Online Appendix to
Tokenizing Loyalty Programs: The Role of Tradability

Junming Hu
School of Business, East China University of Science and Technology, Shanghai, China, junminghu@ecust.edu.cn

Xing Hu
Faculty of Business and Economics, The University of Hong Kong, Hong Kong, China, xinghu@hku.hk

S. Alex Yang
London Business School, London NW1 4SA, United Kingdom, sayang@london.edu

Yi Yang
1. School of Management, Zhejiang University, Hangzhou, China; 2. Laboratory for Human-AI Collaboration and Digital Intelligence Innovation, Zhejiang University, Hangzhou, China, yangyizju@zju.edu.cn

A. Analysis and Technical Results for Section 5.1

This appendix provides the technical details and supplemental results for Section 5.1 (Second-Period Outside Options). The related proofs are presented in Appendix E.3.

A.1. Traditional LP

We first analyze the equilibrium under the traditional LP. In the second period, customers who have not purchased before still choose not to purchase; those who have purchased before choose to repurchase if and only if $U_{L2} = \delta v - p + e - c \geq 0$, i.e., $v \geq \frac{p-e+c}{\delta}$. In the first period, customers choose to buy if and only if $U_{L1}^P \geq U_{L1}^N$, where $U_{L1}^P = v - p + \alpha \max\{\delta v - p + e - c, 0\}$ and $U_{L1}^N = \alpha \max\{v - p - c, 0\}$. By analyzing the utilities, customer purchase behavior can be summarized in Lemma A1.

Lemma A1. *Under the traditional LP, a customer purchases in period $i \in \{1,2\}$ if and only if $v \geq \tau_{Li}(p,e)$, where τ_{Li} is given by:*

$$(\tau_{L1}, \tau_{L2}) = \begin{cases} (1,1) & \text{if } p \geq 1 \cap e \leq p - \delta + c + \frac{p-1}{\alpha}, \\ (p,1) & \text{if } p \leq 1 \cap e \leq p - \delta + c, \\ (p, \frac{p-e+c}{\delta}) & \text{if } p - \delta + c \leq e \leq (1-\delta)p + c, \\ (\frac{p+\alpha(p-e+c)}{1+\alpha\delta}, \frac{p+\alpha(p-e+c)}{1+\alpha\delta}) & \text{if } \max\{(1-\delta)p + c, p - \delta + c + \frac{p-1}{\alpha}\} \leq e \leq p + c + \frac{p}{\alpha}, \\ (0,0) & \text{if } e \geq p + c + \frac{p}{\alpha}. \end{cases}$$

Anticipating customers' behavior, the firm maximizes its total profit as follows.

$$\max_{p,e} \pi_L = p(1 - \tau_{L1}(p,e)) + (p-e)(1 - \tau_{L2}(p,e)).$$

Solving this optimization problem yields the following equilibrium results.

Lemma A2. *Under the traditional LP, the equilibrium results are as follows.*

(i) ***No Repurchase*** *($0 \leq \delta \leq c$). The firm's optimal decisions are $p_L^* = \frac{1}{2}$ and $e_L^* \leq \frac{1+2(c-\delta)}{2}$; customers' purchase decisions are $(\tau_{L1}^*, \tau_{L2}^*) = (\frac{1}{2}, 1)$; the reward redemption rate is $r_L^* = 0$; the firm's profit is $\pi_L^* = \frac{1}{4}$.*

(ii) ***Partial Repurchase*** *($c \le \delta \le 1$). The firm's optimal decisions are $(p_L^*, e_L^*) = (\frac{1}{2}, \frac{1-\delta+c}{2})$; customers' purchase decisions are $(\tau_{L1}^*, \tau_{L2}^*) = (\frac{1}{2}, \frac{\delta+c}{2\delta})$; the reward redemption rate is $r_L^* = 1 - \frac{c}{\delta}$; the firm's profit is $\pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$.*

A.2. Tradable LP

We next analyze the equilibrium under the tradable LP. In the second period, for customers who have not purchased before, their second-period utility is $U_{T2}^F = \max\{v-p-c, v-p+e-e_M-c, 0\}$. For customers who purchased before, their second-period utility is $U_{T2}^S = \max\{\delta v - p + e - c, e_M, 0\}$. In the first period, their long-term utility is $U_{T1} = \max\{v - p + \alpha U_{T2}^S, \alpha U_{T2}^F\}$.

By analyzing the utilities, we define the following thresholds: $\hat{e}_1 = p - 1 + c + \frac{p-1}{\alpha}$, $\hat{e}_2 = p - \delta + c$, $\hat{e}_3 = \frac{1+c-p+\alpha(p+c-1)}{1+\alpha}$, $\hat{e}_4 = \frac{-1+\alpha+2\alpha c-\delta-3\alpha\delta+2(1+\alpha)p}{2\alpha}$, $\hat{e}_5 = c + \frac{(1-\alpha)(1-\delta)p}{1+\delta-\alpha(1-\delta)}$, $\hat{e}_6 = p + c + \frac{p}{\alpha}$. Then, customer purchase behavior can be summarized in Lemma A3.

Lemma A3. *Under the tradable LP, the consumer segmentation is given by*

$$(\tau_{T1}, \tau_{T2}^L, \tau_{T2}^H) = \begin{cases} (1,1,1) & \text{if } p \ge 1 \text{ and } e \le \hat{e}_1, \\ (p,p,1) & \text{if } p \le 1 \text{ and } e \le \min\{\hat{e}_2, c\}, \\ (\frac{p+\alpha(2+c-e+p)}{1+3\alpha}, \frac{-1+\alpha+2\alpha c-2\alpha e+2(1+\alpha)p}{1+3\alpha}, 1) & \text{if } \max\{\hat{e}_1, \hat{e}_3\} \le e \le \hat{e}_4, \\ (p + \frac{\alpha(e-c)}{1-\alpha}, p-e+c, 1) & \text{if } c \le e \le \min\{\hat{e}_2, \hat{e}_3\}, \\ (p, p, \frac{p-e+c}{\delta}) & \text{if } \hat{e}_2 \le e \le c, \\ (p + \frac{\alpha(e-c)}{1-\alpha}, p-e+c, \frac{p-e+c}{\delta}) & \text{if } \max\{\hat{e}_2, c\} \le e \le \hat{e}_5, \\ (\frac{(1+\delta)(p+\alpha(p-e+c))}{1+\delta+\alpha(3\delta-1)}, \frac{2\delta(p+\alpha(c-e+p))}{1+\delta+\alpha(3\delta-1)}, \frac{2(p+\alpha(c-e+p))}{1+\delta+\alpha(3\delta-1)}) & \text{if } \max\{\hat{e}_4, \hat{e}_5\} \le e \le \hat{e}_6, \\ (0,0,0) & \text{if } e \ge \hat{e}_6. \end{cases}$$

Anticipating customer behavior, the firm maximizes its total profit.

$$\max_{p,e} \pi_T = p(1-\tau_{T1}) + p(1 - \tau_{T2}^H + \tau_{T1} - \tau_{T2}^L) - e\min\{1-\tau_{T1}, 1 - \tau_{T2}^H + \tau_{T1} - \tau_{T2}^L\}.$$

Solving this optimization problem yields thresholds $\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4$, and $\hat{c}_1$, which characterize the equilibrium results summarized in Lemma A4 and illustrated in Figure A-1.

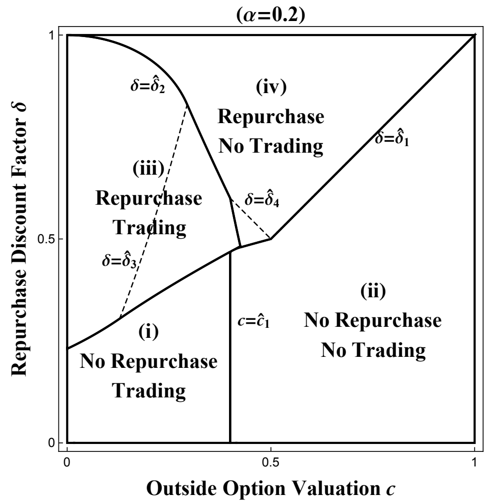
Lemma A4. *Under the tradable LP, there exist thresholds $\hat{\delta}_1$, $\hat{\delta}_2$, $\hat{\delta}_3$, $\hat{\delta}_4$, $\hat{c}_1$ such that the equilibrium results are as follows.*

(i) ***No Repurchase, Trading*** *($\delta \le \hat{\delta}_1$ and $c \le \hat{c}_1$). The firm's optimal decisions are $(p_T^*, e_T^*) = (\frac{2-c}{2+\alpha}, \frac{1+c-\alpha(1-c)}{3+\alpha})$; customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{2-c-\alpha(1+\alpha+c)}{(1-\alpha)(3+\alpha)}, \frac{1+\alpha+c}{3+\alpha}, 1)$; the reward redemption rate is $r_T^* = \frac{1-\alpha-2c}{1+c-\alpha(1-c)}$; the firm's profit is $\pi_T^* = \frac{1-\alpha(1-c)-c(1-c)}{(1-\alpha)(3+\alpha)}$.*

(ii) ***No Repurchase, No Trading*** *($\delta \le \hat{\delta}_1$ and $c \ge \hat{c}_1$). The firm's optimal decisions are $p_T^* = \frac{1}{2}$ and $e_T^* \le \min\{\frac{1+2(c-\delta)}{2}, c\}$; customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{1}{2}, \frac{1}{2}, 1)$; the reward redemption rate is $r_T^* = 0$; the firm's profit is $\pi_T^* = \frac{1}{4}$.*

(iii) ***Repurchase, Trading*** *($\hat{\delta}_1 \le \delta \le \hat{\delta}_2$). There exists a threshold $\hat{\delta}_3$ such that:*

Figure A-1 Equilibrium Results under Tradable LP with Outside Options



(a) *When* $\delta \le \hat{\delta}_3$, *the firm's optimal decisions are* $(p_T^*, e_T^*) = (\frac{(1-\alpha)(2-c)+\delta(3+\alpha-c)}{4+\delta(3+\alpha)}, \frac{2(1-\alpha)+c(1+\alpha)(1+\delta)}{4+\delta(3+\alpha)})$; *customers' purchase decisions are* $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{(1-\alpha)(2+\delta(3+\alpha))-c(1+\alpha)(1+\delta)}{(1-\alpha)(4+\delta(3+\alpha))}, \frac{2c+\delta(3+\alpha+c)}{4+\delta(3+\alpha)}, \frac{2c+\delta(3+\alpha+c)}{\delta(4+\delta(3+\alpha))})$; *the reward redemption rate is* $r_T^* = \frac{\delta(1-\alpha)(3-\alpha+\delta(3+\alpha))-2c(1-\alpha+\delta)(1+\delta)}{\delta(2(1-\alpha)+c(1+\alpha)(1+\delta)}$; *the firm's profit is* $\pi_T^* = \frac{c^2(1-\alpha+\delta)(1+\delta)+\delta(1-\alpha)(1-\alpha(1-\delta)+3\delta)-c\delta(1-\alpha)(3-\alpha+\delta(3+\alpha))}{\delta(1-\alpha)(4+\delta(3+\alpha))}$.

(b) *When* $\delta \ge \hat{\delta}_3$, *the firm's optimal decisions are* $(p_T^*, e_T^*) = (\frac{(1+\delta-\alpha(1-\delta))(1-\alpha+c+\delta(3+\alpha+c))}{2(1+\delta)(1+3\delta-\alpha(1-\delta))}, \frac{(1-\alpha)(1-\delta)(1+3\delta-\alpha(1-\delta))+c(1+\delta)(3+5\delta-3\alpha(1-\delta))}{2(1+\delta)(1+3\delta-\alpha(1-\delta))})$; *customers' purchase decisions are* $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{1-\alpha+c+\delta(3+\alpha+c)}{2(1-\alpha)+2\delta(3+\alpha)}, \frac{\delta(1-\alpha+c+\delta(3+\alpha+c)}{(1+\delta)(1+3\delta-\alpha(1-\delta))}, \frac{1}{1+\delta} + \frac{c}{1-\alpha+\delta(3+\alpha)})$; *the reward redemption rate is* $r_T^* = 1$; *the firm's profit is* $\pi_T^* = \frac{(1-\alpha-c+\delta(3+\alpha-c))^2}{4(1+\delta)(1-\alpha+\delta(3+\alpha))}$.

(iv) ***Repurchase, No Trading*** *(*$\delta \ge \max\{\hat{\delta}_1, \hat{\delta}_2\}$*).*

(a) *When* $\delta \le \hat{\delta}_4$, *the firm's optimal decisions are* $(p_T^*, e_T^*) = (\frac{2\delta+c}{2(1+\delta)}, c)$; *customers' purchase decisions are* $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{2\delta+c}{2(1+\delta)}, \frac{2\delta+c}{2(1+\delta)}, \frac{2\delta+c}{2\delta(1+\delta)})$; *the reward redemption rate is* $r_T^* = \frac{2\delta^2-c}{\delta(2-c)}$; *the firm's profit is* $\pi_T^* = \frac{4(1-c)\delta^2+c^2}{4\delta(1+\delta)}$.

(b) *When* $\delta \ge \hat{\delta}_4$, *the firm's optimal decisions are* $(p_T^*, e_T^*) = (\frac{1}{2}, \frac{1+c-\delta}{2})$; *customers' purchase decisions are* $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{1}{2}, \frac{1}{2}, \frac{\delta+c}{2\delta})$; *the reward redemption rate is* $r_T^* = 1 - \frac{c}{\delta}$; *the firm's profit is* $\pi_T^* = \frac{(c-\delta)^2+\delta}{4\delta}$. □

B. Analysis and Technical Results for Section 5.2

This appendix provides the technical details and supplemental results for Section 5.2 (New Customer Arrival with Heterogeneous Valuations). The related proofs are presented in Appendix E.4.

B.1. Traditional LP

We begin by analyzing the equilibrium for the traditional LP. For customers arriving in the first period, their purchase behavior follows Lemma A1. For the new cohort arriving in the second period, the marginal consumer is given by $\zeta_L(p) := \min\{p, \theta\}$. Anticipating these responses, the firm maximizes its total profit.

$$\max_{p,e} \pi_L = p(1-\tau_{L1}(p,e)) + (p-e)(1-\tau_{L2}(p,e)) + p(\theta - \zeta_L(p)). \tag{B1}$$

Solving this problem yields the following equilibrium results (also shown in Figure A-2(a)).

Lemma B5. *Under the traditional LP, there exist thresholds $\hat{\theta}_{L1}$ and $\hat{\theta}_{L2}$ such that:*

(i) When $0 \le \theta \le \hat{\theta}_{L1}$, the firm's optimal decisions are $(p_L^*, e_L^*) = (\frac{1}{2}, \frac{c}{2})$; customers' purchase decisions are $(\tau_{L1}^*, \tau_{L2}^*, \zeta_L^*) = (\frac{1}{2}, \frac{1+c}{2}, \theta)$; the reward redemption rate is $r_L^* = 1-c$; the firm's profit is $\pi_L^* = \frac{c^2-2c+2}{4}$.

(ii) When $\hat{\theta}_{L1} \le \theta \le \hat{\theta}_{L2}$, the firm's optimal decisions are $(p_L^*, e_L^*) = (\frac{2+\theta-c}{6}, 0)$; customers' purchase decisions are $(\tau_{L1}^*, \tau_{L2}^*, \zeta_L^*) = (\frac{2+\theta-c}{6}, \frac{2+\theta+5c}{6}, \frac{2+\theta-c}{6})$; the reward redemption rate is $r_L^* = \frac{4-5c-\theta}{4+c-\theta}$; the firm's profit is $\pi_L^* = \frac{(2+\theta-c)^2}{12}$.

(iii) When $\max\{\hat{\theta}_{L1}, \hat{\theta}_{L2}\} \le \theta \le 1$, the firm's optimal decisions are $(p_L^*, e_L^*) = (\frac{1+\theta}{4}, \frac{\theta+2c-1}{4})$; customers' purchase decisions are $(\tau_{L1}^*, \tau_{L2}^*, \zeta_L^*) = (\frac{1+\theta}{4}, \frac{1+c}{2}, \frac{1+\theta}{4})$; the reward redemption rate is $r_L^* = \frac{2(1-c)}{3-\theta}$; the firm's profit is $\pi_L^* = \frac{\theta^2+2\theta+2c^2-4c+3}{8}$.

B.2. Tradable LP

Next, we analyze the equilibrium under the tradable LP. For customers arriving in the second period, a purchase occurs if and only if $v - p + e - e_M \ge 0$. Thus, the marginal consumer is defined as $\zeta_T(p) := \min\{p + e - e_M, \theta\}$. Lemma B6 summarizes customers' purchase behaviors.

Lemma B6. *Under the tradable LP, the first-period demand threshold is $\tau_{T1} = p$. The second-period thresholds (τ_{T2}, ζ_T) are given as follows.*

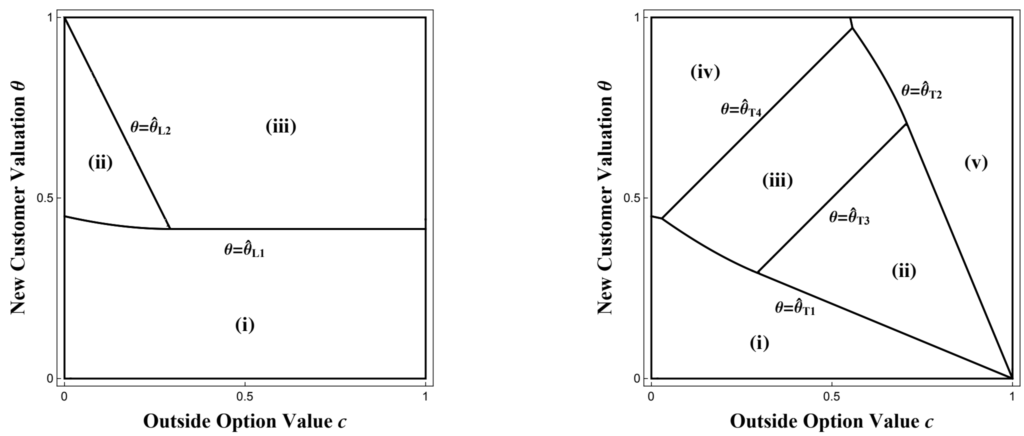
$$(\tau_{T2}, \zeta_T) = \begin{cases} (p+c, p) & \text{if } \ p \le \theta - c, \\ (1, \theta) & \text{if } \ p \le 1 \cap e \le \min\{p+c-1, p-\theta\}, \\ (p-e+c, \theta) & \text{if } \ p+c-1 \le e \le \min\{p-\theta, c\}, \\ (p-e+c, p-e) & \text{if } \ \max\{p+c-1, p-\theta\} \le e \le \frac{p-\theta+c}{2}, \\ (\frac{p+\theta+c}{2}, \frac{p+\theta-c}{2}) & \text{if } \ \theta - c \le p \le \min\{\theta+c, 2-\theta-c\} \cap e \ge \frac{p-\theta+c}{2}, \\ (1, p-e) & \text{if } \ p-\theta \le e \le \min\{p+c-1, 1-\theta\}, \\ (1, p+\theta-1) & \text{if } \ 2-\theta-c \le p \le 1 \cap e \ge 1-\theta, \\ (p, \theta) & \text{if } \ \theta + c \le p \le 1 \cap e \ge c. \end{cases}$$

Anticipating these behaviors, the firm solves the following maximization problem:

$$\max \pi_T = p(1-\tau_{T1}) + p(1-\tau_{T2}+\theta-\zeta_T) - e\min\{1-\tau_{T1}, 1-\tau_{T2}+\theta-\zeta_T\}. \tag{B2}$$

The equilibrium results are characterized as follows (also shown in Figure A-2(b)).

