Electronic Companion

EC.1. State Transition Diagram for Section 4

Figure EC.1 displays a state transition diagram for the age-based model corresponding to Section 4.

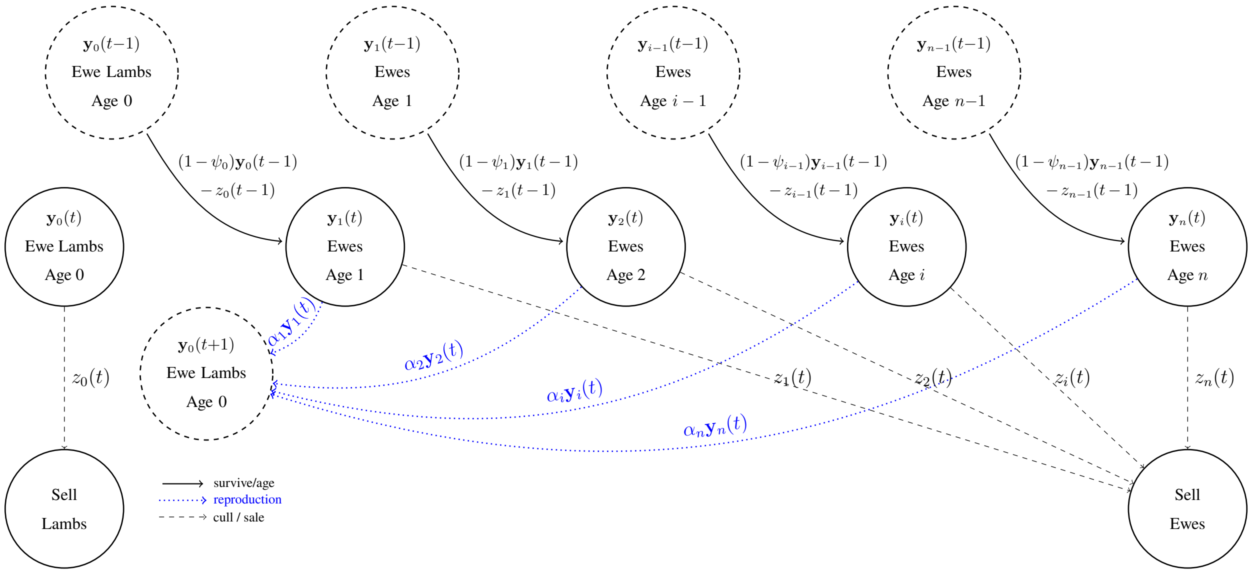

Figure EC.1 **Annual state–transition diagram of ewes in the flock for the deterministic age-structured ewe model (uniform node size). For convenience, let $\mathbf{y}_0(t+1) := \sum_{j=1}^{n} \alpha_j \mathbf{y}_j(t)$.**

EC.2. Proofs from Sections 3 and 4

Proof of Proposition 1. The function $g(\mathbf{x})$ is the sum of the linear, and thus concave, term $-\sum_{i=1}^{n} c_i \mathbf{x}_i$ and the optimal objective function value of the LP (2) where we omit the $\mathbf{x}$ term from the objective function. The condition in that LP that $\mathbf{x} \geq 0$ is simply a restriction of the domain of $\mathbf{x}$. Thus, we can view the LP (2) without the linear term in $\mathbf{x}$ in the objective function as a standard LP

$$\begin{aligned} \max \quad & c^T u \\ \text{s/t} \quad & Au \leq b \\ & u \geq 0, \end{aligned}$$

where the initial procurement vector $\mathbf{x}$ enters only through the right-hand-side vector b. By standard results in LP, the optimal value is a concave and piecewise affine function of the right-hand-side vector b, e.g., see Chapter 5.2 of Bertsimas and Tsitsiklis 1997. ■

