Online Appendix – *Service Systems with On-Demand and Reserved Servers*

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Appendix A Proofs of the Results in Section 2.1

Proof of Lemma 1: We show that $V_\alpha^*(m+1) \le V_\alpha^*(m) + \frac{c_o}{\alpha+\mu}$. We consider two systems A and B, both of which use s_r reserved servers. At time $t=0$, there are $m+1$ jobs in A and m jobs in B, with no on-demand servers hired in both systems. We decompose System A as a system with m jobs and s_r reserved servers, and another system with one job and no reserved servers at the beginning of the time horizon. We use the optimal policy in System B to manage the system with m jobs and s_r reserved servers, which incurs the same expected cost as in System B. The system with one job and no reserved servers is served by an on-demand server, and the expected cost is $\frac{c_o}{\alpha+\mu}$. Then, the expected infinite-horizon discounted cost incurred in System A under the optimal policy is no more than that incurred in System B under the optimal policy plus $\frac{c_o}{\alpha+\mu}$, i.e., $V_\alpha^*(m+1) \le V_\alpha^*(m) + \frac{c_o}{\alpha+\mu}$. ■

The proof of Lemma 2 (establishing $V_\alpha^*(m+1)+V_\alpha^*(m-1) \ge V_\alpha^*(m)+V_\alpha^*(m)$) presents a formidable challenge. To address this, we employ innovative *sample-path* comparison arguments by meticulously constructing and analyzing four coupled systems (A, B, C, and D) which start with states (m_A, m_B, m_C, m_D) that are "well-arranged" (a condition ensuring $m_A + m_B = m_C + m_D$ and m_C, m_D are "sandwiched" between m_A and m_B). Our proof demonstrates that if the systems start in a well-arranged state vector, they remain well-arranged after any transition (Claim A.3). The core of the argument lies in constructing specific history-dependent policies, π_1 and π_2, for systems C and D, based on the optimal policies assumed for A and B. These constructed policies dynamically determine the number of active servers in C and D at each transition such that the total number of active servers in C and D is equal to that in A and B, and the server counts in C and D are "sandwiched" between those in A and B (Claim A.1). This careful construction ensures that the instantaneous cost rate in systems A and B is greater than or equal to that in systems C and D (Claim A.2). Through a meticulously designed coupling mechanism for state transitions (arrivals, departures, or self-transitions) across all four systems, we demonstrate that this cost inequality, and the well-arranged nature of states, persists over time. This ultimately establishes the desired convexity for every sample path.

Proof of Lemma 2: To prove the lemma, observe that it is sufficient to show that, given two systems A and B that start with $m+1$ and $m-1$ jobs, respectively, and are operated under an optimal policy, there exist policies π_1 and π_2 such that the sum of the expected infinite-horizon discounted costs in systems C and D that both start with m jobs and are operated by π_1 and π_2, respectively, is (weakly) smaller than the corresponding sum in A and B. In fact, it is convenient to allow the four systems A, B, C, and D to start with m_A, m_B, m_C, and m_D jobs, respectively, where $\boldsymbol{m} = (m_A, m_B, m_C, m_D)$ is *well-arranged*, a notion we define below; this is sufficient because the original choice for $\boldsymbol{m}$ was $(m+1, m-1, m, m)$, which is easily seen to be well-arranged.

Definition A.1 *A four-dimensional vector of non-negative integers (m_A, m_B, m_C, m_D) is said to be well-arranged if it satisfies at least one of the following conditions (a), (b), or (c).*

(a) $m_C = m_A$ *and* $m_D = m_B$*;*

(b) $m_C = m_B$ *and* $m_D = m_A$*;*

(c) $m_C + m_D = m_A + m_B$ *and* $m_A > m_C > m_B$*.*

Before we proceed, we preview some important features of our analysis.

1. Recall that a policy is defined to be a function that specifies the number of servers in use at any time as a function of the number of jobs in the system at that time. One could think of an expanded policy class which includes history-dependent functions – in fact, the policies π_1 and π_2 we construct below are history-dependent; such a construction is legitimate since the original, restrictive definition of a policy is without loss of optimality due to the memoryless properties of the Poisson arrival process and the exponential service-time distribution.

2. We will show $V_\alpha^*(m_A)+V_\alpha^*(m_B) \geq V_\alpha^{\pi_1}(m_C)+V_\alpha^{\pi_2}(m_D)$ for any well-arranged vector (m_A, m_B, m_C, m_D) in the stronger sample-path sense through a careful coupling of the four systems A, B, C, and D. This coupling ensures that state transitions (some of which could be fictitious self-transitions) occur simultaneously in the four systems. To define a sample path of transitions, we use a sequence of i.i.d. $U[0,1]$ random variables which define, for each system, whether a transition corresponds to an arrival or departure (or is a self-transition).

3. Let $s_{A,t}^*$, $s_{B,t}^*$, $s_{C,t}^{\pi_1}$, and $s_{D,t}^{\pi_2}$ denote the numbers of servers in use in Systems A, B, C, and D at time t, respectively. Our policies π_1 and π_2 will be such that $s_{A,t}^* + s_{B,t}^* = s_{C,t}^{\pi_1} + s_{D,t}^{\pi_2}$ and $\max\{s_{A,t}^*, s_{B,t}^*\} \geq s_{C,t}^{\pi_1}, s_{D,t}^{\pi_2} \geq \min\{s_{A,t}^*, s_{B,t}^*\}$ for all $t \geq 0$, for every sample path (Claim A.1). Further, these two policies will ensure that the vector of the number of jobs in the four systems is well-arranged at any instant of time for every sample path (Claim A.3). The two claims together enable us to show $V_\alpha^*(m_A)+V_\alpha^*(m_B) \geq V_\alpha^{\pi_1}(m_C) + V_\alpha^{\pi_2}(m_D)$ in the stronger sample-path sense.

Next, we preview the construction of policies π_1 and π_2 under which Systems C and D are operated. The coupling we employ in our analysis below ensures that all four systems experience state transitions (a transition could be real, i.e., an arrival or departure, in some systems or a fictitious self-transition in other systems) simultaneously. Consider any such transition epoch. Let $\boldsymbol{m}$ denote the vector of states (recall that the state in a system at any time is the number of jobs in that system at that time) in (A, B, C, D). Note that System A would employ (immediately after the transition epoch) $s_A(\boldsymbol{m}) := s^*(m_A)$ servers whereas System B would employ $s_B(\boldsymbol{m}) := s^*(m_B)$ servers. Recall from (5) that $s^*(m) \in \{m, s_r\}$. If $\boldsymbol{m}$ is well-arranged, then the number of servers employed in Systems C and D, denoted by $s_C(\boldsymbol{m})$ and $s_D(\boldsymbol{m})$, respectively, at this transition epoch are defined as follows:

If condition (a) holds, then let $s_C(\boldsymbol{m}) = s_A(\boldsymbol{m})$ and $s_D(\boldsymbol{m}) = s_B(\boldsymbol{m})$.

If condition (b) holds, then let $s_C(\boldsymbol{m}) = s_B(\boldsymbol{m})$ and $s_D(\boldsymbol{m}) = s_A(\boldsymbol{m})$.

If condition (c) holds, then we define $s_C(\boldsymbol{m})$ and $s_D(\boldsymbol{m})$ in the four cases below:

- If $s_A(\boldsymbol{m}) = m_A$ and $s_B(\boldsymbol{m}) = m_B$, then let $s_C(\boldsymbol{m}) = m_C$ and $s_D(\boldsymbol{m}) = m_D$.
- If $s_A(\boldsymbol{m}) = m_A$ and $s_B(\boldsymbol{m}) = s_r$, then let $s_C(\boldsymbol{m}) = m_A + s_r - m_D$ and $s_D(\boldsymbol{m}) = m_D$.
- If $s_A(\boldsymbol{m}) = s_r$ and $s_B(\boldsymbol{m}) = m_B$, then we consider the following subcases.

- If $m_C \geq s_r$, then let $s_C(\boldsymbol{m}) = s_r$ and $s_D(\boldsymbol{m}) = m_B$.
- If $m_C < s_r$, then let $s_C(\boldsymbol{m}) = m_C$ and $s_D(\boldsymbol{m}) = m_B + s_r - m_C$.

- If $s_A(\boldsymbol{m}) = s_r$ and $s_B(\boldsymbol{m}) = s_r$, then let $s_C(\boldsymbol{m}) = s_r$ and $s_D(\boldsymbol{m}) = s_r$.

Claim A.1 *For any well-arranged vector $\boldsymbol{m} = (m_A, m_B, m_C, m_D)$, the decision vector*

$$\boldsymbol{s}(\boldsymbol{m}) := (s_A(\boldsymbol{m}), s_B(\boldsymbol{m}), s_C(\boldsymbol{m}), s_D(\boldsymbol{m}))$$

satisfies the set of conditions (A-1) *below:*

$$\begin{aligned} &s_C(\boldsymbol{m}) \leq m_C, s_D(\boldsymbol{m}) \leq m_D, s_C(\boldsymbol{m}) + s_D(\boldsymbol{m}) = s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}), \text{ and} \\ &\max\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m})\} \geq s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) \geq \min\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m})\}. \end{aligned} \tag{A-1}$$

We will now see that the instantaneous cost rate incurred by Systems A and B put together exceeds that in Systems C and D put together. Recall that, for all $s \leq m$, $\gamma(s; m) = w(m - s) + c_o(s - s_r)^+$ denotes the instantaneous cost rate incurred by a system when there are m jobs and s servers in use.

Claim A.2 *For any well-arranged vector $\boldsymbol{m} = (m_A, m_B, m_C, m_D)$,*

$$\gamma(s_A(\boldsymbol{m}); m_A) + \gamma(s_B(\boldsymbol{m}); m_B) \geq \gamma(s_C(\boldsymbol{m}); m_C) + \gamma(s_D(\boldsymbol{m}); m_D).$$

Next, we explain the main coupling construction we employ.