Figure A-2 Equilibrium Results with Heterogeneous New Customer Arrival



(a) Traditional LP (b) Tradable LP

LEMMA B7. *Under the tradable LP, there exist thresholds $\hat{\theta}_{T1}$, $\hat{\theta}_{T2}$, $\hat{\theta}_{T3}$, and $\hat{\theta}_{T4}$ such that:*

(i) When $0 \le \theta \le \hat{\theta}_{T1}$, the firm's optimal decisions are $(p_T^*, e_T^*) = (\frac{1}{2}, \frac{c}{2})$; customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^*, \zeta_T^*) = (\frac{1}{2}, \frac{1+c}{2}, \theta)$; the market token price is $e_M^* = 0$; the reward redemption rate is $r_T^* = 1-c$; the firm's profit is $\pi_T^* = \frac{c^2-2c+2}{4}$.

(ii) When $\hat{\theta}_{T1} \le \theta \le \min\{\hat{\theta}_{T2}, \hat{\theta}_{T3}\}$, the firm's optimal decisions are $(p_T^*, e_T^*) = (\frac{1}{2}, \frac{1-\theta+c}{4})$; customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^*, \zeta_T^*) = (\frac{1}{2}, \frac{1+\theta+3c}{4}, \frac{1+\theta-c}{4})$; the market token price is $e_M^* = 0$; the reward redemption rate is $r_T^* = 1+\theta-c$; the firm's profit is $\pi_T^* = \frac{c^2-2c+3+\theta^2-2c\theta+2\theta}{8}$.

(iii) When $\max\{\hat{\theta}_{T1}, \hat{\theta}_{T3}\} \le \theta \le \min\{\hat{\theta}_{T2}, \hat{\theta}_{T4}\}$, the firm's optimal decisions are $(p_T^*, e_T^*) = (\frac{3-\theta+c}{6}, \frac{3-7\theta+7c}{12})$; customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^*, \zeta_T^*) = (\frac{3-\theta+c}{6}, \frac{3+5\theta+7c}{12}, \frac{3+5\theta-5c}{12})$; the market token price is $e_M^* = 0$; the reward redemption rate is $r_T^* = 1$; the firm's profit is $\pi_T^* = \frac{(3+\theta-c)^2}{24}$.

(iv) When $\max\{\hat{\theta}_{T1}, \hat{\theta}_{T4}\} \le \theta \le \hat{\theta}_{T2}$, the firm's optimal decisions are $(p_T^*, e_T^*) = (\frac{2+\theta-c}{6}, 0)$; customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^*, \zeta_T^*) = (\frac{2+\theta-c}{6}, \frac{2+\theta+5c}{6}, \frac{2+\theta-c}{6})$; the market token price is $e_M^* = 0$; the reward redemption rate is $r_T^* = 1$; the firm's profit is $\pi_T^* = \frac{(2+\theta-c)^2}{12}$.

(v) When $\hat{\theta}_{T2} \le \theta \le 1$, the firm's optimal decisions are $(p_T^*, e_T^*) = (\frac{1}{2}, \frac{1-\theta}{2})$; customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^*, \zeta_T^*) = (\frac{1}{2}, 1, \frac{\theta}{2})$; the market token price is $e_M^* = 0$; the reward redemption rate is $r_T^* = \theta$; the firm's profit is $\pi_T^* = \frac{1+\theta^2}{4}$.

C. Analysis and Technical Results for Section 5.3

This appendix presents the analysis and supplemental results for Section 5.3 (Speculation with Market Uncertainty). The related proofs are presented in Appendix E.5.

C.1. Traditional LP

We first analyze the equilibrium results under traditional LPs. For customers arriving in the first period, their purchase behavior is characterized by Lemma A1. In the second period, new customers arrive with probability β. They purchase if and only if $v - p \geq 0$, i.e., $v \geq \zeta_L(p) := \min\{p, 1\}$. Anticipating customers' purchase behavior, the firm chooses p and e in the first period to maximize the long-term profit.

$$\max_{p,e} \pi_L = p(1-\tau_{L1}(p,e)) + (p-e)(1-\tau_{L2}(p,e)) + \beta p(1-\zeta_L).$$

The equilibrium results are summarized as follows.

Lemma C8. *Under the traditional LP, the firm's optimal decisions are* $(p_L^*, e_L^*) = (\frac{1}{2}, \frac{1-\delta}{2})$*; customers' purchase decisions are* $(\tau_{L1}^*, \tau_{L2}^*, \zeta_L^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$*, and the firm's expected profit is* $\pi_L^* = \frac{1+\delta+\beta}{4}$.

C.2. Tradable LP

Given the complexity of the analytical solution under the tradable LP, we provide a proof sketch that characterizes the equilibrium conditions. These conditions form the basis for the numerical analysis presented in the main text.

We first establish that all customers who purchase in the first period will sell their loyalty tokens to speculators. If a customer sells the token in the speculation stage at price e_F, their expected future payoff is $e_F + E[\max\{\delta v - p + e - e_M, 0\}]$. In a competitive speculation market, the token price equals the expected future price, i.e., $e_F = E[e_M]$. Thus, the expected future payoff from selling can be written as: $U_{sell} = E[e_M] + E[\max\{\delta v - p + e - e_M, 0\}] = E[\max\{\delta v - p + e, e_M\}]$. In contrast, if the customer holds the token, their expected payoff is discounted by α: $U_{hold} = \alpha E[\max\{\delta v - p + e, e_M\}]$. Since $\alpha \leq 1$, selling tokens to speculators dominates (or is equivalent to) holding them. Thus, all first-period buyers enter the second period without tokens but with cash e_F.

Next, we analyze customers' utility in the second period. For customers who did not purchase before, their second-period utility is $U_{T2}^F = \max\{v - p + e - e_M, 0\}$. They purchase if and only if $v \geq \hat{\tau}_{T2}^L(e_M) := p - e + e_M$. For customers who purchased in the first period, their second-period utility is $U_{T2}^S = \max\{\delta v - p + e - e_M, 0\}$. They purchase if and only if $v \geq \hat{\tau}_{T2}^H(e_M) := \frac{p-e+e_M}{\delta}$. The token price e_M in the second period is stochastic, depending on the arrival of new customers. Let e_M^H denote the token price when new customers arrive (probability β) and e_M^L when they do not (probability $1-\beta$).

Then, we analyze customers' purchase decision in the first period. A customer purchases in the first period if the long-term utility of buying exceeds that of waiting:

$$v - p + e_F + \alpha E[U_{T2}^S] \geq \alpha E[U_{T2}^F].$$

Substituting $e_F = E[e_M]$, the above inequality is equivalent to $v - p + E[g(e_M, v)] \geq 0$, where $g(e_M, v) = e_M + \alpha(U_{T2}^S - U_{T2}^F)$. The function $g(e_M, v)$ can be expanded as:

$$g(e_M, v) = \begin{cases} e_M - \alpha(1-\delta)v & \text{if } e_M \leq \delta v - p + e, \\ (1+\alpha)e_M - \alpha(v - p + e) & \text{if } \delta v - p + e < e_M \leq v - p + e, \\ e_M & \text{if } e_M > v - p + e. \end{cases}$$

We can prove that there always exists a threshold $\hat{\tau}_{T1}$ such that customers choose to purchase in the first period if and only if $v \geq \hat{\tau}_{T1}$. Define $f(v) = v - p + E[g(e_M, v)]$. We analyze the monotonicity of $f(v)$ by considering the relative positions of v to the thresholds $\hat{\tau}_{T2}^L(e_M) = p - e + e_M$ and $\hat{\tau}_{T2}^H(e_M) = \frac{p-e+e_M}{\delta}$. Since e_M takes values in $\{e_M^L, e_M^H\}$, we identify five intervals for v:

(1) If $v \leq \hat{\tau}_{T2}^L(e_M^L)$, then $f(v) = v - p + E[e_M]$. The derivative is $f'(v) = 1 > 0$. Solving $f(v) = 0$ yields the threshold $\hat{\tau}_{T1} = p - E[e_M]$.

(2) If $\hat{\tau}_{T2}^L(e_M^L) < v \leq \min\{\hat{\tau}_{T2}^L(e_M^H), \hat{\tau}_{T2}^H(e_M^L)\}$, then $f(v) = (1-\alpha(1-\beta))(v-p) - \alpha(1-\beta)e + \beta e_M^H + (1+\alpha)(1-\beta)e_M^L$ and $f'(v) = 1 - \alpha(1-\beta) \geq 0$. Solving $f(v) = 0$ yields the threshold $\hat{\tau}_{T1} = p + \frac{\alpha(1-\beta)e - \beta e_M^H - (1+\alpha)(1-\beta)e_M^L}{1-\alpha(1-\beta)}$.

(3) If $\hat{\tau}_{T2}^L(e_M^H) < v \leq \hat{\tau}_{T2}^H(e_M^L)$, then $f(v) = (1-\alpha)v - (1-\alpha)p - \alpha e + (1+\alpha)E[e_M]$ and $f'(v) = 1 - \alpha \geq 0$. Solving $f(v) = 0$ yields the threshold $\hat{\tau}_{T1} = p + \frac{\alpha e - (1+\alpha)E[e_M]}{1-\alpha}$.

(4) If $\max\{\hat{\tau}_{T2}^L(e_M^H), \hat{\tau}_{T2}^H(e_M^L)\} < v \leq \hat{\tau}_{T2}^H(e_M^H)$, then $f(v) = (1-\alpha(1-(1-\beta)\delta))v - (1-\alpha\beta)p - \alpha\beta e + (1+\alpha)\beta e_M^H + (1-\beta)e_M^L$ and $f'(v) = 1 - \alpha(1-(1-\beta)\delta) \geq 0$. Solving $f(v) = 0$ yields the threshold $\hat{\tau}_{T1} = \frac{(1-\alpha\beta)p + \alpha\beta e - (1+\alpha)\beta e_M^H - (1-\beta)e_M^L}{1-\alpha(1-(1-\beta)\delta)}$.

(5) If $v > \hat{\tau}_{T2}^H(e_M^H)$, then $f(v) = (1-\alpha(1-\delta))v - p + E[e_M]$ and $f'(v) = 1 - \alpha(1-\delta) \geq 0$. Solving $f(v) = 0$ yields the threshold $\hat{\tau}_{T1} = \frac{p - E[e_M]}{1-\alpha(1-\delta)}$.

Therefore, given the firm's decisions (p, e), since $f(v)$ is monotonically increasing in all cases, there exists a unique threshold $\hat{\tau}_{T1}$ such that customers purchase in the first period if and only if $v \geq \hat{\tau}_{T1}$, where $\hat{\tau}_{T1}$ is the solution to $f(v) = 0$ (bounded within $[0, 1]$).

In summary, customers' purchase behavior is fully characterized by the purchase thresholds $(\hat{\tau}_{T1}, \hat{\tau}_{T2}^L, \hat{\tau}_{T2}^H)$. These thresholds determine the market-clearing prices e_M^H and e_M^L according to the supply and demand conditions in the second period. Given the piecewise structure of $f(v)$ and the complex interdependence between customer decisions and token prices, we employ numerical methods to solve for the equilibrium and compare profits. To provide a comparative benchmark, we also conduct numerical experiments for the scenario characterized by market uncertainty but without speculation. As the theoretical analysis is analogous to the cases derived above, we omit the details here.

D. Analysis and Technical Results for Section 5.4

This appendix presents the analysis and supplemental results for Section 5.4 (Dynamic Pricing). The related proofs are presented in Appendix E.6.

D.1. Traditional LP

Under a traditional loyalty program, we can show that, given the firm's first-period decisions (p_1, e), customers adopt a threshold purchasing policy in the first period. That is, they purchase in the first period *if and only if* v is greater than a threshold τ_{L1}.

Moving to the second period, the firm observes customers' first-period purchase decisions and sets the second-period price p_2. In response, customers who have not purchased in the first period make their initial purchases if their utility $U^F_{L2} = v - p_2$ is positive, that is, $v \geq p_2$. Meanwhile, customers who have purchased in the first period will repurchase if the utility $U^S_{L2} = \delta v - p_2 + e$ is positive, that is, $v \geq \frac{p_2 - e}{\delta}$. Let $\tau^L_{L2} = \min\{p_2, \tau_{L1}\}$ and $\tau^H_{L2} = \min\{\max\{\frac{p_2-e}{\delta}, \tau_{L1}\}, 1\}$. Under this notation, customers with valuation $v \in [\tau^L_{L2},\, \tau_{L1}]$ make initial purchases in the second period, and those with $v \in [\tau^H_{L2},\, 1]$ will make repeated purchases in the second period.

Anticipating customer behavior, the firm sets the price p_2 to maximize its second-period profit.

$$\max_{p_2}\ \pi_{L2}(p_2) = p_2(\tau_{L1} - \tau^L_{L2}) + (p_2 - e)(1 - \tau^H_{L2}), \tag{D3}$$

where the first term represents the profit from initial purchases in the second period, and the second term accounts for the profit from repeat purchases, considering the loyalty reward discount. Solving this optimization problem yields the optimal second-period price $p_2^*(\tau_{L1}, e)$ and the corresponding customers purchase thresholds in the second period $(\tau^{L^*}_{L1}(p_2^*, \tau_{L1}, e),\, \tau^{H^*}_{L2}(p_2^*, \tau_{L1}, e))$.

In the first period, customers choose to purchase only if $v \geq \tau_{L1}(p_1, e)$. Anticipating that, the firm decides (p_1, e) to maximize the total profit over both periods:

$$\max_{p_1, e}\ \pi_L(p_1, e) = p_1(1 - \tau_{L1}) + p_2^*(\tau_{L1} - \tau^{L^*}_{T2}) + (p_2^* - e)(1 - \tau^{H^*}_{L2}). \tag{D4}$$

The equilibrium results are described as follows.

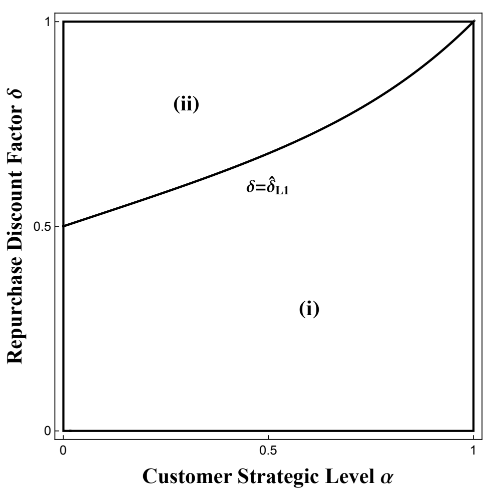
Lemma D9. *Under the traditional LP with dynamic pricing, there exists a threshold $\hat{\delta}_{L1}(\alpha)$ such that the equilibrium results are as follows.*

(i) When $0 \leq \delta \leq \hat{\delta}_{L1}$, the firm's unique optimal decisions are $(e^*_L, p^*_{L1}, p^*_{L2}) = (\frac{1-\alpha^2+(\alpha(2+\alpha)-1)\delta-2\delta^2}{3+4\delta-\alpha(2+\alpha)}, \frac{2(1+\delta)-\alpha(2+\delta(1-\alpha))}{3+4\delta-\alpha(2+\alpha)}, \frac{\delta p^*_{L1}+(1-\alpha)e^*_L}{1-\alpha(1-\delta)})$; customers' purchase decisions are $(\tau^*_{L1}, \tau^{L^*}_{L2}, \tau^{H^*}_{L2}) = (\frac{p^*_{L1}-\alpha p^*_{L2}}{1-\alpha}, p^*_{L2}, \frac{p^*_{L1}-\alpha p^*_{L2}}{1-\alpha})$; the firm's profit is $\pi^*_L = \frac{(1+\delta)(1-\alpha+\delta)}{3+4\delta-\alpha(2+\alpha)}$.

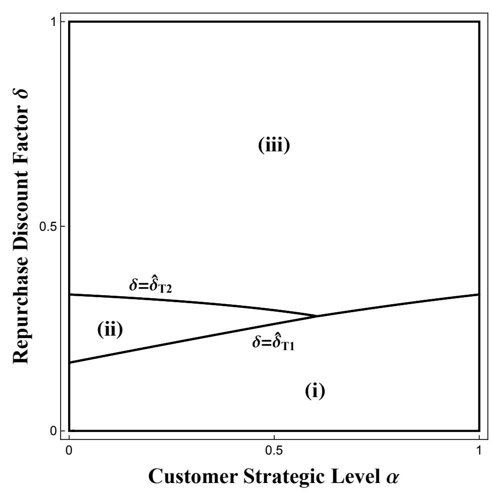
(ii) When $\hat{\delta}_{L1} < \delta \leq 1$, the firm's unique optimal decisions are $(e^*_L, p^*_{L1}, p^*_{L2}) = (0, \frac{(1-\alpha(1-\delta))(1+\delta-\alpha(1-\delta))}{2(1+\delta^2-\alpha(1-\delta))}, \frac{\delta p^*_{L1}}{1-\alpha(1-\delta)})$; customers' purchase decisions are $(\tau^*_{L1}, \tau^{L^*}_{L2}, \tau^{H^*}_{L2}) = (\frac{p^*_{L1}-\alpha p^*_{L2}}{1-\alpha}, p^*_{L2}, \frac{p^*_{L1}-\alpha p^*_{L2}}{1-\alpha})$; the firm's profit is $\pi^*_L = \frac{(1+\delta-\alpha(1-\delta))^2}{4(1+\delta^2-\alpha(1-\delta))}$.

