Electronic Companion for “Preparing for the Next Health Emergency: The Effect of Facility Specialization in Concurrent Management of Pandemic and Non-Pandemic Demand”

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EC.1. Complete Formulation and Proof of Proposition 1

The optimal solution to problem (13) is given by the following proposition, which is an expanded version of Proposition 1 in the main text of the paper.

PROPOSITION EC.1. *Given $r_p, r_n > 0$ and $r_p + r_n < 2h$, the optimal solution to problem* (13) *depends on the parameters' values as follows:*

- *When $r_p < h \le r_n$, or $r_p < h$, $r_n < h$, and*

$$\frac{\tau_p}{\tau_n} < \frac{r_n}{r_p} \cdot \frac{h - r_p}{(h - r_n)^2} \cdot \frac{\lambda_n (2h - r_n)(h - r_p) - (h - r_n)^2 \lambda_p}{\lambda_n (h - r_p) + \lambda_p r_n}, \tag{EC.1}$$

$m = 1(n)$ will be the optimal configuration. The optimal demand allocation, in this case, is given by $y_p^* = 0$ and

$$y_n^* = \frac{h\left(\kappa^{m=1(n)} - (h - r_p - r_n)\right)}{r_n\left(\kappa^{m=1(n)} + h\right)}, \tag{EC.2}$$

and the corresponding objective value is

$$\begin{aligned} \tilde{Z} &= \tilde{Z}^{m=1(n)} = g(0, y_n^*) = g(1, 1 - y_n^*) \\ &= \frac{\left(\beta_p h + \kappa^{m=1(n)}\left(\beta_p + \beta_m + \beta_n\right)\right) r_n^2 + \beta_m h\left(h - r_p - \kappa^{m=1(n)}\right) r_n + \beta_n h\left(h - r_p - \kappa^{m=1(n)}\right)^2}{2h(2h - r_p - r_n)\kappa^{m=1(n)} {r_n}^2}, \end{aligned} \tag{EC.3}$$

where

$$\kappa^{m=1(n)} = \sqrt{s_1^{m=1(n)} s_2^{m=1(n)}},$$

with

$$s_1^{m=1(n)} = h - r_p + r_n \frac{\lambda_p}{\lambda_n}, \quad \text{and} \quad s_2^{m=1(n)} = h - r_p + r_p \frac{\tau_p}{\tau_n}. \tag{EC.4}$$

- *When $r_n < h \le r_p$, or $r_p < h$, $r_n < h$, and*

$$\frac{\tau_n}{\tau_p} < \frac{r_p}{r_n} \cdot \frac{h - r_n}{(h - r_p)^2} \cdot \frac{\lambda_p (2h - r_p)(h - r_n) - (h - r_p)^2 \lambda_n}{\lambda_p (h - r_n) + \lambda_n r_p}, \tag{EC.5}$$

$m = 1(p)$ will be the optimal configuration. The optimal demand allocation, in this case, is given by $y_n^* = 0$ and

$$y_p^* = \frac{h\left(\kappa^{m=1(p)} - (h - r_p - r_n)\right)}{r_p\left(\kappa^{m=1(p)} + h\right)}, \tag{EC.6}$$

and the optimal objective value is

$$\begin{aligned} \tilde{Z} &= \tilde{Z}^{m=1(p)} = g(y_p^*, 0) = g(1 - y_p^*, 1) \\ &= \frac{\left(\beta_n h + \kappa^{m=1(p)}\left(\beta_p + \beta_m + \beta_n\right)\right) r_p^2 + h\beta_m\left(h - r_n - \kappa^{m=1(p)}\right) r_p + \beta_p h\left(h - r_n - \kappa^{m=1(p)}\right)^2}{2h(2h - r_p - r_n)\kappa^{m=1(p)} {r_p}^2} \end{aligned} \tag{EC.7}$$

where

$$\kappa^{m=1(p)} = \sqrt{s_1^{m=1(p)} s_2^{m=1(p)}},$$

with

$$s_1^{m=1(p)} = h - r_n + r_p \frac{\lambda_n}{\lambda_p}, \quad \textit{and} \quad s_2^{m=1(p)} = h - r_n + r_n \frac{\tau_n}{\tau_p}. \tag{EC.8}$$

- *When $r_p < h$, $r_n < h$, and neither of inequalities* (EC.1) *and* (EC.5) *hold, $m = 0$ will be the optimal configuration; thus $y_p^* = 0$ and $y_n^* = 1$. The corresponding optimal objective value is*

$$\tilde{Z} = \tilde{Z}^{m=0} = g(0,1) = g(1,0) = \frac{\beta_p}{2h\,(h - r_p)} + \frac{\beta_n}{2h\,(h - r_n)}. \tag{EC.9}$$

Proof First note that, since $\mathcal{N}$ is open and convex, the objective function and constraints are differentiable in $\mathcal{N}$, and the constraints are linear, the necessity of KKT conditions apply, i.e., if $\mathbf{y}^* = (y_p^*, y_n^*)$ solves problem (13), there exists a vector $\boldsymbol{\pi}^*$ so that $(\mathbf{y}^*, \boldsymbol{\pi}^*)$ meets the KKT conditions; see Theorem 5.4 in Carter (2001, p. 577). To examine KKT conditions, we define the Lagrangian as follows:

$$L(y_p, y_n, \pi_1, \pi_2) = -g(y_p, y_n) - \pi_1(y_p - 1) - \pi_2(y_n - 1). \tag{EC.10}$$

The KKT conditions are then obtained as:

$$\frac{\partial L(y_p, y_n, \pi_1, \pi_2)}{\partial y_p} \le 0, \qquad y_p \ge 0, \qquad y_p \frac{\partial L(y_p, y_n, \pi_1, \pi_2)}{\partial y_p} = 0, \tag{EC.11}$$

$$\frac{\partial L(y_p, y_n, \pi_1, \pi_2)}{\partial y_n} \le 0, \qquad y_n \ge 0, \qquad y_n \frac{\partial L(y_p, y_n, \pi_1, \pi_2)}{\partial y_n} = 0, \tag{EC.12}$$

$$y_p \le 1, \qquad \pi_1 \ge 0, \qquad \pi_1(y_p - 1) = 0, \tag{EC.13}$$

$$y_n \le 1, \qquad \pi_2 \ge 0, \qquad \pi_2(y_n - 1) = 0. \tag{EC.14}$$

We find all the solutions satisfying KKT conditions and compare their corresponding objective values to find the optimal solution. To simplify this, we consider five sets of potential solutions, each corresponding to one of the configurations given in Figure 1.

EC.1.1. Scenario 1: $y_p = 1, y_n = 0$

This scenario is equivalent to configuration $m = 0$. From the inequalities in $\mathcal{N}$, it is clear that this scenario is feasible only if $r_p < h$ and $r_n < h$. Given these two inequalities, we find values of π_1 and π_2 meeting KKT conditions. Since $y_p = 1$, the third condition in (EC.11) yields

$$\pi_1 = -\frac{\partial g(y_p, y_n)}{\partial y_p}\bigg|_{y_p=1, y_n=0} = -\frac{(2h - r_p)\,\beta_p}{2h\,(h - r_p)^2} + \frac{\beta_m h - r_n \beta_m + \beta_n r_p}{2h\,(h - r_n)^2}. \tag{EC.15}$$

From (9), we derive the expression

$$\beta_m = \frac{\lambda_p}{\lambda_n}\beta_n + \frac{\lambda_n}{\lambda_p}\beta_p, \tag{EC.16}$$

for β_m, inserting which in (EC.15) yields

$$\pi_1 = \frac{\beta_p \lambda_n}{2\,(h - r_n)\,h\lambda_p} - \frac{(2h - r_p)\,\beta_p}{2h\,(h - r_p)^2} + \frac{\beta_n r_p}{2h\,(h - r_n)^2} + \frac{\lambda_p \beta_n}{2\,(h - r_n)\,h\lambda_n},$$

which must be non-negative by the second condition in (EC.13). This gives the inequality

$$\frac{(2h - r_p)\,\beta_p}{(h - r_p)^2} \le \frac{\lambda_p^2 \beta_n\,(h - r_n) + \beta_n r_p \lambda_p \lambda_n + \beta_p \lambda_n^2\,(h - r_n)}{(h - r_n)^2\,\lambda_p \lambda_n}.$$

Replacing β_p and β_n by their equivalent expressions given in (9) and rearranging the terms, we arrive at

$$\frac{\tau_n}{\tau_p} \geq \frac{r_p}{r_n} \cdot \frac{h-r_n}{(h-r_p)^2} \cdot \frac{\lambda_p (h-r_n)(2h-r_p) - (h-r_p)^2 \lambda_n}{\lambda_p \cdot (h-r_n) + \lambda_n r_p}. \tag{EC.17}$$

On the other hand, from the last condition in (EC.14), we obtain $\pi_2 = 0$. Inserting this in the first condition in (EC.12) and replacing β_m by (EC.16) in the resulting expression, we get

$$\frac{(2h-r_n)\beta_n}{(h-r_n)^2} \leq \frac{\lambda_p^2 \beta_n (h-r_p) + r_n \beta_p \lambda_p \lambda_n + \beta_p \lambda_n^2 (h-r_p)}{(h-r_p)^2 \lambda_p \lambda_n}.$$

Replacing β_p and β_n by their corresponding expressions given in (9) and rearranging the terms, we arrive at

$$\frac{\tau_p}{\tau_n} \geq \frac{r_n}{r_p} \cdot \frac{h-r_p}{(h-r_n)^2} \cdot \frac{\lambda_n (h-r_p)(2h-r_n) - (h-r_n)^2 \lambda_p}{\lambda_n \cdot (h-r_p) + \lambda_p r_n}. \tag{EC.18}$$

In summary, when $r_p < h$, $r_n < h$, and the inequalities in (EC.17) and (EC.18) hold, the solution vector $(\mathbf{y}^{m=0}, \boldsymbol{\pi}^{m=0}) = ((1,0),(\pi_1,0))$ with π_1 given in (EC.15) meets the KKT conditions as well as the inequalities in $\mathcal{N}$, and thus can potentially be optimal. The resulting objective value would be

$$\tilde{Z}^{m=0} = g(1,0) = \frac{\beta_p}{2h(h-r_p)} + \frac{\beta_n}{2h(h-r_n)}. \tag{EC.19}$$

Note that, since two facilities have equal capacities, the scenario with $y_p = 0$ and $y_n = 1$ gives exactly the same results as Scenario 1.