Proof of Proposition 2. Let $(\mathbf{y},\mathbf{z})$ be a feasible solution for the full formulation (3). Define

$$(\bar{y},\bar{z}) = \left(T^{-1}\sum_{t=0}^{T-1}\mathbf{y}(t), T^{-1}\sum_{t=0}^{T-1}\mathbf{z}(t)\right)$$

as the average of the time-dependent variables over the full cycle. Then, $(\bar{y},\bar{z})$ is feasible for the steady-state formulation (4). To see why this holds, consider just the first constraint in (2). Averaging over $t=1,2,\ldots,T$ yields

$$(1-\psi_0)\sum_{j=1}^{n}\alpha_j\bar{y}_j - \bar{z}_0 = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_1(t) = \frac{1}{T}\sum_{t=0}^{T-1}\mathbf{y}_1(t) = \bar{y}_1,$$

because of the constraint that $\mathbf{y}_1(T)=\mathbf{y}_1(0)$. A similar argument holds for other constraints. Moreover, the objective function values in the two formulations match. Likewise, given a feasible solution for (4), (y,z), simply define $\mathbf{y}(t)=y$ for $0\le t\le T$ and $\mathbf{z}(t)=z$ for $0\le t<T$, which is then a feasible solution for (2) with the same objective function value. Thus, the optimal objective function value in the two formulations match. ■

Proof of Theorem 1. In the LP formulation (4), the decisions $\{z_j\}_{j=0}^n$ are fully determined by $\{y_j\}_{j=1}^n$, because both the objective function and constraints can be rewritten solely in terms of y, yielding the following reformulation:

$$\max_{y_1,\ldots,y_n} \sum_{i=1}^{n}\Big[\left(p_0+p^R\right)(1-\psi_0)\,\alpha_i + p_i\,(1-\psi_i) - p_{i-1}\Big]y_i$$

$$\text{s/t}\quad (1-\psi_0)\sum_{j=1}^{n}\alpha_j y_j \ge y_1, \tag{EC.1}$$

$$(1-\psi_{i-1})y_{i-1}\ge y_i,\quad i=2,3,\ldots,n, \tag{EC.2}$$

$$\sum_{i=1}^{n} y_i = \ell^*, \tag{EC.3}$$

$$y_i\ge 0,\quad i=1,2,\ldots,n.$$

Let us ignore constraint (EC.1) for the moment. Under the assumptions that $\{p_i\}_{i=0}^n$ are decreasing and convex, $\{\alpha_i\}_{i=1}^n$ are increasing, and $\{\psi_i\}_{i=0}^n$ are decreasing, one can verify that the coefficients $\left\{\left(p_0+p^R\right)(1-\psi_0)\,\alpha_i + p_i\,(1-\psi_i) - p_{i-1}\right\}_{i=1}^n$ in the objective function are increasing in i. Thus, all constraints (EC.2) are tight at the optimal solution y^*, because it is optimal to allocate as much as possible to y_i with a larger coefficient, and $\{y_i^*\}_{i=1}^n$ can be determined from (EC.3). Finally, under the assumption that $(1-\psi_0)\sum_{j=1}^n\alpha_j\zeta_{j-1}-1\ge 0$, the previously obtained solution $\{y_i^*\}_{i=1}^n$ also satisfies (EC.1), making it optimal for the full problem and yielding $z_1^*=\ldots=z_{n-1}^*=0$.

Since $z_1^* = \cdots = z_{n-1}^* = 0$, we have $y_i^* = (1-\psi_{i-1})y_{i-1}^*$ for each $i = 2,\ldots,n$. Recalling that $\zeta_0 = 1$ and for $i = 1,2,\ldots,n$, $\zeta_i = (1-\psi_1)(1-\psi_2)\cdots(1-\psi_i)$ it follows that

$$\begin{aligned}\ell^* &= y_1^* + y_2^* + \ldots + y_n^* \\ &= y_1^*\left[\zeta_0 + \zeta_1 + \cdots + \zeta_{n-1}\right] \\ &= y_1^* R_n,\end{aligned}$$

which implies $y_1^* = R_n^{-1}\ell^*$. Thus,

$$\begin{aligned}z_0^* &= (1-\psi_0)\sum_{j=1}^n \alpha_j y_j^* - y_1^* = \left((1-\psi_0)\sum_{j=1}^n \alpha_j \zeta_{j-1} - 1\right) R_n^{-1}\ell^*, \\ z_n^* &= (1-\psi_n)y_n^* = \zeta_n y_1^* = \zeta_n R_n^{-1}\ell^*,\end{aligned}$$