Coupling: Given four systems A, B, C, and D and a well-arranged vector of starting states $\boldsymbol{m}$, let $\boldsymbol{s}(\boldsymbol{m})$ denote the vector $(s_A(\boldsymbol{m}), s_B(\boldsymbol{m}), s_C(\boldsymbol{m}), s_D(\boldsymbol{m}))$ as in Claim A.1. Let

$$\nu := \lambda + \max\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m}), s_C(\boldsymbol{m}), s_D(\boldsymbol{m})\}\mu.$$

All four systems experience a transition at time τ that is exponentially distributed with rate ν. We will use a $U[0, 1]$ random variable, denoted by U, to specify the nature of the transitions in the four systems as follows.

Note that when the transition occurs at time τ, for each system $x \in \{A, B, C, D\}$, it is a job arrival with probability $\frac{\lambda}{\nu}$, a job departure with probability $\frac{s_x(\boldsymbol{m})\mu}{\nu}$, or a self-transition with probability $\frac{\nu - \lambda - s_x(\boldsymbol{m})\mu}{\nu}$. We index the four systems in the decreasing order of their number of servers in use. In particular, let $S_j \in \{A, B, C, D\}$ denote the system with the j^{th} largest number of servers in use and s_j denote the j^{th} largest number of $\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m}), s_C(\boldsymbol{m}), s_D(\boldsymbol{m})\}$ for $j \in \{1, 2, 3, 4\}$. When there are ties, we let $\{S_1, S_4\} = \{A, B\}$ and $\{S_2, S_3\} = \{C, D\}$ following (A-1). Thus, $\nu = \lambda + s_1\mu$. Note that $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$ by (A-1). Thus, we have $s_1 + s_4 = s_2 + s_3$.

Then, when the transition occurs at time τ, the nature of the transition in each of the four systems is specified as follows.

- If $0 \leq U \leq \frac{\lambda}{\lambda + s_1\mu}$, then a job arrives in each of the four systems.
- If $\frac{\lambda}{\lambda + s_1\mu} < U \leq \frac{\lambda + s_4\mu}{\lambda + s_1\mu}$, then a job departs from each of the four systems.
- If $\frac{\lambda + s_4\mu}{\lambda + s_1\mu} < U \leq \frac{\lambda + s_1\mu}{\lambda + s_1\mu} = 1$, then a job departs from System S_1, no job departs from System S_4, and a job departs from one of the systems S_2 and S_3. In particular,

- if $\frac{\lambda+s_4\mu}{\lambda+s_1\mu} < U \le \frac{\lambda+s_2\mu}{\lambda+s_1\mu}$, then a job departs from System S_2 and a self-transition occurs in System S_3;
- if $\frac{\lambda+s_2\mu}{\lambda+s_1\mu} < U \le 1$, then a job departs from System S_3 and a self-transition occurs in System S_2.

Note that a job departs from System S_2 if $\frac{\lambda}{\lambda+s_1\mu} < U \le \frac{\lambda+s_2\mu}{\lambda+s_1\mu}$, which occurs with probability $\frac{s_2\mu}{\lambda+s_1\mu}$. A job departs from System S_3 if $\frac{\lambda}{\lambda+s_1\mu} < U \le \frac{\lambda+s_4\mu}{\lambda+s_1\mu}$ or $\frac{\lambda+s_2\mu}{\lambda+s_1\mu} < U \le 1$, which occurs with probability $\frac{s_4\mu}{\lambda+s_1\mu} + \frac{(s_1-s_2)\mu}{\lambda+s_1\mu} = \frac{s_3\mu}{\lambda+s_1\mu}$ using $s_1 + s_4 = s_2 + s_3$.

Let $\hat{\boldsymbol{m}} = (\hat{m}_A, \hat{m}_B, \hat{m}_C, \hat{m}_D)$ denote the resulting state vector immediately after the transition. Then, m_{S_j} and $\hat{m}_{S_j}$ are the number of jobs in System S_j before and immediately after the transition, respectively, for $j = 1, 2, 3, 4$, where

$$\begin{aligned}
\hat{m}_{S_j} &= m_{S_j} + \mathbb{I}\left\{U \le \frac{\lambda}{\lambda+s_1\mu}\right\} - \mathbb{I}\left\{\frac{\lambda}{\lambda+s_1\mu} < U \le \frac{\lambda+s_j\mu}{\lambda+s_1\mu}\right\}, \text{ for } j = 1,2,4 \\
\hat{m}_{S_3} &= m_{S_3} + \mathbb{I}\left\{U \le \frac{\lambda}{\lambda+s_1\mu}\right\} - \mathbb{I}\left\{\frac{\lambda}{\lambda+s_1\mu} < U \le \frac{\lambda+s_4\mu}{\lambda+s_1\mu}\right\} - \mathbb{I}\left\{\frac{\lambda+s_2\mu}{\lambda+s_1\mu} < U \le 1\right\}.
\end{aligned} \tag{A-2}$$

We can now show

Claim A.3 *Given any well-arranged vector $\boldsymbol{m}$ and the decision vector $\boldsymbol{s(m)}$, the resulting state vector $\hat{\boldsymbol{m}}$, defined above, is also well-arranged.*

Let t_i denote the time of the i^{th} transition in the four systems for $i \ge 1$ and $(U_1, U_2, \cdots)$ be i.i.d. $U[0,1]$ random variables, where U_i is used to specify the nature of the i^{th} transition in the four systems as in (A-2). Then, the history until time t_i (i.e., immediately after the i^{th} transition), denoted by H_i, is $H_i = (t_1, U_1, \cdots, t_i, U_i)$. Let $\boldsymbol{m(H_i)} = (m_A^*(H_i), m_B^*(H_i), m_C^{\pi_1}(H_i), m_D^{\pi_2}(H_i))$ denote the vector of the numbers of jobs in Systems A, B, C, and D at time t_i. For any well-arranged vector $\boldsymbol{m(H_i)}$, we define the number of servers in use in C under policy π_1 and that in D under policy π_2 at time t_i as $s_C(\boldsymbol{m(H_i)})$ and $s_D(\boldsymbol{m(H_i)})$, respectively. Let $H_0 = \emptyset$ denote the vacuous history at time $t = 0$. Note that $\boldsymbol{m(H_0)} = (m+1, m-1, m, m)$ is well-arranged. By a repeated application of Claim A.3, $\boldsymbol{m(H_i)}$ is well-arranged for any $H_i, i \ge 0$. Then, by Claim A.2, the instantaneous cost rate at time t_i incurred by Systems A and B put together exceeds that in Systems C and D put together. Moreover, the instantaneous cost rate in each system is a constant in $[t_i, t_{i+1})$. Thus, the sum of the total cost incurred by A and B in $[t_i, t_{i+1})$ exceeds that in C and D. Therefore, the sum of the expected infinite-horizon discounted costs in Systems C and D is smaller than the corresponding sum in A and B, i.e., $V_\alpha^*(m+1) + V_\alpha^*(m-1) \ge V_\alpha^{\pi_1}(m) + V_\alpha^{\pi_2}(m) \ge V_\alpha^*(m) + V_\alpha^*(m)$. ■

Proof of Claim A.1: Recall that any well-arranged vector $\boldsymbol{m}$ satisfies at least one of (a), (b), or (c). If condition (a) or (b) holds, then it is easy to verify that (A-1) holds. If condition (c) holds, then we show that (A-1) holds in the four cases below:

- If $\boldsymbol{s(m)} = (m_A, m_B, m_C, m_D)$, then $s_C(\boldsymbol{m}) = m_C$ and $s_D(\boldsymbol{m}) = m_D$. Since $m_A + m_B = m_C + m_D$ and $m_A > m_C > m_B$, we have $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$ and $s_A(\boldsymbol{m}) > s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) > s_B(\boldsymbol{m})$, i.e., both $s_C(\boldsymbol{m})$ and $s_D(\boldsymbol{m})$ are between $s_A(\boldsymbol{m})$ and $s_B(\boldsymbol{m})$.
- If $\boldsymbol{s(m)} = (m_A, s_r, m_A + s_r - m_D, m_D)$, then $s_C(\boldsymbol{m}) = m_A + s_r - m_D = m_C + s_r - m_B \le m_C$, $s_D(\boldsymbol{m}) = m_D$, and $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$. Note that $m_A > m_D > m_B \ge s_r$. Thus, $s_A(\boldsymbol{m}) > s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) > s_B(\boldsymbol{m})$.

