiv G(s, g_j)$.

Because the constraints in (4)-(7) define a non-empty (when $M$ is not too negative, as discussed after (7)), convex, and bounded feasible region, and the objective in (3) is strictly concave, the first-best solution exists and is unique. The first-best solution must satisfy the Karush–Kuhn–Tucker (KKT) conditions. We leverage a subset of the KKT conditions for the purpose of this proof.

If at the first-best solution, the individual rationality constraint is non-binding for consumer group $(i,j)$, then $\xi_{ij} = 0$ and the first-best resource allocation $s^{\mathsf{FB}}_{ij}$ must satisfy the following subset of KKT conditions:

$$N\theta_{ij}\big(G'_j(s^{\mathsf{FB}}_{ij}) - \lambda\big) + \mu_{ij} = 0, \tag{34}$$

$$\mu_{ij}s^{\mathsf{FB}}_{ij} = 0, \tag{35}$$

$$\mu_{ij} \geq 0, \tag{36}$$

$$s^{\mathsf{FB}}_{ij} \geq 0. \tag{37}$$

If $\lambda < G'_j(0)$, then $s^{\mathsf{FB}}_{ij}$ cannot be 0, as zero allocation would violate either (34) or (36). Therefore, in this case, we must have $s^{\mathsf{FB}}_{ij} > 0$, which implies $\mu_{ij} = 0$ due to (35). Hence, $s^{\mathsf{FB}}_{ij}$ must satisfy $\lambda = G'_j(s^{\mathsf{FB}}_{ij})$ due to (34).

If $\lambda \geq G'_j(0)$, then $\lambda > G'_j(s)$ for all $s > 0$, because $G'_j(s)$ strictly decreasing in $s$. In this case, $s^{\mathsf{FB}}_{ij}$ cannot be positive, because $s^{\mathsf{FB}}_{ij} > 0$ implies $\lambda > G'_j(s^{\mathsf{FB}}_{ij})$, which requires $\mu_{ij} > 0$ to satisfy (34), which then violates (35). Hence, in this case, $s^{\mathsf{FB}}_{ij} = 0$ and $\mu_{ij} = N\theta_{ij}\big(\lambda - G'_j(0)\big)$.

To summarize, the solution for $s_{ij}^{\mathsf{FB}}$ is:

$$s_{ij}^{\mathsf{FB}} = \begin{cases} {G'_j}^{-1}(\lambda), & \text{if } \lambda < G'_j(0), \\ 0, & \text{if } \lambda \geq G'_j(0), \end{cases}$$

which is exactly (8).

Clearly, $s_{ij}^{\mathsf{FB}}$ depends only on $j$. To see that $s_{ij}^{\mathsf{FB}}$ increases in $j$, note the following properties. First, if $j$ is small such that $\lambda \geq G'_j(0)$, then $s_{ij}^{\mathsf{FB}} = 0$. Second, $G'_j(s)$ decreases in $s$ (convexity) and increases in $j$ (supermodularity of $G(s, g)$). Thus, its inverse ${G'_j}^{-1}(\lambda)$ decreases in $\lambda$ and increases in $j$. □

**Proof of Proposition 1**: We prove a stronger result that any feasible solution to (9)-(11) satisfies the conditions in the proposition.

(i) The incentive compatibility constraints in (11) ensure that consumers within the same income group $i$, but with different resource utility parameters $g_j$ and $g_k$, have no incentive to mimic each other. That is, $\forall i \in \mathcal{I}$, $\forall j, k \in \mathcal{J}$, a feasible solution must satisfy:

$$G(s_{ik}, g_j) - G(s_{ij}, g_j) \leq F(f_i + m_{ij}) - F(f_i + m_{ik}) \leq G(s_{ik}, g_k) - G(s_{ij}, g_k). \tag{38}$$

Consider $j < k$ and thus $g_j < g_k$. We aim to prove $s_{ij} \leq s_{ik}$ by contradiction. Suppose $s_{ij} > s_{ik}$ holds. Then, the strict supermodularity of $G(s, g)$ (Assumption 1) implies that

$$G(s_{ik}, g_j) - G(s_{ij}, g_j) > G(s_{ik}, g_k) - G(s_{ij}, g_k),$$

which contradicts (38). Therefore, $s_{ij} \leq s_{ik}$ for $j < k$.