EC.1.2. Scenario 2: $0 < y_p < 1$ and $0 < y_n < 1$

This corresponds to configuration $m = 2$. From the last conditions in (EC.13) and (EC.14), we must have $\pi_1 = \pi_2 = 0$. Combining these with the last conditions in (EC.11) and (EC.12), we obtain

$$\begin{aligned}
\frac{\partial g(y_p,y_n)}{\partial y_p} = &\frac{2\beta_p y_p + \beta_m y_n}{2h(h - y_p r_p - y_n r_n)} + r_p \frac{\beta_p y_p^2 + \beta_m y_p y_n + \beta_n y_n^2}{2h(h - y_p r_p - y_n r_n)^2} \\
&- \frac{2\beta_p(1-y_p) + \beta_m(1-y_n)}{2h(h-(1-y_p)r_p - (1-y_n)r_n)} - r_p \frac{\beta_p(1-y_p)^2 + \beta_m(1-y_p)(1-y_n) + \beta_n(1-y_n)^2}{2h(h-(1-y_p)r_p-(1-y_n)r_n)^2} = 0, \\
\frac{\partial g(y_p,y_n)}{\partial y_n} = &\frac{2\beta_n y_n + \beta_m y_p}{2h(h-y_p r_p - y_n r_n)} + r_n \frac{\beta_p {y_p}^2 + \beta_m y_p y_n + \beta_n {y_n}^2}{2h(h-y_p r_p - y_n r_n)^2} \\
&- \frac{2\beta_n(1-y_n) + \beta_m(1-y_p)}{2h(h - r_p(1-y_p) - r_n(1-y_n))} - r_n \frac{\beta_p(1-y_p)^2 + \beta_m(1-y_p)(1-y_n) + \beta_n(1-y_n)^2}{2h(h-(1-y_p)r_p-(1-y_n)r_n)^2} = 0.
\end{aligned}$$

The unique real solution to the system of equations above is $y_p = y_n = \frac{1}{2}$, which meets the inequalities in $\mathcal{N}$ as we have $r_p + r_n < 2h$ by assumption. Hence, the solution vector $(\mathbf{y}^{m=2}, \boldsymbol{\pi}^{m=2}) = ((1/2,1/2),(0,0))$ can always be potentially optimal. The corresponding objective function value is

$$\tilde{Z}^{m=2} = g\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\beta_p + \beta_m + \beta_n}{2h(2h - r_p - r_n)}. \tag{EC.20}$$

EC.1.3. Scenario 3: $y_p = 0$ and $0 < y_n < 1$

This corresponds to configuration $m = 1(n)$. From the inequalities in $\mathcal{N}$, we obtain $r_p + r_n(1-y_n) < h$, which implies that we must have $r_p < h$. This is an intuitive requirement for this configuration as the entire pandemic workload will be allocated to one hospital. Given $r_p < h$, we find values of y_n, π_1, and π_2 meeting

KKT conditions. From the last conditions in (EC.13) and (EC.14), we obtain $\pi_1 = \pi_2 = 0$. Combining these with the last condition in (EC.12), we arrive at

$$\left.\frac{\partial g(y_p, y_n)}{\partial y_n}\right|_{y_p=0} = \frac{\beta_n y_n (2h - y_n r_n)}{2h(h - y_n r_n)^2} - \frac{(2h - 2r_p - (1-y_n) r_n)(1-y_n)\beta_n + \beta_p r_n + \beta_m (h - r_p)}{2h(h - r_p - (1-y_n) r_n)^2} = 0, \tag{EC.21}$$

solving which gives two solutions for y_n. The first solution is

$$y_n = \frac{h(\Delta^{m=1(n)} - \beta_n(h - r_p - r_n))}{r_n(\Delta^{m=1(n)} + h\beta_n)}, \tag{EC.22}$$

where

$$\Delta^{m=1(n)} = \sqrt{\beta_n\left(r_n(h-r_p)\beta_m + (h-r_p)^2\beta_n + \beta_p r_n^2\right)}. \tag{EC.23}$$

Note that, since $r_p < h$ by assumption, $\Delta^{m=1(n)}$ is a real constant. To simplify Equation (EC.22) further, we replace β_m in Equation (EC.23) with its equivalent expression provided in (EC.16) and simplify to obtain

$$\begin{aligned}\Delta^{m=1(n)} &= \sqrt{\left(\beta_n(h-r_p) + \beta_n r_n \frac{\lambda_p}{\lambda_n}\right)\left(\beta_n(h-r_p) + \beta_p r_n \frac{\lambda_n}{\lambda_p}\right)} \\ &= \beta_n\sqrt{\left(h - r_p + r_n\frac{\lambda_p}{\lambda_n}\right)\left(h - r_p + r_n\frac{\beta_p\lambda_n}{\beta_n\lambda_p}\right)} = \beta_n\sqrt{\left(h - r_p + r_n\frac{\lambda_p}{\lambda_n}\right)\left(h - r_p + r_p\frac{\tau_p}{\tau_n}\right)},\end{aligned}$$

where the third equality is obtained by replacing β_p and β_n inside the square root on the right-hand side (rhs) of the second equality by $\lambda_p r_p \tau_p$ and $\lambda_n r_n \tau_n$, respectively. Defining

$$s_1^{m=1(n)} = h - r_p + r_n\frac{\lambda_p}{\lambda_n}, \quad \text{and} \quad s_2^{m=1(n)} = h - r_p + r_p\frac{\tau_p}{\tau_n}, \tag{EC.24}$$

we can now write

$$\Delta^{m=1(n)} = \beta_n \kappa^{m=1(n)}, \tag{EC.25}$$

where

$$\kappa^{m=1(n)} = \sqrt{s_1^{m=1(n)} s_2^{m=1(n)}}. \tag{EC.26}$$

The solution y_n given in (EC.22) can then be simplified as

$$y_n^{m=1(n)} = \frac{h\left(\kappa^{m=1(n)} - (h - r_p - r_n)\right)}{r_n\left(\kappa^{m=1(n)} + h\right)}, \tag{EC.27}$$

which is the same as Equation (EC.2).

We show that $y_n^{m=1(n)}$ given in (EC.27) is positive, meets the first condition in Equation (EC.11), is less than one when $h \le r_n$, or $h > r_n$ and inequality (EC.1) holds, and meets the inequalities in $\mathcal{N}$. To show that y_n is positive, we note that

$$\kappa^{m=1(n)} = \sqrt{\left(h - r_p + r_n\frac{\lambda_p}{\lambda_n}\right)\left(h - r_p + r_p\frac{\tau_p}{\tau_n}\right)} > \sqrt{(h-r_p)^2} = h - r_p > h - r_p - r_n, \tag{EC.28}$$

which implies that $y_n^{m=1(n)}$ is positive. To show that the first condition of (EC.11) holds, we evaluate

$$\begin{aligned}\left.\frac{\partial L(y_p, y_n, \pi_1, \pi_2)}{\partial y_p}\right|_{y_p=0, y_n=y_n^{m=1(n)}, \pi_1=0, \pi_2=0} &= -\left.\frac{\partial g(y_p, y_n)}{\partial y_p}\right|_{y_p=0, y_n=y_n^{m=1(n)}} \\ = \frac{h + \kappa^{m=1(n)}}{2hr_n(2h - r_p - r_n)\kappa^{m=1(n)}}&\left[\left(h - r_p - \kappa^{m=1(n)}\right)\beta_m + 2\beta_p r_n\right] + \frac{h + \kappa^{m=1(n)} r_p}{2r_n^2 h(\kappa^{m=1(n)})^2(2h - r_p - r_n)^2} \\ &\times\left[r_n\left(h(h-r_p) - \kappa^{m=1(n)}(h - r_n)\right)\beta_m + r_n^2\left(h + \kappa^{m=1(n)}\right)\beta_p\right. \\ &\left.+ (h - r_p - \kappa_n)\left(\left(\kappa^{m=1(n)}\right)^2 - (2h - r_p - 2r_n)\kappa^{m=1(n)} + h(h - r_p)\right)\beta_n\right].\end{aligned} \tag{EC.29}$$