- If $s_A(\boldsymbol{m}) = s_r$ and $s_B(\boldsymbol{m}) = m_B$, then we consider the following subcases.
  - If $m_C \geq s_r$, then $s_C(\boldsymbol{m}) = s_r$ and $s_D(\boldsymbol{m}) = m_B < m_D$. It is easy to verify that (A-1) holds.
  - If $m_C < s_r$, then $s_C(\boldsymbol{m}) = m_C$, $s_D(\boldsymbol{m}) = m_B + s_r - m_C = m_D + s_r - m_A \leq m_D$, and $s_C(\boldsymbol{m}) + s_D(\boldsymbol{m}) = s_A(\boldsymbol{m}) + s_B(\boldsymbol{m})$. Note that $s_r > m_C > m_B$. Thus, $s_A(\boldsymbol{m}) > s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) > s_B(\boldsymbol{m})$.
- If $\boldsymbol{s}(\boldsymbol{m}) = (s_r, s_r, s_r, s_r)$, then it is easy to verify that (A-1) holds. ■

Proof of Claim A.2: $m_A + m_B = m_C + m_D$ and $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$ together imply

$$w(m_A + m_B - s_A(\boldsymbol{m}) - s_B(\boldsymbol{m})) = w(m_C + m_D - s_C(\boldsymbol{m}) - s_D(\boldsymbol{m})). \tag{A-3}$$

By $\max\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m})\} \geq s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) \geq \min\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m})\}$ and the convexity of x^+, we have

$$c_o(s_A(\boldsymbol{m}) - s_r)^+ + c_o(s_B(\boldsymbol{m}) - s_r)^+ \geq c_o(s_C(\boldsymbol{m}) - s_r)^+ + c_o(s_D(\boldsymbol{m}) - s_r)^+. \tag{A-4}$$

Then, (A-3) and (A-4) together imply that

$$\gamma(s_A(\boldsymbol{m}); m_A) + \gamma(s_B(\boldsymbol{m}); m_B) \geq \gamma(s_C(\boldsymbol{m}); m_C) + \gamma(s_D(\boldsymbol{m}); m_D). \quad ■$$

Proof of Claim A.3: If $\boldsymbol{m}$ is a well-arranged vector, then it satisfies at least one of (a), (b), or (c). If $\boldsymbol{m}$ satisfies condition (a), it is easy to verify that $\hat{\boldsymbol{m}}$ satisfies condition (a). If $\boldsymbol{m}$ satisfies condition (b), it is easy to verify that $\hat{\boldsymbol{m}}$ satisfies condition (b). If $\boldsymbol{m}$ satisfies condition (c), it is easy to verify $\hat{\boldsymbol{m}}$ satisfies condition (c) if $U \leq \frac{\lambda + s_4\mu}{\lambda + s_1\mu}$, i.e., a job arrival or a job departure occurs in each of the four systems when the transition occurs. Then, we show that $\hat{\boldsymbol{m}}$ satisfies at least one of (a), (b), or (c) when $U > \frac{\lambda + s_4\mu}{\lambda + s_1\mu}$. Since $\max\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m})\} \geq s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) \geq \min\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m})\}$ by (A-1), we have $\{S_1, S_4\} = \{A, B\}$ and $\{S_2, S_3\} = \{C, D\}$. This implies that a job departure occurs in only one of A and B (resp., C and D).

- If $m_A = m_C + 1$, then $m_B = m_D - 1$. We consider the following subcases:
  - If $\hat{\boldsymbol{m}} = (m_A - 1, m_B, m_C - 1, m_D)$, then $\hat{\boldsymbol{m}}$ satisfies condition (b) if $m_A = m_D + 1$ and satisfies condition (c) if $m_A > m_D + 1$.
  - If $\hat{\boldsymbol{m}} = (m_A - 1, m_B, m_C, m_D - 1)$, then $\hat{\boldsymbol{m}}$ satisfies condition (a).
  - If $\hat{\boldsymbol{m}} = (m_A, m_B - 1, m_C - 1, m_D)$ or $\hat{\boldsymbol{m}} = (m_A, m_B - 1, m_C, m_D - 1)$, then $\hat{\boldsymbol{m}}$ satisfies condition (c).
- If $m_A = m_D + 1$, then $m_B = m_C - 1$. We consider the following subcases:
  - If $\hat{\boldsymbol{m}} = (m_A - 1, m_B, m_C - 1, m_D)$, then $\hat{\boldsymbol{m}}$ satisfies condition (b).
  - If $\hat{\boldsymbol{m}} = (m_A - 1, m_B, m_C, m_D - 1)$, then $\hat{\boldsymbol{m}}$ satisfies condition (a) if $m_A = m_C + 1$ and satisfies condition (c) if $m_A > m_C + 1$.
  - If $\hat{\boldsymbol{m}} = (m_A, m_B - 1, m_C - 1, m_D)$ or $\hat{\boldsymbol{m}} = (m_A, m_B - 1, m_C, m_D - 1)$, then $\hat{\boldsymbol{m}}$ satisfies condition (c).
- If $m_A > m_C + 1$ and $m_A > m_D + 1$, then it is easy to verify that $\hat{\boldsymbol{m}}$ satisfies condition (c). ■

Using Lemmas 1 and 2, we first show that $g(m)$ defined in (4) decreases in m, which will be used to show the optimality of the threshold-based bang-bang policy.

Lemma A.1 *The function $g(m)$ is decreasing in m for $m > s_r$.*

Proof of Lemma A.1: For $m > s_r$, we show that $g(m+1) < g(m)$:

$$\begin{aligned}
&g(m+1) - g(m)\\
=& -\mu w + (\alpha+\lambda)\mu\Delta V_\alpha^*(m) - \lambda\mu\Delta V_\alpha^*(m+2)\\
=& -\mu w + \alpha\mu\Delta V_\alpha^*(m) + \lambda\mu(\Delta V_\alpha^*(m) - \Delta V_\alpha^*(m+1)) + \lambda\mu(\Delta V_\alpha^*(m+1) - \Delta V_\alpha^*(m+2))\\
\leq& -\mu w + \alpha\mu\frac{c_o}{\alpha+\mu} < 0.
\end{aligned} \tag{A-5}$$

The first inequality holds by Lemmas 1 and 2. The second inequality holds since $c_o < \frac{\alpha+\mu}{\alpha} w$. ■

Proof of Theorem 1: Recall from (5) that the optimal policy for $m > s_r$ is

$$s^*(m) = \begin{cases} m & \text{if } g(m) < 0 \\ s_r & \text{otherwise} \end{cases}.$$

Since $g(m)$ decreases in m by Lemma A.1, there exists a threshold $M_\alpha^* \geq s_r + 1$ such that $g(m) < 0$ for $m \geq M_\alpha^*$ and $g(m) \geq 0$ for $s_r + 1 \leq m < M_\alpha^*$. Therefore, the optimal number of servers that are currently in use is

$$s^*(m) = \begin{cases} m & \text{if } m \geq M_\alpha^* \\ s_r & \text{if } s_r < m < M_\alpha^* \end{cases}.$$ ■

Appendix B Upper Bound on the Threshold $M_\alpha^*(s_r)$ and Proof of Lemma 3

To derive an upper bound on the optimal threshold $M_\alpha^*(s_r)$, we first consider and analyze an auxiliary problem. In this section, for notational simplicity, we suppress the dependence on s_r and write M_α^*, $\bar{M}_\alpha$, and $\tilde{M}_\alpha$ in place of $M_\alpha^*(s_r)$, $\bar{M}_\alpha(s_r)$, and $\tilde{M}_\alpha(s_r)$, respectively.

Definition A.2 (An Auxiliary Problem) *A system starts with s_r reserved servers and $m+1 \geq s_r$ jobs, which are labeled as $0, 1, \cdots, m$. There are no further job arrivals. Jobs $1, 2, \cdots, m$ are only allowed to access the s_r reserved servers while job 0 is allowed to access the s_r reserved servers and an on-demand server. However, jobs $1, 2, \cdots, m$ have priority over job 0 for accessing the reserved servers. The auxiliary problem is that of minimizing the expected infinite-horizon discounted cost associated with job 0 (i.e., expected discounted sum of the on-demand server cost and waiting cost incurred towards job 0). Let $\Gamma^*(m)$ denote this minimum expected cost.*

Note that when $m = s_r - 1$, job 0 is assigned to a reserved server; thus, $\Gamma^*(s_r - 1) = 0$. For all $m \geq s_r$,

$$\begin{aligned}
\Gamma^*(m) =& \min\left\{ \begin{array}{l} \int_0^\infty (s_r\mu+\mu)e^{-(s_r\mu+\mu)t}\left[\int_0^t e^{-\alpha\tau}c_o d\tau + e^{-\alpha t}\frac{s_r\mu}{s_r\mu+\mu}\Gamma^*(m-1)\right]dt, \\ \int_0^\infty s_r\mu e^{-s_r\mu t}\left[\int_0^t e^{-\alpha\tau}w d\tau + e^{-\alpha t}\Gamma^*(m-1)\right]dt \end{array} \right\}\\
=& \min\left\{\frac{c_o + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu+\mu}, \frac{w + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu}\right\}.
\end{aligned} \tag{A-6}$$

Lemma A.2 (Optimal Policy in the Auxiliary Problem) *The optimal policy in the auxiliary problem is defined by a threshold $\hat{M}_\alpha \geq s_r$ such that, for all $s_r \leq m < \hat{M}_\alpha$, we make job 0 wait, and, for all $m \geq \hat{M}_\alpha$, we use an on-demand server to serve job 0.*

Proof of Lemma A.2: Note that if the special job is always served by an on-demand server until it completes its service, then the expected on-demand capacity cost is $\frac{c_o}{\alpha+\mu}$. Thus, we have

$$\Gamma^*(m) \leq \frac{c_o}{\alpha+\mu} < \frac{w}{\alpha} \text{ for } m \geq s_r - 1. \tag{A-7}$$

Using (A-7), we show that $\Gamma^*(m)$ increases in m by showing that $\frac{c_o+s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu+\mu} \geq \Gamma^*(m-1)$ and $\frac{w+s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu} \geq \Gamma^*(m-1)$ for $m \geq s_r$:

$$\frac{c_o + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu+\mu} \geq \frac{(\alpha+\mu)\Gamma^*(m-1) + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu+\mu} = \Gamma^*(m-1),$$

$$\frac{w + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu} \geq \frac{\alpha\Gamma^*(m-1) + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu} = \Gamma^*(m-1).$$

Thus, $\Gamma^*(m) \geq \Gamma^*(m-1)$ by (A-6). Note that it is optimal to hire an on-demand server to serve the special job at state m if and only if

$$\frac{c_o + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu+\mu} < \frac{w + s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu}$$
$$\Leftrightarrow \left[\frac{s_r\mu^2}{(\alpha+s_r\mu+\mu)(\alpha+s_r\mu)}\right]\Gamma^*(m-1) > \frac{c_o}{\alpha+s_r\mu+\mu} - \frac{w}{\alpha+s_r\mu}, \tag{A-8}$$