Lemma D9 reveals that the firm adopts a markdown pricing strategy ($p^*_{L2} \leq p^*_{L1}$). This strategy effectively implements intertemporal price discrimination, inducing some customers who don't purchase in the first period to make purchases in the second period. In addition, the effectiveness of traditional LPs in this setting hinges upon the magnitude of δ. When δ is low (Case (i)), even under dynamic pricing, the firm continues to find a traditional LP valuable. This aligns with the purchase-history-based price discrimination role of traditional LPs, which remains valuable when customers heavily discount repeated purchases. In this case, dynamic pricing and traditional loyalty programs target different customer segments, enabling more effective price discrimination. However, in the case of high δ (Case (ii)), the traditional LP becomes redundant ($e^*_{L1} = 0$) in the presence of dynamic pricing. Instead, the firm simply lowers the second-period price, which effectively incentivizes first-time and repeated purchases in the second period.

Figure A-3 Equilibrium Results with Dynamic Pricing



(a) Traditional LP



(b) Tradable LP

D.2. Tradable LP

The analysis for the tradable LP under dynamic pricing parallels that of the traditional LP. We first establish that customers adopt a threshold purchasing policy. In the first period, a customer with valuation v purchases if $U^P_{T1} \geq U^N_{T1}$. The utility difference is $U^P_{T1} - U^N_{T1} = v - p_1 + \alpha \max\{\delta v - p_2 + e, e_M\} - \alpha \max\{v - p_2 + e - e_M, 0\}$. It can be shown that $U^P_{T1} - U^N_{T1}$ is strictly increasing in v. Therefore, there exists a threshold τ_{T1} such that customers purchase in the first period if and only if $v \geq \tau_{T1}$.

We now solve the problem backward starting from the second period. For customers who did not purchase before, they choose to purchase if and only if $U^F_{T2} = v - p_2 + e - e_M \geq 0$, i.e., $v \geq$

$\hat{\tau}^L_{T2} := p_2 - e + e_M$. For customers who purchased before, they choose to repurchase if and only if $U^S_{T2} = \delta v - p_2 + e \geq e_M$. Otherwise, they choose to sell their tokens. The threshold is $\hat{\tau}^H_{T2} := \frac{p_2 - e + e_M}{\delta}$. Let $\tau^L_{T2} = \max\{\min\{\hat{\tau}^L_{T2}, \tau_{T1}\}, 0\}$ and $\tau^H_{T2} = \min\{\max\{\hat{\tau}^H_{T2}, \tau_{T1}\}, 1\}$. The token market price e_M is determined by the market clearing condition $S_t(e_M) = D_t(e_M)$, where the supply S_t comes from customers with $v \in [\tau_{T1}, \tau^H_{T2}]$ and the demand D_t comes from customers with $v \in [\tau^L_{T2}, \tau_{T1}]$. Given customers' purchase behavior, the firm sets the price p_2 to maximize its second-period profit.

$$\max_{p_2} \ \pi_{T2}(p_2) = p_2(\tau_{T1} - \tau^L_{T2} + 1 - \tau^H_{T2}) - e\min\{\tau_{T1} - \tau^L_{T2} + 1 - \tau^H_{T2}, 1 - \tau_{T1}\}, \tag{D5}$$

where the first term represents the revenue from both initial and repeat purchases and the second term represents the total costs associated with reward token redemption. Solving the optimization problem leads to the optimal price $p_2^*(\tau_{T1}, e)$ and the customers' decision $(\tau^{L^*}_{T2}(p_2^*, \tau_{T1}, e), \tau^{H^*}_{T2}(p_2^*, \tau_{T1}, e))$.

In the first period, the firm sets the first-period price and the reward value (p_1, e) to maximize the total profit across both periods.

$$\max_{p_1, e} \ \pi_T = p_1(1 - \tau_{T1}) + p_2^*(\tau_{T1} - \tau^{L^*}_{T2} + 1 - \tau^{H^*}_{T2}) - e\min\{\tau_{T1} - \tau^{L^*}_{T2} + 1 - \tau^{H^*}_{T2}, 1 - \tau_{T1}\}. \tag{D6}$$

The equilibrium results are described as follows s (also shown in Figure A-3(b)).

LEMMA D10. *Under the tradable LP with dynamic pricing, there exist thresholds $\hat{\delta}_{T1}(\alpha)$, $\hat{\delta}_{T2}(\alpha)$ and $\hat{e}_{T1}(\delta, \alpha)$ such that the equilibrium results are as follows.*

(i) When $\delta \in [0, \hat{\delta}_{T1}]$, the firm's decisions are optimal if and only if $e^*_{T1} \geq \hat{e}_{T1}(\delta, \alpha)$, and $(p^*_{T1}, p^*_{T2} - e^*_{T1}) = (\frac{2}{3+\alpha}, \frac{1+\alpha}{3+\alpha})$; customers' purchase decisions are $(\tau^*_{T1}, \tau^{L^*}_{T2}, \tau^{H^*}_{T2}) = (\frac{2+\alpha}{3+\alpha}, \frac{1+\alpha}{3+\alpha}, 1)$; the firm's profit is $\pi^*_T = \frac{1}{3+\alpha}$.

(ii) When $\delta \in (\hat{\delta}_{T1}, \hat{\delta}_{T2}]$, the firm's unique optimal decisions are $(e^*_{T1}, p^*_{T1}, p^*_{T2}) = (0, \frac{\alpha^2(2+\delta) - 2\alpha(1+\delta)(2+\delta) + (1+\delta)(2+3\delta)}{(1+\delta)(4+3\delta-2\alpha(2+\delta))}, \frac{\delta(p^*_{T1}+1-\alpha)}{2(1+\delta)-\alpha(2+\delta)})$; customers' purchase decisions are $(\tau^*_{T1}, \tau^{L^*}_{T2}, \tau^{H^*}_{T2}) = (\frac{p^*_{T1} - \alpha p^*_{T2}}{1-\alpha}, p^*_{T2}, \frac{p^*_{T2}}{\delta})$; the firm's profit is $\pi^*_T = \frac{\alpha^2 - 2\alpha(1+\delta)^2 + (1+\delta)(1+3\delta)}{(1+\delta)(4+3\delta-2\alpha(2+\delta))}$.

(iii) When $\delta \in (\hat{\delta}_{T2}, 1]$, the firm's unique optimal decisions are $(e^*_{T1}, p^*_{T1}, p^*_{T2}) = (0, \frac{(1-\alpha(1-\delta))(1+\delta-\alpha(1-\delta))}{2(1+\delta^2-\alpha(1-\delta))}, \frac{\delta p^*_{T1}}{1-\alpha(1-\delta)})$; customers' purchase decisions are $(\tau^*_{T1}, \tau^{L^*}_{T2}, \tau^{H^*}_{T2}) = (\frac{p^*_{T1} - \alpha p^*_{T2}}{1-\alpha}, p^*_{T2}, \tau^*_{T1})$; the firm's profit is $\pi^*_T = \frac{(1+\delta-\alpha(1-\delta))^2}{4(1+\delta^2-\alpha(1-\delta))}$.

When δ is low (Case (i)), the tradable LP leads to the same firm profit and customer purchase behavior under dynamic pricing as it does under static pricing. In this case, dynamic pricing adds no additional value because the tradable LP alone effectively manages intertemporal discrimination. However, compared to dynamic pricing, tradable LPs can mitigate strategic purchase delays by rewarding early buyers with tokens that can be sold in the secondary market. Conversely, when δ is high (Cases (ii) and (iii)), the tradable LP loses its value ($e^*_T = 0$) under dynamic pricing. This mirrors the logic behind Lemma D9 (Case (ii)), where dynamic pricing alone is sufficient to induce purchases in the second period.

D.3. Dynamic Reward Value

In this subsection, we examine a setting where the firm cannot commit to the reward value in advance. The timeline proceeds as follows. In the first period, the firm announces a product price p and an initial reward face value e_1. Note that the announcement of e_1 is non-binding, as the firm retains the flexibility to adjust the redemption value to e_2 in the second period. Consequently, rational customers base their first-period purchase decisions on their expectations of e_2 rather than the announced e_1. Finally, in the second period, the firm observes the realized first-period sales and chooses the actual redemption value e_2 to maximize its ex-post profit.

Let e_1^* denote the optimal reward value derived in the base model under full commitment (see Propositions 2 and 3). Let e_2^* denote the firm's optimal ex-post reward value chosen in the second period. We can obtain the following results.

Lemma D11. *For both traditional and tradable LPs, the firm's optimal ex-post reward value e_2^* in the second period is identical to the optimal committed value e_1^* derived in the base model (i.e., $e_2^* = e_1^*$). Consequently, the optimal firm profits under dynamic reward values are identical to those of the base model with commitment.*

E. Proofs

E.1. Proofs of Results in Section 4.

Proof of Proposition 1. Based on the demand function in (3), the firm's profit function can be formulated as

$$\pi_N(p) = \begin{cases} p(2-p-\frac{p}{\delta}) & \text{if } 0 \le p \le \delta, \\ p(1-p) & \text{if } \delta \le p \le 1. \end{cases}$$

We first respectively analyze the optimal price in the two feasible regions.

(1) If $0 \le p \le \delta$, the firm's profit function is concave. The first-order condition yields the local optimal price $p_1^* = \frac{\delta}{1+\delta}$, with corresponding profit $\pi_1^* = \frac{\delta}{\delta+1}$. Note that $p_1^* \le \delta$ always holds.

(2) If $\delta < p \le 1$, the firm's profit function is concave. By solving the profit maximization problem with the constraint, the local optimal price and corresponding firm profit are

$$(p_2^*, \pi_2^*) = \begin{cases} (\frac{1}{2}, \frac{1}{4}) & \text{if } 0 \le \delta \le \frac{1}{2}, \\ (\delta, \delta(1-\delta)) & \text{if } \frac{1}{2} \le \delta \le 1. \end{cases}$$

By comparing the local maximum profits π_1^* and π_2^*, we obtain the following equilibrium outcomes.

$$(p_N^*, \pi_N^*, \tau_{N1}^*, \tau_{N2}^*) = \begin{cases} \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 1\right) & \text{if } 0 \le \delta \le \frac{1}{3}, \\ \left(\frac{\delta}{\delta+1}, \frac{\delta}{\delta+1}, \frac{\delta}{\delta+1}, \frac{1}{\delta+1}\right) & \text{if } \frac{1}{3} < \delta \le 1; \end{cases}$$

as desired. □

Proof of Lemma 1. As discussed in Section 4.2.1, a customer purchases in the first period if $U_{L1}^P \ge U_{L1}^N$, where $U_{L1}^P = v - p + \alpha \max\{\delta v - p + e, 0\}$ and $U_{L1}^N = \alpha \max\{v-p, 0\}$.

We first derive the first-period threshold $\hat{\tau}_{L1}$, implying that customers with $v \ge \hat{\tau}_{L1}$ make purchases in the first period. We discuss the following two cases: $e \le (1-\delta)p$ and $e > (1-\delta)p$.

(1) If $e \le (1-\delta)p$, for $v \le p$, we have $U_{L1}^P = v - p \le 0 \le U_{L1}^N$; for $v \ge p$, we have $U_{L1}^P = v - p + \alpha \max\{\delta v - p + e, 0\} \ge v - p \ge U_{L1}^N$. Thus, the threshold is $\hat{\tau}_{L1} = p$.

(2) If $e > (1-\delta)p$, for $v \le \frac{p-e}{\delta}$, we have $U_{L1}^P = v - p \le 0 \le U_{L1}^N$; for $\frac{p-e}{\delta} \le v \le p$, we have $U_{L1}^P = v - p + \alpha(\delta v - p + e)$ and $U_{L1}^N = 0$, implying that $U_{L1}^P \ge U_{L1}^N$ if $v \ge \frac{(1+\alpha)p - \alpha e}{1+\alpha\delta}$; for $v \ge p$, we have $U_{L1}^P = v - p + \alpha \max\{\delta v - p + e, 0\} \ge v - p \ge U_{L1}^N$. Thus, the threshold is $\hat{\tau}_{L1} = \frac{(1+\alpha)p - \alpha e}{1+\alpha\delta}$.

Combining the above cases, the first-period threshold is

$$\hat{\tau}_{L1} = \begin{cases} p & \text{if } e \le (1-\delta)p, \\ \frac{(1+\alpha)p - \alpha e}{1+\alpha\delta} & \text{if } e > (1-\delta)p. \end{cases}$$

We next derive the second-period threshold $\hat{\tau}_{L2}$, implying that customers with $v \ge \hat{\tau}_{L2}$ make purchases in the second period. Customers purchase in the second period only if they purchased in the first period ($v \ge \tau_{L1}$) and the second-period utility is non-negative ($U_{T2} \ge 0, i.e., v \ge \frac{p-e}{\delta}$). Thus, the threshold is $\hat{\tau}_{L2} = \max\{\hat{\tau}_{L1}, \frac{p-e}{\delta}\}$.

Considering the valuation constraint $v \in [0,1]$, the final thresholds are $\tau_{Li} = \min\{\max\{\hat{\tau}_{Li}, 0\}, 1\}$, leading to the following complete characterization.

- If $0 \le p < \frac{\alpha e}{1+\alpha}$, $(\tau_{L1}, \tau_{L2}) = (0,0)$.
- If $\frac{\alpha e}{1+\alpha} \le p \le \max\{1, \frac{1+\alpha(e+\delta)}{1+\alpha}\}$,

$$(\tau_{L1}, \tau_{L2}) = \begin{cases} (p, 1) & \text{if } \ e \le p - \delta, \\ (p, \frac{p-e}{\delta}) & \text{if } \ p - \delta \le e \le (1-\delta)p, \\ (\frac{(1+\alpha)p - \alpha e}{1+\alpha\delta}, \frac{(1+\alpha)p - \alpha e}{1+\alpha\delta}) & \text{if } \ e \ge (1-\delta)p. \end{cases}$$

- If $p > \max\{1, \frac{1+\alpha(e+\delta)}{1+\alpha}\}$, $(\tau_{L1}, \tau_{L2}) = (1,1)$. $\square$

Proof of Proposition 2. By equation (7), the firm's profit function is $\pi_L = p(1-\tau_{L1}) + (p - e)(1-\tau_{L2})$, where τ_{L1} and τ_{L2} are given in the proof of Lemma 1. We analyze the maximization problem across the following three feasible regions for p.

(1) If $0 \le p \le \frac{\alpha e}{1+\alpha}$, then $(\tau_{L1}, \tau_{L2}) = (0,0)$ and $\pi_L = 2p - e$. The profit strictly increases in p. Thus, the optimal solution must lie on the upper boundary, merging into Case (3).

(2) If $p \ge \max\{1, \frac{1+\alpha(e+\delta)}{1+\alpha}\}$, then $(\tau_{L1}, \tau_{L2}) = (1,1)$ and $\pi_L = 0$, which is suboptimal.

(3) If $\frac{\alpha e}{1+\alpha} \le p \le \max\{1, \frac{1+\alpha(e+\delta)}{1+\alpha}\}$, we examine the following three subcases.

(3.1) If $e \le p - \delta$, then $(\tau_{L1}, \tau_{L2}) = (p, 1)$. The profit is $\pi_L = p(1-p)$, yielding a maximum profit of $\frac{1}{4}$ at $p = \frac{1}{2}$.

(3.2) If $p - \delta \le e \le (1-\delta)p$, then $(\tau_{L1}, \tau_{L2}) = (p, \frac{p-e}{\delta})$. The profit function is $\pi_L(p,e) = -p^2 + 2p - e - \frac{(p-e)^2}{\delta}$. The Hessian matrix is $H = \begin{bmatrix} -\frac{2(1+\delta)}{\delta} & \frac{2}{\delta} \\ \frac{2}{\delta} & -\frac{2}{\delta} \end{bmatrix}$. Since $H_{11} = -\frac{2(1+\delta)}{\delta} \le 0$ and the determinant is $\det(H) = \frac{4}{\delta} \ge 0$, H is negative definite and π_L is jointly concave. By solving the first-order conditions $\frac{\partial \pi_L}{\partial p} = 0$ and $\frac{\partial \pi_L}{\partial e} = 0$, we have $p^* = \frac{1}{2}$ and $e^* = \frac{1-\delta}{2}$, satisfying the region constraints. The corresponding profit is $\pi_L^* = \frac{1+\delta}{4}$.

(3.3) If $e \ge (1-\delta)p$, then $\tau_{L1} = \tau_{L2} = \frac{(1+\alpha)p - \alpha e}{1+\alpha\delta}$ and $\pi_L = \frac{(2p-e)(1+\alpha\delta+\alpha e-(1+\alpha)p)}{1+\alpha\delta}$. Since $\frac{\partial^2 \pi_L}{\partial p^2} = \frac{-4(1+\alpha)}{1+\alpha\delta} \le 0$, we have $p^*(e) = \frac{2(1+\alpha\delta)+(1+3\alpha)e}{4(1+\alpha)}$ by the first order condition. Substituting $p^*(e)$ back into the profit function, it can be shown that $\pi_L(p^*(e), e)$ decreases in e. Thus, the optimal e lies on the lower boundary $e = (1-\delta)p$, which reduces to Case (3.2).

Comparing the local optima, the global optimal solution is derived in Case (3.2). The equilibrium results are $(p^*, e^*) = (\frac{1}{2}, \frac{1-\delta}{2})$, $(\tau_{L1}^*, \tau_{L2}^*) = (\frac{1}{2}, \frac{1}{2})$, and $\pi_L^* = \frac{1+\delta}{4}$. $\square$

Proof of Lemma 2. We first establish the bounds for the equilibrium token price, $e_M \in [0, e]$. If $e_M < 0$, token demand would be infinite while supply is zero, driving the price up. Conversely, if $e_M > e$, the price exceeds the face value, reducing demand to zero while supply remains positive, driving the price down. Thus, $e_M \in [0, e]$ must hold.

Next, we derive the token demand and supply in the secondary market based on the utility analysis in Section 4.3.1. In the second period, for first-time buyers, their utility is $U_{T2}^F = \max\{0, v -$

$p, v - p + e - e_M\}$, and thus they choose to buy if and only if $v \geq \hat{\tau}^L_{T2}(e_M) := p - e + e_M$; for second-time buyers, their utility is $U^S_{T2} = \max\{0, \delta v - p + e, e_M\}$, and thus they choose to repurchase if and only if $v \geq \hat{\tau}^H_{T2}(e_M) := \frac{p-e+e_M}{\delta}$.