The fraction in the first term as well as the one in the second term of the equation above are clearly positive. We show that the coefficients of both fractions are non-positive. We first evaluate the coefficient of the first fraction in (EC.29) as

$$\begin{aligned}
&\left(h-r_p-\kappa^{m=1(n)}\right)\beta_m+2\beta_p r_n \\
&\quad= r_p\tau_p\lambda_n\left(h-r_p+r_n\frac{\lambda_p}{\lambda_n}\right)-r_p\tau_p\lambda_n\kappa^{m=1(n)}+r_n\tau_n\lambda_p\left(h-r_p+\frac{r_p\tau_p}{\tau_n}\right)-r_n\tau_n\lambda_p\kappa^{m=1(n)} \\
&\quad= r_p\tau_p\lambda_n s_1^{m=1(n)}-r_p\tau_p\lambda_n\sqrt{s_1^{m=1(n)}s_2^{m=1(n)}}+r_n\tau_n\lambda_p s_2^{m=1(n)}-r_n\tau_n\lambda_p\sqrt{s_1^{m=1(n)}s_2^{m=1(n)}} \\
&\quad= \left(\sqrt{s_2^{m=1(n)}}-\sqrt{s_1^{m=1(n)}}\right)\left(r_n\tau_n\lambda_p\sqrt{s_2^{m=1(n)}}-r_p\tau_p\lambda_n\sqrt{s_1^{m=1(n)}}\right),
\end{aligned} \tag{EC.30}$$

where the first equality is obtained by replacing β_p, β_n, and β_m with their equivalent expressions given in (9), and the second equality is obtained using the parameters defined in (EC.24) and (EC.26). The rhs of Equation (EC.30) is non-positive because $\sqrt{s_2^{m=1(n)}} > \sqrt{s_1^{m=1(n)}}$ implies that $\tau_p/\tau_n > \mu_p/\mu_n$, whereas $r_n\tau_n\lambda_p\sqrt{s_2^{m=1(n)}} > r_p\tau_p\lambda_n\sqrt{s_1^{m=1(n)}}$ implies that $\tau_p/\tau_n < \mu_p/\mu_n$, hence the product $\left(\sqrt{s_2^{m=1(n)}}-\sqrt{s_1^{m=1(n)}}\right)\left(\lambda_p\tau_n r_n\sqrt{s_2^{m=1(n)}}-\lambda_n\tau_p r_p\sqrt{s_1^{m=1(n)}}\right)$ cannot be positive. We next evaluate the coefficient of the second fraction in (EC.29) as

$$\begin{aligned}
&r_n\left(h(h-r_p)-\kappa^{m=1(n)}(h-r_n)\right)\beta_m+r_n^2\left(h+\kappa^{m=1(n)}\right)\beta_p \\
&\quad+\left(h-r_p-\kappa^{m=1(n)}\right)\left(\left(\kappa^{m=1(n)}\right)^2-(2h-r_p-2r_n)\kappa^{m=1(n)}+h\left(h-r_p\right)\right)\beta_n = r_n(2h-r_p-r_n) \\
&\quad\times\left(2\lambda_n\tau_n\left(h-r_p+r_n\frac{\lambda_p}{\lambda_n}\right)\left(h-r_p+r_p\frac{\tau_p}{\tau_n}\right)-\lambda_n\tau_n\left(h-r_p+r_n\frac{\lambda_p}{\lambda_n}+h-r_p+r_p\frac{\tau_p}{\tau_n}\right)\kappa^{m=1(n)}\right) \\
&\quad= r_n(2h-r_p-r_n)\left(2\lambda_n\tau_n s_1^{m=1(n)}s_2^{m=1(n)}-\lambda_n\tau_n\left(s_1^{m=1(n)}+s_2^{m=1(n)}\right)\sqrt{s_1^{m=1(n)}s_2^{m=1(n)}}\right) \\
&\quad= r_n(2h-r_p-r_n)\lambda_n\tau_n\sqrt{s_1^{m=1(n)}s_2^{m=1(n)}}\left(2\sqrt{s_1^{m=1(n)}s_2^{m=1(n)}}-s_1^{m=1(n)}-s_2^{m=1(n)}\right),
\end{aligned} \tag{EC.31}$$

where the first equality is obtained by replacing β_p, β_n, and β_m with their equivalent expressions given in (9), and the second equality is obtained using the parameters defined in (EC.24) and (EC.26). The expression given in (EC.31) is non-positive because $\left(2\sqrt{s_1^{m=1(n)}s_2^{m=1(n)}}-s_1^{m=1(n)}-s_2^{m=1(n)}\right) > 0$ implies that $\left(s_1^{m=1(n)}-s_2^{m=1(n)}\right)^2 < 0$, which cannot be true. As such, the derivative given in Equation (EC.29) is non-positive.

To have $y_n^{m=1(n)} < 1$, we must have $h(\kappa^{m=1(n)}-(h-r_p-r_n)) < r_n(h+\kappa^{m=1(n)})$. Simplifying this, we arrive at

$$\kappa^{m=1(n)}(h-r_n) < h(h-r_p), \tag{EC.32}$$

which holds if $h \le r_n$. For $h > r_n$, dividing both sides of (EC.32) by $(h-r_n)$, we get $\kappa^{m=1(n)} < h(h-r_p)/(h-r_n)$. Replacing $\kappa^{m=1(n)}$ with $\sqrt{s_1^{m=1(n)}s_2^{m=1(n)}}$, where $s_1^{m=1(n)}$ and $s_2^{m=1(n)}$ are defined in (EC.24), we obtain

$$\sqrt{\left(h-r_p+r_n\frac{\lambda_p}{\lambda_n}\right)\left(h-r_p+r_p\frac{\tau_p}{\tau_n}\right)} < h\frac{h-r_p}{h-r_n}, \tag{EC.33}$$

which gives

$$\left(h-r_p+r_n\frac{\lambda_p}{\lambda_n}\right)\left(h-r_p+r_p\frac{\tau_p}{\tau_n}\right) < h^2\left(\frac{h-r_p}{h-r_n}\right)^2. \tag{EC.34}$$

Rearranging the above, we obtain

$$\frac{\tau_p}{\tau_n} < \frac{r_n}{r_p} \cdot \frac{h-r_p}{(h-r_n)^2} \cdot \frac{\lambda_n(2h-r_n)(h-r_p)-(h-r_n)^2\lambda_p}{\lambda_n(h-r_p)+\lambda_p r_n}, \tag{EC.35}$$

which is the same as inequality (EC.1) and must be met for $y_n^{m=1(n)}$ to be less than one when $h > r_n$. Finally, since

$$y_n^{m=1(n)} r_n - h = -\frac{h\,(2h-r_p-r_n)}{h+\kappa^{m=1(n)}} < 0, \tag{EC.36}$$

and

$$y_n^{m=1(n)} r_n - (r_p + r_n - h) = \frac{\kappa^{m=1(n)}\,(2h-r_p-r_n)}{h+\kappa^{m=1(n)}} > 0, \tag{EC.37}$$

the inequalities $r_p + r_n - h < y_n r_n + y_p r_p < h$ in $\mathcal{N}$ are met for $y_p = 0$ and $y_n = y_n^{m=1(n)}$.

The second solution of Equation (EC.21) is

$$y_n' = \frac{h\left(\kappa^{m=1(n)} + h - r_p - r_n\right)}{r_n\left(\kappa^{m=1(n)} - h\right)}. \tag{EC.38}$$

To check if the inequalities in $\mathcal{N}$ hold for this solution, we evaluate

$$y_n' r_n - h = \frac{h\,(2h-r_p-r_n)}{\kappa^{m=1(n)} - h}, \tag{EC.39}$$

and

$$y_n' r_n - (r_p + r_n - h) = \frac{(2h-r_p-r_n)\,\kappa^{m=1(n)}}{\kappa^{m=1(n)} - h}. \tag{EC.40}$$

For the rhs of Equation (EC.39) to be negative, we must have $\kappa^{m=1(n)} - h < 0$, which implies that the rhs of Equation (EC.40) will also be negative. Hence, the inequalities in $\mathcal{N}$ will not be met for y_n'.

In summary, $(\mathbf{y}^{m=1(n)}, \boldsymbol{\pi}^{m=1(n)}) = ((0, y_n^{m=1(n)}), (0,0))$ with $y_n^{m=1(n)}$ given in Equation (EC.27) meets all KKT conditions as well as the inequalities in $\mathcal{N}$ as long as $r_p < h \le r_n$ or $r_p < h$, $r_n < h$, and the inequality in (EC.35) holds. The corresponding objective value is

$$\begin{aligned}\tilde{Z}^{m=1(n)} &= g\left(0, y_n^{m=1(n)}\right) = g\left(1, 1 - y_n^{m=1(n)}\right)\\ &= \frac{\left(\beta_p h + \kappa^{m=1(n)}\left(\beta_p + \beta_m + \beta_n\right)\right) r_n^2 + \beta_m h\left(h - r_p - \kappa^{m=1(n)}\right) r_n + \beta_n h\left(h - r_p - \kappa^{m=1(n)}\right)^2}{2h(2h-r_p-r_n)\kappa^{m=1(n)} {r_n}^2},\end{aligned} \tag{EC.41}$$

where the second equality is obtained by swapping facilities one and two.

It is interesting to know the optimal solution when $r_p < h$ and $r_n < h$, but the inequality in (EC.35) is not met. Note that setting y_p to 0 and y_n to any value in the interval $[\max\{(r_p + r_n - h)/r_n, 0\}, 1]$ yields a feasible solution for configuration $m = 1(n)$ when $r_p < h$ and $r_n < h$. However, as we show in Lemma EC.1, the derivative of $g(y_p, y_n)$ with respect to y_n will be negative for $y_p = 0$ and all values of $y_n \in [0,1]$ when inequality (EC.35) is not met. This implies that the optimal solution occurs at $y_n = 1$, which corresponds to configuration $m = 0$.