Since $\Gamma^*(m)$ increases in m, there exists $\hat{M}_\alpha \geq s_r$ such that it is optimal to hire an on-demand server to serve the special job if $m \geq \hat{M}_\alpha$ and let the special job wait if $m < \hat{M}_\alpha$. ■

Note that for $s_r = 0$, $\frac{c_o+s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu+\mu} = \frac{c_o}{\alpha+\mu} < \frac{w}{\alpha} = \frac{w+s_r\mu\Gamma^*(m-1)}{\alpha+s_r\mu}$ for all m. Thus, it is always optimal to serve job 0 on an on-demand server, so $\hat{M}_\alpha = 0$ and

$$\Gamma^*(m) = \frac{c_o}{\alpha+\mu} \text{ for } m \geq 0. \tag{A-9}$$

We now show an upper bound on $\hat{M}_\alpha$ for $s_r \geq 1$. Recall that

$$\tilde{M}_\alpha = \max\left\{s_r, s_r + 1 + \left\lfloor \frac{\ln\left(1 - \frac{\alpha[c_o(\alpha+s_r\mu) - w(\alpha+s_r\mu+\mu)]}{s_r\mu^2 w}\right)}{\ln\left(\frac{s_r\mu}{s_r\mu+\alpha}\right)} \right\rfloor\right\}, \tag{A-10}$$

where $c_o < \frac{\alpha+\mu}{\alpha}w$ ensures that $\frac{\alpha[c_o(\alpha+s_r\mu)-w(\alpha+s_r\mu+\mu)]}{s_r\mu^2 w} < 1$. Then, we show

Lemma A.3 (Upper Bound on Threshold $\hat{M}_\alpha$ in Auxiliary Problem) *For $s_r \geq 1$, the threshold $\hat{M}_\alpha$ satisfies $\hat{M}_\alpha \leq \tilde{M}_\alpha$.*

Proof of Lemma A.3: It is easy to show by induction on m that for $s_r - 1 \leq m \leq \hat{M}_\alpha - 1$,

$$\Gamma^*(m) = \frac{w}{\alpha}\left[1 - \left(\frac{s_r\mu}{\alpha+s_r\mu}\right)^{m-s_r+1}\right]. \tag{A-11}$$

We now show that $\hat{M}_\alpha \le \tilde{M}_\alpha$. Suppose not, i.e., $\hat{M}_\alpha \ge \tilde{M}_\alpha + 1$, then by (A-11), we have $\Gamma^*(\tilde{M}_\alpha - 1) = \frac{w}{\alpha}\left[1 - \left(\frac{s_r\mu}{\alpha+s_r\mu}\right)^{\tilde{M}_\alpha - s_r}\right]$. It is easy to verify that for $\tilde{M}_\alpha$ in (A-10), we have

$$\left[\frac{s_r\mu^2}{(\alpha+s_r\mu+\mu)(\alpha+s_r\mu)}\right]\Gamma^*\left(\tilde{M}_\alpha - 1\right) > \frac{c_o}{\alpha+s_r\mu+\mu} - \frac{w}{\alpha+s_r\mu}.$$

Then, by (A-8), it is optimal to hire an on-demand server to serve the special job in state $\tilde{M}_\alpha$. This contradicts the assumption that $\hat{M}_\alpha \ge \tilde{M}_\alpha + 1$ under which it is optimal to let the special job wait in state $\tilde{M}_\alpha$. Thus, we have $\hat{M}_\alpha \le \tilde{M}_\alpha$. ■

Our next result connects $\Gamma^*(\cdot)$, the optimal cost function in the auxiliary problem, with $V^*_\alpha(\cdot)$, the optimal cost function in the original problem.

Lemma A.4 (Connection Between the Original and Auxiliary Problems) *For any $m \ge s_r$ and $g(m) \ge 0$, we have $\Delta V^*_\alpha(m) \ge \Gamma^*(m-1)$.*

Proof of Lemma A.4: Consider two systems each with s_r reserved servers: System A has m jobs and System B has $m-1$ jobs at time $t=0$. We select an arbitrary job from the m jobs in A as a "special" job and consider the remaining $m-1$ jobs and all newly arrived jobs as "regular" jobs. Without loss of optimality, we assume that regular jobs have priority over the special job for accessing the reserved servers in System A. Note that, at time $t=0$, the number of regular jobs in System A equals the number of jobs in System B, namely $m-1$; moreover, the arrival process of jobs in System B is identical to that of regular jobs in System A. We consider a policy π in System B which mimics the actions that the optimal policy in System A takes on its regular jobs. Then the expected cost in System A under the optimal policy minus the expected cost in System B under policy π (i.e., $V^*_\alpha(m) - V^\pi_\alpha(m-1)$) equals the expected cost (i.e., on-demand capacity cost plus waiting cost) incurred by the special job in System A.

We call the system in the auxiliary problem in which the starting number of jobs, excluding job 0, is $m-1$ as System C. In System C, job 0 is called the "special" job and all other jobs are called "regular" jobs. Next, we show that $V^*_\alpha(m) - V^\pi_\alpha(m-1) \ge \Gamma^*(m-1)$. That is, the expected discounted cost incurred by the special job in System A is at least as large as the minimum expected discounted cost incurred by the special job in System C. To prove this, we show a stronger result that there exists policy ψ that we will construct such that the expected cost incurred by the special job in System A is (weakly) greater than that in System C under policy ψ. We will show this result in the stronger sample-path sense through a careful coupling of Systems A and C.

We first explain the evolution process of the number of jobs in each of the two systems. Let $s^*_{A,t}$ and $s^\psi_{C,t}$ denote the number of servers in use in System A under the optimal policy and that in System C under policy ψ at time t, respectively. Our policy ψ will ensure that $s^*_{A,t} \ge s^\psi_{C,t}$. We assume transitions occur in both systems simultaneously; for each system, a transition occurs at time $t \ge 0$ with rate $\nu_t = \lambda + s^*_{A,t}\mu$. Let t_i denote the time of the i^{th} transition and $t_0 = 0$. With a minor abuse of notation, let $s^*_{A,i} := s^*_{A,t_i}$ denote the number of servers that are currently in use, $m^*_{A,i}$ denote the number of regular jobs, and $n^*_{A,i} \in \{0,1\}$ denote the number of special jobs in System A at time t_i (these three numbers remain the same throughout the interval $[t_i, t_{i+1})$). Let $s^\psi_{C,i} := s^\psi_{C,t_i}$ denote the number of servers that are currently in use, $m^\psi_{C,i}$ denote the number of regular jobs, and $n^{\psi}_{C,i} \in \{0,1\}$ denote the number of special jobs in System C at time t_i (these three numbers remain the same throughout the interval $[t_i, t_{i+1})$). Let $(U_1, U_2, \cdots)$ be i.i.d. random variables distributed uniformly in $[0,1]$. Note that the special job in System A will be served by a (reserved or on-demand) server if and only if $s^*_{A,i} = m^*_{A,i} + 1$. Thus, the evolution process of $m^*_{A,i}$ and $n^*_{A,i}$ is

$$m^*_{A,i+1} = \begin{cases} m^*_{A,i} + \mathbb{I}\left\{U_{i+1} \le \frac{\lambda}{\lambda+s^*_{A,i}\mu}\right\} - \mathbb{I}\left\{\frac{\lambda}{\lambda+s^*_{A,i}\mu} < U_{i+1} \le \frac{\lambda+s^*_{A,i}\mu-\mu}{\lambda+s^*_{A,i}\mu}\right\} & \text{if } s^*_{A,i} = m^*_{A,i}+1, \\ m^*_{A,i} + \mathbb{I}\left\{U_{i+1} \le \frac{\lambda}{\lambda+s^*_{A,i}\mu}\right\} - \mathbb{I}\left\{\frac{\lambda}{\lambda+s^*_{A,i}\mu} < U_{i+1} \le 1\right\} & \text{otherwise} \end{cases}$$

$$n^*_{A,i+1} = \begin{cases} n^*_{A,i} - \mathbb{I}\left\{\frac{\lambda+s^*_{A,i}\mu-\mu}{\lambda+s^*_{A,i}\mu} < U_{i+1} \le 1\right\}, & \text{if } s^*_{A,i} = m^*_{A,i}+1, \\ n^*_{A,i} & \text{otherwise} \end{cases}$$

Note that the special job in System C will be served by a (reserved or on-demand) server if and only if $s^{\psi}_{C,i} \in \{s_r+1, m^{\psi}_{C,i}+1\}$. Thus, the evolution process of $m^{\psi}_{C,i}$ and $n^{\psi}_{C,i}$ is

$$m^{\psi}_{C,i+1} = \begin{cases} m^{\psi}_{C,i} - \mathbb{I}\left\{\frac{\lambda}{\lambda+s^*_{A,i}\mu} < U_{i+1} \le \frac{\lambda+s^{\psi}_{C,i}\mu-\mu}{\lambda+s^*_{A,i}\mu}\right\} & \text{if } s^{\psi}_{C,i} \in \{s_r+1, m^{\psi}_{C,i}+1\}, \\ m^{\psi}_{C,i} - \mathbb{I}\left\{\frac{\lambda}{\lambda+s^*_{A,i}\mu} < U_{i+1} \le \frac{\lambda+s^{\psi}_{C,i}\mu}{\lambda+s^*_{A,i}\mu}\right\} & \text{otherwise} \end{cases}$$

$$n^{\psi}_{C,i+1} = \begin{cases} n^{\psi}_{C,i} - \mathbb{I}\left\{\frac{\lambda+s^*_{A,i}\mu-\mu}{\lambda+s^*_{A,i}\mu} < U_{i+1} \le 1\right\} & \text{if } s^{\psi}_{C,i} \in \{s_r+1, m^{\psi}_{C,i}+1\}, \\ n^{\psi}_{C,i} & \text{otherwise} \end{cases}$$