In the first period, customers maximize their long-term total utility, $U_{T1} = \max\{\alpha U^F_{T2}, v - p + \alpha U^S_{T2}\}$. For an interior equilibrium, the indifference condition is established by equating the utility of purchasing to sell the token $(v - p + \alpha e_M)$ and the utility of waiting to buy the token $(\alpha(v - p + e - e_M))$. Solving for v yields the threshold $\hat{\tau}_{T1}(e_M) = p + \frac{\alpha(e-2e_M)}{1-\alpha}$, implying that customers choose to purchase in the first period if and only if $v \geq \hat{\tau}_{T1}$.

We now analyze the equilibrium token price in the secondary market. The token supply, S_t, comes from customers who buy in the first period but do not repurchase ($\hat{\tau}_{T1} \leq v < \hat{\tau}^H_{T2}$), while the demand, D_t, comes from customers who only buy in the second period ($\hat{\tau}^L_{T2} \leq v < \hat{\tau}_{T1}$). Considering the valuation constraint $v \in [0,1]$, we solve the market clearing condition $S_t(e_M) = D_t(e_M)$ by analyzing the following cases.

(1) If $\hat{\tau}^L_{T2} \geq 1$, no purchases occur in the second period, resulting in $D_t = 0$. Solving the constraints yields $0 \leq e \leq \frac{(1+\alpha)(p-1)}{\alpha}$.

(2) If $\hat{\tau}^H_{T2} \leq 0$, all customers repurchase, resulting in $S_t = 0$. Solving the constraints yields $e \geq \frac{(1+\alpha)p}{\alpha}$.

(3) If $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq \hat{\tau}^H_{T2} \leq 1$, then $S_t = \hat{\tau}^H_{T2} - \hat{\tau}_{T1}$ and $D_t = \hat{\tau}_{T1} - \hat{\tau}^L_{T2}$. Solving $S_t = D_t$ yields the equilibrium token price $e_M = \frac{(1+\delta-\alpha(1-\delta))e-(1-\alpha)(1-\delta)p}{1+\delta-\alpha(1-3\delta)}$.

  (3.1) Substituting this e_M into the constraints $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq \hat{\tau}^H_{T2} \leq 1$ and price bounds $0 \leq e_M \leq e$ yields the range: $\max\left\{\frac{(1-\alpha)(1-\delta)p}{1+\delta-\alpha(1-\delta)}, \frac{2(1+\alpha)p-(1+\delta)-\alpha(3\delta-1)}{2\alpha}\right\} \leq e \leq \frac{(1+\alpha)p}{\alpha}$.

  (3.2) Substituting $e_M = 0$ into constraints $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq \hat{\tau}^H_{T2} \leq 1$ and $S_t \geq D_t$ yields the range: $p - \delta \leq e \leq \frac{(1-\alpha)(1-\delta)p}{1+\delta-\alpha(1-\delta)}$.

  (3.3) Substituting $e_M = e$ into constraints $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq \hat{\tau}^H_{T2} \leq 1$ and $D_t \geq S_t$ derives contradictions.

(4) If $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq 1 \leq \hat{\tau}^H_{T2}$, then $S_t = 1 - \hat{\tau}_{T1}$ and $D_t = \hat{\tau}_{T1} - \hat{\tau}^L_{T2}$. Solving $S_t = D_t$ yields $e_M = \frac{(1+\alpha)e+(1-\alpha)(p-1)}{1+3\alpha}$.

  (4.1) Substituting $e_M = \frac{(1+\alpha)e+(1-\alpha)(p-1)}{1+3\alpha}$ into constraints $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq 1 \leq \hat{\tau}^H_{T2}$ and $0 \leq e_M \leq e$ yields the range: $\max\{\frac{(1-\alpha)(1-p)}{1+\alpha}, \frac{(1+\alpha)(p-1)}{\alpha}\} \leq e \leq \frac{2(1+\alpha)p-(1+\delta)-\alpha(3\delta-1)}{2\alpha}$.

  (4.2) Substituting $e_M = 0$ into constraints $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq 1 \leq \hat{\tau}^H_{T2}$ and $S_t \geq D_t$ yields the range: $e \leq \min\{p - \delta, \frac{(1-\alpha)(1-p)}{1+\alpha}\}$.

  (4.3) Substituting $e_M = e$ into constraints $0 \leq \hat{\tau}^L_{T2} \leq \hat{\tau}_{T1} \leq 1 \leq \hat{\tau}^H_{T2}$ and $D_t \geq S_t$ derives contradictions.

Combining the above cases, we summarize the results as follows.

- If $0 \leq e \leq \frac{(1+\alpha)(p-1)}{\alpha}$, no customers purchase; i.e., $(\tau_{T1}, \tau^L_{T2}, \tau^H_{T2}) = (1, 1, 1)$.

• If $e \geq \frac{(1+\alpha)p}{\alpha}$, all customers purchase in both periods, and all reward tokens are redeemed; i.e., $(\tau_{T1}, \tau_{T2}^L, \tau_{T2}^H) = (0,0,0)$.

• If $\frac{(1+\alpha)(p-1)}{\alpha} \leq e \leq \frac{(1+\alpha)p}{\alpha}$, the secondary token market has positive transaction volume. Customers with $v \in [\tau_{T2}^H, 1]$ purchase in both periods; those with $v \in [\tau_{T1}, \tau_{T2}^H)$ purchase only in the first period; those with $v \in [\tau_{T2}^L, \tau_{T1}]$ purchase only in the second period; and those with $v \in [0, \tau_{T2}^L)$ do not purchase. Specifically, defining the thresholds $\hat{e}_1 = \frac{(1-\alpha)(1-p)}{1+\alpha}$, $\hat{e}_2 = \frac{(1-\alpha)(1-\delta)p}{1+\delta-\alpha(1-\delta)}$, and $\hat{e}_3 = \frac{2(1+\alpha)p-(1+\delta)-\alpha(3\delta-1)}{2\alpha}$, the results are given by:

$$(e_M, \tau_{T1}, \tau_{T2}^L, \tau_{T2}^H) = \begin{cases} (0, p+\frac{\alpha e}{1-\alpha}, p-e, 1) & \text{if } 0 \leq e \leq \min\{p-\delta, \hat{e}_1\}, \\ (0, p+\frac{\alpha e}{1-\alpha}, p-e, \frac{p-e}{\delta}) & \text{if } p-\delta \leq e \leq \hat{e}_2, \\ (\frac{(1-\alpha)(p-1)+(1+\alpha)e}{3\alpha+1}, \frac{(1+\alpha)p+\alpha(2-e)}{3\alpha+1}, p-e+e_M, 1) & \text{if } \max\{\hat{e}_1, \frac{(1+\alpha)(p-1)}{\alpha}\} \leq e \leq \hat{e}_3, \\ (\frac{(1+\delta-\alpha(1-\delta))e-(1-\delta)(1-\alpha)p}{1+\delta+\alpha(3\delta-1)}, \frac{(1+\delta)((1+\alpha)p-\alpha e)}{1+\delta+\alpha(3\delta-1)}, p-e+e_M, \frac{p-e+e_M}{\delta}) & \text{if } \max\{\hat{e}_2, \hat{e}_3\} \leq e \leq \frac{(1+\alpha)p}{\alpha}. \end{cases} \quad \square$$

Proof of Proposition 3. Based on Equation (11) and the demand thresholds derived in the proof of Lemma 2, the firm's profit function is:

$$\pi_T(p,e) = p(1-\tau_{T1}) + p(1-\tau_{T2}^H + \tau_{T1} - \tau_{T2}^L) - e\min\{1-\tau_{T1}, 1-\tau_{T2}^H + \tau_{T1} - \tau_{T2}^L\}.$$

We analyze the profit maximization problem by discussing the customer demand determined by p and e. The boundary cases where $e \leq \frac{(1+\alpha)(p-1)}{\alpha}$ (resulting in zero demand) or $e \geq \frac{(1+\alpha)p}{\alpha}$ (resulting in full demand) yield suboptimal profits compared to the interior solutions. Thus, we focus on the interior regime $\frac{(1+\alpha)(p-1)}{\alpha} \leq e \leq \frac{(1+\alpha)p}{\alpha}$, which is further divided into four subcases based on the secondary market equilibrium derived in the previous proof.

(1) If $e \leq \min\{p-\delta, \hat{e}_1\}$, then $e_M = 0$, $(\tau_{T1}, \tau_{T2}^L, \tau_{T2}^H) = (p+\frac{\alpha e}{1-\alpha}, p-e, 1)$, and the profit function simplifies to $\pi_T = (1+e-p)p - \frac{e^2}{1-\alpha}$. The Hessian Matrix of the profit function is $H = \begin{bmatrix} -2 & 1 \\ 1 & \frac{-2}{1-\alpha} \end{bmatrix}$. Since $\det(H) = \frac{3+\alpha}{1-\alpha} \geq 0$ and $H_{11} = -2 \leq 0$, H is negative definite, ensuring π_L is jointly concave. Solving the first-order conditions yields the candidate solution $(p_1^*, e_1^*) = (\frac{2}{3+\alpha}, \frac{1-\alpha}{3+\alpha})$ with profit $\pi_1^* = \frac{1}{3+\alpha}$. Substituting the solution into the constraints $e \leq \min\{p-\delta, \hat{e}_1\}$ yields the valid range: $0 \leq \delta \leq \frac{1+\alpha}{3+\alpha}$. Otherwise, if $\frac{1+\alpha}{3+\alpha} \leq \delta \leq 1$, the constraint $e \leq \hat{e}_1$ becomes active, pushing the optimum to the boundary of Case 2.

(2) If $p-\delta \leq e \leq \hat{e}_2$, then $e_M = 0$, $(\tau_{T1}, \tau_{T2}^L, \tau_{T2}^H) = (p+\frac{\alpha e}{1-\alpha}, p-e, \frac{p-e}{\delta})$, and the profit function simplifies to $\pi_T = -\frac{e^2}{1-\alpha} - \frac{(e-p)^2}{\delta} + e(p-1) + 2p - p^2$. The Hessian Matrix is $H = \begin{bmatrix} -\frac{2(\delta+1)}{\delta} & \frac{\delta+2}{\delta} \\ \frac{\delta+2}{\delta} & -\frac{2}{1-\alpha} - \frac{2}{\delta} \end{bmatrix}$. Since $\det(H) = \frac{(3+\alpha)\delta+4}{\delta(1-\alpha)} \geq 0$ and $H_{11} = -\frac{2(\delta+1)}{\delta} \leq 0$, H is negative definite, ensuring π_L is jointly concave. Solving the first-order conditions yields the candidate solution $(p_2^*, e_2^*) = (\frac{(1+\alpha)\delta+1-\alpha}{2(\delta+1)}, \frac{(1-\delta)(1-\alpha)}{2(\delta+1)})$ with profit $\pi_2^* = \frac{(3+\alpha)\delta+1-\alpha}{4(\delta+1)}$. We verify that the solution always satisfies the constraint $p-\delta \leq e \leq \hat{e}_2$.

(3) If $\max\{\hat{e}_1, \frac{(1+\alpha)(p-1)}{\alpha}\} \le e \le \hat{e}_3$, then $e_M = \frac{(1-\alpha)(p-1)+(1+\alpha)e}{3\alpha+1}$, $(\tau_{T1}, \tau_{T2}^L, \tau_{T2}^H) = (p + \frac{\alpha(e-2e_M)}{1-\alpha}, p - e + e_M, 1) = (\frac{(\alpha+1)p-\alpha e+2\alpha}{3\alpha+1}, \frac{2(\alpha+1)p-2\alpha e-(1-\alpha)}{3\alpha+1}, 1)$, and the profit function simplifies to $\pi_T = \frac{(2p-e)(1+\alpha+\alpha e-(1+\alpha)p)}{1+3\alpha}$. Since $\frac{\partial^2 \pi_L}{\partial p^2} \le 0$, we have $p^*(e) = \frac{2(1+\alpha)+(1+3\alpha)e}{4(1+\alpha)}$ by the first-order condition. Substituting $p^*(e)$ back into the profit function yields $\pi_L(p^*(e), e)$, which can be shown to be strictly decreasing in e. Thus, we can easily prove that the optimal e lies on the lower bound $e \ge \hat{e}_1$, reducing this case to the boundary of Case 1.

(4) If $\max\{\hat{e}_2, \hat{e}_3\} \le e \le \frac{(1+\alpha)p}{\alpha}$, then $e_M = \frac{(1+\delta-\alpha(1-\delta))e-(1-\delta)(1-\alpha)p}{1+\delta+\alpha(3\delta-1)}$, $(\tau_{T1}, \tau_{T2}^L, \tau_{T2}^H) = (p + \frac{\alpha(e-2e_M)}{1-\alpha}, p - e + e_M, \frac{p-e+e_M}{\delta}) = (\frac{(1+\delta)((1+\alpha)p-\alpha e)}{1+\delta+\alpha(3\delta-1)}, \frac{2\delta((1+\alpha)p-\alpha e)}{1+\delta+\alpha(3\delta-1)}, \frac{2((1+\alpha)p-\alpha e)}{1+\delta+\alpha(3\delta-1)})$, and the profit function simplifies to $\pi_T = \frac{(2p-e)(1+\delta-\alpha(1-3\delta)+\alpha(1+\delta)e-(1+\alpha)(1+\delta)p)}{1+\delta-\alpha(1-3\delta)}$. Since $\frac{\partial^2 \pi}{\partial p^2} = \frac{-4(1+\alpha)(1+\delta)}{1+\delta+\alpha(3\delta-1)} \le 0$, we have $p^*(e) = \frac{2(1+\delta+\alpha(3\delta-1))+(1+3\alpha)(1+\delta)e}{4(1+\alpha)(1+\delta)}$ by the first-order condition. Substituting $p^*(e)$ into the profit function yields $\pi_L(p^*(e), e)$, which can be shown to be strictly decreasing in e. Thus, we can easily prove that the optimal e lies on the lower bound $e \ge \hat{e}_2$, reducing this case to the boundary of Case 2.

To sum up, the global optimal solution must be either (p_1^*, e_1^*) from Case (1) or (p_2^*, e_2^*) from Case (2). Comparing the optimal profits π_1^* and π_2^* yields the following results.

- If $0 \le \delta \le \frac{(1+\alpha)}{5+\alpha}$, the optimal solutions and the optimal profit are $(p^*, e^*, \pi_L^*) = (\frac{2}{3+\alpha}, \frac{1-\alpha}{3+\alpha}, \frac{1}{3+\alpha})$; the customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{2+\alpha}{3+\alpha}, \frac{1+\alpha}{3+\alpha}, 1)$.
- If $\frac{(1+\alpha)}{5+\alpha} \le \delta \le 1$, the optimal solutions and the optimal profit are $(p^*, e^*, \pi_L^*) = (\frac{1+\delta-\alpha(1-\delta)}{2(1+\delta)}, \frac{(1-\alpha)(1-\delta)}{2(1+\delta)}, \frac{1+3\delta-\alpha(1-\delta)}{4(1+\delta)})$; the customers' purchase decisions are $(\tau_{T1}^*, \tau_{T2}^{L^*}, \tau_{T2}^{H^*}) = (\frac{1}{2}, \frac{\delta}{1+\delta}, \frac{1}{1+\delta})$. □

Proof of Corollary 1. We compare the optimal solutions under the tradable LP (Proposition 3) with those under the traditional LP (Proposition 2). Recall that the traditional LP equilibrium yields $(p_L^*, e_L^*) = (\frac{1}{2}, \frac{1-\delta}{2})$ and $(D_{L1}^{F^*}, D_{L2}^{F^*}, D_{L2}^{S^*}) = (\frac{1}{2}, 0, \frac{1}{2})$.

(1) If $0 \le \delta \le \frac{\alpha+1}{\alpha+5}$, the price comparison results are $p_T^* - p_L^* = \frac{2}{\alpha+3} - \frac{1}{2} = \frac{1-\alpha}{2(\alpha+3)} \ge 0$, $e_T^* - e_L^* = \frac{1-\alpha}{\alpha+3} - \frac{1-\delta}{2} = -\frac{3\alpha+1-(\alpha+3)\delta}{2(\alpha+3)} \le 0$, $(p_T^* - e_T^*) - (p_L^* - e_L^*) = \frac{\alpha+1}{\alpha+3} - \frac{\delta}{2} \ge 0$; and the demand comparison results are $D_{T1}^{F^*} - D_{L1}^{F^*} = \frac{1}{\alpha+3} - \frac{1}{2} \le 0$, $D_{T2}^{F^*} - D_{L2}^{F^*} = \frac{1}{\alpha+3} - 0 \ge 0$, $D_{T2}^{S^*} - D_{L2}^{S^*} = 0 - \frac{1}{2} \le 0$.

(2) If $\frac{\alpha+1}{\alpha+5} \le \delta \le 1$, the price comparison results are $p_T^* - p_L^* = \frac{(1+\alpha)\delta+1-\alpha}{2(\delta+1)} - \frac{1}{2} = -\frac{\alpha(1-\delta)}{2(\delta+1)} \le 0$, $e_T^* - e_L^* = \frac{(1-\delta)(1-\alpha)}{2(\delta+1)} - \frac{1-\delta}{2} = -\frac{(1-\delta)(\alpha+\delta)}{2(\delta+1)} \le 0$, $(p_T^* - e_T^*) - (p_L^* - e_L^*) = \frac{\delta}{\delta+1} - \frac{\delta}{2} = \frac{\delta(1-\delta)}{2(\delta+1)} \ge 0$; and the demand comparison results are $D_{T1}^{F^*} - D_{L1}^{F^*} = \frac{1}{2} - \frac{1}{2} = 0$, $D_{T2}^{F^*} - D_{L2}^{F^*} = (\frac{1}{2} - \frac{\delta}{\delta+1}) - 0 = \frac{1-\delta}{2(\delta+1)} \ge 0$, $D_{T2}^{S^*} - D_{L2}^{S^*} = (1 - \frac{1}{\delta+1}) - \frac{1}{2} = \frac{\delta-1}{2(\delta+1)} \le 0$. □

Proof of Proposition 4. By the optimal profits derived in Propositions 2 and 3, the profit difference between the traditional and tradable LPs can be written as:

$$\pi_T^* - \pi_L^* = \begin{cases} \frac{-(\alpha+3)\delta+1-\alpha}{4(\alpha+3)} & \text{if } 0 \le \delta \le \frac{\alpha+1}{\alpha+5}, \\ \frac{(1-\delta)(\delta-\alpha)}{4(\delta+1)} & \text{if } \frac{\alpha+1}{\alpha+5} \le \delta \le 1. \end{cases}$$

(1) If $0 \le \delta \le \frac{\alpha+1}{\alpha+5}$, $\pi_T^* - \pi_L^* \ge 0$ requires $\delta \le \frac{1-\alpha}{\alpha+3}$. Thus, the tradable LP dominates when $\delta \in [0, \min\{\frac{\alpha+1}{\alpha+5}, \frac{1-\alpha}{\alpha+3}\}]$.