EC.1.4. Scenario 4: $0 < y_p < 1$ and $y_n = 0$

This corresponds to configuration $m = 1(p)$. Following the same argument as in Section EC.1.3, we can show that $(\mathbf{y}^{m=1(p)}, \boldsymbol{\pi}^{m=1(p)}) = ((y_p^{m=1(p)}, 0), (0,0))$, with $y_p^{m=1(p)}$ defined as

$$y_p^{m=1(p)} = \frac{h\left(\kappa^{m=1(p)} - (h - r_p - r_n)\right)}{r_p\left(\kappa^{m=1(p)} + h\right)}, \tag{EC.42}$$

where

$$\kappa^{m=1(p)} = \sqrt{s_1^{m=1(p)} s_2^{m=1(p)}}, \tag{EC.43}$$

with

$$s_1^{m=1(p)} = h - r_n + r_p \frac{\lambda_n}{\lambda_p}, \quad \text{and} \quad s_2^{m=1(p)} = h - r_n + r_n \frac{\tau_n}{\tau_p}, \tag{EC.44}$$

meets all KKT conditions as well as the conditions of $\mathcal{N}$ as long as $r_n < h \leq r_p$ or $r_p < h$, $r_n < h$, and the inequality

$$\frac{\tau_n}{\tau_p} < \frac{r_p}{r_n} \cdot \frac{h - r_n}{(h - r_p)^2} \cdot \frac{\lambda_p(2h - r_p)(h - r_n) - (h - r_p)^2 \lambda_n}{\lambda_p(h - r_n) + \lambda_n r_p}, \tag{EC.45}$$

holds. The value of the objective function is given by

$$\begin{aligned}
\tilde{Z}^{m=1(p)} &= g\left(y_p^{m=1(p)}, 0\right) = g\left(1 - y_p^{m=1(p)}, 1\right) \\
&= \frac{\left(\beta_n h + \kappa^{m=1(p)}\left(\beta_p + \beta_m + \beta_n\right)\right) r_p^2 + h\beta_m\left(h - r_n - \kappa^{m=1(p)}\right) r_p + h\beta_p\left(h - r_n - \kappa^{m=1(p)}\right)^2}{2h(2h - r_p - r_n)\kappa^{m=1(p)} {r_p}^2}.
\end{aligned} \tag{EC.46}$$

Following an argument similar to Scenario 3, we can show that when $r_p < h$ and $r_n < h$, but inequality (EC.45) is not met, the optimal solution will occur at $y_p = 1$, which corresponds to configuration $m = 0$.

EC.1.5. Scenario 5: $y_p = 1$ and $y_n = 1$

This corresponds to configuration $m = 1(u)$. From the inequalities in $\mathcal{N}$, this scenario is feasible only if $r_p + r_n < h$. Since $y_p = 1$ and $y_n = 1$, from the third condition in (EC.11) and (EC.12), we obtain

$$\pi_1 = -\frac{\partial g(y_p, y_n)}{\partial y_p}\bigg|_{y_p=1, y_n=1} = -\frac{\beta_n\left(h - r_n\right)\lambda_p^2 + \lambda_n\left(\left(2h - r_p - 2r_n\right)\beta_p + \beta_n r_p\right)\lambda_p + \beta_p \lambda_n^2\left(h - r_n\right)}{2\lambda_n\lambda_p h\left(h - r_p - r_n\right)^2}, \tag{EC.47}$$

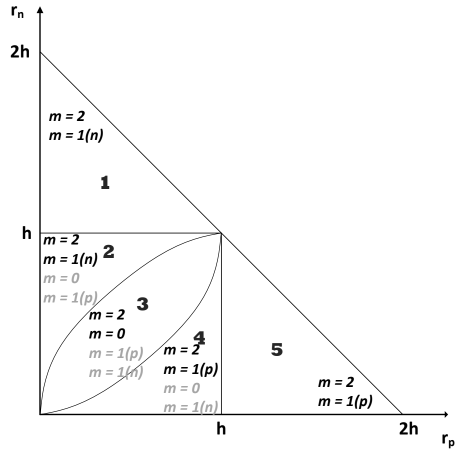
and

$$\pi_2 = -\frac{\partial g(y_p, y_n)}{\partial y_n}\bigg|_{y_p=1, y_n=1} = -\frac{\beta_p\left(h - r_p\right)\lambda_n^2 + \lambda_p\left(\left(2h - r_n - 2r_p\right)\beta_n + \beta_p r_n\right)\lambda_n + \beta_n \lambda_p^2\left(h - r_p\right)}{2\lambda_n\lambda_p h\left(h - r_p - r_n\right)^2}, \tag{EC.48}$$

respectively. By the second condition in (EC.13) and (EC.14), we must have $\pi_1, \pi_2 \geq 0$. However, since $r_p + r_n < h$, Equations (EC.47) and (EC.48) imply that π_1 and π_2 are always non-positive. Further, π_1 and π_2 are equal to zero only when $\lambda_p = \lambda_n = 0$, and so configuration $m = 1(u)$ is never optimal.

The five scenarios discussed above cover all solution vectors meeting KKT conditions. By the necessity of KKT conditions, one (or more) of these solutions should be optimal depending on the values of parameters. Based on the conditions we derived for potential optimality of each of the five scenarios, we divide the set of (r_p, r_n) values into five regions as follows: $r_p < h \leq r_n$ in region (1); $r_p < h$, $r_n < h$, and inequality (EC.1) holds in region (2); $r_p < h$, $r_n < h$, and neither inequality (EC.1) nor (EC.5) holds in region (3); $r_p < h$, $r_n < h$,

Figure EC.1 Different regions and the corresponding feasible policies.



and inequality (EC.5) holds in region (4); and $r_n < h \leq r_p$ in region (5). A visual representation of these regions is provided in Figure EC.1.

The borders between regions (1) and (2) and regions (4) and (5) are trivial. The borders between regions (2) and (3) and regions (3) and (4) in Figure EC.1 are illustrative and obtained by knowing that: i) by Lemma EC.2, for a given value of r_p (r_n), inequality (EC.1) (inequality (EC.5)) holds for values of r_n (r_p) bigger than a threshold, which is smaller than h and increases with r_p (r_n); and ii) by Lemma EC.3, regions (2) and (4) do not overlap. Note that inequalities (EC.1) and (EC.5) are the opposite of inequalities (EC.18) and (EC.17), respectively. This implies that both (EC.17) and (EC.18) are met in region (3), whereas one of them is not met in regions (2) and (4).

Inside each region in Figure EC.1, a list of feasible configurations is given. For regions (2) and (4), configuration $m = 0$ is feasible, i.e., we can dedicate one facility to pandemic demand and the other to non-pandemic demand. However, the corresponding solution vector does not meet one of the inequalities given in (EC.17) and (EC.18), and thus cannot be optimal. Similarly, configuration $m = 1(p)$ (configuration $m = 1(n)$) is feasible in regions (2) and (3) (regions (4) and (3)), but no value of $y_p < 1$ ($y_n < 1$) can be found in these regions satisfying KKT conditions as explained in Section EC.1.4 (Section EC.1.3). The feasible but non-optimal configurations are identified in grey in Figure EC.1. Configuration $m = 1(u)$ is omitted from Figure EC.1 as it is never optimal.

To identify the optimal configuration for each region of Figure EC.1, we compare the objective values of the corresponding feasible and potentially optimal configurations. This can be accomplished by only two comparisons; one for regions (1) and (2), and the other for region (3). The comparisons for regions (4) and (5) are similar to those for regions (1) and (2) as we will explain.

Starting with regions (1) and (2), we evaluate the difference between $\tilde{Z}^{m=2}$ and $\tilde{Z}^{m=1(n)}$, given in Equations (EC.20) and (EC.41), respectively, to obtain

$$\tilde{Z}^{m=2} - \tilde{Z}^{m=1(n)} = \frac{-\left(h - r_p - \kappa^{m=1(n)}\right)^2 \beta_n - \left(\beta_p r_n + \beta_m \left(h - r_p - \kappa^{m=1(n)}\right)\right) r_n}{2\kappa^{m=1(n)} \left(2h - r_p - r_n\right) r_n^2}.$$

Inserting the expression for β_m given in (EC.16) and simplifying, we arrive at

$$\tilde{Z}^{m=2} - \tilde{Z}^{m=1(n)} = -\frac{\left(\lambda_p r_n + \lambda_n \left(h - r_p - \kappa^{m=1(n)}\right)\right) \left(\beta_n \left(h - r_p - \kappa^{m=1(n)}\right) \lambda_p + r_n \beta_p \lambda_n\right)}{2\kappa^{m=1(n)} \left(2h - r_p - r_n\right) r_n^2 \lambda_n \lambda_p}$$

$$
\begin{aligned}
&= -\frac{\left(h - r_p + r_n \frac{\lambda_p}{\lambda_n} - \kappa^{m=1(n)}\right)\beta_n\left(h - r_p + r_n \frac{\beta_p \lambda_n}{\lambda_p \beta_n} - \kappa^{m=1(n)}\right)}{2\kappa^{m=1(n)}\left(2h - r_p - r_n\right) r_n^2} \\
&= -\frac{\left(s_1^{m=1(n)} - \sqrt{s_1^{m=1(n)} s_2^{m=1(n)}}\right)\beta_n\left(s_2^{m=1(n)} - \sqrt{s_1^{m=1(n)} s_2^{m=1(n)}}\right)}{2\left(2h - r_p - r_n\right)\sqrt{s_1^{m=1(n)} s_2^{m=1(n)}}\, r_n^2} \\
&= \beta_n \frac{\left(\sqrt{s_2^{m=1(n)}} - \sqrt{s_1^{m=1(n)}}\right)^2}{2\left(2h - r_p - r_n\right) r_n^2},
\end{aligned} \tag{EC.49}
$$

where the second equality is obtained by dividing the numerator and denominator in the rhs of the first equality by $\lambda_p \lambda_n$, and the third equality is obtained by using the parameters defined in (EC.24) and (EC.26). Following the same argument for the difference between $\tilde{Z}_{m=2}$ and $\tilde{Z}_{m=1(p)}$ in regions (4) and (5), we arrive at

$$
\tilde{Z}^{m=2} - \tilde{Z}^{m=1(p)} = \beta_p \frac{\left(\sqrt{s_2^{m=1(p)}} - \sqrt{s_1^{m=1(p)}}\right)^2}{2\left(2h - r_p - r_n\right) r_p^2}, \tag{EC.50}
$$

where $s_1^{m=1(p)}$ and $s_2^{m=1(p)}$ are defined in (EC.44). Both $\tilde{Z}^{m=2} - \tilde{Z}^{m=1(n)}$ and $\tilde{Z}^{m=2} - \tilde{Z}^{m=1(p)}$, given in Equations (EC.49) and (EC.50), respectively, are always non-negative, indicating that configuration $m = 1(n)$ is optimal in regions (1) and (2), and configuration $m = 1(p)$ is optimal in regions (4) and (5). They also imply that configurations $m = 1(n)$ and $m = 2$ ($m = 1(p)$ and $m = 2$) will perform the same in regions (1) and (2) (regions (4) and (5)) when $s_1^{m=1(n)} = s_2^{m=1(n)}$ ($s_1^{m=1(p)} = s_2^{m=1(p)}$), or equivalently,

$$
\frac{1 + \theta_p}{1 + \theta_n} = \left(\frac{\mu_p}{\mu_n}\right)^2. \tag{EC.51}
$$