(ii) From part (i), for $j < k$, we have $s_{ij} \leq s_{ik}$ and thus $G(s_{ik}, g_j) - G(s_{ij}, g_j) \geq 0$. Then, (38) implies that $F(f_i + m_{ij}) - F(f_i + m_{ik}) \geq 0$, which holds only if $m_{ij} \geq m_{ik}$. Thus, we conclude that $m_{ij}$ decreases in $j$ for any feasible solution.

(iii) If $s_{ij} < s_{ik}$, we have $G(s_{ik}, g_j) - G(s_{ij}, g_j) > 0$, because $G(s, g)$ strictly increases in $s$. Then, (38) implies that $F(f_i + m_{ij}) - F(f_i + m_{ik}) > 0$, which holds only if $m_{ij} > m_{ik}$.

Conversely, if $m_{ij} > m_{ik}$, we have $F(f_i + m_{ij}) - F(f_i + m_{ik}) > 0$, since $F(\cdot)$ is a strictly increasing function. Then, the second inequality in (38) implies that $G(s_{ik}, g_k) - G(s_{ij}, g_k) > 0$, which holds only if $s_{ij} < s_{ik}$. □

**Proof of Proposition 2**: For a given income group $i \in \mathcal{I}$ and a feasible resource allocation $\{s_{ij}, j \in \mathcal{J}\}$, we define

$$L_j = G(s_{i,j+1}, g_j) - G(s_{ij}, g_j), \qquad U_j = G(s_{i,j+1}, g_{j+1}) - G(s_{ij}, g_{j+1}), \qquad j \in \mathcal{J}\backslash\{J\}.$$

The proof of Proposition 1 shows that the feasible $s_{ij}$ must be (weakly) increasing in $j$. This, along with the supermodularity of $G(s,g)$, implies $0 \leq L_j \leq U_j$.

Given $\{s_{ij}, j \in \mathcal{J}\}$ and the corresponding $L_j$'s and $U_j$'s, we optimize the monetary allocation $\{m_{ij}, j \in \mathcal{J}\}$. Equivalently, we change decision variables and optimize $w_j = F(f_i + m_{ij})$.

The incentive compatibility constraints (14) for the given $i$ can be written as

$$L_j \ \leq \ w_j - w_{j+1} \ \leq \ U_j, \quad \forall j \in \mathcal{J}\backslash\{J\}. \tag{39}$$

The structural property we aim to prove can be restated as: There exists $\widehat{j} \in \mathcal{J} \cup \{0\}$, such that

(i) $\forall j > \widehat{j}$, $w_j^* - w_{j+1}^* = L_j$;

(ii) $\forall j < \widehat{j}$, $w_j^* - w_{j+1}^* = U_j$.

Note that part (iii) of the property in Proposition 2 is implied by (i) and (ii).

If a feasible monetary allocation does not have the above structure, it must have the following property: There exist $k$ and $l$ with $1 \leq k < l < J$, such that

$$w_k - w_{k+1} < U_k \qquad \text{and} \qquad w_l - w_{l+1} > L_l. \tag{40}$$

In what follows, we show that such a solution can be strictly improved.

We construct another feasible solution as follows:

$$\widetilde{w}_j = w_j, \qquad \text{for } j = 1, \ldots, k, \tag{41}$$

$$\widetilde{w}_j = w_j + \delta_n, \quad \text{for } j = k+1, \ldots, l, \tag{42}$$

$$\widetilde{w}_j = w_j + \delta_p, \quad \text{for } j = l+1, \ldots, J, \tag{43}$$

where $\delta_n < 0$ and $\delta_p > 0$ are chosen so that the solution remains feasible, as detailed below. First, the individual rationality constraint (13) is satisfied, because $w_1$ remains unchanged by (41). Second, to verify that $\widetilde{w}_j$'s satisfy the incentive compatibility constraints (14) or (39), note that $\widetilde{w}_j - \widetilde{w}_{j+1} = w_j - w_{j+1}$ for $j \neq k, l$, so we only need to check (39) for $j = k, l$. Because of (40), we can choose $\delta_n < 0$ and $\delta_p > 0$ small enough so that (39) remains satisfied:

$$\widetilde{w}_k - \widetilde{w}_{k+1} = w_k - w_{k+1} - \delta_n \qquad < U_k,$$

$$\widetilde{w}_l - \widetilde{w}_{l+1} = w_l - w_{l+1} + \delta_n - \delta_p > L_l.$$