(2) If $\frac{\alpha+1}{\alpha+5} \le \delta \le 1$, $\pi_T^* - \pi_L^* \ge 0$ requires $\delta \ge \alpha$. Thus, the tradable LP dominates when $\delta \in [\max\{\frac{\alpha+1}{\alpha+5}, \alpha\}, 1]$.

To sum up, we have the following results.

- If $0 \le \alpha \le \sqrt{5}-2$, $\pi_T^* \ge \pi_L^*$ holds for all $\delta \in [0,1]$.
- If $\sqrt{5}-2 \le \alpha \le 1$, $\pi_T^* \ge \pi_L^*$ if and only if $\delta \in [0, \frac{1-\alpha}{\alpha+3}] \cup [\alpha, 1]$. $\square$

Proof of Proposition 5. Based on Proposition 2, the consumer surplus C_L and social welfare S_L under the traditional LP are given by:

$$C_L = \int_{\tau_{L1}^*}^1 (v - p_L^*)dv + \alpha \int_{\tau_{L2}^*}^1 (\delta v - p_L^* + e_L^*)dv = \frac{\alpha\delta + 1}{8} \quad \text{and} \quad S_L = C_L + \pi_L^* = \frac{(\alpha+2)\delta + 3}{8}.$$

Similarly, based on Proposition 3, we derive the consumer surplus C_T and social welfare S_T under the tradable LP as follows.

$$C_T = \int_{\tau_{T1}^*}^1 (v - p_T^*)dv + \alpha \left[\int_{\tau_{T2}^{L*}}^{\tau_{T1}^*} (v - p_T^* + e_T^* - e_M^*)dv + \int_{\tau_{T1}^*}^{\tau_{T2}^{H*}} e_M^* dv + \int_{\tau_{T2}^{H*}}^1 (\delta v - p_T^* + e_T^*)dv \right].$$

(1) If $0 \le \delta \le \frac{\alpha+1}{\alpha+5}$, we have $C_T = \frac{3\alpha+1}{(\alpha+3)^2}$ and $S_T = C_T + \pi_T^* = \frac{5\alpha+7}{2(\alpha+3)^2}$. In this case, the differences between the two LPs are given by $C_T - C_L = -\frac{\alpha(3+\alpha)^2\delta + (5-\alpha)(1-\alpha)}{8(\alpha+3)^2}$ and $S_T - S_L = \frac{(1-\alpha)(1+3\alpha) - (2+\alpha)(3+\alpha)^2\delta}{8(\alpha+3)^2}$. It follows that $C_T - C_L \le 0$ in this region; and $S_T - S_L \ge 0$ if and only if $0 \le \delta \le \frac{1+2\alpha-3\alpha^2}{(\alpha+2)(\alpha+3)^2}$.

(2) If $\frac{\alpha+1}{\alpha+5} \le \delta \le 1$, we have $C_T = \frac{4\alpha\delta^2 + (1-5\alpha)\delta + 3\alpha + 1}{8(\delta+1)}$ and $S_T = C_T + \pi_T^* = \frac{4\alpha\delta^2 + (7-3\alpha)\delta + \alpha + 3}{8(\delta+1)}$. In this case, the differences between the two LPs are given by $C_T - C_L = \frac{3\alpha(1-\delta)^2}{8(\delta+1)}$ and $S_T - S_L = \frac{(1-\delta)((2-3\alpha)\delta + \alpha)}{8(\delta+1)}$. It follows that $C_T - C_L \ge 0$ and $S_T - S_L \ge 0$ in this region.

To sum up, we have the following results.

- The tradable LP leads to higher consumer surplus if and only if $\delta \in [\frac{\alpha+1}{\alpha+5}, 1]$;
- The tradable LP leads to higher social welfare if and only if $\delta \in [0, \frac{1+2\alpha-3\alpha^2}{(\alpha+2)(\alpha+3)^2}] \cup [\frac{\alpha+1}{\alpha+5}, 1]$. $\square$

E.2. Proofs of Results in Section 5.

Proof of Proposition 6. In Appendix A, we have characterized the equilibria for both the traditional and tradable LPs considering competition with outside options. We now proceed to analyze their profit difference $\pi_T^*(\delta) - \pi_L^*(\delta)$ as follows.

(1) When $c \in [0, \frac{1-\alpha}{2}]$, we discuss the following cases.

(1.1) If $0 \le \delta \le c$, $\pi_L^* = \frac{1}{4}$ and $\pi_T^* = \frac{1-\alpha(1-c)-c(1-c)}{(1-\alpha)(3+\alpha)}$. The profit difference is $\pi_T^* - \pi_L^* = \frac{1-\alpha(1-c)-c(1-c)}{(1-\alpha)(3+\alpha)} - \frac{1}{4} \ge 0$.

(1.2) If $c \le \delta \le \hat{\delta}_1$, $\pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$ and $\pi_T^* = \frac{1-\alpha(1-c)-c(1-c)}{(1-\alpha)(3+\alpha)}$. Solving $\pi_T^* = \pi_L^*$ in this interval yields the threshold $\hat{\delta}_{C1} = \frac{(1-\alpha)^2+2(1-\alpha^2)c+4c^2+\sqrt{(1-\alpha-2c)^2((1-\alpha)^2+4(2-\alpha-\alpha^2)c+4c^2)}}{2(1-\alpha)(3+\alpha)}$. It can be verified that $\pi_T^* - \pi_L^* \ge 0$ for $\delta \le \hat{\delta}_{C1}$.

(1.3) If $\hat{\delta}_1 \le \delta \le \hat{\delta}_2$, $\pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$ and π_T^* splits at $\hat{\delta}_3$.

(a) For $\delta \le \hat{\delta}_3$, $\pi_T^* = \frac{c^2(1-\alpha+\delta)(1+\delta)+\delta(1-\alpha)(1-\alpha(1-\delta)+3\delta)-c\delta(1-\alpha)(3-\alpha+\delta(3+\alpha))}{\delta(1-\alpha)(4+\delta(3+\alpha))}$. Solving the equation $\pi_T^* = \pi_L^*$ in this interval yields the unique threshold $\tilde{\delta}_1 = \frac{5-\alpha(2+3\alpha)-2(1-\alpha)(3+\alpha)c+4c^2-(1-\alpha-2c)\sqrt{(1-\alpha)(25+7\alpha)-4(2-\alpha-\alpha^2)c+4c^2}}{2(1-\alpha)(3+\alpha)}$. It can be verified that $\pi_T^* - \pi_L^* \ge 0$ if $\delta \ge \tilde{\delta}_1$.

(b) For $\delta \ge \hat{\delta}_3$, $\pi_T^* = \frac{(1-\alpha-c+\delta(3+\alpha-c))^2}{4(1+\delta)(1-\alpha+\delta(3+\alpha))}$. Solving the equation $\pi_T^* = \pi_L^*$ in this interval yields the threshold $\tilde{\delta}_2$. It can be verified that $\pi_T^* - \pi_L^* \ge 0$ if $\delta \ge \tilde{\delta}_2$.

(1.4) If $\delta \ge \max\{\hat{\delta}_1, \hat{\delta}_2\}$, $\pi_L^* = \pi_T^* = \frac{(c-\delta)^2+\delta}{4\delta}$.

To sum up case (1), defining $\hat{\delta}_{C2} = \begin{cases} \tilde{\delta}_2 & \text{if } 0 \le c \le \hat{c}_1, \\ \tilde{\delta}_1 & \text{if } \hat{c}_1 \le c \le \frac{1-\alpha}{2}, \end{cases}$ where c_1 is the solution of equation $\tilde{\delta}_2 = \tilde{\delta}_3$, and $\hat{\delta}_{C3} = \hat{\delta}_2$, we obtain the result described in Proposition 6(i).

(2) When $c \in [\frac{1-\alpha}{2}, \frac{1}{2}]$, we discuss the following cases.

(2.1) If $0 \le \delta \le c$, $\pi_L^* = \pi_T^* = \frac{1}{4}$.

(2.2) If $c \le \delta \le \hat{\delta}_1$, $\pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$ and $\pi_T^* = \frac{1}{4}$. The profit difference is $\pi_T^* - \pi_L^* = \frac{1}{4} - \frac{(\delta-c)^2+\delta}{4\delta} \le 0$.

(2.3) If $\hat{\delta}_1 \le \delta \le \hat{\delta}_2$, $\pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$ and $\pi_T^* = \frac{c^2(1-\alpha+\delta)(1+\delta)+\delta(1-\alpha)(1-\alpha(1-\delta)+3\delta)-c\delta(1-\alpha)(3-\alpha+\delta(3+\alpha))}{\delta(1-\alpha)(4+\delta(3+\alpha))}$. It can be verified that $\pi_T^* - \pi_L^* \le 0$.

(2.4) If $\delta \ge \max\{\hat{\delta}_1, \hat{\delta}_2\}$, $\pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$. For $\delta \le \hat{\delta}_4$, $\pi_T^* = \frac{4(1-c)\delta^2+c^2}{4\delta(1+\delta)}$ and it can be verified that $\pi_T^* - \pi_L^* \le 0$. For $\delta \ge \hat{\delta}_4$, $\pi_T^* = \pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$.

To sum up case (2), $\pi_T^* - \pi_L^* \le 0$ for $\delta \in [c, 1-c]$ and $\pi_T^* = \pi_L^*$ for $\delta \in [0, c] \cup [1-c, 1]$. Thus, Proposition 6(ii) is proved.

(3) When $c \in [\frac{1}{2}, 1]$, for $0 \le \delta \le c$, $\pi_L^* = \pi_T^* = \frac{1}{4}$; and for $c \le \delta \le 1$, $\pi_L^* = \pi_T^* = \frac{(\delta-c)^2+\delta}{4\delta}$. Thus, Proposition 6(iii) is proved. □

It can be verified that $\hat{\delta}_{C1}$ and $\hat{\delta}_{C2}$ increase with c, while $\hat{\delta}_{C3}$ decreases with c. □

Proof of Lemma 3. In Appendix A, we have characterized the equilibria for both the traditional and tradable LPs considering competition with outside options. We now proceed to analyze the difference $r_T^*(\delta) - r_L^*(\delta)$ in their reward redemption rates.

(1) When $c \in [0, \frac{1-\alpha}{2}]$, we discuss the following cases.

(1.1) If $0 \le \delta \le c$, $r_L^* = 0$ and $r_T^* = \frac{1-\alpha-2c}{1+c-\alpha(1-c)}$. Thus, $r_T^* - r_L^* \ge 0$.

(1.2) If $c \le \delta \le \hat{\delta}_1$, $r_L^* = 1 - \frac{c}{\delta}$ and $r_T^* = \frac{1-\alpha-2c}{1+c-\alpha(1-c)}$. Solving the equation $r_T^* = r_L^*$ in this interval yields the threshold $\tilde{\delta}_{C1} = \frac{1+c-(1-c)\alpha}{3+\alpha}$. It can be verified that $r_T^* - r_L^* \ge 0$ if $\delta \le \tilde{\delta}_{C1}$.

(1.3) If $\hat{\delta}_1 \le \delta \le \hat{\delta}_2$, $r_L^* = 1 - \frac{c}{\delta}$ and r_T^* splits at $\hat{\delta}_3$.

(a) For $\delta \le \hat{\delta}_3$, $r_T^* = \frac{\delta(1-\alpha)(3-\alpha+\delta(3+\alpha))-2c(1-\alpha+\delta)(1+\delta)}{\delta(2(1-\alpha)+c(1+\alpha)(1+\delta)}$. Solving $r_T^* = r_L^*$ yields the threshold $\tilde{\delta}_1 = \frac{-(1-\alpha)^2+(5-\alpha)c-(1+\alpha)c^2+\sqrt{((1-\alpha)^2-(5-\alpha)c+(1+\alpha)c^2)^2-4(1+\alpha)(3+\alpha)(1-\alpha-c)c^2}}{2(3+\alpha)(1-\alpha-c)}$. It can be verified that $r_T^* - r_L^* \ge 0$ if $\delta \ge \tilde{\delta}_1$.

(b) For $\delta \ge \hat{\delta}_3$, $r_T^* = 1$ and $r_T^* - r_L^* \ge 0$.

(1.4) If $\delta \ge \max\{\hat{\delta}_1, \hat{\delta}_2\}$, $r_T^* = r_L^* = 1 - \frac{c}{\delta}$.

To sum up case (1), defining $\tilde{\delta}_{C2} = \begin{cases} \hat{\delta}_2 & \text{if } 0 \le c \le \hat{c}_1, \\ \tilde{\delta}_1 & \text{if } \hat{c}_1 \le c \le \frac{1-\alpha}{2}, \end{cases}$ where c_1 is the larger solution of equation $((1-\alpha)^2 - (5-\alpha)c + (1+\alpha)c^2)^2 - 4(1+\alpha)(3+\alpha)(1-\alpha-c)c^2 = 0$, we obtain the result described in Lemma 3(i).

(2) When $c \in [\frac{1-\alpha}{2}, \frac{1}{2}]$, we discuss the following cases.

(2.1) If $0 \le \delta \le c$, $r_L^* = r_T^* = 0$.

(2.2) If $c \le \delta \le \hat{\delta}_1$, $r_L^* = 1 - \frac{c}{\delta}$ and $r_T^* = 0$. $r_T^* - r_L^* \le 0$.

(2.3) If $\hat{\delta}_1 \le \delta \le \hat{\delta}_2$, $r_L^* = 1 - \frac{c}{\delta}$ and $r_T^* = \frac{\delta(1-\alpha)(3-\alpha+\delta(3+\alpha))-2c(1-\alpha+\delta)(1+\delta)}{\delta(2(1-\alpha)+c(1+\alpha)(1+\delta)}$. It can be verified that $r_T^* - r_L^* \le 0$ in this interval.

(2.4) If $\delta \ge \max\{\hat{\delta}_1, \hat{\delta}_2\}$, $r_L^* = 1 - \frac{c}{\delta}$. For $\delta \le \hat{\delta}_4$, $r_T^* = \frac{2\delta^2-c}{\delta(2-c)}$ and it can verified that $r_T^* - r_L^* \le 0$. For $\delta \ge \hat{\delta}_4$, $r_T^* = r_L^* = 1 - \frac{c}{\delta}$.

To sum up case (2), $r_T^* - r_L^* \le 0$ for $\delta \in [c, 1-c]$ and $r_T^* = r_L^*$ for $\delta \in [0, c] \cup [1-c, 1]$. Thus, Lemma 3(ii) is proved.

(3) When $c \in [\frac{1}{2}, 1]$, for $0 \le \delta \le c$, $r_L^* = r_T^* = 0$; and for $c \le \delta \le 1$, $r_L^* = r_T^* = 1 - \frac{c}{\delta}$. Thus, Lemma 3(iii) is proved.

Finally, taking derivatives confirms that $\tilde{\delta}_{C1}$ and $\tilde{\delta}_{C2}$ increase with c. Comparing these with the profit thresholds in Proposition 6, we verify that $\tilde{\delta}_{C1} \le \hat{\delta}_{C1}$ and $\tilde{\delta}_{C2} \le \hat{\delta}_{C2}$. $\square$

Proof of Proposition 7. In Appendix B, we have characterized the equilibria for both the traditional and tradable LPs considering heterogeneous new customer arrival in the second period. We now analyze their profit difference $\pi_T^*(\theta) - \pi_L^*(\theta)$.

(1) When $0 \le \theta \le \hat{\theta}_{L1}$, the firm's profit under the traditional LP is $\pi_L^* = \frac{c^2-2c+2}{4}$.

(1.1) If $\theta \le \hat{\theta}_{T1}$, since token trading does not occur under the tradable LP, the equilibrium outcomes are equivalent, i.e, $\pi_T^* = \pi_L^* = \frac{c^2-2c+2}{4}$. We define the threshold as $\hat{\theta}_1 = \hat{\theta}_{T1}$.

(1.2) If $\theta \ge \hat{\theta}_{T1}$, the firm's profit under the tradable LP is given by: $\pi_T^* = \frac{c^2-2c+3+\theta^2-2c\theta+2\theta}{8}$ for $\theta \le \min\{\hat{\theta}_{T2}, \hat{\theta}_{T3}\}$; $\pi_T^* = \frac{1+\theta^2}{4}$ for $\theta \ge \hat{\theta}_{T2}$; and $\pi_T^* = \frac{(3+\theta-c)^2}{24}$ for $\theta \ge \hat{\theta}_{T3}$. We can prove that $\pi_T^* - \pi_L^* \ge 0$ always holds.

(2) When $\hat{\theta}_{L1} \le \theta \le \hat{\theta}_{L2}$, the firm's profit under the traditional LP is $\pi_L^* = \frac{(2+\theta-c)^2}{12}$.

(2.1) If $\theta \le \hat{\theta}_{T4}$, the firm's profit under the tradable LP is $\pi_T^* = \frac{(3+\theta-c)^2}{24}$. It can be verified that $\pi_T^* - \pi_L^* \ge 0$ in this interval.

(2.2) If $\theta \geq \hat{\theta}_{T4}$, since the firm does not issue rewards under both LPs, the profits are equivalent, i.e., $\pi_T^* = \pi_L^* = \frac{(2+\theta-c)^2}{12}$.

(3) When $\max\{\hat{\theta}_{L1}, \hat{\theta}_{L2}\} \leq \theta \leq 1$, the firm's profit under the traditional LP is $\pi_L^* = \frac{\theta^2+2\theta+2c^2-4c+3}{8}$.

(3.1) If $\theta \leq \min\{\hat{\theta}_{T2}, \hat{\theta}_{T3}\}$, we have $\pi_T^* = \frac{c^2-2c+3+\theta^2-2c\theta+2\theta}{8}$. We find that $\pi_T^* - \pi_L^* \geq 0$.

(3.2) If $\hat{\theta}_{T3} \leq \theta \leq \min\{\hat{\theta}_{T2}, \hat{\theta}_{T4}\}$, we have $\pi_T^* = \frac{(3+\theta-c)^2}{24}$. Solving $\pi_T^* - \pi_L * \geq 0$ in this interval yields the condition $\theta \leq \frac{\sqrt{3c(4-3c)}-c}{2}$.