completing the proof. ■

Proof of Proposition EC.1. We first bound the optimal LP value from above. From (4),

$$\begin{aligned}\sum_{j=0}^n p_j z_j + p^R(z_0+y_1) &\le p_0\sum_{j=0}^n z_j + p_0(1-\psi_0)\sum_{j=1}^n \alpha_j y_j \\ &= p_0\Bigg[\left(2(1-\psi_0)\sum_{j=1}^n \alpha_j y_j - y_1\right) + ((1-\psi_1)y_1 - y_2) + \cdots \\ &\qquad + ((1-\psi_{n-1})y_{n-1} - y_n) + (1-\psi_n)y_n\Bigg] \\ &= p_0\left[2(1-\psi_0)\sum_{j=1}^n \alpha_j y_j - \sum_{j=1}^n \psi_j y_j\right] \\ &\le p_0\left[\max_{j=1,\ldots,n}\{2(1-\psi_0)\alpha_j - \psi_j\}\sum_{j=1}^n y_j\right] \\ &= p_0 \max_{j=1,\ldots,n}\{2(1-\psi_0)\alpha_j - \psi_j\}\,\ell^*.\end{aligned}$$

In addition, the value of the ML policy equals

$$\begin{aligned}p_0 z_0 + p_n z_n + p^R(z_0+y_1) &\ge p_0\left(2z_0 + y_1\right) \\ &= p_0\left(2(1-\psi_0)\sum_{j=1}^n \alpha_j\zeta_{j-1} - 1\right) R_n^{-1}\ell^*,\end{aligned}$$

where we use the fact that Theorem 1 gives an expression for z_0 and its proof establishes that $y_1 = R_n^{-1}\ell^*$. The two bounds, above, yield the result. ■

EC.3. Proofs from Section 5

Proof of Theorem 2. For $t \geq 0$, let $\mathcal{F}_t$ be the sigma field generated by $(\mathbf{Y}(k), \mathbf{L}(k), N(k), \mathbf{Z}(k), \mathcal{L}^R(k) : k = 0, 1, \ldots, t)$ and let $\mathcal{F} = (\mathcal{F}_t : t \geq 0)$ be the associated filtration. Fix any $i \in \{1, 2, \ldots, n\}$. For $t \geq 0$, define $M_t = \sum_{k=0}^t (\psi_i \mathbf{Y}_i(k) - \mathbf{L}_i(k))$. Then, $M = (M_t : t \geq 0)$ is a martingale with respect to the filtration $\mathcal{F}$. Our assumptions ensure that the martingale differences are bounded in absolute value by K, and hence we can apply Azuma's inequality, e.g., Ross (1996, p. 307), to get that for any $T \geq 1$ and any $\delta > 0$,

$$\begin{aligned} \mathbb{P}\left(|\psi_i \bar{\mathbf{Y}}_i(T) - \bar{\mathbf{L}}_i(T)| > \delta\right) &= \mathbb{P}(|M_{T-1}| > \delta T) \\ &\leq 2 \exp\left(\frac{-\delta^2 T}{2K^2}\right). \end{aligned}$$

The right-hand side of this bound is summable in T, and thus by the Borel-Cantelli Lemma (see, e.g., Ross 1996, p. 4),

$$\psi_i \bar{\mathbf{Y}}_i(T) - \bar{\mathbf{L}}_i(T) \to 0 \tag{EC.4}$$

as $T \to \infty$ a.s. Combining this fact for $i = n$ with our assumption that age-n sheep are sold, which implies that $\bar{\mathbf{Y}}_n(T) - \bar{\mathbf{L}}_n(T) = \bar{\mathbf{Z}}_n(T)$, gives

$$(1 - \psi_n) \bar{\mathbf{Y}}_n(T) - \bar{\mathbf{Z}}_n(T) \to 0 \tag{EC.5}$$

as $T \to \infty$ a.s. In other words, the third constraint of the LP (4) holds asymptotically.