We now define policy ψ formally. Let $H_i = (t_1, U_1, \cdots, t_i, U_i)$ denote the history until the i^{th} transition. For a given history H_i until the i^{th} transition, let $s^*_A(H_i)$ and $m^*_A(H_i)$ (resp., $n^*_A(H_i)$) denote the number of servers in use and the number of regular (resp., special) jobs in System A at time $t \in [t_i, t_{i+1})$ under the optimal policy, respectively. Let $m^{\psi}_C(H_i)$ and $n^{\psi}_C(H_i)$ denote the number of regular jobs and the number of special jobs in System C at time $t \in [t_i, t_{i+1})$ under policy ψ, respectively. Then we define the non-anticipating deterministic policy ψ which is a function $s^{\psi}_C(H_i)$ that specifies the number of servers in use in System C at time $t \in [t_i, t_{i+1})$ as follows: For $i \ge 0$,

$$s^{\psi}_C(H_i) = \begin{cases} m^{\psi}_C(H_i) + n^{\psi}_C(H_i) & \text{if } m^{\psi}_C(H_i) < s_r, \\ s_r + 1 & \text{if } m^{\psi}_C(H_i) \ge s_r,\ n^{\psi}_C(H_i) = 1, \text{ and } s^*_A(H_i) = m^*_A(H_i) + 1, \\ s_r & \text{otherwise.} \end{cases}$$

For any given sample path $(t_1, U_1, t_2, U_2, \cdots)$ and $i \ge 0$, we show that $m^{\psi}_C(H_i)$ and $n^{\psi}_C(H_i)$ satisfy

$$m^{\psi}_C(H_i) \le m^*_A(H_i) \text{ and } n^{\psi}_C(H_i) \le n^*_A(H_i). \tag{A-12}$$

It is clear that (A-12) holds for $i = 0$. Suppose (A-12) holds for i. Next, following an approach similar to that in the proof of Claim A.3, by analyzing all possible scenarios after the $(i+1)^{th}$ transition, we show in Claim A.4 that (A-12) holds for $i+1$. The proof of Claim A.4 is omitted for brevity.

Claim A.4 *Consider any $i \ge 0$ and a fixed history H_{i+1}. If $m^{\psi}_C(H_i) \le m^*_A(H_i)$ and $n^{\psi}_C(H_i) \le n^*_A(H_i)$, then $m^{\psi}_C(H_{i+1}) \le m^*_A(H_{i+1})$ and $n^{\psi}_C(H_{i+1}) \le n^*_A(H_{i+1})$.*

It is easy to verify that $s^{\psi}_C(H_i) \le s^*_A(H_i)$ when (A-12) holds.

We will now see that the instantaneous cost rate incurred by the special job at any time t in System A, denoted by $\varphi_{A,t}$, exceeds that in System C, denoted by $\varphi_{C,t}$.

Claim A.5 *$\varphi_{A,t} \geq \varphi_{C,t}$ for all $t \geq 0$.*

Recall that the expected cost incurred by the special job in System A is equal to $V_\alpha^*(m) - V_\alpha^\pi(m-1)$. Thus, $V_\alpha^*(m) - V_\alpha^\pi(m-1) = \mathbb{E}[\int_0^\infty e^{-\alpha t}\varphi_{A,t}dt]$. Let $\Gamma^\psi(m-1) := \mathbb{E}[\int_0^\infty e^{-\alpha t}\varphi_{C,t}dt]$ denote the expected cost incurred by the special job in System C. Then, we have

$$V_\alpha^*(m) - V_\alpha^\pi(m-1) \geq \Gamma^\psi(m-1) \geq \Gamma^*(m-1),$$

which implies the claimed result since $V_\alpha^\pi(m-1) \geq V_\alpha^*(m-1)$, by definition of $V_\alpha^*(\cdot)$. ■

Proof of Claim A.5: For $i \geq 0$, we show that $\varphi_{A,t} \geq \varphi_{C,t}$ for $t \in [t_i, t_{i+1})$. When $m_C^\psi(H_i) < s_r$ or $n_C^\psi(H_i) = 0$, it is straightforward that $\varphi_{C,t} = 0 \leq \varphi_{A,t}$. Next, we show that $\varphi_{A,t} \geq \varphi_{C,t}$ when $m_C^\psi(H_i) \geq s_r$ and $n_C^\psi(H_i) = 1$ in the following two cases. Note that $m_A^*(H_i) \geq m_C^\psi(H_i) \geq s_r$ and $n_A^*(H_i) \geq n_C^\psi(H_i) = 1$ by (A-12).

- If $s_A^*(H_i) = m_A^*(H_i) + 1$, then $s_C^\psi(H_i) = s_r + 1$. Thus, $\varphi_{A,t} = \varphi_{C,t} = c_o$.
- Otherwise, $s_A^*(H_i) = s_C^\psi(H_i) = s_r$. Thus, $\varphi_{A,t} = \varphi_{C,t} = w$. ■

Using the upper bound $\tilde{M}_\alpha$ on $\hat{M}_\alpha$ (Lemma A.3) and the connection between the original and auxiliary problems (Lemma A.4), we establish an upper bound $\bar{M}_\alpha$ on the optimal threshold M_α^* for the original problem.

Proof of Lemma 3: We first show that $M_\alpha^* = 1$ for $s_r = 0$. Suppose not, i.e., $g(1) \geq 0$. By the definition of $g(\cdot)$, we have

$$g(1) = (c_o - w)(\alpha + \lambda) - \mu w + (\alpha + \lambda)\mu V_\alpha^*(0) - \lambda\mu V_\alpha^*(2).$$

By $g(1) \geq 0$, we have $s^*(1) = 0$ and

$$V_\alpha^*(1) = \frac{w + \lambda V_\alpha^*(2)}{\alpha + \lambda}. \tag{A-13}$$

Thus, we can rewrite $g(1)$ as

$$\begin{aligned} g(1) &= (c_o - w)(\alpha + \lambda) - \mu w + (\alpha + \lambda)\mu V_\alpha^*(0) - \lambda\mu V_\alpha^*(2) \\ &= (c_o - w)(\alpha + \lambda) - \mu w + (\alpha + \lambda)\mu V_\alpha^*(0) - (\alpha + \lambda)\mu V_\alpha^*(1) + \mu w \\ &= (c_o - w - \mu\Delta V_\alpha^*(1))(\alpha + \lambda). \end{aligned}$$

The second equality holds by (A-13). Then, we have

$$\begin{aligned} g(1) &= (c_o - w - \mu\Delta V_\alpha^*(1))(\alpha + \lambda) \\ &\leq (c_o - w - \mu\Gamma^*(0))(\alpha + \lambda) \\ &= \left(c_o - w - \mu\frac{c_o}{\alpha + \mu}\right)(\alpha + \lambda) < 0. \end{aligned}$$

The first inequality holds by Lemma A.4. The second equality holds by (A-9). The second inequality holds since $c_o < \frac{\alpha+\mu}{\alpha}w$. This contradicts the assumption that $g(1) \geq 0$. Thus, we have $g(1) < 0$ and $M_\alpha^* = 1$ for $s_r = 0$.

Next, we show that $M_\alpha^* \leq \bar{M}_\alpha$ for $s_r \geq 1$. To do this, we show that $g(\bar{M}_\alpha) < 0$ by contradiction. Suppose not, i.e., $g(\bar{M}_\alpha) \geq 0$. Since $g(m)$ decreases in m by Lemma A.1, we have $g(m) \geq 0$ for all $m \leq \bar{M}_\alpha$. Therefore, $s^*(m) = s_r$ for all $m \in \{s_r + 1, \cdots, \bar{M}_\alpha\}$. By the definition of $g(\cdot)$, we have

$$g(\bar{M}_\alpha) = (c_o - w)(\alpha + \lambda) + \mu c_o s_r - \mu w\bar{M}_\alpha + (\alpha + \lambda)\mu V_\alpha^*(\bar{M}_\alpha - 1) - \lambda\mu V_\alpha^*(\bar{M}_\alpha + 1).$$

Examining the R.H.S. above, we see a need to analyze two quantities $V_\alpha^*(\bar{M}_\alpha - 1)$ and $V_\alpha^*(\bar{M}_\alpha + 1)$. To do this, we use the fact that $s^*(\bar{M}_\alpha) = s_r$ to write

$$V_\alpha^*(\bar{M}_\alpha) = \frac{w(\bar{M}_\alpha - s_r) + \lambda V_\alpha^*(\bar{M}_\alpha + 1) + s_r\mu V_\alpha^*(\bar{M}_\alpha - 1)}{\alpha + \lambda + s_r\mu}. \tag{A-14}$$

This equation allows us to connect $V_\alpha^*(\bar{M}_\alpha - 1)$ and $V_\alpha^*(\bar{M}_\alpha + 1)$ through $V_\alpha^*(\bar{M}_\alpha)$. We proceed to make this connection by establishing the following claim using Lemma A.4.