For region (3), we evaluate the difference between $Z^{m=0}$ and $Z^{m=2}$, given in Equations (EC.19) and (EC.20), respectively. This yields

$$
\tilde{Z}^{m=0} - \tilde{Z}^{m=2} = \frac{\beta_p}{2h\left(h - r_p\right)} + \frac{\beta_n}{2h\left(h - r_n\right)} - \frac{\beta_p + \beta_m + \beta_n}{2h\left(h - r_p - r_n\right)}.
$$

Replacing β_m with its equivalent expression given in (EC.16) and simplifying, we arrive at

$$
\tilde{Z}^{m=0} - \tilde{Z}^{m=2} = \frac{\left(-\beta_n\left(h - r_p\right)\lambda_p + \beta_p \lambda_n\left(h - r_n\right)\right)\left(\left(h - r_n\right)\lambda_p - \lambda_n\left(h - r_p\right)\right)}{2h\left(h - r_p\right)\left(h - r_n\right)\lambda_n \lambda_p\left(2h - r_p - r_n\right)}. \tag{EC.52}
$$

The denominator of (EC.52) is always positive because $r_p < h$ and $r_n < h$ in region (3). Hence, the rhs of Equation (EC.52) will be positive when both terms of the numerator are either positive or negative. This implies that $\tilde{Z}^{m=2}$ would be smaller than $\tilde{Z}^{m=0}$ when either

$$
\frac{h - r_p}{h - r_n} < \frac{\lambda_p}{\lambda_n} < \frac{h - r_n}{h - r_p}\frac{\beta_p}{\beta_n}, \tag{EC.53}
$$

or

$$
\frac{h - r_n}{h - r_p}\frac{\beta_p}{\beta_n} < \frac{\lambda_p}{\lambda_n} < \frac{h - r_p}{h - r_n}, \tag{EC.54}
$$

occurs. Now, by Lemma EC.4, neither (EC.53) nor (EC.54) holds in region (3). Thus, configuration $m = 0$ is optimal in that region. Equation (EC.52) also implies that configurations $m = 0$ and $m = 2$ will perform the same and thus, both will be optimal in region (3) when

$$
\frac{\lambda_p}{\lambda_n} = \frac{h - r_p}{h - r_n}, \tag{EC.55}
$$

or

$$\frac{\tau_n r_n}{\tau_p r_p} = \frac{h - r_n}{h - r_p}. \tag{EC.56}$$

We finally note the symmetry between our results for regions (1) and (2) and those for regions (4) and (5); one can be obtained by swapping "p" and "n" in all notations in the other. □

LEMMA EC.1. *When $r_p < h$, $r_n < h$, and inequality* (EC.1) *is not met, the derivative of $g(y_p, y_n)$ with respect to y_n will be negative for $y_p = 0$ and any value of $y_n \in [0,1]$. Similarly, when $r_p < h$, $r_n < h$, and inequality* (EC.5) *is not met, the derivative of $g(y_p, y_n)$ with respect to y_p will be negative for any value of $y_p \in [0,1]$ and $y_n = 0$.*

Proof Setting $y_n = 1$ in Equation (EC.21), we obtain

$$\begin{aligned}\left.\frac{\partial g(y_p,y_n)}{\partial y_n}\right|_{y_p=0,y_n=1} &= \frac{-r_n (h-r_n)^2 \beta_p + (h-r_p)^2 (2h-r_n)\beta_n - \beta_m (h-r_n)^2 (h-r_p)}{2h (h-r_n)^2 (h-r_p)^2} \\ &= \frac{-r_n (h-r_n)^2 \lambda_p \tau_p r_p + (h-r_p)^2 (2h-r_n)\lambda_n \tau_n r_n - (\lambda_n \tau_p r_p + \lambda_p \tau_n r_n)(h-r_n)^2 (h-r_p)}{2h (h-r_n)^2 (h-r_p)^2},\end{aligned} \tag{EC.57}$$

where the second equality is obtained by replacing β_p, β_n, and β_m with their equivalent expressions given in (9). The expression above will be non-positive when its numerator is non-positive, which will occur when

$$\frac{\tau_p}{\tau_n} \geq \frac{r_n}{r_p} . \frac{h-r_p}{(h-r_n)^2} . \frac{\lambda_n (2h-r_n)(h-r_p) - (h-r_n)^2 \lambda_p}{\lambda_n (h-r_p) + \lambda_p r_n},$$

which is the same as the inequality given in (EC.1) but with an opposite direction. Now, since

$$\left.\frac{\partial^2 g(y_p,y_n)}{\partial y_n^2}\right|_{y_p=0} = \frac{\beta_n h}{(h-y_n r_n)^3} + \frac{\beta_p r_n^2 + \beta_m (h-r_p) r_n + (h-r_p)^2 \beta_n}{(h-(1-y_n) r_n - r_p)^3 h}, \tag{EC.58}$$

is always positive, the derivative given in Equation (EC.57) will be non-positive for any value of $y_n \in [0,1]$ as long as inequality (EC.1) is not met. A similar argument applies to the derivative of $g(y_p, y_n)$ with respect to y_p evaluated at $y_p \in [0,1]$ and $y_n = 0$.

□

LEMMA EC.2. *For any given value of $r_p < h$ ($r_n < h$), inequality* (EC.1) *(inequality* (EC.5)*) holds for all values of r_n (r_p) bigger than a threshold. This threshold is less than h and increases with r_p (r_n).*

Proof Denoting the rhs of inequality (EC.1) by $t(r_p, r_n)$, we have

$$\begin{aligned} t(r_p, r_n) &= \frac{r_n (h-r_p)\left(\lambda_n (h-r_p)(2h-r_n) - (h-r_n)^2 \lambda_p\right)}{r_p (h-r_n)^2 (\lambda_n (h-r_p) + \lambda_p r_n)} \\ &= \frac{r_n (h-r_p)\left(r_n \mu_n (h-r_p)(2h-r_n) - (h-r_n)^2 r_p \mu_p\right)}{r_p (h-r_n)^2 (r_n \mu_n (h-r_p) + r_p \mu_p r_n)},\end{aligned}$$

where the second equality is obtained by replacing λ_p and λ_n in the rhs of the first equality by $r_p \mu_p$ and $r_n \mu_n$, respectively. Taking derivative from $t(r_p, r_n)$ with respect to r_n, we arrive at

$$\frac{\partial t(r_p, r_n)}{\partial r_n} = \frac{2\mu_n h^2 (h-r_p)^2}{r_p (\mu_n (h-r_p) + \mu_p r_p)(h-r_n)^3}, \tag{EC.59}$$

which is always positive. This implies that, for a given value of $r_p < h$, the rhs of inequality (EC.1) increases as r_n increases, and so the inequality will hold for all r_n bigger than a threshold. This threshold is always less than h because $\lim_{r_n \to h} t(r_p, r_n) = +\infty$. Also, taking derivative from expression (EC.59) with respect to r_p, we obtain

$$\frac{\partial^2 t(r_p, r_n)}{\partial r_n \partial r_p} = -\frac{2\mu_n h^3 (h - r_p)(\mu_n(h - r_p) + 2\mu_p r_p)}{r_p^2 ((\mu_p - \mu_n) r_p + \mu_n h)^2 (h - r_n)^3}, \tag{EC.60}$$

which is always negative. This implies that, for larger values of r_p, the rhs of inequality (EC.1) increases in smaller steps as r_n increases, and so the threshold for r_n above which the inequality holds increases as well. The proof for inequality (EC.5) follows a similar argument. □

LEMMA EC.3. *When $r_n < h$ and $r_p < h$, inequalities* (EC.1) *and* (EC.5) *cannot hold at the same time.*