Now, by the conservation of the flow of ewes from (6), for $i = 2, 3, \ldots, n$, we have

$$\bar{\mathbf{Y}}_{i-1}(T) - \bar{\mathbf{L}}_{i-1}(T) - \bar{\mathbf{Z}}_{i-1}(T) + a_i \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{I}(D(t)) - \bar{\mathbf{Y}}_i(T) + \frac{\mathbf{Y}_i(0) - \mathbf{Y}_i(T)}{T} = 0.$$

We therefore get

$$\limsup_{T \to \infty} \left| \bar{\mathbf{Y}}_{i-1}(T) - \bar{\mathbf{L}}_{i-1}(T) - \bar{\mathbf{Z}}_{i-1}(T) - \bar{\mathbf{Y}}_i(T) \right| \in [0, a_i].$$

Then, using (EC.4), for $i = 2, 3, \ldots, n$, almost surely

$$\limsup_{T \to \infty} \left| (1 - \psi_{i-1}) \bar{\mathbf{Y}}_{i-1}(T) - \bar{\mathbf{Z}}_{i-1}(T) - \bar{\mathbf{Y}}_i(T) \right| \in [0, a_i].$$

We can show, using the same martingale argument above, that

$$\sum_{j=1}^n \alpha_j \bar{\mathbf{Y}}_j(T) - \bar{N}(T) \to 0, \text{ and}$$

$$\psi_0 \sum_{j=1}^{n} \alpha_j \bar{\mathbf{Y}}_j(T) - \bar{\mathbf{L}}_0(T) \to 0$$

as $T \to \infty$ a.s. Combining these observations with conservation of ewe lambs in (7) gives

$$\limsup_{T\to\infty} \left| (1-\psi_0) \sum_{j=1}^{n} \alpha_j \bar{\mathbf{Y}}_j(T) - \bar{\mathbf{Z}}_0(T) - \bar{\mathbf{Y}}_1(T) \right| \in [0, a_1]$$

as $T \to \infty$ a.s. The same argument for ewe lambs can be applied to ram lambs to establish that

$$\limsup_{T\to\infty} \left| (1-\psi_0) \sum_{j=1}^{n} \alpha_j \bar{\mathbf{Y}}_j(T) - \bar{\mathcal{L}}^R(T) \right| \in [0, a_1],$$

which implies that

$$\limsup_{T\to\infty} \left| p^R(\bar{\mathbf{Z}}_0(T) + \bar{\mathbf{Y}}_1(T)) - p^R \bar{\mathcal{L}}^R(T) \right| \le p^R a_1. \tag{EC.6}$$

We have thus shown that for $i = 2, 3, \ldots, n$, almost surely,

$$\limsup_{T\to\infty} \left| (1-\psi_{i-1}) \bar{\mathbf{Y}}_{i-1}(T) - \bar{\mathbf{Z}}_{i-1}(T) - \bar{\mathbf{Y}}_i(T) \right| \in [0, a_i] \quad \text{and} \tag{EC.7}$$

$$\limsup_{T\to\infty} \left| (1-\psi_0) \sum_{j=1}^{n} \alpha_j \bar{\mathbf{Y}}_j(T) - \bar{\mathbf{Z}}_0(T) - \bar{\mathbf{Y}}_1(T) \right| \in [0, a_1]. \tag{EC.8}$$

From (EC.6), the contribution to revenue from ram lamb sales coincides with the corresponding term in (4) almost surely as $T \to \infty$. From (5), (EC.5), (EC.7) and (EC.8), almost surely as $T \to \infty$, $(\bar{\mathbf{Y}}(T), \bar{\mathbf{Z}}(T))$ satisfies the constraints of the LP (4) except for an error in the right-hand side values that is bounded by $g_1(\ell^*) + \|a\|$. Hoffman's inequality (Hoffman 1952) then implies that, almost surely as $T \to \infty$, $(\bar{\mathbf{Y}}(T), \bar{\mathbf{Z}}(T))$ is at most a distance $c_0(g_1(\ell^*) + \|a\|)$ away from the feasible region of the LP for some constant c_0 that does not depend on ℓ^*. Thus, since the objective function is linear, we conclude that, almost surely as $T \to \infty$, the objective function of $(\bar{\mathbf{Y}}(T), \bar{\mathbf{Z}}(T))$ is at most the optimal objective value of the LP plus $c_3(g_1(\ell^*) + \|a\|)$, where