(3.3) If $\theta \geq \hat{\theta}_{T2}$, we have $\pi_T^* = \frac{1+\theta^2}{4}$. Solving $\pi_T^* - \pi_L^* \geq 0$ in this interval yields the condition $\theta \leq 1 - \sqrt{2}(1-c)$.

(3.4) If $\theta \geq \hat{\theta}_{T4}$, we have $\pi_T^* = \frac{(2+\theta-c)^2}{12}$. The traditional LP dominates, i.e., $\pi_T^* - \pi_L^* \leq 0$.

Combining the conditions where $\pi_T^* - \pi_L^* \geq 0$ derived above, we define the upper threshold as

$$\hat{\theta}_2 = \begin{cases} \sqrt{3(c^2-2c+2)} + c - 2 & \text{if } 0 \leq c \leq \frac{3-\sqrt{6\sqrt{2}}}{3}, \\ c + \sqrt{2} - 1 & \text{if } \frac{3-\sqrt{6\sqrt{2}}}{3} \leq c \leq \frac{2-\sqrt{2}}{3}, \\ \frac{\sqrt{3c(4-3c)}-c}{2} & \text{if } \frac{2-\sqrt{2}}{3} \leq c \leq \frac{2(4-\sqrt{2})}{7}, \\ 1-\sqrt{2}(1-c) & \text{if } \frac{2(4-\sqrt{2})}{7} \leq c \leq 1. \end{cases}$$

The comparison results are summarized as follows.

- When $0 \leq \theta \leq \hat{\theta}_1$, the tradable LP is equivalent to the traditional LP ($\pi_T^* = \pi_L^*$).
- When $\hat{\theta}_1 < \theta < \hat{\theta}_2$, the tradable LP yields strictly higher profits ($\pi_T^* > \pi_L^*$).
- When $\hat{\theta}_2 \leq \theta \leq 1$, the traditional LP yields equal or higher profits. Specifically, if $0 \leq c \leq \frac{1-\theta}{2}$,

both firms refrain from issuing rewards and achieve equivalent profits; otherwise, if $\frac{1-\theta}{2} < c \leq 1$, the traditional LP is strictly superior. □

Proof of Proposition 8. In Appendix D, we characterized the equilibria for both traditional and tradable LPs under dynamic pricing. We now compare their performance regarding firm profit, consumer surplus, and social welfare.

The consumer surplus under the traditional LP (C_L) and the tradable LP (C_T) are defined as:

$$C_L = \int_{\tau_{L1}^*}^{1} (v - p_{L1}^*) dv + \alpha \left[\int_{\tau_{L2}^{L*}}^{\tau_{L1}^*} (v - p_{L2}^*) dv + \int_{\tau_{L2}^{H*}}^{1} (\delta v - p_{L2}^* + e_L^*) dv \right];$$

$$C_T = \int_{\tau_{T1}^*}^{1} (v - p_{T_1}^*) dv + \alpha \left[\int_{\tau_{T2}^{L*}}^{\tau_{T1}^*} (v - p_{T2}^* + e_T^* - e_M^*) dv + \int_{\tau_{T1}^*}^{\tau_{T2}^{H*}} e_M^* dv + \int_{\tau_{T2}^{H*}}^{1} (\delta v - p_{T2}^* + e_T^*) dv \right].$$

Since $\hat{\delta}_{T1} \leq \hat{\delta}_{T2} \leq \hat{\delta}_{L1} \leq \delta \leq 1$, we analyze the comparison by discussing the following cases.

(1) When $0 \leq \delta \leq \hat{\delta}_{L1}$, the traditional LP yields the firm's profit $\pi_L^* = \frac{(1+\delta)(1-\alpha+\delta)}{3+4\delta-\alpha(2+\alpha)}$ and consumer surplus $C_L^* = \frac{\alpha^3(3+\delta(5+\delta)) - \alpha^2(5+2\delta(7+5\delta)) + \alpha(1+\delta(1+\delta)(5+4\delta)) + (1+2\delta)^2}{2(3+4\delta-\alpha(2+\alpha))^2}$.

(1.1) If $\delta \leq \hat{\delta}_{T1}$, the results under the tradable LP are $\pi_T^* = \frac{1}{\alpha+3}$ and $C_T^* = \frac{3\alpha+1}{2(\alpha+3)^2}$. We can prove $\pi_T^* - \pi_L^* \leq 0$ and $C_T^* - C_L^* \leq 0$. Thus, the traditional LP dominates in both profit and consumer surplus.

(1.2) If $\hat{\delta}_{T1} \le \delta \le \hat{\delta}_{T2}$, the results under the tradable LP are $\pi_T^* = \frac{\alpha^2-2\alpha(1+\delta)^2+(1+\delta)(1+3\delta)}{(1+\delta)(4+3\delta-2\alpha(2+\delta))}$ and $C_T^* = \frac{\alpha^3(12+5\delta+8\delta^2+12\delta^3+4\delta^4)-2\alpha^2(10+7\delta+7\delta^2+14\delta^3+6\delta^4)+\alpha(1+\delta)^2(4-3\delta+9\delta^2)+4(1+\delta)}{2(1+\delta)(4+3\delta-2\alpha(2+\delta))^2}$. Comparing the expressions yields $\pi_T^* - \pi_L^* \le 0$ and $C_T^* - C_L^* \ge 0$. Thus, the traditional LP yields higher profit, while the tradable LP yields higher consumer surplus.

(1.3) If $\delta \ge \hat{\delta}_{T2}$, the results under the tradable LP are $\pi_T^* = \frac{(1+\delta-\alpha(1-\delta))^2}{4(1+\delta^2-\alpha(1-\delta))}$ and $C_T^* = \frac{\alpha^3(1-\delta)^2(3-(3-\delta)\delta)-\alpha^2(1-\delta)(5-3\delta+2\delta^3)+\alpha(1+3\delta-8\delta^2+11\delta^3-7\delta^4+4\delta^5)+(1-\delta(1-2\delta))^2}{8(1-\alpha(1-\delta)+\delta^2)^2}$. Comparing the expressions yields $\pi_T^* - \pi_L^* \le 0$ and $C_T^* - C_L^* \ge 0$. Thus, the traditional LP yields higher profit, while the tradable LP yields higher consumer surplus.

(2) When $\hat{\delta}_{L1} \le \delta \le 1$, the firm does not issue rewards under both LPs ($e_L^* = e_T^* = 0$), resulting in the same results ($\pi_T^* = \pi_L^*$ and $C_T^* = C_L^*$).

Let $\hat{\delta}_{D1} = \hat{\delta}_{T1}$ and $\hat{\delta}_{D2} = \hat{\delta}_{L1}$. Thus, Proposition 8 is proved. □

E.3. Proofs of Results in Appendix A

Proof of Lemma A1. Following Appendix A.1, a customer purchases in the first period if $U_{L1}^P \ge U_{L1}^N$. Substituting $U_{L1}^P = v - p + \alpha\max\{\delta v - p + e - c, 0\}$ and $U_{L1}^N = \alpha\max\{v - p - c, 0\}$ yields the first-period threshold:

$$\hat{\tau}_{L1} = \begin{cases} p & \text{if } e \le (1-\delta)p + c, \\ \frac{p+\alpha(p-e+c)}{1+\alpha\delta} & \text{if } e > (1-\delta)p + c. \end{cases}$$

In the second period, customers purchase only if they have purchased in the first period ($v \ge \tau_{L1}$) and their second-period utility is non-negative ($U_{T2} \ge 0$, i.e., $v \ge \frac{p-e+c}{\delta}$). This implies a second-period threshold of $\hat{\tau}_{L2} = \max\left\{\hat{\tau}_{L1}, \frac{p-e+c}{\delta}\right\}$.

Restricting these thresholds to the valuation support $v \in [0,1]$ gives $\tau_{Li} = \min\{\max\{\hat{\tau}_{Li}, 0\}, 1\}$ for $i \in \{1,2\}$, which establishes Lemma A1. □

Proof of Lemma A2. The firm's profit function is $\pi_L = p(1-\tau_{L1}) + (p-e)(1-\tau_{L2})$, where τ_{L1} and τ_{L2} are given in Lemma A1. We first rule out the cases $(\tau_{L1}, \tau_{L2}) = (1,1)$ and $(\tau_{L1}, \tau_{L2}) = (0,0)$ as they yield non-positive profits, and the boundary case $\tau_{L1} = \tau_{L2}$ as it is dominated by other interior solutions. We thus focus on the following two cases.

(1) **No Repurchase** $((\tau_{L1}, \tau_{L2}) = (p, 1))$. The profit simplifies to $\pi_L(p) = p(1-p)$. The unconstrained first-order condition yields $p_L^* = \frac{1}{2}$, resulting in $\pi_L^* = \frac{1}{4}$. Substituting p_L^* into the constraint $e \le p - \delta + c$ yields the feasibility condition $e_L^* \le \frac{1+2(c-\delta)}{2}$.

(2) **Partial Repurchase** $((\tau_{L1}, \tau_{L2}) = (p, \frac{p-e+c}{\delta}))$. The profit is $\pi_L(p,e) = p(1-p) + (p-e)(1 - \frac{p-e+c}{\delta})$, which is jointly concave. Solving the first-order conditions yields the candidate interior solution $(p^*, e^*) = (\frac{1}{2}, \frac{1-\delta+c}{2})$. Substituting (p^*, e^*) into the constraints $p - \delta + c \le e \le (1-\delta)p + c$ yields the following results.

(2.1) If $c \leq \delta \leq 1$, the candidate solution satisfies all constraints. The optimal decisions are $(p_L^*, e_L^*) = (p^*, e^*)$, yielding $\pi_L^* = \frac{(\delta-c)^2+\delta}{4\delta}$.

(2.2) If $0 \leq \delta < c$, the constraint $e \geq p - \delta + c$ is active. The optimal solution binds at the lower boundary $e = p - \delta + c$, implying $\tau_{L2} = 1$. This reduces the problem to Case (1). □

Proof of Lemma A3. Based on the utility analysis in Appendix A.2, the purchase thresholds are derived as follows. In the first period, customers purchase if and only if $v \geq \hat{\tau}_{T1}(e_M)$, where

$$\hat{\tau}_{T1} = \begin{cases} p - \alpha e_M & \text{if } e \leq c, \\ p + \frac{\alpha(e-2e_M-c)}{1-\alpha} & \text{if } e > c. \end{cases}$$

In the second period, first-time buyers purchase if and only if $v \geq \hat{\tau}_{T2}^L(e_M) := p - e + e_M + c$; and second-time buyers repurchase if and only if $v \geq \hat{\tau}_{T2}^H(e_M) := \frac{p-e+e_M+c}{\delta}$. Restricting these thresholds to the valuation support $v \in [0,1]$, we have $\tau_{T1} = \min\{\max\{\hat{\tau}_{T1}, 0\}, 1\}$, $\tau_{T2}^L = \min\{\max\{\hat{\tau}_{T2}^L, 0\}, \tau_{T1}\}$, and $\tau_{T2}^H = \min\{\max\{\hat{\tau}_{T2}^H, \tau_{T1}\}, 1\}$.

In the secondary market, the token supply is $S_t = \tau_{T2}^H - \tau_{T1}$ and the demand is $D_t = \tau_{T1} - \tau_{T2}^L$. By the market clearing rule, we analyze the following cases.

(1) **Interior Token Price ($0 < e_M^* < e$).** When supply and demand intersect at a positive price, we solve $S_t(e_M) = D_t(e_M)$ for the candidate price e_M°.

(1.1) If $0 \leq \hat{\tau}_{T2}^L \leq \hat{\tau}_{T1} \leq \hat{\tau}_{T2}^H \leq 1$, then $S_t = \hat{\tau}_{T2}^H - \hat{\tau}_{T1}$ and $D_t = \hat{\tau}_{T1} - \hat{\tau}_{T2}^L$. Solving $S_t = D_t$ yields the candidate price $e_M^\circ = \frac{(1+\delta-\alpha(1-\delta))(e-c)-(1-\alpha)(1-\delta)p}{1+\delta-\alpha(1-3\delta)}$. Substituting this into the constraints yields the segmentation vector corresponding to Case 7 in Lemma A3.

(1.2) If $0 \leq \hat{\tau}_{T2}^L \leq \hat{\tau}_{T1} \leq 1 \leq \hat{\tau}_{T2}^H$, then $S_t = 1 - \hat{\tau}_{T1}$ while $D_t = \hat{\tau}_{T1} - \hat{\tau}_{T2}^L$. Solving $S_t = D_t$ yields $e_M^\circ = \frac{(1+\alpha)e-(1-\alpha)(1-p)-(1+\alpha)c}{1+3\alpha}$. This yields the segmentation vector corresponding to the Case 3 in Lemma A3.

(2) **Corner Token Price ($e_M^* = 0$).** If the candidate interior price $e_M^\circ \leq 0$, or if demand is naturally zero (i.e., $\tau_{T2}^L \geq \tau_{T1}$ at $e_M = 0$), the market clears at the lower bound $e_M^* = 0$. Substituting $e_M^* = 0$ into the thresholds yields the segmentation vectors corresponding Cases 2, 4, 5, 6 in Lemma A3.

(3) **Trivial Cases**. If $p \geq 1$ and $e \leq p - 1 + c + \frac{p-1}{\alpha}$, no one purchases (Case 1). If $e \geq p + c + \frac{p}{\alpha}$, all customers repurchase (Case 8). □

Proof of Lemma A4. The firm's profit function is $\pi_T = p(1-\tau_{T1}) + p(1-\tau_{T2}^H + \tau_{T1} - \tau_{T2}^L) - e\min\{1-\tau_{T1}, 1-\tau_{T2}^H + \tau_{T1} - \tau_{T2}^L\}$, where τ_{T1}, τ_{T2}^L, and τ_{T2}^H are given in Lemma A3. For brevity, the algebraic derivations for the dominated cases (Cases 1, 3, 7, 8 in Lemma A3) are omitted, and we focus on the four dominant regions (Cases 2, 4, 5, 6).

(1) **No Repurchase, Trading (Case 4 in Lemma A3).** The purchase thresholds are $(\tau_{T1}, \tau_{T2}^L, \tau_{T1}^H) = (p + \frac{\alpha(e-c)}{1-\alpha}, p-e+c, 1)$, and the profit function simplifies to $\pi_T(p,e) = p(1-p+e-c) - \frac{e(e-c)}{1-\alpha}$, which is jointly concave. Solving the first-order conditions yields the interior solution $(p^*, e^*) = (\frac{2-c}{2+\alpha}, \frac{1+c-\alpha(1-c)}{3+\alpha})$, which is feasible if $c \le \frac{1-\alpha}{2}$ and $\delta \le \frac{1+\alpha+c}{\alpha+3}$.

(2) **No Repurchase, No Trading (Case 2 in Lemma A3).** The purchase thresholds are $(\tau_{T1}, \tau_{T2}^L, \tau_{T1}^H) = (p, p, 1)$, and the profit function simplifies to $\pi_T(p) = p(1-p)$. The unconstrained optimal decision is $p^* = \frac{1}{2}$, resulting in $\pi_T^* = \frac{1}{4}$. Substituting p^* into the constraints yields the feasibility condition $e^* \le \min\{\frac{1+2(c-\delta)}{2}, c\}$.

(3) **Repurchase, Trading (Case 6 in Lemma A3).** The purchase thresholds are $(\tau_{T1}, \tau_{T2}^L, \tau_{T1}^H) = (p + \frac{\alpha(e-c)}{1-\alpha}, p-e+c, \frac{p-e+c}{\delta})$, and the profit function is $\pi_T(p,e) = p(1-p-\frac{\alpha(e-c)}{1-\alpha}) + \frac{(p-e)((1-\alpha+\delta)(e-c)-(1-\alpha)(p-\delta))}{\delta(1-\alpha)}$, which is jointly concave. Solving the first-order conditions yields the candidate interior solution $(p^*, e^*) = (\frac{(1-\alpha)(2-c)+\delta(3+\alpha-c)}{4+\delta(3+\alpha)}, \frac{2(1-\alpha)+c(1+\alpha)(1+\delta)}{4+\delta(3+\alpha)})$. Let $\hat{\delta}_3 = \frac{\alpha+c-1+\sqrt{(1-\alpha)^2+2(11+5\alpha)c+c^2}}{2(3+\alpha)}$.

  (3.1) If $\hat{\delta}_3 \le \delta \le \min\{\frac{(5-\alpha)c-(1-\alpha)^2+\sqrt{(1-\alpha)^4+2(1-\alpha)^2(7+5\alpha)c+(c+3\alpha c)^2}}{2(3+\alpha)(1-\alpha-c)}, \frac{2-\alpha(2-c)-3c}{2c}\}$, the interior solution is feasible.

  (3.2) If $c \le \frac{2(1-\alpha)}{5-\alpha}$ and $\delta \ge \hat{\delta}_3$, the constraint $e \le \hat{e}_5$ binds, resulting in $(p_T^*, e_T^*) = (\frac{(1+\delta-\alpha(1-\delta))(1-\alpha+c+\delta(3+\alpha+c))}{2(1+\delta)(1+3\delta-\alpha(1-\delta))}, \frac{(1-\alpha)(1-\delta)(1+3\delta-\alpha(1-\delta))+c(1+\delta)(3+5\delta-3\alpha(1-\delta))}{2(1+\delta)(1+3\delta-\alpha(1-\delta))})$.

(4) **Repurchase, No Trading (Case 5 in Lemma A3).** The purchase thresholds are $(\tau_{T1}, \tau_{T2}^L, \tau_{T1}^H) = (p, p, \frac{p-e+c}{\delta})$, and the profit function is $\pi_T = \frac{\delta((2-p)p-e)-(e-p)^2-c(p-e))}{\delta}$, which is jointly concave. Solving the first-order conditions yields the candidate interior solution $(p^*, e^*) = (\frac{1}{2}, \frac{1-\delta+c}{2})$. Let $\hat{\delta}_4 = 1-c$.

  (4.1) If $\delta \ge \max\{c, \hat{\delta}_4\}$, the interior solution is feasible.

  (4.2) If $\sqrt{\frac{c}{2}} \le \delta \le \hat{\delta}_4$, the constraint $e \le c$ binds, resulting in $(p^*, e^*) = (\frac{2\delta+c}{2(1+\delta)}, c)$.

Comparing the optimal profits across these regions defines the boundaries $\hat{\delta}_1$, $\hat{\delta}_2$, and $\hat{c}_1$. $\square$

E.4. Proofs of Results in Appendix B.