Claim A.6 $V_\alpha^*(\bar{M}_\alpha - 1)$ *and* $V_\alpha^*(\bar{M}_\alpha + 1)$ *satisfy*

$$(\alpha+\lambda)V_\alpha^*(\bar{M}_\alpha - 1) - \lambda V_\alpha^*(\bar{M}_\alpha + 1) \le w(\bar{M}_\alpha - s_r) - \frac{\alpha+\lambda+s_r\mu}{\alpha+\mu}c_o + c_o\frac{\alpha+\lambda+s_r\mu}{\alpha+\mu}\left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{\bar{M}_\alpha - \tilde{M}_\alpha}.$$

We now use the inequality in Claim A.6 and the definition of $\bar{M}_\alpha$ in (6) to write

$$\begin{aligned}
g(\bar{M}_\alpha) &= (c_o - w)(\alpha+\lambda) + \mu c_o s_r - \mu w \bar{M}_\alpha + (\alpha+\lambda)\mu V_\alpha^*(\bar{M}_\alpha - 1) - \lambda\mu V_\alpha^*(\bar{M}_\alpha + 1) \\
&\le (c_o - w)(\alpha+\lambda) + \mu c_o s_r - \mu w \bar{M}_\alpha + \mu w(\bar{M}_\alpha - s_r) - \frac{\alpha+\lambda+s_r\mu}{\alpha+\mu}\mu c_o + \\
&\quad c_o\mu\frac{\alpha+\lambda+s_r\mu}{\alpha+\mu}\left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{\bar{M}_\alpha - \tilde{M}_\alpha} \\
&= (\frac{\alpha}{\alpha+\mu}c_o - w)(\alpha+\lambda+\mu s_r) + c_o\mu\frac{\alpha+\lambda+s_r\mu}{\alpha+\mu}\left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{\bar{M}_\alpha - \tilde{M}_\alpha} < 0.
\end{aligned}$$

This contradicts the assumption that $g(\bar{M}_\alpha) \ge 0$. Thus, we have $g(\bar{M}_\alpha) < 0$. Recall from Lemma A.1 that $g(m)$ decreases in m; also recall from the proof of Theorem 1 that $g(m) < 0$ for $m \ge M_\alpha^*$ and $g(m) \ge 0$ for $s_r + 1 \le m < M_\alpha^*$. These facts imply that $M_\alpha^* \le \bar{M}_\alpha$. ■

Proof of Claim A.6: By contradiction assumption, $g(\bar{M}_\alpha) \ge 0$. Thus, by Lemma A.4, we have $V_\alpha^*(\bar{M}_\alpha) - V_\alpha^*(\bar{M}_\alpha - 1) \ge \Gamma^*(\bar{M}_\alpha - 1)$. We derive a lower bound on $\Gamma^*(m)$ for $m \ge \tilde{M}_\alpha$:

$$\begin{aligned}
\Gamma^*(m) &= \frac{c_o}{\alpha+\mu}\left[1 - \left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{m-\hat{M}_\alpha+1}\right] + \left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{m-\hat{M}_\alpha+1}\frac{w}{\alpha}\left[1 - \left(\frac{s_r\mu}{\alpha+s_r\mu}\right)^{\hat{M}_\alpha - s_r}\right] \\
&\ge \frac{c_o}{\alpha+\mu}\left[1 - \left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{m-\tilde{M}_\alpha+1}\right].
\end{aligned}$$

The equality holds by induction on m. The inequality holds since $\tilde{M}_\alpha \ge \hat{M}_\alpha$ by Lemma A.3. Thus,

$$V_\alpha^*(\bar{M}_\alpha) - V_\alpha^*(\bar{M}_\alpha - 1) \ge \Gamma^*(\bar{M}_\alpha - 1) \ge \frac{c_o}{\alpha+\mu}\left[1 - \left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{\bar{M}_\alpha - \tilde{M}_\alpha}\right]. \tag{A-15}$$

Using (A-14) and (A-15), we have

$$\begin{aligned}
&(\alpha+\lambda+s_r\mu)\left\{V_\alpha^*(\bar{M}_\alpha - 1) + \frac{c_o}{\alpha+\mu} - \frac{c_o}{\alpha+\mu}\left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{\bar{M}_\alpha - \tilde{M}_\alpha}\right\} \\
&\le (\alpha+\lambda+s_r\mu)V_\alpha^*(\bar{M}_\alpha) = w(\bar{M}_\alpha - s_r) + \lambda V_\alpha^*(\bar{M}_\alpha + 1) + s_r\mu V_\alpha^*(\bar{M}_\alpha - 1),
\end{aligned}$$

or, equivalently,

$$(\alpha+\lambda)V_\alpha^*(\bar{M}_\alpha - 1) - \lambda V_\alpha^*(\bar{M}_\alpha + 1) \le w(\bar{M}_\alpha - s_r) - \frac{\alpha+\lambda+s_r\mu}{\alpha+\mu}c_o + c_o\frac{\alpha+\lambda+s_r\mu}{\alpha+\mu}\left(\frac{s_r\mu}{s_r\mu+\mu+\alpha}\right)^{\bar{M}_\alpha - \tilde{M}_\alpha}.$$

■

Appendix C Proofs of Other Results in Section 2.2

To establish the monotonicity in Proposition 1, we first present two properties (Lemmas A.5 and A.6) regarding the optimal value function $V_\alpha^*(m; s_r)$ with respect to changes in s_r and m.

Lemma A.5 *For any $m \geq 0$ and $s_r \geq 0$, we have $V_\alpha^*(m; s_r) - V_\alpha^*(m; s_r + 1) \leq \frac{c_o}{\alpha}$.*

Proof of Lemma A.5: Consider the system with $s_r + 1$ (resp., s_r) reserved servers and m jobs at time $t = 0$ as System A (resp., B). We use the optimal policy in System A. We construct a policy π for System B:

$$s^\pi(m; s_r) = s^*(m; s_r + 1).$$

Let m_t^A and m_t^B denote the number of jobs in System A (under the optimal policy) and System B (under policy π) at time t, respectively. Note that at any given state m, the number of servers in use in System A under the optimal policy and that in System B under policy π are the same. Thus, $\{m_t^A : t \geq 0\}$ and $\{m_t^B : t \geq 0\}$ follow the same distribution. When $\{m_t^A : t \geq 0\} = \{m_t^B : t \geq 0\}$, it is easy to verify that the cost rate (incurred by hiring on-demand servers and waiting) in System B at any time t minus the cost rate in System A at that time is no more than c_o. Thus, $V_\alpha^*(m; s_r) - V_\alpha^*(m; s_r + 1) \leq V_\alpha^\pi(m; s_r) - V_\alpha^*(m; s_r + 1) \leq c_o \int_0^\infty e^{\alpha t} dt = \frac{c_o}{\alpha}$. ■

Lemma A.6 *For any $s_r \geq 0$, we have $V_\alpha^*(m_1; s_r) + V_\alpha^*(m_2; s_r + 1) \geq V_\alpha^*(m_3; s_r) + V_\alpha^*(m_4; s_r + 1)$ for $m_1 + m_2 = m_3 + m_4$ and $m_1 \geq m_3 \geq m_2$.*

Proof of Lemma A.6: The proof of this lemma is similar to the proof of Lemma 2. Differences between these two proofs are due to the fact that the numbers of reserved servers in the four systems in this proof are different, whereas in the proof of Lemma 2, all four systems use the same number of reserved servers. We begin with an overview of our proof. Consider four systems A, B, C, and D with $s_r, s_r + 1, s_r$, and $s_r + 1$ reserved servers and m_1, m_2, m_3, and m_4 jobs at time $t = 0$, respectively, where $m_1 + m_2 = m_3 + m_4$ and $m_1 \geq m_3 \geq m_2$. To prove the lemma, observe that it is sufficient to show that, given Systems A and B operated under an optimal policy, there exist policies π_1 and π_2 such that the sum of the expected infinite-horizon discounted costs in Systems C and D operated by π_1 and π_2, respectively, is (weakly) smaller than the corresponding sum in A and B. Similar to the proof of Lemma 2, we will show this result in the stronger sample-path sense through a careful coupling of the four systems. In fact, it is convenient to allow the four systems A, B, C, and D to start with m_A, m_B, m_C, and m_D jobs, respectively, where $\boldsymbol{m} = (m_A, m_B, m_C, m_D)$ is *well-arranged*, a notion we define below; this is sufficient because the originally choice for $\boldsymbol{m}$ was (m_1, m_2, m_3, m_4), which is easily seen to be well-arranged.

Definition A.3 *A four-dimensional vector of non-negative integers (m_A, m_B, m_C, m_D) is said to be well-arranged if it satisfies either*

(a) $m_C = m_A$ and $m_D = m_B$; or

(b) $m_C + m_D = m_A + m_B$ and $m_A > m_C \geq m_B - 1$.

Note that condition (b) implies that $m_A \geq m_D - 1 > m_B - 1$.

Next, we preview the construction of policies π_1 and π_2 under which Systems C and D are operated. The coupling we employ in our analysis below ensures that all four systems experience state transitions simultaneously. Consider any such transition epoch. Let $\boldsymbol{m}$ denote the vector of states (recall that the state in a system at any time is the number of jobs in that system at that time) in (A, B, C, D). Note that System A would employ (immediately after the transition epoch) $s_A(\boldsymbol{m}) := s^*(m_A; s_r)$ servers whereas System B would employ $s_B(\boldsymbol{m}) := s^*(m_B; s_r + 1)$ servers. Recall from (5) that $s^*(m; s_r) \in \{m, s_r\}$. If $\boldsymbol{m}$ is well-arranged, the number of servers employed in Systems C and D, denoted by $s_C(\boldsymbol{m})$ and $s_D(\boldsymbol{m})$, respectively, at this transition epoch are defined as follows:

If condition (a) holds, then let $s_C(\boldsymbol{m}) = s_A(\boldsymbol{m})$ and $s_D(\boldsymbol{m}) = s_B(\boldsymbol{m})$.