$$c_3 = c_0 \max\left\{ \max_{i=0}^{n} p_i,\ p^R \right\}.$$

■

The proof of Theorem 3 requires a series of lemmas. Our first result proves that, starting from the target state, there is a very large probability that we will return to the target state in one transition.

LEMMA EC.1. *Assume A4-A14. Then, there exist $\theta_1, \theta_2 > 0$ and a constant $\theta_3 \in (0,1)$ that do not depend on ℓ^* such that*

$$\mathbb{P}(Y(1) = \xi | Y(0) = \xi) \ge 1 - \theta_1 \exp(-\theta_2 (\ell^*)^{\theta_3})$$

for sufficiently large ℓ^*.

Proof: Fix $\theta_3 \in (0, 1-2r_1)$, where r_1 was chosen in Assumption A11. Suppose $Y(0) = \xi$. For each $i = 1, 2, \ldots, n-1$, the number of age-i ewes that are lost, L_i say, has a binomial distribution with parameters $\xi_i = \lceil \ell^*(\tilde{y}_i - (i-1)\epsilon) \rceil$ and ψ_i, and thus mean $\psi_i \lceil \ell^*(\tilde{y}_i - (i-1)\epsilon) \rceil$. Suppose first that $z_0^* = 0$ so that $\tilde{y}_i = y_i^* + b\zeta_{i-1}$. Using the facts that y^* is feasible for (4) and the fact that for any real numbers x_1 and x_2, $(1-\psi_i)(\lceil x_1 \rceil - x_1) - (\lceil x_2 \rceil - x_2) \geq -1$,

$$
\begin{aligned}
&\mathbb{P}(Y_{i+1}(1) = \xi_{i+1} | Y(0) = \xi) \\
&= \mathbb{P}(L_i \leq \xi_i - \xi_{i+1} | Y_i(0) = \xi_i) \\
&= \mathbb{P}(L_i - \psi_i \xi_i \leq (1-\psi_i)\xi_i - \xi_{i+1} | Y_i(0) = \xi_i) \\
&= \mathbb{P}\left(L_i - \psi_i \xi_i \leq (1-\psi_i)\lceil \ell^*(y_i^* + b\zeta_{i-1} - (i-1)\epsilon) \rceil - \lceil \ell^*(y_{i+1}^* + b\zeta_i - i\epsilon) \rceil | Y_i(0) = \xi_i\right) \\
&\geq \mathbb{P}(L_i - \psi_i \xi_i \leq \ell^* z_i^* + \ell^* \epsilon[i - (1-\psi_i)(i-1)] - 1 | Y_i(0) = \xi_i) \\
&\geq \mathbb{P}(L_i - \psi_i \xi_i \leq \ell^* \epsilon - 1 | Y_i(0) = \xi_i) \\
&\geq 1 - \exp\left(\frac{-2(\ell^*\epsilon - 1)^2}{\lceil \ell^*(\tilde{y}_i - (i-1)\epsilon) \rceil}\right) \\
&\geq 1 - \exp(-\beta_i (\ell^*)^{\theta_3})
\end{aligned}
$$

for an appropriately defined constant β_i, where the second-to-last step uses Hoeffding's inequality. The same argument applies for the case when $z_0^* > 0$, except that the terms involving b that cancel, above, are not present.