Finally, satisfying the budget constraint (5) by changing $\{m_{ij}, j \in \mathcal{J}\}$ for the given $i$ requires keeping $\sum_{j\in\mathcal{J}} m_{ij}\theta_{ij}$ constant. Since $w_j = F(f_i + m_{ij})$, we equivalently keep $\sum_{j\in\mathcal{J}} \big(H(w_j) - f_i\big)\theta_{ij}$ constant, where $H(w) = F^{-1}(w)$, or equivalently, we must ensure

$$\sum_{j\in\mathcal{J}} H(\widetilde{w}_j)\theta_{ij} = \sum_{j\in\mathcal{J}} H(w_j)\theta_{ij}. \tag{44}$$

Assumption 1 ensures sufficient differentiability for applying Taylor's expansion: $H(\widetilde{w}_j) = H(w_j) + \delta_n H'(w_j) + O(\delta_n^2), \forall j = k+1, \ldots, l$, and $H(\widetilde{w}_j) = H(w_j) + \delta_p H'(w_j) + O(\delta_p^2), \forall j = l+1, \ldots, J$. Substituting the expanded expressions into (44), canceling terms on both sides, we have

$$\delta_n \sum_{j=k+1}^{l} H'(w_j)\theta_{ij} + \delta_p \sum_{j=l+1}^{J} H'(w_j)\theta_{ij} = O(\delta_p^2).$$

Therefore, we can choose a small value of $\delta_p > 0$ and set

$$\delta_n = -\delta_p \frac{\sum_{j=l+1}^{J} H'(w_j)\theta_{ij}}{\sum_{j=k+1}^{l} H'(w_j)\theta_{ij}} + O(\delta_p^2),$$

so that the values $\{\widetilde{w}_j : j \in \mathcal{J}\}$ in (41)-(43) correspond to a feasible monetary allocation.

It remains to be shown that $\{\widetilde{w}_j : j \in \mathcal{J}\}$ strictly improves the objective (12) under $\{w_j : j \in \mathcal{J}\}$. Note that this perturbation affects the objective only through $\sum_{j\in\mathcal{J}} w_j\theta_{ij}$ for the given $i$. The perturbed value of this term is:

$$\sum_{j=1}^{J} \widetilde{w}_j\theta_{ij} = \sum_{j=1}^{k} w_j\theta_{ij} + \sum_{j=k+1}^{l} (w_j + \delta_n)\theta_{ij} + \sum_{j=l+1}^{J} (w_j + \delta_p)\theta_{ij} = \sum_{j=1}^{J} w_j\theta_{ij} + \delta_n \sum_{j=k+1}^{l} \theta_{ij} + \delta_p \sum_{j=l+1}^{J} \theta_{ij}.$$

Thus, the change in the objective value is

$$\delta_n \sum_{j=k+1}^{l} \theta_{ij} + \delta_p \sum_{j=l+1}^{J} \theta_{ij} = -\delta_p \frac{\sum_{j=l+1}^{J} H'(w_j)\theta_{ij}}{\sum_{j=k+1}^{l} H'(w_j)\theta_{ij}} \sum_{j=k+1}^{l} \theta_{ij} + \delta_p \sum_{j=l+1}^{J} \theta_{ij} + O(\delta_p^2). \tag{45}$$

Because $H(w) = F^{-1}(w)$ is strictly convex and increasing in $w$, and $w_j$ is (weakly) decreasing in $j$ ($w_j - w_{j+1} \geq L_j \geq 0$ in (39)), we have $H'(w_j) \geq H'(w_{j+1})$ for all $j < J$. Furthermore, the strict inequality holds at $j = l$, i.e., $H'(w_l) > H'(w_{l+1})$, because $w_l - w_{l+1} > L_l$ is the assumed property of the feasible allocation and $L_l \geq 0$. Therefore,

$$\frac{\sum_{j=l+1}^{J} H'(w_j)\theta_{ij}}{\sum_{j=k+1}^{l} H'(w_j)\theta_{ij}} \leq \frac{H'(w_{l+1}) \sum_{j=l+1}^{J} \theta_{ij}}{H'(w_l) \sum_{j=k+1}^{l} \theta_{ij}} < \frac{\sum_{j=l+1}^{J} \theta_{ij}}{\sum_{j=k+1}^{l} \theta_{ij}}. \tag{46}$$