If condition (b) holds, then we define $s_C(\boldsymbol{m})$ and $s_D(\boldsymbol{m})$ in the four cases below:

- If $s_A(\boldsymbol{m}) = m_A$ and $s_B(\boldsymbol{m}) = m_B$, then let $s_C(\boldsymbol{m}) = m_C$ and $s_D(\boldsymbol{m}) = m_D$.
- If $s_A(\boldsymbol{m}) = m_A$ and $s_B(\boldsymbol{m}) = s_r + 1$, then let $s_C(\boldsymbol{m}) = m_A + s_r + 1 - m_D$ and $s_D(\boldsymbol{m}) = m_D$.
- If $s_A(\boldsymbol{m}) = s_r$ and $s_B(\boldsymbol{m}) = m_B$, then we consider the following subcases.
  - If $m_C \geq s_r$, then let $s_C(\boldsymbol{m}) = s_r$ and $s_D(\boldsymbol{m}) = m_B$.
  - If $m_C < s_r$, then let $s_C(\boldsymbol{m}) = m_C$ and $s_D(\boldsymbol{m}) = m_B + s_r - m_C$.
- If $s_A(\boldsymbol{m}) = s_r$ and $s_B(\boldsymbol{m}) = s_r + 1$, then let $s_C(\boldsymbol{m}) = s_r$ and $s_D(\boldsymbol{m}) = s_r + 1$.

Claim A.7 *For any well-arranged vector $\boldsymbol{m} = (m_A, m_B, m_C, m_D)$, the decision vector*

$$\boldsymbol{s}(\boldsymbol{m}) := (s_A(\boldsymbol{m}), s_B(\boldsymbol{m}), s_C(\boldsymbol{m}), s_D(\boldsymbol{m}))$$

satisfies the set of conditions (A-16) *below:*

$$\begin{aligned} s_C(\boldsymbol{m}) &\leq m_C, \\ s_D(\boldsymbol{m}) &\leq m_D, \\ s_C(\boldsymbol{m}) + s_D(\boldsymbol{m}) &= s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}), \text{ and} \\ \max\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m}) - 1\} \geq s_C(\boldsymbol{m}),\ & s_D(\boldsymbol{m}) - 1 \geq \min\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m}) - 1\}. \end{aligned} \tag{A-16}$$

We will now see that the instantaneous cost rate incurred by Systems A and B put together exceeds that in Systems C and D put together. For $(x, y) \in \{(A, B), (C, D)\}$ where $s_x \leq m_x$ and $s_y \leq m_y$, let $\theta(m_x, m_y, s_x, s_y) = w(m_x - s_x + m_y - s_y) + c_o(s_x - s_r)^+ + c_o(s_y - s_r - 1)^+$ denote the total instantaneous cost rate incurred by Systems x and y when there are m_x and m_y jobs and s_x and s_y servers in use in the two systems, respectively.

Claim A.8 *For any well-arranged vector $\boldsymbol{m} = (m_A, m_B, m_C, m_D)$,*

$$\theta(m_A, m_B, s_A(\boldsymbol{m}), s_B(\boldsymbol{m})) \geq \theta(m_C, m_D, s_C(\boldsymbol{m}), s_D(\boldsymbol{m})).$$

We employ the same coupling construction as in (A-2) to Systems A, B, C, and D. Note that when $s_A(\boldsymbol{m}) = s_C(\boldsymbol{m})$ (and $s_B(\boldsymbol{m}) = s_D(\boldsymbol{m})$) or $s_A(\boldsymbol{m}) = s_D(\boldsymbol{m})$ (and $s_B(\boldsymbol{m}) = s_C(\boldsymbol{m})$), we let $\{S_1, S_4\} = \{A, B\}$ and $\{S_2, S_3\} = \{C, D\}$. Let $\hat{\boldsymbol{m}} = (\hat{m}_A, \hat{m}_B, \hat{m}_C, \hat{m}_D)$ denote the resulting state vector immediately after the transition. Then, we show

Claim A.9 *Given any well-arranged vector $\boldsymbol{m}$ and the decision vector $\boldsymbol{s}(\boldsymbol{m})$, the resulting state vector $\hat{\boldsymbol{m}}$ is also well-arranged.*

Using Claims A.8 and A.9 and following a similar argument as in the proof of Lemma 2, we show that the sum of the instantaneous cost rates incurred in A and B exceeds that in C and D at any time. Therefore, the sum of the expected infinite-horizon discounted costs in C and D is smaller than the corresponding sum in A and B, i.e., $V_\alpha^*(m_1; s_r) + V_\alpha^*(m_2; s_r+1) \geq V_\alpha^{\pi_1}(m_3; s_r) + V_\alpha^{\pi_2}(m_4; s_r+1) \geq V_\alpha^*(m_3; s_r) + V_\alpha^*(m_4; s_r+1)$. ■

Proof of Claim A.7: Recall that any well-arranged vector $\boldsymbol{m}$ satisfies exactly one of (a) or (b). If condition (a) holds, then it is easy to verify that (A-16) holds. If condition (b) holds, then we show that (A-16) holds in the four cases below:

- If $\boldsymbol{s}(\boldsymbol{m}) = (m_A, m_B, m_C, m_D)$, then $s_C(\boldsymbol{m}) = m_C$ and $s_D(\boldsymbol{m}) = m_D$. Since $m_A + m_B = m_C + m_D$ and $m_A > m_C \geq m_B - 1$, we have $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$ and $s_A(\boldsymbol{m}) \geq s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) - 1 \geq s_B(\boldsymbol{m}) - 1$.
- If $\boldsymbol{s}(\boldsymbol{m}) = (m_A, s_r+1, m_A+s_r+1-m_D, m_D)$, then $s_C(\boldsymbol{m}) = m_A+s_r+1-m_D = m_C+s_r+1-m_B \leq m_C$, $s_D(\boldsymbol{m}) = m_D$, and $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$. Note that $m_A \geq m_D - 1 > m_B - 1 \geq s_r$. Thus, $s_A(\boldsymbol{m}) \geq s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) - 1 \geq s_B(\boldsymbol{m}) - 1$.
- If $s_A(\boldsymbol{m}) = s_r$ and $s_B(\boldsymbol{m}) = m_B$, then we consider the following subcases.
  - If $m_C \geq s_r$, then $s_C(\boldsymbol{m}) = s_r$ and $s_D(\boldsymbol{m}) = m_B < m_D$. It is easy to verify that (A-16) holds.
  - If $m_C < s_r$, then $s_C(\boldsymbol{m}) = m_C$, $s_D(\boldsymbol{m}) = m_B + s_r - m_C = m_D + s_r - m_A \leq m_D$, and $s_C(\boldsymbol{m}) + s_D(\boldsymbol{m}) = s_A(\boldsymbol{m}) + s_B(\boldsymbol{m})$. Note that $s_r > m_C \geq m_B - 1$. Thus, $s_A(\boldsymbol{m}) \geq s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) - 1 \geq s_B(\boldsymbol{m}) - 1$.
- If $\boldsymbol{s}(\boldsymbol{m}) = (s_r, s_r+1, s_r, s_r+1)$, then $m_C \geq m_B - 1 \geq s_r = s_C(\boldsymbol{m})$ and $m_D > m_B \geq s_r + 1 = s_D(\boldsymbol{m})$. It is easy to verify that (A-16) holds. ■

Proof of Claim A.8: Note that $m_A + m_B = m_C + m_D$ and $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$ by (A-16). Thus, we have

$$w(m_A + m_B - s_A(\boldsymbol{m}) - s_B(\boldsymbol{m})) = w(m_C + m_D - s_C(\boldsymbol{m}) - s_D(\boldsymbol{m})). \tag{A-17}$$

In addition, $\max\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m}) - 1\} \geq s_C(\boldsymbol{m}), s_D(\boldsymbol{m}) - 1 \geq \min\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m}) - 1\}$ by (A-16) and x^+ is convex in x. Thus, we have

$$c_o(s_A(\boldsymbol{m}) - s_r)^+ + c_o(s_B(\boldsymbol{m}) - s_r - 1)^+ \geq c_o(s_C(\boldsymbol{m}) - s_r)^+ + c_o(s_D(\boldsymbol{m}) - s_r - 1)^+. \tag{A-18}$$

Then, (A-17) and (A-18) together imply that $\theta(m_A, m_B, s_A(\boldsymbol{m}), s_B(\boldsymbol{m})) \geq \theta(m_C, m_D, s_C(\boldsymbol{m}), s_D(\boldsymbol{m}))$. ■

Proof of Claim A.9: If $\boldsymbol{m}$ is a well-arranged vector, then it satisfies exactly one of (a) or (b). If $\boldsymbol{m}$ satisfies condition (a), it is easy to verify that $\hat{\boldsymbol{m}}$ satisfies condition (a). If $\boldsymbol{m}$ satisfies condition (b), it is easy to verify that $\hat{\boldsymbol{m}}$ satisfies condition (b) if $U \leq \frac{\lambda+s_4\mu}{\lambda+s_1\mu}$, i.e., a job arrival or a job departure occurs in each of the four systems when the transition occurs. Then, we show that $\hat{\boldsymbol{m}}$ satisfies exactly one of (a) or (b) when $U > \frac{\lambda+s_4\mu}{\lambda+s_1\mu}$.

- If $m_A = m_B = m$, then $m_C = m-1$ and $m_D = m+1$. It is easy to verify that $s_C(\boldsymbol{m}) \leq s_A(\boldsymbol{m}), s_B(\boldsymbol{m}) \leq s_D(\boldsymbol{m})$, i.e., both $s_A(\boldsymbol{m})$ and $s_B(\boldsymbol{m})$ are between $s_C(\boldsymbol{m})$ and $s_D(\boldsymbol{m})$. Thus, we only need to consider the following two cases:

- If $\hat{\boldsymbol{m}} = (m-1, m, m-1, m)$, then condition (a) holds.
- If $\hat{\boldsymbol{m}} = (m, m-1, m-1, m)$, then condition (b) holds.