Next, starting from the target state, the number of Year 1 ewes, $Y_1(1)$, in one transition, will equal its target value provided that B, the number of ewe lambs that are born and survive their first year, is sufficiently large. The random variable B is a sum over all ewes of the (bounded by c_2) random number of their ewe offspring that survive their first year, and has mean $\mu_B := (1-\psi_0)\sum_{i=1}^n \alpha_i \xi_i$. In what follows, we again apply Hoeffding's inequality to show that B is sufficiently large with high probability. For the case where $z_0^* = 0$, using the facts that $\xi_i = \lceil \ell^*(y_i^* + b\zeta_{i-1} - (i-1)\epsilon) \rceil$, $y_1^* = (1-\psi_0)\sum_{i=1}^n \alpha_i y_i^*$ and for any x, $0 \leq \lceil x \rceil - x \leq 1$,

$$
\begin{aligned}
&\mathbb{P}(Y_1(1) = \xi_1 | Y(0) = \xi) = \mathbb{P}(B - \mu_B \geq \xi_1 - \mu_B) \\
&\geq \mathbb{P}\left(B - \mu_B \geq \ell^*(y_1^* + b) + 1 - (1-\psi_0)\sum_{i=1}^n \alpha_i \xi_i\right) \\
&\geq \mathbb{P}\left(B - \mu_B \geq \ell^*\left(y_1^* - (1-\psi_0)\sum_{i=1}^n \alpha_i y_i^*\right) + \ell^* b\left(1 - (1-\psi_0)\sum_{i=1}^n \alpha_i \zeta_{i-1}\right)\right.
\end{aligned}
$$

$$+\ell^*\epsilon(1-\psi_0)\sum_{i=1}^{n}(i-1)\alpha_i+1\Bigg)$$

$$=\mathbb{P}\left(B-\mu_B\geq -\ell^*b(\eta-1)+\ell^*\epsilon(1-\psi_0)\sum_{i=1}^{n}(i-1)\alpha_i+1\right).$$

Now, the coefficient of b in the last expression is negative due to the stability condition that $\eta>1$ in Assumption A13. Moreover, we chose $\epsilon(\ell^*)=o(b(\ell^*))$ as $\ell^*\to\infty$, so, for sufficiently large ℓ^*,

$$-\ell^*b(\eta-1)+\ell^*\epsilon(1-\psi_0)\sum_{i=1}^{n}(i-1)\alpha_i+1\leq -\ell^*\delta:=-\ell^*b(\eta-1)/2.$$

For such ℓ^*, Hoeffding's inequality then yields

$$\mathbb{P}(Y_1(1)=\xi_1|Y(0)=\xi)\geq 1-\exp\left(\frac{-2{\ell^*}^2\delta^2}{[\ell^*(\sum_{i=1}^{n}\tilde{y}_i)+n]c_2^2}\right).$$

Now, $\delta^2=b^2(\eta-1)^2/4={\ell^*}^{-2r_2}(\eta-1)^2/4$. Thus, for ℓ^* sufficiently large, we can define a constant β_0 that does not depend on ℓ^* such that

$$\mathbb{P}(Y_1(1)=\xi_1|Y(0)=\xi)\geq 1-\exp(-\beta_0{\ell^*}^{\theta_3}).$$

For the case where $z_0^*>0$ the argument is similar, but we retain the (positive) value of z_0^* in the probability calculation, i.e., $y_1^*=(1-\psi_0)\sum_{i=1}^{n}\alpha_i y_i^*-z_0^*$ and $b=0$, and then for large enough ℓ^*, ϵ is so small that $\epsilon(1-\psi_0)\sum_{i=1}^{n}(i-1)\alpha_i-z_0^*=-\delta<0$, and the remainder of the argument is the same.

Using conditional independence of the various expressions above, conditional on $Y(0)=\xi$, it follows that

$$\mathbb{P}(Y(1)=\xi|Y(0)=\xi)\geq\prod_{i=0}^{n-1}(1-e^{-\beta_i(\ell^*)^{\theta_3}})\geq 1-\theta_1 e^{-\theta_2(\ell^*)^{\theta_3}}$$

for appropriately defined positive constants θ_1 and θ_2 that do not depend on ℓ^*. ■

Next we show that the expected revenue received over one transition when we start in the target state is near optimal.