Inequality (46) implies that the value in (45) is positive when $\delta_p$ is small enough. Therefore, $\{w_j : j \in \mathcal{J}\}$ satisfying $w_k - w_{k+1} < U_k$ and $w_l - w_{l+1} > L_l$ with $k < l$ cannot be optimal. This completes the proof of the properties (i)-(iii) stated in the proposition.

The final statement in the proposition is that if the individual rationality constraint is not binding for a given $i \in \mathcal{I}$, then property (i) holds for the given $i$ and $\forall j \in \mathcal{J}$. We prove this statement by contradiction. Suppose the optimal solution is such that $F(f_i+m^*_{ij})-F(f_i+m^*_{i,j+1}) > G(s^*_{i,j+1}, g_j) - G(s^*_{ij}, g_j)$ for some $j \in \mathcal{J}\backslash\{J\}$. Then, we can construct a solution such that

$$\widetilde{m}_{ij'} = m^*_{ij'} - \epsilon_1, \text{ for } j' \le j, \qquad \text{and} \qquad \widetilde{m}_{ij'} = m^*_{ij'} + \epsilon_2, \text{ for } j' > j,$$

where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are small enough so that this new solution satisfies the individual rationality constraint in (13) and the incentive compatibility constraints in (14). In addition, $\epsilon_1$ and $\epsilon_2$ are such that the budget constraint in (5) is satisfied: $\sum_{j' \in \mathcal{J}} \widetilde{m}_{ij'}\theta_{ij'} = \sum_{j' \in \mathcal{J}} m^*_{ij'}\theta_{ij'}$. Then, by the Karamata's inequality and the strict concavity of the financial utility function, we can conclude that

$$\sum_{j' \in \mathcal{J}} F(f_i + \widetilde{m}_{ij'})\theta_{ij} > \sum_{j' \in \mathcal{J}} F(f_i + m^*_{ij'})\theta_{ij}.$$

Note that the original Karamata's inequality applies to cases with equal weights, but its generalization to unequal weights is intuitive. This can be seen by refining the unequal weights into many small, equal weights. A formal proof can follow the structure of the original proof of Karamata's inequality. □

**Proof of Lemma 3**: Consider any two different pairs $(S^a, M^a)$ and $(S^b, M^b)$ for which the problem (15)-(20) is feasible. Because the problem is a convex optimization on a compact convex set, the optimal solution exists, which is denoted as $\{v^a_{ij}, w^a_{ij}\}$ and $\{v^b_{ij}, w^b_{ij}\}$, respectively.

Next, we show that the problem (15)-(20) with $(S, M) = \big(\frac{S^a+S^b}{2}, \frac{M^a+M^b}{2}\big)$ is feasible and a feasible solution is $\{v^m_{ij}, w^m_{ij}\} = \big\{\frac{v^a_{ij}+v^b_{ij}}{2}, \frac{w^a_{ij}+w^b_{ij}}{2}\big\}$. It is immediate to see that $\{v^m_{ij}, w^m_{ij}\}$ satisfy the linear constraints (18)-(20). It remains to verify the nonlinear constraints (16) and (17):

$$N \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \bar{G}^{-1}\big(\tfrac{1}{2}(v^a_{ij} + v^b_{ij})/g_j\big)\theta_{ij} \le N \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \tfrac{1}{2}\big(\bar{G}^{-1}(v^a_{ij}/g_j) + \bar{G}^{-1}(v^b_{ij}/g_j)\big)\theta_{ij} = S,$$

$$N \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \big(F^{-1}\big(\tfrac{1}{2}(w^a_{ij} + w^b_{ij}) - f_i\big)\theta_{ij} \le N \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \tfrac{1}{2}\big(F^{-1}(w^a_{ij}) - f_i + F^{-1}(w^b_{ij}) - f_i\big)\theta_{ij} = M,$$