Proof of Lemma B5. The firm maximizes the profit function $\pi_L(p,e) = p(1-\tau_{L1}(p,e)) + (p-e)(1-\tau_{L2}(p,e)) + p(\theta - \zeta_L(p))$, where τ_{L1} and τ_{L2} can be derived from Lemma A1 as

$$(\tau_{L1}, \tau_{L2}) = \begin{cases} (1,1) & \text{if } \ p \ge 1, \\ (p,1) & \text{if } \ p \le 1 \cap e \le p-1+c, \\ (p, p-e+c) & \text{if } \ p-1+c \le e \le c, \\ (p,p) & \text{if } \ p \le 1 \cap e \ge c; \end{cases}$$

and $\zeta_L(p) := \min\{p, \theta\}$. For brevity, we focus on the potential dominant case where the firm induces partial repurchase, corresponding to the region $p-1+c \le e \le c$. Then we analyze the following two sub-cases based on the price relative to θ.

(1) If $p \le \theta$, the profit function is $\pi_L(p,e) = p(1+\theta-2p) + (p-e)(1-c+e-p)$, which is jointly concave. Solving the first-order conditions yields the interior solution $(p^*, e^*) = (\frac{1+\theta}{4}, \frac{\theta+2c-1}{4})$.

(1.1) If $\max\{1-2c, \frac{1}{3}\} \le \theta \le 1$, the interior solution is feasible and $\pi_L^* = \frac{\theta^2+2\theta+2c^2-4c+3}{8}$.

(1.2) If $\frac{2-c}{5} \le \theta \le 1-2c$, the constraint $e \ge 0$ binds, resulting in $(p^*, e^*) = (\frac{2+\theta-c}{6}, 0)$ and the firm's profits $\pi_L^* = \frac{(2+\theta-c)^2}{12}$.

(2) If $p \ge \theta$, the profit function is $\pi_L(p,e) = p(1-p) + (p-e)(1-c+e-p)$, which is jointly concave. Solving the first-order conditions yields the interior solution $(p^*, e^*) = (\frac{1}{2}, \frac{c}{2})$ and the firm's profits $\pi_L^* = \frac{c^2-2c+2}{4}$, which is feasible if $\theta \le \frac{1}{2}$.

Comparing the optimal profits across these cases yields the following thresholds:

$$\hat{\theta}_{L1} = \begin{cases} \sqrt{3(c^2-2c+2)} + c - 2 & \text{if } 0 \le c \le \frac{2-\sqrt{2}}{2}, \\ \sqrt{2} - 1 & \text{if } \frac{2-\sqrt{2}}{2} \le c \le 1, \end{cases}$$

and $\hat{\theta}_{L2} = 1-2c$, which establishes Lemma B5. $\square$

Proof of Lemma B6. Referring to Proof of Lemma A3, for customers arriving in the first period, they purchase in the first period if and only if $v \ge \hat{\tau}_{T1} := p$; and purchase in the second period if and only if $v \ge \hat{\tau}_{T2} := p - e + e_M + c$. For customers arriving in the second period, they purchase if and only if $v \ge \hat{\zeta}_T := p + e - e_M$. Restricting these thresholds to the valuation support v, we have $\tau_{T1} = \min\{\max\{\hat{\tau}_{T1}, 0\}, 1\}$, $\tau_{T2} = \min\{\max\{\hat{\tau}_{T2}, 0\}, 1\}$, and $\zeta_T = \min\{\max\{\hat{\zeta}_T, 0\}, \theta\}$.

In the secondary market, the token supply is $S_t = \tau_{T2} - \tau_{T1}$ and the demand is $D_t = \theta - \hat{\zeta}_T$. By the market clearing rule, we analyze the following cases.

(1) **Interior Token Price ($0 < e_M^* < e$).** When supply and demand intersect at a positive price, we solve $S_t(e_M) = D_t(e_M)$ for the candidate price e_M°.

(1.1) If $0 \le \hat{\tau}_{T1} \le \hat{\tau}_{T2} \le 1$ and $0 \le \hat{\zeta}_T \le \theta$, then $S_t = \hat{\tau}_{T2} - \hat{\tau}_{T1}$ and $D_t = \theta - \hat{\zeta}_T$. Solving $S_t = D_t$ yields the candidate price $e_M^\circ = \frac{\theta - p + 2e - c}{2}$. Substituting this into the constraints yields the segmentation vector corresponding to Case 5 in Lemma B6.

(1.2) If $0 \le \hat{\tau}_{T1} \le 1 \le \hat{\tau}_{T2}$ and $0 \le \hat{\zeta}_T \le \theta$, then $S_t = 1 - \hat{\tau}_{T1}$ and $D_t = \theta - \hat{\zeta}_T$. Solving $S_t = D_t$ yields the candidate price $e_M^\circ = \theta + 1 - e$. Substituting this into the constraints yields the segmentation vector corresponding to Case 7 in Lemma B6.

(1.3) If $0 \le \hat{\tau}_{T1} \le \hat{\tau}_{T2} \le 1$ and $\hat{\zeta}_T \ge \theta$, then $S_t = \hat{\tau}_{T2} - \hat{\tau}_{T1}$ and $D_t = 0$. Solving $S_t = D_t$ yields the candidate price $e_M^\circ = e - c$. Substituting this into the constraints yields the segmentation vector corresponding to Case 8 in Lemma B6.

(2) **Corner Token Price ($e_M^* = 0$).** If the candidate interior price $e_M^\circ \le 0$, or if demand is naturally zero (i.e., $\tau_{T2}^L \ge \tau_{T1}$ at $e_M = 0$), the market clears at the lower bound $e_M^* = 0$. Substituting $e_M^* = 0$ into the thresholds yields the segmentation vectors corresponding Cases 2, 3, 4, 6 in Lemma B6.

(3) **Corner Token Price ($e_M^* = e$).** If the candidate interior price $e_M^\circ \geq e$, the market clears at the upper bound $e_M^* = e$. Substituting $e_M^* = e$ into the thresholds yields the segmentation vectors corresponding Case 1 in Lemma B6. $\square$

Proof of Lemma B7. The firm's profit function is $\pi_T = p(1-\tau_{T1}) + p(1-\tau_{T2}+\theta-\zeta_T) - e\min\{1-\tau_{T1}, 1-\tau_{T2}+\theta-\zeta_T\}$, where τ_{T1}, τ_{T2}, and ζ_T are given in Lemma B6. For brevity, the algebraic derivations for the dominated cases (Cases 2, 5, 7, 8 in Lemma B6) are omitted, and we focus on the four dominant regions (Cases 1, 3, 4, 6).

(1) **Case 1 in Lemma B6**. The purchase thresholds are $(\tau_{T1}, \tau_{T2}, \zeta_T) = (p, p+c, p)$, and the profit function simplifies to $\pi_T(p,e) = p(2-3p+\theta-c) - e(1-p)$, decreasing with e. Solving the maximization problem yields $(p^*, e^*) = (\frac{2+\theta-c}{6}, 0)$, which is feasible if $\frac{5c+2}{5} \leq \theta \leq 1$. The results correspond to Lemma B7(iv).

(2) **Case 3 in Lemma B6**. The purchase thresholds are $(\tau_{T1}, \tau_{T2}, \zeta_T) = (p, p-e+c, \theta)$, and the profit function is $\pi_T(p,e) = -e^2 + p(2-c-2p) - e(1-c-2p)$, which is jointly concave. Solving the first-order conditions yields the candidate interior solution $(p^*, e^*) = (\frac{1}{2}, \frac{c}{2})$, which is feasible if $\theta \leq \frac{1-c}{2}$. The results correspond to Lemma B7(i).

(3) **Case 4 in Lemma B6**. The purchase thresholds are $(\tau_{T1}, \tau_{T2}, \zeta_T) = (p, p-e+c, p-e)$, and the profit function is $\pi_T(p,e) = -2e^2 + e(p-\theta+c-1) + p(2-3p+\theta-c)$, which is jointly concave. Solving the first-order conditions yields the candidate interior solution $(p^*, e^*) = (\frac{1}{2}, \frac{1-\theta+c}{4})$.

   (3.1) If $\frac{1-c}{3} \leq \theta \leq \min\{c, 3(1-c)\}$, the interior solution is feasible. The results correspond to Lemma B7(ii).

   (3.2) If $\max\{\frac{3-5c}{7}, c\} \leq \theta \leq \min\{\frac{7c+3}{7}, \frac{9-7c}{5}\}$, the constraint $e \leq \frac{p-\theta+c}{2}$ binds, resulting in $(p_T^*, e_T^*) = (\frac{3-\theta+c}{6}, \frac{3-7\theta+7c}{12})$. The results correspond to Lemma B7(iii).

(4) **Case 6 in Lemma B6**. The purchase thresholds are $(\tau_{T1}, \tau_{T2}, \zeta_T) = (p, 1, p-e)$, and the profit function is $\pi_T(p,e) = -e^2 + e(2p-\theta) + p(1-2p+\theta)$, which is jointly concave. Solving the first-order conditions yields the candidate interior solution $(p^*, e^*) = (\frac{1}{2}, \frac{1-\theta}{2})$, which is feasible if $2(1-c) \leq \theta \leq 1$. The results correspond to Lemma B7(v).

Comparing the optimal profits across these regions defines the following thresholds:

$$\hat{\theta}_{T1} = \begin{cases} \sqrt{3(c^2-2c+2)}+c-2 & \text{if } 0 \leq c \leq \frac{3-\sqrt{6\sqrt{2}}}{3}, \\ \sqrt{6(c^2-2c+2)}+c-3 & \text{if } \frac{3-6\sqrt{2}}{3} < c \leq \frac{2-\sqrt{2}}{2}, \\ (\sqrt{2}-1)(1-c) & \text{if } \frac{2-\sqrt{2}}{2} < c \leq 1, \end{cases} \quad \hat{\theta}_{T2} = \begin{cases} \frac{\sqrt{3(c^2-4c+2)}-c+2}{2} & \text{if } 0 \leq c \leq \frac{3-3\sqrt{2}+\sqrt{6\sqrt{2}}}{3}, \\ \frac{\sqrt{6(c^2-6c+4)}-c+3}{5} & \text{if } \frac{3-3\sqrt{2}+\sqrt{6\sqrt{2}}}{3} < c \leq \frac{\sqrt{2}}{2}, \\ (\sqrt{2}+1)(1-c) & \text{if } \frac{\sqrt{2}}{2} < c \leq 1, \end{cases}$$

$\hat{\theta}_{T3} = c$, and $\hat{\theta}_{T4} = c + \sqrt{2} - 1$, which establishes Lemma B7. $\square$

E.5. Proofs for Results in Appendix C

Proof of Lemma C8. The profit function can be decomposed into two components: the profit from the initial cohort, denoted as $\pi_{\text{initial}}(p,e) = p(1-\tau_{L1}(p,e)) + (p-e)(1-\tau_{L2}(p,e))$, and the profit from the new second-period arrivals, denoted as $\pi_{\text{new}}(p) = \beta p(1-\zeta_L)$. From Proposition 2, $\pi_{\text{initial}}(p,e)$ is maximized at $(p^*,e^*) = (\frac{1}{2}, \frac{1-\delta}{2})$, yielding a profit of $\frac{1+\delta}{4}$. For the new cohort, the profit term $\pi_{\text{new}}(p) = \beta p(1-p)$ is strictly concave in p and is maximized at $p = \frac{1}{2}$. Therefore, the global optimal solution for the combined profit function remains $(p_L^*, e_L^*) = (\frac{1}{2}, \frac{1-\delta}{2})$, resulting in $\pi_L^* = \frac{1+\delta+\beta}{4}$. $\square$

E.6. Proofs for Results in Appendix D.

Proof of Lemma D9. We solve the problem backward, starting from the second period. Given the firm's decisions (p_1, e) and customers' choice τ_{L1} in the first period, in the second period, customers' purchase thresholds are $\tau_{L2}^L = \min\{p_2, \tau_{L1}\}$ and $\tau_{L2}^H = \min\{\max\{\frac{p_2-e_2}{\delta}, \tau_{L1}\}, 1\}$, which can be detailed in Table E1. The firm decides p_2 to maximize the profit in period 2, given by $\pi_{L2}(p_2) = p_2(\tau_{L1} - \tau_{L2}^L) + (p_2 - e)(1-\tau_{L2}^H)$.

Table E1 Customer Purchase Behavior in the Second Period

	τ_{L2}^L	τ_{L2}^H
(1) $0 \le p_2 \le \min\{\tau_{L1}, e+\delta\tau_{L1}\}$	p_2	τ_{L1}
(2) $\tau_{L1} \le p_2 \le e+\delta\tau_{L1}$	τ_{L1}	τ_{L1}
(3) $e+\delta\tau_{L1} \le p_2 \le \min\{\tau_{L1}, \delta+e\}$	p_2	$\frac{p_2-e}{\delta}$
(4) $\max\{e+\delta\tau_{L1}, \tau_{L1}\} \le p_2 \le \delta+e$	τ_{L1}	$\frac{p_2-e}{\delta}$
(5) $\delta+e \le p_2 \le \tau_{L1}$	p_2	1
(6) $p_2 \ge \max\{\delta+e, \tau_{L1}\}$	τ_{L1}	1

Given that the demand function is piecewise defined, we solve the constrained maximization problem in the second period for each case, deriving the corresponding interior or boundary solutions. The specific intervals for τ_{L1} associated with each sub-case are determined by the active constraints of the respective demand segments. The results are summarized in Table E2.

In the first period, the firm decides (p_1, e_1) to maximize the total profit over both periods. We can rewrite the maximization problem as $\max_{\tau_{L1},e} \pi_L(\tau_{L1}, e) = p_1(\tau_{L1}, e)(1-\tau_{L1}) + p_2^*(\tau_{L1} - \tau_{L2}^{L^*}) + (p_2^* - e)(1-\tau_{L2}^{H^*})$. We analyze the profit function under each candidate case from Table E2 to derive candidate optimal solutions. We then compare these candidates to determine the unique global

Table E2 Candidate Sub-equilibrium Results in the Second Period

	p_2^*	Implied $p_1(\tau_{L1},e)$	$\tau_{L2}^{L^*}$	$\tau_{L2}^{H^*}$
(1) $\max\{\frac{1}{2},\frac{1-2e}{2\delta}\}\le\tau_{L1}\le 1$	$\frac{1}{2}$	$\alpha(1-\delta)\tau_{L1}+\alpha e$	$\frac{1}{2}$	τ_{L1}
(2) $\frac{\delta+2e}{2+\delta}\le\tau_{L1}\le\frac{1-2e}{1+2\delta}$	$\frac{2e+\delta(1+\tau_{L1})}{2(1+\delta)}$	$\tau_{L1}+\frac{\alpha(\delta+2e-(2+\delta)\tau_{L1})}{2(1+\delta)}$	$\frac{2e+\delta(1+\tau_{L1})}{2(1+\delta)}$	$\frac{1-2e+\tau_{L1}}{2(1+\delta)}$
(3) $\max\{\frac{1-2e}{1+2\delta},\frac{e}{1-\delta}\}\le\tau_{L1}\le\frac{1-2e}{2\delta}$	$e+\delta\tau_{L1}$	$\tau_{L1}+\alpha(e-(1-\delta)\tau_{L1})$	$e+\delta\tau_{L1}$	τ_{L1}
(4) $0\le\tau_{L1}\le\min\{\frac{\delta}{2}+e,\frac{1}{2}\}$	$\frac{\delta}{2}+e$	τ_{L1}	τ_{L1}	$\frac{1}{2}$
(5) $\frac{1}{2}\le\tau_{L1}\le\min\{\frac{e}{1-\delta},1\}$	$e+\delta\tau_{L1}$	τ_{L1}	τ_{L1}	τ_{L1}
(6) $2(e+\delta)\le\tau_{L1}\le 1$	$\frac{\tau_{L1}}{2}$	$\frac{(2-\alpha)\tau_{L1}}{2}$	$\frac{\tau_{L1}}{2}$	1

optimum. For brevity, the algebraic derivations for the dominated cases are omitted, and we detail the dominant Case (3) as follows.

In Case (3), the maximization problem is $\pi(\tau_{L1},e)=-e^2+\alpha(1-\tau_{L1})(e-(1-\delta)\tau_{L1})+e(1-2\delta)\tau_{L1}+\tau_{L1}(1+\delta-(1+\delta^2)\tau_{L1})$, which is jointly concave. By KKT conditions, we obtain the following results.

- If $0\le\delta\le\frac{-1+2\alpha+\alpha^2+\sqrt{9-4\alpha-6\alpha^2+4\alpha^3+\alpha^4}}{4}$, the optimal solutions and profits are $(\tau_{L1}^*,e^*,\pi_L^*)=(\frac{2(1+\delta)-\alpha(1+\alpha)}{4\delta+3-\alpha(2+\alpha)},\frac{1-\alpha^2-\delta+\alpha(2+\alpha)\delta-2\delta^2}{4\delta+3-\alpha(2+\alpha)},\frac{(1+\delta)(1-\alpha+\delta)}{3+4\delta-\alpha(2+\alpha)})$.
- If $\frac{-1+2\alpha+\alpha^2+\sqrt{9-4\alpha-6\alpha^2+4\alpha^3+\alpha^4}}{4}\le\delta\le 1$, the optimal solutions and profits are $(\tau_{L1}^*,e^*,\pi_L^*)=(\frac{1+\delta-\alpha(1-\delta)}{2(1+\delta^2-\alpha(1-\delta))},0,\frac{(1+\delta-\alpha(1-\delta))^2}{4(1+\delta^2-\alpha(1-\delta))})$.

Substituting the optimal solutions (τ_{L1}^*,e^*) into the conditions in Table E2, we can easily prove that the sub-equilibrium consistently aligns with Case (3) and does not deviate to other cases in the second period. Therefore, the optimal solutions (τ_{L1}^*,e^*) in case (3) are unique equilibrium solutions. Define $\hat{\delta}_{L1}(\alpha)=\frac{-1+2\alpha+\alpha^2+\sqrt{9-4\alpha-6\alpha^2+4\alpha^3+\alpha^4}}{4}$, and thus Lemma D9 is proved. □

Proof of Lemma D10. We solve the problem backward starting from the second period. Given the firm's decisions (p_1,e) and customers' choice τ_{T1} in the first period, in the second period, customers' purchase thresholds are $\tau_{T2}^L=\max\{\min\{p_2-e+e_M,\tau_{T1}\},0\}$ and $\tau_{T2}^H=\min\{\max\{\frac{p_2-e+e_M}{\delta},\tau_{T1}\},1\}$. By the market clearing rule, we can solve the equilibrium token e_M^*. Table E3 summarizes the resulting purchase behaviors and reward redemption amount $d_r(e,\tau_{T1},p_2)=\min\{\tau_{T1}-\tau_{T2}^L+1-\tau_{T2}^H,1-\tau_{T1}\}$.