Proof Suppose both inequalities are met. Then we should have

$$\frac{r_p}{r_n} \cdot \frac{(h - r_n)^2}{(h - r_p)} \cdot \frac{\lambda_n(h - r_p) + \lambda_p r_n}{\lambda_n(2h - r_n)(h - r_p) - (h - r_n)^2 \lambda_p} < \frac{\tau_n}{\tau_p} < \frac{r_p}{r_n} \cdot \frac{(h - r_n)}{(h - r_p)^2} \cdot \frac{\lambda_p(h - r_n)(2h - r_p) - (h - r_p)^2 \lambda_n}{\lambda_p(h - r_n) + \lambda_n r_p}, \tag{EC.61}$$

which implies that

$$\frac{(\lambda_n(h - r_p) + \lambda_p r_n)(h - r_n)}{\lambda_n(2h - r_n)(h - r_p) - (h - r_n)^2 \lambda_p} < \frac{\lambda_p(h - r_n)(2h - r_p) - (h - r_p)^2 \lambda_n}{(\lambda_p(h - r_n) + \lambda_n r_p)(h - r_p)}, \tag{EC.62}$$

or equivalently,

$$\frac{h(2h - r_p - r_n)(\lambda_p(h - r_n) - \lambda_n h + r_p \lambda_n)^2}{(\lambda_p(h - r_n) + \lambda_n r_p)(h - r_p)((h - r_n)^2 \lambda_p - \lambda_n(2h - r_n)(h - r_p))} > 0. \tag{EC.63}$$

Since $r_n < h$ and $r_p < h$, the inequality above holds as long as $(h - r_n)^2 \lambda_p - \lambda_n(2h - r_n)(h - r_p)$ is positive. However, if this happens, the numerator of the third fraction in the rhs of inequality (EC.1) will be negative implying that inequality (EC.1) will not hold as τ_p and τ_n are always positive. □

LEMMA EC.4. *Inequalities* (EC.53) *and* (EC.54) *do not hold in region (3).*

Proof We show that for region (3), i.e., the set of parameter values for which neither inequality (EC.1) nor (EC.5) holds, $\frac{\lambda_p}{\lambda_n} > \frac{h - r_n}{h - r_p} \frac{\beta_p}{\beta_n}$ when $\frac{\lambda_p}{\lambda_n} > \frac{h - r_p}{h - r_n}$, and $\frac{\lambda_p}{\lambda_n} < \frac{h - r_n}{h - r_p} \frac{\beta_p}{\beta_n}$ when $\frac{\lambda_p}{\lambda_n} < \frac{h - r_p}{h - r_n}$.

We first assume that $\frac{\lambda_p}{\lambda_n} > \frac{h - r_p}{h - r_n}$. Since $\frac{h - r_p}{2h - r_p} < 1$, we then have

$$\frac{\lambda_p}{\lambda_n} > \frac{h - r_p}{h - r_n} > \frac{(h - r_p)^2}{(h - r_n)(2h - r_p)}, \tag{EC.64}$$

which implies that $\lambda_p(2h - r_p)(h - r_n) - \lambda_n(h - r_p)^2 > 0$, and so the rhs of inequality (EC.5) is a positive value. Now, since inequality (EC.5) does not hold in region (3), we must have

$$\begin{aligned}
\frac{\tau_n}{\tau_p} &\geq \frac{r_p}{r_n} \cdot \frac{h - r_n}{(h - r_p)^2} \cdot \frac{\lambda_p(2h - r_p)(h - r_n) - (h - r_p)^2 \lambda_n}{\lambda_p(h - r_n) + \lambda_n r_p} \\
&> \frac{r_p}{r_n} \cdot \frac{h - r_n}{(h - r_p)^2} \cdot \frac{(2h - r_p)(h - r_n) - (h - r_p)(h - r_n)}{(h - r_n) + \frac{h - r_n}{h - r_p} r_p} = \frac{r_p}{r_n} \frac{h - r_n}{h - r_p},
\end{aligned}$$

where the second inequality is obtained by dividing the numerator and denominator of the third term in the rhs of first inequality by λ_p, and replacing $\frac{\lambda_n}{\lambda_p}$ by $\frac{h - r_n}{h - r_p}$ (Note that $\frac{\lambda_n}{\lambda_p} < \frac{h - r_n}{h - r_p}$ by assumption.) Hence, we have

$$\frac{\tau_p}{\tau_n} < \frac{r_n}{r_p} \cdot \frac{h - r_p}{h - r_n}, \tag{EC.65}$$

multiplying both sides of which by λ_p/λ_n and rearranging the terms, we obtain

$$\frac{\lambda_p}{\lambda_n} > \frac{h-r_n}{h-r_p} \cdot \frac{\beta_p}{\beta_n}, \tag{EC.66}$$

and so inequality (EC.53) is not met. Following a similar argument, we can show that

$$\frac{\lambda_p}{\lambda_n} < \frac{h-r_n}{h-r_p} \cdot \frac{\beta_p}{\beta_n}, \tag{EC.67}$$

when $\frac{\lambda_p}{\lambda_n} < \frac{h-r_p}{h-r_n}$. □

EC.2. Proof of Corollary 1

The objective function in (20) is not differentiable due to its piecewise-defined IM cost component. Consequently, standard KKT conditions, applicable to smooth optimization problems, cannot directly be employed here as they were in the queue-length minimization model. To address this, we take a different approach: we evaluate each feasible configuration separately, determine its optimal allocation, and then compare the corresponding objective function values to identify the configuration that minimizes the overall cost.

Before proceeding, recall from Proposition 1 that configuration $m = 2$ is never optimal in the absence of IM costs. Since this configuration always incurs higher or equivalent IM costs compared to all other configurations, it remains non-optimal in the cost-augmented case as well. We therefore exclude $m = 2$ from the analysis that follows.

Configuration $m = 0$: This configuration is feasible when $r_p < h$ and $r_n < h$. Under these conditions, the optimal allocation is $y_p = 1, y_n = 0$. Hence, $Z^{m=0} = c_w g(1.0, 0.0) + hc_p$

Configuration $m = 1(n)$: This configuration is feasible when $r_p < h$. The corresponding optimization model is $Z^{m=1(n)} = \min_{y_n \in \mathcal{N}'} \{c_w g(0, y_n) + hc_m : 0 < y_n < 1\}$, where $\mathcal{N}' = \{y_n \in \mathbb{R};\, r_p + r_n - h < y_n r_n < h\}$. Applying the KKT conditions, we obtain the unique solution $y_n^{m=1(n)}$, given in Equation (EC.27). Following the same argument as in Section EC.1.3, we observe that this optimal allocation is strictly positive and meets the inequalities in $\mathcal{N}'$. It is also less than 1 when either $h \le r_n$, or $h > r_n$ and inequality (EC.35) holds. Hence, $Z^{m=1(n)} = c_w g(0, y_n^{m=1(n)}) + hc_m$.

Configuration $m = 1(p)$: This configuration is feasible when $r_n < h$. Utilizing the same logic as in configuration $m = 1(n)$, the optimal solution is obtained as $y_p^{m=1(p)}$ given in Equation (EC.42). Thus, $Z^{m=1(p)} = c_w g(y_p^{m=1(p)}, 0) + h(c_m + c_p)$.

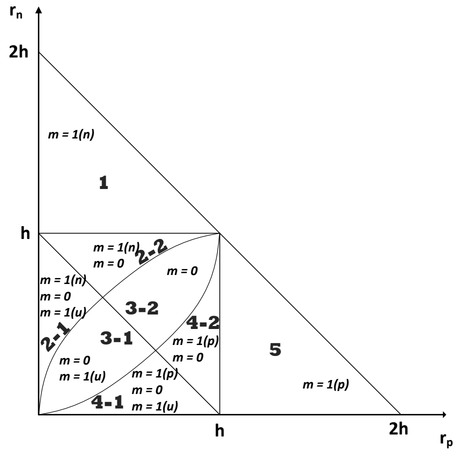
Configuration $m = 1(u)$: This configuration is feasible when $r_p + r_n < h$. The optimal allocation is $y_p = y_n = 1$, yielding $Z^{m=1(u)} = c_w g(1.0, 1.0) + hc_m$.

Integrating feasibility and optimality results, the parameter space (r_p, r_n) naturally partitions into eight distinct regions, as depicted in Figure EC.2. Note that, compared to Figure EC.1, we have divided each of the regions (2)-(4) into two sub-regions, depending on whether $r_p + r_n$ lies below or above h. With the exception of regions (1), (3-2), and (5), every region contains at least two potentially optimal configurations. To determine the true optimum in these cases, we proceed by directly comparing their objective values as outlined below.

First, note that

$$Z^{m=1(u)} - Z^{m=0} = c_w \left(\frac{r_p \left(\mu_p \left(h - r_n\right) + r_n \mu_n\right) \beta_n}{2 r_n \mu_n h \left(h - r_n - r_p\right)\left(h - r_n\right)} + \frac{\beta_p r_n \left(\mu_n \left(h - r_p\right) + r_p \mu_p\right)}{2 r_p \mu_p h \left(h - r_n - r_p\right)\left(h - r_p\right)} \right) + h(c_m - c_p),$$

Figure EC.2 Different parameter regions and the corresponding potentially optimal configurations for the model with IM costs.



which is always positive in the region where $m = 1(u)$ is feasible. This implies that $m = 1(u)$ is dominated by $m = 0$. Eliminating $m = 1(u)$ from Figure EC.2 then reveals that, for regions (1), (3), and (5), the optimal configuration and the corresponding allocation remain exactly as characterized in Proposition 1. In contrast, within region (2) either configuration $m = 1(n)$ or $m = 0$ could be optimal, and within region (4) either configuration $m = 1(p)$ and $m = 0$ may be optimal, depending on parameter values. Starting with region (2), we evaluate

$$Z^{m=1(n)} - Z^{m=0} = \frac{(A\tau_p + B\tau_n + C) + 2\sqrt{s_1^{m=1(n)} s_2^{m=1(n)}}\lambda_n c_w h \tau_n (h - r_n)(h - r_p)}{2r_n (2h - r_n - r_p) h (h - r_p)(h - r_n)}, \tag{EC.68}$$

where

$$\begin{aligned}
A &= -s_1^{m=1(n)} \lambda_n (h - r_n)^2 r_p c_w, \\
B &= -\lambda_n \left((h - r_n)^2 s_1^{m=1(n)} + (h - r_p) h^2 \right) (h - r_p) c_w, \\
C &= 2h^2 r_n (h - r_p)(h - r_n)(2h - r_n - r_p)(c_m - c_p).
\end{aligned}$$