where the inequalities follow from the convexity of $\bar{G}^{-1}(\cdot)$ and $F^{-1}(\cdot)$, and the equalities are due to the optimality of $\{v^a_{ij}, w^a_{ij}\}$ and $\{v^b_{ij}, w^b_{ij}\}$. Therefore, we have

$$W_{\mathcal{M}}\big(\tfrac{S^a+S^b}{2}, \tfrac{M^a+M^b}{2}\big) \ge N \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \tfrac{1}{2}(v^a_{ij} + v^b_{ij} + w^a_{ij} + w^b_{ij})\,\theta_{ij} = \tfrac{1}{2}\big(W_{\mathcal{M}}(S^a, M^a) + W_{\mathcal{M}}(S^b, M^b)\big),$$

where the inequality follows from the feasibility of $\{v^m_{ij}, w^m_{ij}\}$ and the equality is due to the optimality of $\{v_{ij}^a, w_{ij}^a\}$ and $\{v_{ij}^b, w_{ij}^b\}$. □

**Proof of Lemma 4**: In the lower-level problem, we allocate $S_i$ and $M_i$ within income group $i$ that has a population of $N\theta_i$, which is a special case of (15)-(20) with one income group. Therefore, the optimal aggregate welfare of income group $i$, $W_{\mathcal{M}_i}(S_i, M_i)$, is concave in $(S_i, M_i)$ according to Lemma 3. Hence, the objective function in (21) is concave in $(S_i, M_i : i \in \mathcal{I})$. □

**Proof of Proposition 3**: (i) The resource allocation problem is:

$$\max_{\{s_i \geq 0\}} \sum_{i \in \mathcal{I}, j \in \mathcal{J}} G(s_i, g_j)\, \theta_{ij}, \quad \text{subject to: } N \sum_{i \in \mathcal{I}} s_i \theta_i = S.$$

Consider a feasible solution in which allocation are not equal, i.e., $s_i < s_k$ for some $i, k \in \mathcal{I}$. Define $\bar{s} = \frac{s_i \theta_i + s_k \theta_n}{\theta_i + \theta_k}$. Replacing $s_i$ and $s_k$ by $\bar{s}$ clearly respects the resource constraint $\sum_{i \in \mathcal{I}} s_i \theta_i = S$. Because $\bar{s}$ is a convex combination of $s_i$ and $s_k$, the strict concavity of $G(\cdot, g_j)$ implies

$$G(\bar{s}, g_j) > G(s_i, g_j) \frac{\theta_i}{\theta_i + \theta_k} + G(s_k, g_j) \frac{\theta_k}{\theta_i + \theta_k}, \qquad \forall j \in \mathcal{J}. \tag{47}$$

By the proposition's assumption, we have that $\theta_{ij}/\theta_{kj} = \theta_i/\theta_k$. Thus, (47) implies

$$G(\bar{s}, g_j)\,(\theta_{ij} + \theta_{kj}) > G(s_i, g_j)\,\theta_{ij} + G(s_k, g_j)\,\theta_{kj}, \qquad \forall j \in \mathcal{J}.$$

Summing over $j$, we conclude that the solution with $s_i < s_k$ can be strictly improved because

$$\sum_{j \in \mathcal{J}} G(\bar{s}, g_j)\,\theta_{ij} + \sum_{j \in \mathcal{J}} G(\bar{s}, g_j)\,\theta_{kj} > \sum_{j \in \mathcal{J}} G(s_i, g_j)\,\theta_{ij} + \sum_{j \in \mathcal{J}} G(s_k, g_j)\,\theta_{kj}.$$

Therefore, the optimal allocation are equal.

(ii) The cost allocation problem is:

$$\max_{\{m_i \geq 0\}} \sum_{i \in \mathcal{I}} F(f_i + m_i)\,\theta_i, \quad \text{subject to: } N \sum_{i \in \mathcal{I}} m_i \theta_i = M.$$

The proof is parallel to part (i) and abbreviated. If a feasible solution has $m_i < m_k$ for $i < k$, then increasing $m_i$ and decreasing $m_k$ while maintaining the budget constraint will strictly improve the objective value due to the strict concavity of the financial utility function $F(\cdot)$. Thus, at optimality, $m_i$ must decrease in $i$. □