Given customers' purchase behavior, the firm sets p_2 to maximize its second-period profit.

$$\max_{p_2}\ \pi_{T2}(p_2)=p_2(\tau_{T1}-\tau_{T2}^L+1-\tau_{T2}^H)-e\cdot d_r.$$

Solving the optimization problem across all market regimes yields the candidate sub-equilibrium results summarized in Table E4.

Table E3 Customer Purchase Behavior in the Second Period

	e_M^*	τ_{T2}^L	τ_{T2}^H	d_r
(1) $0 \le p_2 \le \delta\tau_{T1}$	e	p_2	τ_{T1}	$1-\tau_{T1}$
(2) $\delta\tau_{T1} \le p_2 \le \min\{\frac{2\delta\tau_{T1}}{1+\delta}, \delta\}$	e	p_2	$\frac{p_2}{\delta}$	$1-\tau_{T1}$
(3) $\delta \le p_2 \le 2\tau_{T1}-1$	e	p_2	1	$1-\tau_{T1}$
(4) $0 \le \tau_{T1} \le \frac{1+\delta}{2} \cap \frac{2\delta\tau_{T1}}{1+\delta} \le p_2 \le e+\frac{2\delta\tau_{T1}}{1+\delta}$	$e-p_2+\frac{2\delta\tau_{T1}}{1+\delta}$	$\frac{2\delta\tau_{T1}}{1+\delta}$	$\frac{2\tau_{T1}}{1+\delta}$	$1-\tau_{T1}$
(5) $\frac{1+\delta}{2} \le \tau_{T1} \le 1 \cap 2\tau_{T1}-1 \le p_2 \le e+2\tau_{T1}-1$	$e-p_2+2\tau_{T1}-1$	$2\tau_{T1}-1$	1	$1-\tau_{T1}$
(6) $e+\frac{2\delta\tau_{T1}}{1+\delta} \le p_2 \le \min\{e+\tau_{T1}, e+\delta\}$	0	p_2-e	$\frac{p_2-e}{\delta}$	$\tau_{T1}-\tau_{T2}^L+1-\tau_{T2}^H$
(7) $\max\{e+\delta, e+2\tau_{T1}-1\} \le p_2 \le e+\tau_{T1}$	0	p_2-e	1	$\tau_{T1}-\tau_{T2}^L$
(8) $e+\tau_{T1} \le p_2 \le e+\delta$	0	τ_{T1}	$\frac{p_2-e}{\delta}$	$1-\tau_{T2}^H$
(9) $p_2 \ge \max\{e+\tau_{T1}, e+\delta\}$	0	τ_{T1}	1	0

Table E4 Candidate Sub-equilibrium Results in the Second Period

	p_2^*	Implied $p_1(\tau_{T1}, e)$	$\tau_{T2}^{L^*}$	$\tau_{T2}^{H^*}$	d_r^*	e_M^*
(1) $\frac{1}{1+2\delta} \le \tau_{T1} \le \frac{1}{2\delta}$	$\delta\tau_{T1}$	$\tau_{T1}+\alpha(e-(1-\delta)\tau_{T1})$	p_2	τ_{T1}	$1-\tau_{T1}$	e
(2) $\frac{1}{2\delta} \le \tau_{T1} \le 1$	$\frac{1}{2}$	$\tau_{T1}+\alpha(e-(1-\delta)\tau_{T1})$	p_2	τ_{T1}	$1-\tau_{T1}$	e
(3) $\frac{1}{3} \le \tau_{T1} \le \frac{1}{1+2\delta}$	$\frac{\delta(1+\tau_{T1})}{2(1+\delta)}$	$\tau_{T1}+\frac{\alpha(\delta+2(1+\delta)e-(2+\delta)\tau_{T1})}{2(1+\delta)}$	p_2	$\frac{p_2}{\delta}$	$1-\tau_{T1}$	e
(4) $\max\{\frac{2}{3}, 2\delta\} \le \tau_{T1} \le 1$	$\frac{\tau_{T1}}{2}$	$\tau_{T1}+\alpha(e-\frac{\tau_{T1}}{2})$	p_2	1	$1-\tau_{T1}$	e
(5) $\frac{\delta}{2+\delta} \le \tau_{T1} \le \frac{1}{3}$	$e+\frac{\delta(1+\tau_{T1})}{2(1+\delta)}$	$\tau_{T1}+\frac{\alpha(\delta-(2+\delta)\tau_{T1})}{2(1+\delta)}$	p_2-e	$\frac{p_2-e}{\delta}$	$\frac{1+\tau_{T1}}{2}$	0
(6) $\frac{1}{3} \le \tau_{T1} \le \frac{1+\delta}{2}$	$e+\frac{2\delta\tau_{T1}}{1+\delta}$	$\frac{((1+\delta)-\alpha(1-\delta))\tau_{T1}}{1+\delta}$	p_2-e	$\frac{p_2-e}{\delta}$	$1-\tau_{T1}$	0
(7) $2\delta \le \tau_{T1} \le \frac{2}{3}$	$e+\frac{\tau_{T1}}{2}$	$\frac{(2-\alpha)\tau_{T1}}{2}$	p_2-e	1	$\frac{\tau_{T1}}{2}$	0
(8) $\max\{\frac{2}{3}, \frac{1+\delta}{2}\} \le \tau_{T1} \le 1$	$e+2\tau_{T1}-1$	$\tau_{T1}-\alpha(1-\tau_{T1})$	p_2-e	1	$1-\tau_{T1}$	0
(9) $0 \le \tau_{T1} \le \frac{\delta}{2}$	$e+\frac{\delta}{2}$	τ_{T1}	τ_{T1}	$\frac{p_2-e}{\delta}$	$\frac{1}{2}$	0

In the first period, the firm decides (p_1, e_1) to maximize the total profit over both periods. We can rewrite the maximization problem as

$$\max_{\tau_{T1}, e} \pi_T(\tau_{T1}, e) = p_1(\tau_{T1}, e)(1-\tau_{T1}) + p_2^*(\tau_{T1} - \tau_{T2}^{L^*} + 1 - \tau_{T2}^{H^*}) - e d_r^*.$$

We analyze the profit function under each candidate case from Table E4 to derive candidate optimal solutions. We then compare these candidates to determine the unique global optimum. For brevity, we detail the three dominant cases from Table E4 (Cases (1), (3), and (8)) which correspond to the global optima.

(1) **Case (1) from Table E4**. The profit function is $\pi(\tau_{T1}, e) = \delta\tau_{T1}(1-\delta\tau_{T1}) + (1-\tau_{T1})(\tau_{T1} + \alpha(e-(1-\delta)\tau_{T1})) - e(1-\tau_{T1})$. Solving the constrained maximization yields $(\tau_{T1}^*, e^*, \pi_L^*) =$

$(\frac{1+\delta-\alpha(1-\delta)}{2(1+\delta^2-\alpha(1-\delta))}, 0, \frac{(1+\delta-\alpha(1-\delta))^2}{4(1+\delta^2-\alpha(1-\delta))})$. This solution is feasible if $\frac{\sqrt{(9-\alpha)(1-\alpha)}-3(1-\alpha)}{4\alpha} \leq \delta \leq 1$. This corresponds to Lemma D10(iii).

(2) **Case (3) from Table E4**. The profit is $\pi(\tau_{T1}, e) = \frac{\delta+2\alpha\delta(1-\tau_{T1})^2+3\delta(2-\tau_{T1})\tau_{T1}+4(1-\alpha)(1-\tau_{T1})\tau_{T1}}{4(1+\delta)} - e(1-\alpha)(1-\tau_{T1})$. The optimal solution is $(\tau^*_{T1}, e^*, \pi^*_L) = (1 - \frac{2(1-\alpha)}{4+3\delta-2\alpha(2+\delta)}, 0, \frac{\alpha^2-2\alpha(1+\delta)^2+(1+\delta)(1+3\delta)}{(1+\delta)(4+3\delta-2\alpha(2+\delta))})$, feasible if $0 \leq \delta \leq \frac{-1+\alpha+\sqrt{(1-\alpha)(4-3\alpha)}}{3-2\alpha}$. This corresponds to Lemma D10(ii).

(3) **Case (8) from Table E4**. The profit function is $\pi(\tau_{T1}) = (1-\tau_{T1})(-1-\alpha(1-\tau_{T1})+3\tau_{T1})$, which is concave. The optimal solution is $(\tau^*_{T1}, \pi^*_L) = (\frac{2+\alpha}{3+\alpha}, \frac{1}{3+\alpha})$, feasible if $0 \leq \delta \leq \frac{1+\alpha}{3+\alpha}$. This corresponds to Lemma D10(i).

Comparing the profits across these cases defines the thresholds $\tilde{\delta}_{T1}(\alpha)$, $\tilde{\delta}_{T2}(\alpha)$, and $\tilde{\delta}_{T3}(\alpha)$:

$$\tilde{\delta}_{T1}(\alpha) = \frac{-5+2\alpha+4\alpha^2+\sqrt{49-4\alpha^2(2+\alpha)(3+\alpha)(3-2\alpha)}}{2(2+\alpha)(3-2\alpha)},$$
$$\tilde{\delta}_{T2}(\alpha) = \frac{-3+\alpha+3\alpha^2+\alpha^3+2\sqrt{2-\alpha^2}}{-1+7\alpha+5\alpha^2+\alpha^3},$$
$$\tilde{\delta}_{T3}(\alpha) = \min\{\delta | -(1-\alpha)(1+2\alpha)+(1-\alpha)(5+6\alpha)\delta+(3+\alpha)(2\alpha-1)\delta^2-(3+\alpha)(3-2\alpha)\delta^3 = 0\}.$$

There exists a threshold $\hat{\alpha}$ such that $\hat{\delta}_{T1} \leq \tilde{\delta}_{T2} \leq \hat{\delta}_{T3}$ with $\alpha \in [0, \hat{\alpha}]$ and $\hat{\delta}_{T1} \geq \tilde{\delta}_{T2} \geq \hat{\delta}_{T3}$ with $\alpha \in [\hat{\alpha}, 1]$. Accordingly, we define thresholds:

$$(\hat{\delta}_{T1}, \hat{\delta}_{T2}) = \begin{cases} (\tilde{\delta}_{T1}, \tilde{\delta}_{T3}) & \text{if } 0 \leq \alpha \leq \hat{\alpha}, \\ (\tilde{\delta}_{T2}, \tilde{\delta}_{T2}) & \text{if } \hat{\alpha} \leq \alpha \leq 1. \end{cases} \tag{E7}$$

Then, we can obtain the optimal regime for each range of δ, as shown in Lemma D10.

Finally, we verify the no-deviation conditions by substituting the optimal solutions back into the candidate sub-equilibrium results in the second period. For Lemma D10 (i) and (ii), the sub-equilibrium is consistent. For Lemma D10(i), consistency requires $e \geq \hat{e}_T(\alpha, \delta)$, where:

$$\hat{e}_T(\alpha,\delta) = \begin{cases} \frac{\alpha^2}{4(3+\alpha)} & \text{if } 0 \leq \delta \leq \min\left\{\frac{(2+\alpha)^2}{(3+\alpha)(7+3\alpha)}, \frac{3+\alpha-\sqrt{5+2\alpha}}{2(2+\alpha)}\right\}, \\ \frac{(21+4\alpha(4+\alpha))\delta-4(1+\alpha)}{4(3+\alpha)(1+\delta)} & \text{if } \frac{(2+\alpha)^2}{(3+\alpha)(7+3\alpha)} \leq \delta \leq \min\left\{\frac{1}{2(2+\alpha)}, \frac{1+\alpha}{5+\alpha}\right\}, \\ \frac{-1-\alpha+(2+\alpha)(3+\alpha)\delta-(2+\alpha)^2\delta^2}{3+\alpha} & \text{if } \max\left\{\frac{1}{2(2+\alpha)}, \frac{3+\alpha-\sqrt{5+2\alpha}}{2(2+\alpha)}\right\} \leq \delta \leq \frac{1+\alpha}{5+\alpha}. \end{cases} \tag{E8}$$

This completes the proof. $\square$

Proof of Lemma D11. From a mechanism design perspective, the profit under full commitment serves as a theoretical upper bound for the profit under limited commitment. If we can prove that the optimal strategy from the base model constitutes a sub-game perfect equilibrium in the dynamic setting — specifically, that given the customer base generated by e^*_1, the firm has no incentive to deviate ex-post (i.e., $e^*_2 = e^*_1$) — then the dynamic equilibrium coincides with the commitment solution.

We prove this by solving the firm's second-period profit maximization problem, taking the first-period outcomes (prices and demand thresholds) from the base model as given.

(1) **Traditional LP.** In the base model, the optimal committed decisions are $(p_L^*, e_L^*) = (\frac{1}{2}, \frac{1-\delta}{2})$, resulting in a first-period customer base defined by the threshold $\tau_{L1}^* = \frac{1}{2}$.

In the dynamic setting, given p_L^* and τ_{L1}^*, customers repurchase in the second period if $U_{L2} = \delta v - p_L^* + e_2 \geq 0$. The repurchase threshold $\tau_{L2}(e_2)$ can be derived as:

$$\tau_{L2}(e_2) = \begin{cases} 1 & \text{if } e_2 \leq \frac{1-2\delta}{2}, \\ \frac{1-2e_2}{2\delta} & \text{if } \frac{1-2\delta}{2} < e_2 < \frac{1-\delta}{2}, \\ \frac{1}{2} & \text{if } e_2 \geq \frac{1-\delta}{2}. \end{cases}$$

The firm chooses e_2 to maximize the second-period profit $\pi_{L2}(e_2) = (p_L^* - e_2)(1 - \tau_{L2})$. It is straightforward to verify that π_{L2} increases in e_2 when $\tau_{L2} > \tau_{L1}^*$ (market expansion) and decreases when $\tau_{L2} = \tau_{L1}^*$ (margin erosion). The maximum is achieved at the boundary where the constraint binds, $e_2^* = \frac{1-\delta}{2}$. Thus, $e_2^* = e_L^*$.

(2) **Tradable LP.** We analyze the two dominant regions identified in the base model.

(2.1) When $0 \leq \delta \leq \frac{\alpha+1}{\alpha+5}$, the base model yields $(p_T^*, e_T^*) = (\frac{2}{\alpha+3}, \frac{1-\alpha}{\alpha+3})$ and $\tau_{T1}^* = \frac{\alpha+2}{\alpha+3}$. In the dynamic setting, given p_T^* and τ_{T1}^*, customers who did not purchase before choose to purchase if and only if $U_{T2}^F = v - p + e_2 - e_M \geq 0$ $(v \geq \tau_{T2}^L := \max\{\min\{p - e_2 + e_M, \tau_{T1}^*\}, 0\})$, while customers who purchased before choose to purchase if and only if $U_{T2}^S = \delta v - p + e_2 \geq e_M$ $(v \geq \tau_{T2}^H := \min\{\max\{\frac{p-e_2+e_M}{\delta}, \tau_{T1}^*\}, 1\})$. The token price e_M is determined by the supply $S_t = \tau_{T2}^H - \tau_{T1}^*$ and the demand $D_t = \tau_{T1}^* - \tau_{T2}^L$. The resulting thresholds as a function of e_2 are:

$$(\tau_{T2}^L, \tau_{T2}^H, e_M^*) = \begin{cases} (\frac{2}{\alpha+3} - e_2,\, 1,\, 0) & \text{if } e_2 \leq \frac{1-\alpha}{\alpha+3}, \\ (\frac{\alpha+1}{\alpha+3},\, 1,\, e_2 - \frac{1-\alpha}{\alpha+3}) & \text{if } e_2 > \frac{1-\alpha}{\alpha+3}. \end{cases}$$

The firm maximizes $\pi_{T2} = p_T^*(1 - \tau_{T2}^H + \tau_{T1}^* - \tau_{T2}^L) - e_2 \min\{1 - \tau_{T1}^*, 1 - \tau_{T2}^H + \tau_{T1}^* - \tau_{T2}^L\}$. Solving this yields the optimal solution $e_2^* = \frac{1-\alpha}{\alpha+3}$, confirming $e_2^* = e_T^*$.

(2.2) When $\frac{\alpha+1}{\alpha+5} < \delta \leq 1$, the base model yields $(p_T^*, e_T^*) = (\frac{(1+\alpha)\delta+1-\alpha}{2(\delta+1)}, \frac{(1-\alpha)(1-\delta)}{2(1+\delta)})$ and $\tau_{T1}^* = \frac{1}{2}$. In the dynamic setting, similar to Case (2.1), customers' purchase thresholds can be derived as:

$$(\tau_{T2}^L, \tau_{T2}^H, e_M^*) = \begin{cases} (\frac{1}{2} - e_2 - \frac{\alpha(1-\delta)}{2(\delta+1)}, 1, 0) & \text{if } \; e_2 \leq \frac{(1-\alpha)(1-\delta)-2\delta^2}{2(1+\delta)}, \\ (\frac{1}{2} - e_2 - \frac{\alpha(1-\delta)}{2(\delta+1)}, \frac{1}{2\delta} - \frac{e_2}{\delta} - \frac{\alpha(1-\delta)}{2\delta(\delta+1)}, 0) & \text{if } \; \frac{(1-\alpha)(1-\delta)-2\delta^2}{2(1+\delta)} \leq e_2 \leq \frac{(1-\alpha)(1-\delta)}{2(1+\delta)}, \\ (\frac{\delta}{\delta+1}, \frac{1}{\delta+1}, e_2 - \frac{(1-\alpha)(1-\delta)}{2(1+\delta)}) & \text{if } \; e_2 \geq \frac{(1-\alpha)(1-\delta)}{2(1+\delta)}. \end{cases}$$

The firm maximizes $\pi_{T2} = p_T^*(1 - \tau_{T2}^H + \tau_{T1}^* - \tau_{T2}^L) - e_2 \min\{1 - \tau_{T1}^*, 1 - \tau_{T2}^H + \tau_{T1}^* - \tau_{T2}^L\}$. Solving this yields the optimal solution $e_2^* = \frac{(1-\alpha)(1-\delta)}{2(1+\delta)}$, confirming $e_2^* = e_T^*$.

In summary, for both traditional and tradable LPs, the firm's optimal ex-post decision e_2^* is to maintain the reward value e_1^* set in the first period, resulting in the same equilibrium result as the basic model under dynamic reward values. $\square$