- If $m_A \geq m_B + 1$, since $s_A(\boldsymbol{m}) + s_B(\boldsymbol{m}) = s_C(\boldsymbol{m}) + s_D(\boldsymbol{m})$ by (A-16), we have either $\{S_1, S_4\} = \{A, B\}$ or $\{S_1, S_4\} = \{C, D\}$. This implies that a job departure occurs in one of A and B and another job departure occurs in one of C and D. Thus, $\hat{m}_C + \hat{m}_D = \hat{m}_A + \hat{m}_B$. Next, we show that $\hat{\boldsymbol{m}}$ satisfies either (a) or

$$\hat{m}_A > \hat{m}_C \geq \hat{m}_B - 1 \tag{A-19}$$

in the following four cases.

  - If $\hat{\boldsymbol{m}} = (m_A, m_B - 1, m_C - 1, m_D)$, then it is easy to verify that (A-19) holds.
  - If $\hat{\boldsymbol{m}} = (m_A, m_B - 1, m_C, m_D - 1)$, then it is easy to verify that (A-19) holds.
  - If $\hat{\boldsymbol{m}} = (m_A - 1, m_B, m_C - 1, m_D)$, then we consider the following two subcases.
    * If $m_C \geq m_B$, then it is clear that (A-19) holds.
    * If $m_C = m_B - 1$, then it is easy to verify that $s_D(\boldsymbol{m}) \geq \max\{s_A(\boldsymbol{m}), s_B(\boldsymbol{m}), s_C(\boldsymbol{m})\}$, i.e., the number of servers in System D is (weakly) greater than the number of servers in each of Systems A, B, and C. Thus, this case where a self-transition occurs in System D but job departures occur in other systems will not happen.
  - If $\hat{\boldsymbol{m}} = (m_A - 1, m_B, m_C, m_D - 1)$, then we consider the following two subcases.
    * If $m_A > m_C + 1$, then it is clear that (A-19) holds.
    * If $m_A = m_C + 1$, then it is easy to verify that $\hat{\boldsymbol{m}}$ satisfies (a).

Therefore, $\hat{\boldsymbol{m}}$ is a well-arranged vector. ■

Using Lemmas A.5 and A.6, we show that the bang-bang threshold $M^*_\alpha(s_r)$ increases in s_r.

Proof of Proposition 1: Using Lemmas A.5 and A.6, we show that $g(m; s_r + 1) \geq g(m; s_r)$:

$$\begin{aligned}
&g(m; s_r + 1) - g(m; s_r)\\
=&c_o\mu + (\alpha + \lambda)\mu V^*_\alpha(m-1; s_r+1) - \lambda\mu V^*_\alpha(m+1; s_r+1) - (\alpha+\lambda)\mu V^*_\alpha(m-1; s_r) + \lambda\mu V^*_\alpha(m+1; s_r)\\
=&c_o\mu + \alpha\mu[V^*_\alpha(m-1; s_r+1) - V^*_\alpha(m-1; s_r)] +\\
&\lambda\mu[V^*_\alpha(m+1; s_r) + V^*_\alpha(m-1; s_r+1) - V^*_\alpha(m-1; s_r) - V^*_\alpha(m+1; s_r+1)]\\
\geq&c_o\mu + \alpha\mu\frac{-c_o}{\alpha} = 0.
\end{aligned}$$

The inequality holds by Lemmas A.5 and A.6. If $M^*_\alpha(s_r) = s_r + 1$, then $M^*_\alpha(s_r + 1) \geq s_r + 2 = M^*_\alpha(s_r) + 1$. Otherwise, $g(M^*_\alpha(s_r) - 1; s_r) \geq 0$. Then, we have $g(M^*_\alpha(s_r) - 1; s_r + 1) \geq g(M^*_\alpha(s_r) - 1; s_r) \geq 0$. Recall from Lemma A.1 that $g(m; s_r)$ decreases in m. Thus, $M^*_\alpha(s_r + 1) \geq M^*_\alpha(s_r)$. ■

Derivation of N: We show how to find a sufficiently large N such that $(s^*_r(N), \pi^*(N))$ is ϵ-optimal. Let $\tilde{V}^{\pi_M}_\alpha(m, n; s_r) := V^{\pi_M}_\alpha(m; s_r) - V^{\pi_M}_\alpha(m, n; s_r)$ denote the expected discounted cost incurred in hiring on-demand servers and in waiting from the n^{th} arrival until $t = \infty$ under policy π_M when there are m jobs at time $t = 0$ and s_r reserved servers. We derive an upper bound on $\tilde{V}^{\pi_M}_\alpha(m, 0; s_r)$ for all $0 \leq s_r \leq \bar{s}_\alpha$ and $s_r + 1 \leq M \leq \bar{M}_\alpha(s_r)$.

Note that the number of waiting jobs in the system under policy π_M cannot exceed $M-s_r-1$. Thus, the expected discounted cost incurred in waiting is no more than $(M-s_r-1)\cdot\int_0^\infty e^{-\alpha t}wdt=(M-s_r-1)\cdot\frac{w}{\alpha}$. The expected discounted cost incurred in hiring on-demand servers cannot exceed $m\frac{c_o}{\alpha+\mu}+\int_0^\infty \lambda e^{-\alpha t}\frac{c_o}{\alpha+\mu}dt=\left(m+\frac{\lambda}{\alpha}\right)\cdot\frac{c_o}{\alpha+\mu}$. Thus, we have $\tilde{V}_\alpha^{\pi_M}(m,0;s_r)\le(M-s_r-1)\cdot\frac{w}{\alpha}+\left(m+\frac{\lambda}{\alpha}\right)\cdot\frac{c_o}{\alpha+\mu}$.

Let τ_i denote the time between the $(i-1)^{st}$ arrival and the i^{th} arrival. Note that the maximum possible number of jobs in the system after n arrivals is $m+n$ for $n\ge 1$ and $\bar{Q}=\max\{\bar{M}_\alpha(s_r)-s_r-1:0\le s_r\le\bar{s}_\alpha\}$ is an upper bound on the number of jobs waiting in the queue. Then, we have

$$\tilde{V}_\alpha^{\pi_M}(m,n;s_r)\le\mathbb{E}\left[e^{-\alpha(\tau_1+\cdots+\tau_n)}\tilde{V}_\alpha^{\pi_M}(m+n,0;s_r)\right]\le\left(\frac{\lambda}{\alpha+\lambda}\right)^n\left(\bar{Q}\cdot\frac{w}{\alpha}+(m+n)\frac{c_o}{\alpha+\mu}+\frac{\lambda}{\alpha}\cdot\frac{c_o}{\alpha+\mu}\right).$$

Recall from (9) that

$$N:=\min\left\{n:n\in\mathbb{N},\left(\frac{\lambda}{\alpha+\lambda}\right)^n\left(\bar{Q}\frac{w}{\alpha}+(m_0+n)\frac{c_o}{\alpha+\mu}+\frac{\lambda}{\alpha}\frac{c_o}{\alpha+\mu}\right)\le\epsilon\right\}.$$

Thus, for all $0\le s_r\le\bar{s}_\alpha$ and $s_r+1\le M\le\bar{M}_\alpha(s_r)$, we have

$$\tilde{V}_\alpha^{\pi_M}(m_0,N;s_r)\le\left(\frac{\lambda}{\alpha+\lambda}\right)^N\left(\bar{Q}\cdot\frac{w}{\alpha}+(m_0+N)\frac{c_o}{\alpha+\mu}+\frac{\lambda}{\alpha}\cdot\frac{c_o}{\alpha+\mu}\right)\le\epsilon. \tag{A-20}$$

Proof of Lemma 4: Recall that $(s_r^*(N),\pi^*(N))$ minimizes $\frac{c_r s_r}{\alpha}+V_\alpha^\pi(m_0,N;s_r)$ among all solutions where $0\le s_r\le\bar{s}_\alpha$ and $\pi\in\{\pi_M:s_r+1\le M\le\bar{M}_\alpha(s_r)\}$. In addition, (s_r^*,π^*), which minimizes the expected infinite-horizon discounted cost, satisfies $0\le s_r^*\le\bar{s}_\alpha$ and $\pi^*\in\{\pi_M:s_r+1\le M\le\bar{M}_\alpha(s_r)\}$. Thus, we have

$$\frac{c_r s_r^*(N)}{\alpha}+V_\alpha^{\pi^*(N)}(m_0,N;s_r^*(N))\le\frac{c_r s_r^*}{\alpha}+V_\alpha^{\pi^*}(m_0,N;s_r^*)\le W_\alpha^{\pi^*}(m_0,s_r^*)=W_\alpha^*(m_0,s_r^*).$$

Then, we have

$$\begin{aligned}W_\alpha^{\pi^*(N)}(m_0,s_r^*(N))&=\frac{c_r s_r^*(N)}{\alpha}+V_\alpha^{\pi^*(N)}(m_0;s_r^*(N))\\&=\frac{c_r s_r^*(N)}{\alpha}+V_\alpha^{\pi^*(N)}(m_0,N;s_r^*(N))+\tilde{V}_\alpha^{\pi^*(N)}(m_0,N;s_r^*(N))\\&\le W_\alpha^*(m_0,s_r^*)+\tilde{V}_\alpha^{\pi^*(N)}(m_0,N;s_r^*(N)).\end{aligned}$$

Thus, we have

$$W_\alpha^{\pi^*(N)}(m_0,s_r^*(N))-W_\alpha^*(m_0,s_r^*)\le\tilde{V}_\alpha^{\pi^*(N)}(m_0,N;s_r^*(N))\le\epsilon.$$

The second inequality holds by (A-20). Thus, $(s_r^*(N),\pi^*(N))$ is an ϵ-optimal solution. ■

Because of the 16-page limit for the Online Appendix, Appendix D, "Proofs of the Results in Sections 3 and 4," and Appendix E, "Proofs of the Results in Section 5," are included only in the long version of the paper (Feng et al. 2026).

Appendix References

Feng, Z., M. Dawande, G. Janakiraman, and A. Qi. 2026. Service Systems with On-Demand and Reserved Servers (Long Version). *Available at SSRN 4735928*.