Since the denominator on the rhs of Equation (EC.68), as well as the second term in its numerator, are always positive, a necessary condition for $Z^{m=1(n)} - Z^{m=0}$ to be negative is that $A\tau_p + B\tau_n + C < 0$. This gives

$$\tau_p > \frac{2h^2 r_n \delta^{m=1(n)} - \left(\gamma^{m=1(n)} + \lambda_n h^2 (h - r_p)\right)(h - r_p)\tau_n c_w}{\gamma^{m=1(n)} r_p c_w}, \tag{EC.69}$$

where $\delta^{m=1(n)} = (2h - r_n - r_p)(h - r_p)(h - r_n)(c_m - c_p)$ and $\gamma^{m=1(n)} = \lambda_n (h - r_n)^2 s_1^{m=1(n)}$. Now, setting Equation (EC.68) equal to 0, we obtain

$$(A\tau_p + B\tau_n + C) = -2\sqrt{s_1^{m=1(n)} s_2^{m=1(n)}}\lambda_n c_w h \tau_n (h - r_n)(h - r_p).$$

Observing that the lhs of this equation is negative under the condition specified in (EC.69), we proceed by squaring both sides, rearranging terms, and simplifying. This yields

$$A^2 \tau_p^2 + \left(2(B\tau_n + C)A - 4s_1^{m=1(n)} r_p \tau_n \lambda_n^2 c_w^2 h^2 (h - r_n)^2 (h - r_p)^2\right)\tau_p + (B\tau_n + C)^2 - 4s_1^{m=1(n)} (h - r_p)^3 \lambda_n^2 \tau_n^2 c_w^2 h^2 (h - r_n)^2 = 0,$$

which is a quadratic equation in τ_p. Solving this equation for τ_p, noting that the coefficient of τ_p^2 is positive, and substituting the expressions for A, B, and C, we arrive at the following condition

$$\tau_p < \frac{2h^2 r_n \delta^{m=1(n)} - (h - r_p)\left(2h^2\sqrt{2\delta^{m=1(n)}\lambda_n c_w r_n \tau_n} + \left(\gamma^{m=1(n)} - \lambda_n h^2 (h - r_p)\right)\tau_n c_w\right)}{\gamma^{m=1(n)} r_p c_w}, \tag{EC.70}$$

for $Z^{m=1(n)} - Z^{m=0}$ to be negative. Combining inequalities (EC.69) and (EC.70), we obtain the inequality given by (21), which along with $r_p < h$ and $r_n < h$, identifies the updated region (2) where configuration $m = 1(n)$ will be optimal. Following a similar logic for $Z^{m=1(p)} - Z^{m=0}$, we obtain the inequality given by (22). $\square$

EC.3. The Algorithm for a Two-Facility Network with Unequal Capacities

Define

$$\begin{aligned} d_1(y_p, y_n) &= \frac{\partial}{\partial y_p}\left(f\left(y_p, y_n, h^{(1)}\right) + f\left(1 - y_p, 1 - y_n, h^{(2)}\right)\right) \\ d_2(y_p, y_n) &= \frac{\partial}{\partial y_n}\left(f\left(y_p, y_n, h^{(1)}\right) + f\left(1 - y_p, 1 - y_n, h^{(2)}\right)\right) \end{aligned} \tag{EC.71}$$

Algorithm 1 outlines the steps for finding the optimal solution to problem (10). For problem (12), we replace function $f(.)$ with $l(.)$ in Equation (EC.71), and c_w with c_d in Algorithm 1.

Algorithm 1 Numerical method for finding the optimal solution to problems (10) and (12)

Require: $\lambda_n, \lambda_n, \mu_p, \mu_n, \theta_p, \theta_n, h^{(1)}, h^{(2)}, d_1(y_p, y_n), d_2(y_p, y_n), c_p, c_m, c_w$

$r_p \leftarrow \lambda_p/\mu_p,\ r_n \leftarrow \lambda_n/\mu_n$
if $r_p + r_n > h^{(1)} + h^{(2)}$ **then**
  return Unstable
end if
if $r_p < h^{(1)}$ and $r_n < h^{(2)}$ **then**
  $T^{m=0(p)} \leftarrow c_w\left(f\left(1, 0, h^{(1)}\right) + f\left(0, 1, h^{(2)}\right)\right) + h^{(1)}c_p$
end if
if $r_p < h^{(2)}$ and $r_n < h^{(1)}$ **then**
  $T^{m=0(n)} \leftarrow c_w\left(f\left(0, 1, h^{(1)}\right) + f\left(1, 0, h^{(2)}\right)\right) + h^{(2)}c_p$
end if
if $r_p + r_n < h^{(1)}$ **then**
  $T^{m=1(2u)} \leftarrow c_w\left(f\left(1, 1, h^{(1)}\right) + f\left(0, 0, h^{(2)}\right)\right) + h^{(1)}c_m$
end if
if $r_p + r_n < h^{(2)}$ **then**
  $T^{m=1(1u)} \leftarrow c_w\left(f\left(0, 0, h^{(1)}\right) + f\left(1, 1, h^{(2)}\right)\right) + h^{(2)}c_m$
end if
$\mathcal{D} \leftarrow \{(y_p, y_n) \in (0,1)^2;\ d_1(y_p, y_n) = 0;\ d_2(y_p, y_n) = 0;\ r_p + r_n - h^{(2)} < y_p r_p + y_n r_n < h^{(1)}\}$
$T^{m=2} \leftarrow \min\{c_w\left(f\left(y_p, y_n, h^{(1)}\right) + f\left(1 - y_p, 1 - y_n, h^{(2)}\right)\right) + \left(h^{(1)} + h^{(2)}\right)c_m : (y_p, y_n) \in \mathcal{D}\}$
$\mathcal{D} \leftarrow \{y_n \in (0,1); d_2(0, y_n) = 0;\ r_p + r_n - h^{(2)} < y_n r_n < h^{(1)}\}$
$T^{m=1(1n)} \leftarrow \min\left\{c_w\left(f\left(0, y_n, h^{(1)}\right) + f\left(1, 1 - y_n, h^{(2)}\right)\right) + h^{(2)}c_m : y_n \in \mathcal{D}\right\}$
$\mathcal{D} \leftarrow \{y_n \in (0,1); d_2(1, y_n) = 0;\ r_p + r_n - h^{(2)} < r_p + y_n r_n < h^{(1)}\}$
$T^{m=1(2n)} \leftarrow \min\left\{c_w\left(f\left(1, y_n, h^{(1)}\right) + f\left(0, 1 - y_n, h^{(2)}\right)\right) + h^{(1)}c_m : y_n \in \mathcal{D}\right\}$
$\mathcal{D} \leftarrow \{y_p \in (0,1); d_1(y_p, 0) = 0;\ r_p + r_n - h^{(2)} < y_p r_p < h^{(1)}\}$
$T^{m=1(1p)} \leftarrow \min\left\{c_w\left(f\left(y_p, 0, h^{(1)}\right) + f\left(1 - y_p, 1, h^{(2)}\right)\right) + h^{(1)}c_p + h^{(2)}c_m : y_p \in \mathcal{D}\right\}$
$\mathcal{D} \leftarrow \{y_p \in (0,1); d_1(y_p, 1) = 0;\ r_p + r_n - h^{(2)} < y_p r_p + r_n < h^{(1)}\}$
$T^{m=1(2p)} \leftarrow \min\left\{c_w\left(f\left(y_p, 1, h^{(1)}\right) + f\left(1 - y_p, 0, h^{(2)}\right)\right) + h^{(1)}c_m + h^{(2)}c_p : y_p \in \mathcal{D}\right\}$
return $\min\{T^{m=0(p)}, T^{m=0(n)}, T^{m=2}, T^{m=1(1n)}, T^{m=1(2n)}, T^{m=1(1p)}, T^{m=1(2p)}, T^{m=1(1u)}, T^{m=1(2u)}\}$ and the corresponding superscript (which identifies the optimal configuration)

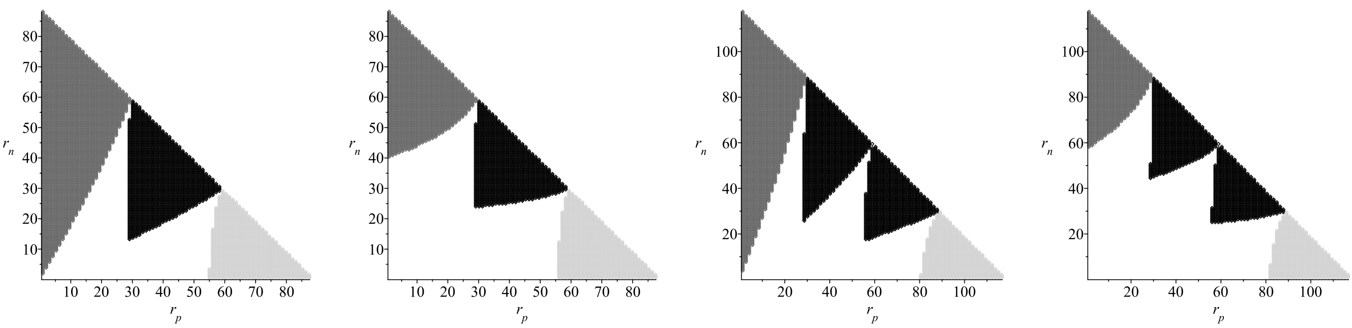
EC.4. Proof of Proposition 2

We prove this by contradiction, starting from the first part of the proposition. Suppose there exist multiple generalized facilities in the optimal solution. Consider any pair of these facilities, thereby forming a sub-network. According to Corollary 1, the optimal allocation for a two-facility network consists of at most one generalized facility. Therefore, within this sub-network, the total cost can be reduced by reallocating pandemic and non-pandemic demand streams according to the optimal allocation. This would lead to a configuration with only one or no generalized facility in the sub-network. Since the objective function for the entire network is the summation of individual facilities costs, reducing the cost of the sub-network would improve the overall objective function. Following the reconfiguration of the sub-network, we focus on the next pair of generalized facilities and continue the same reallocation process until there is only one or no generalized facility in the network. Each reallocation results in an improvement in the objective function, hence contradicting the assumption.

For the second part, suppose at least one empty facility exists in the optimal solution. Pairing this facility with a non-empty one, we can reallocate the load according to Corollary 1 to reduce the total cost in the pair sub-network. Applying the same argument to other empty facilities results in a solution with no empty facility and a better objective function, contradicting the assumption. □

EC.5. Diagrams and Data for Experiments in Section 5

Figure EC.3 The optimal configuration as a function of r_p and r_n with $h = 30$. Top (bottom) panels correspond to networks with $k = 3$ ($k = 4$). Cost parameters are $c_w = 2.0, c_m = 1.5, c_p = 1.5$ in the left panels whereas they are $c_w = 2.0, c_m = 1.75, c_p = 1.5$ in the right panels. The shading indicates the optimal configuration: white for $m = 0$, black for $m = 1(np)$, light grey for $m = 1(p)$, and dark grey for $m = 1(n)$.

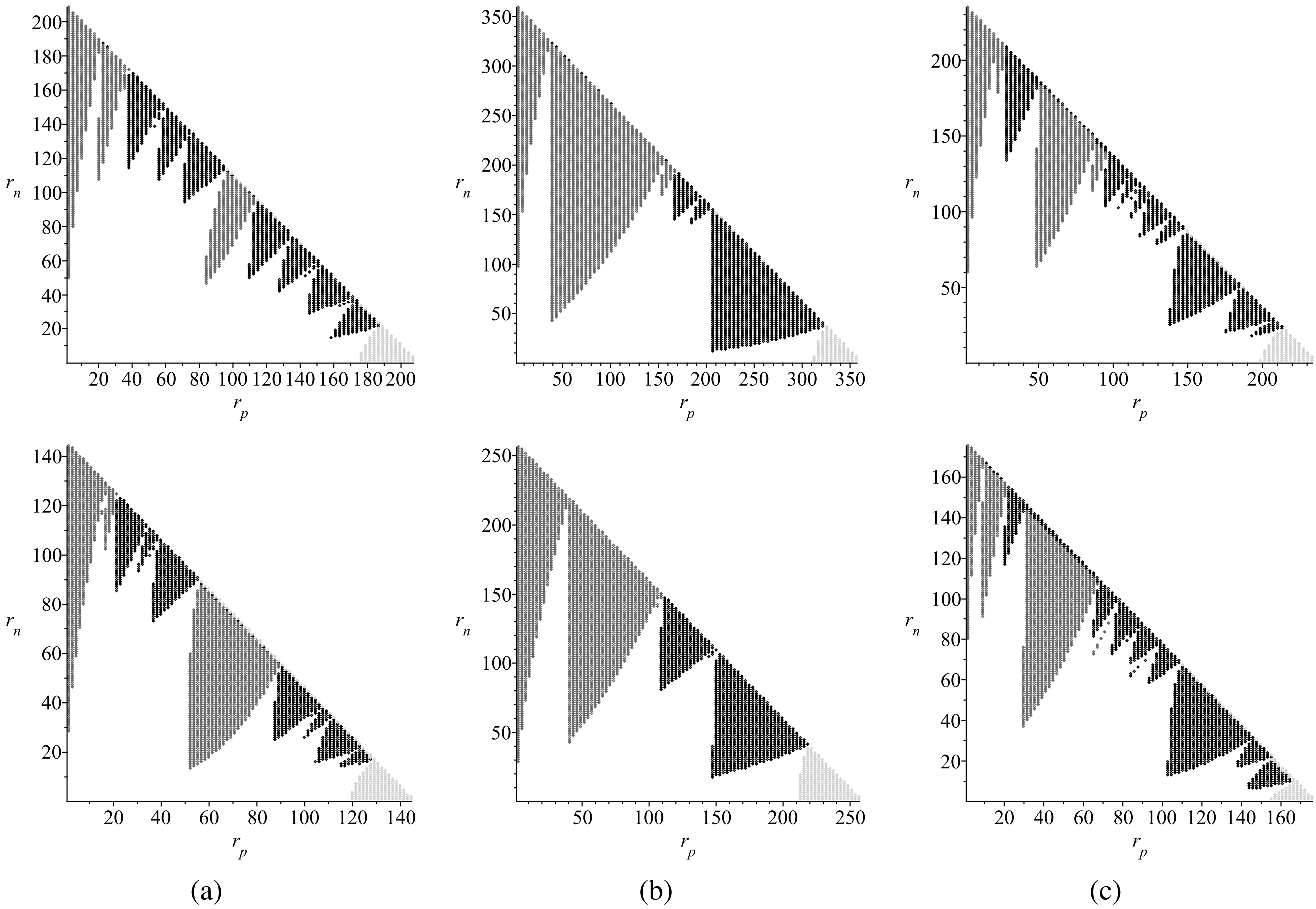


(a) (b) (c) (d)

Table EC.1 The NHS trusts offering intensive care services in London, their ICS, and bed capacity.

No.	NHS Trust	ICS	ICU Capacity Apr 2020	ICU Capacity Dec 2020
1	BARKING, HAVERING AND REDBRIDGE UNIVERSITY HOSPITALS NHS TRUST	NE	90	67
2	BARTS HEALTH NHS TRUST	NE	214	145
3	CHELSEA AND WESTMINSTER HOSPITAL NHS FOUNDATION TRUST	NW	63	31
4	CROYDON HEALTH SERVICES NHS TRUST	SW	37	18
5	EPSOM AND ST HELIER UNIVERSITY HOSPITALS NHS TRUST	SW	39	22
6	GUY'S AND ST THOMAS' NHS FOUNDATION TRUST	SE	159	112
7	HOMERTON UNIVERSITY HOSPITAL NHS FOUNDATION TRUST	NE	30	17
8	IMPERIAL COLLEGE HEALTHCARE NHS TRUST	NW	136	97
9	KING'S COLLEGE HOSPITAL NHS FOUNDATION TRUST	SE	170	110
10	KINGSTON HOSPITAL NHS FOUNDATION TRUST	SW	22	17
11	LEWISHAM AND GREENWICH NHS TRUST	SE	38	41
12	LONDON NORTH WEST UNIVERSITY HEALTHCARE NHS TRUST	NW	81	68
13	NORTH MIDDLESEX UNIVERSITY HOSPITAL NHS TRUST	NC	29	22
14	ROYAL FREE LONDON NHS FOUNDATION TRUST	NC	98	69
15	ST GEORGE'S UNIVERSITY HOSPITALS NHS FOUNDATION TRUST	SW	115	91
16	THE HILLINGDON HOSPITALS NHS FOUNDATION TRUST	NW	10	9
17	UNIVERSITY COLLEGE LONDON HOSPITALS NHS FOUNDATION TRUST	NC	90	79
18	WHITTINGTON HEALTH NHS TRUST	NC	23	10

Figure EC.4 The optimal configuration as a function of r_p and r_n for hospital networks operating under the London SW (a), SE (b), and NC (c) ICSs. The top (bottom) panels are based on first-wave (second-wave) LOS figures. The shading indicates the optimal configuration: white for $m = 0$, black for $m = 1(np)$, light grey for $m = 1(p)$, and dark grey for $m = 1(n)$



EC.6. Proof of Proposition 3

For the maximum saving, note that the optimal static configuration is $m = 1(n)$ when $r_n \to 2h$ and $r_p \to 0$. In this case, we evaluate $\frac{Z^{m=1(n)} - Z^{vp}}{Z^{m=1(n)}}$, replace β_m with its expression from Equation (EC.16), and substitute $\lambda_p = r_p\mu_p$ and $\lambda_n = r_n\mu_n$. Evaluating the resulting expression at $r_n = 2h$ and then taking the limit as $r_p \to 0$ yields $\frac{2c_w - 1}{2c_w}$. By symmetry, applying the same approach shows that $\frac{Z^{m=1(p)} - Z^{vp}}{Z^{m=1(p)}}$ converges to the same value when $r_n \to 0$ and $r_p \to 2h$.

For the minimum saving, observe that the optimal static configuration is $m = 0$ when $r_n, r_p \to 0$, provided $c_m > c_p$. In this case, evaluating $\frac{Z^{m=0} - Z^{vp}}{Z^{m=0}}$ at $r_p = r_n = 0$ gives $\frac{-2c_m + c_p}{c_p}$. When $c_m = c_p$, the optimal static configuration may be either $m = 0$ or $m = 1(n)$. However, taking the limit of $\frac{Z^{m=1(n)} - Z^{vp}}{Z^{m=1(n)}}$ as $r_p, r_n \to 0$ yields -1, which coincides with $\frac{-2c_m + c_p}{c_p}$ evaluated at $c_m = c_p$. $\square$

References

Carter M (2001) *Foundations of mathematical economics* (MIT Press).