LEMMA EC.2. *Under A4-A14, the expected one-step revenue starting from the target state ξ is*

$$r(\xi)\geq\ell^*\rho-O({\ell^*}^{1-r_2}),$$

where ρ is the optimal value of the steady-state LP (4) *with $\ell^*=1$.*

Proof: Suppose that $Y(0)=\xi$. Let $Z=Z(0)$ be the vector of sales decisions. The expected one-step revenue from selling ewes and ewe lambs is then $r(\xi):=\mathbb{E}\left[\sum_{i=0}^n p_i Z_i\right]=\sum_{i=0}^n p_i \mathbb{E} Z_i$. Now,

$$\begin{aligned}
\mathbb{E} Z_n &= (1-\psi_n)\xi_n \\
&= \ell^*(1-\psi_n)y_n^* + \ell^*(1-\psi_n)(\xi_n/\ell^* - y_n^*) \\
&= \ell^* z_n^* - O(\ell^*(b+\epsilon)).
\end{aligned}$$

Similarly, for $i=1,2,\ldots,n-1$, with L_i the number of lost ewes of age i at time 0,

$$\begin{aligned}
\mathbb{E} Z_i &= \mathbb{E}[\xi_i - L_i - \xi_{i+1}]_+ \\
&\geq \mathbb{E}[\xi_i - L_i - \xi_{i+1}] \\
&\geq \ell^*[(1-\psi_i)(\tilde{y}_i - (i-1)\epsilon) - (\tilde{y}_{i+1} - i\epsilon)] - 1 \\
&= \ell^*[(1-\psi_i)y_i^* - y_{i+1}^* - O(\epsilon)] - 1 \\
&= \ell^* z_i^* - O(\ell^*\epsilon).
\end{aligned}$$

Also, let B be the number of ewe lambs in year 0 that are born and not lost, so that $\mathbb{E}B = (1-\psi_0)\sum_{i=1}^n \alpha_i \xi_i$. Then

$$\begin{aligned}
\mathbb{E} Z_0 &= \mathbb{E}[B - \xi_1]_+ \\
&\geq \mathbb{E}[B - \xi_1] \\
&\geq \ell^*\left[(1-\psi_0)\sum_{i=1}^n \alpha_i(\tilde{y}_i - (i-1)\epsilon) - \tilde{y}_1\right] - 1 \\
&= \ell^*\left[(1-\psi_0)\sum_{i=1}^n \alpha_i y_i^* - y_1^* - O(b+\epsilon)\right] - 1 \\
&= \ell^* z_0^* - O(\ell^*(b+\epsilon)).
\end{aligned}$$

The same argument for ewe lambs can be applied for ram lambs to establish that the expected number of non-lost ram lambs that are sold is $\mathbb{E}\mathcal{L}^R(0) \geq \ell^*(z_0^* + y_1^* - O(b+\epsilon))$. Combining the above observations with A12 yields the result. ■

To proceed further, we first need the following technical result on random walks.

LEMMA EC.3. *Suppose that* $(X_n : n \geq 1)$ *is an i.i.d. sequence with* $\mathrm{var}(X_1) < \infty$ *and* $\mathbb{E}[X_1] = 0$. *If we define the random walk* $\chi_n := X_1 + X_2 + \cdots + X_n$ *for* $n \geq 1$, *then*

$$\inf_{n\geq 1} \mathbb{P}(\chi_n \geq 0) > 0.$$

Proof of Lemma EC.3. If $\mathrm{var}(X_1) = 0$, then the result holds trivially since the infimum is 1. So assume that $\mathrm{var}(X_1) > 0$. Since $\mathbb{E}\chi_n = 0$ for all $n \geq 1$ and $\mathrm{var}(\chi_n) > 0$, it follows that $\mathbb{P}(\chi_n \geq 0) > 0$ for all $n \geq 1$. The central limit theorem proves that $\mathbb{P}(\chi_n \geq 0) \to 1/2$ as $n \to \infty$. Thus, $\mathbb{P}(\chi_n \geq 0)$ is a (strictly) positive sequence that converges to 1/2, and is therefore bounded away from 0. ■

From Lemma EC.3, it follows that there exists $q_0 > 0$ such that the (random) number of ewe-lamb births that survive their first year exceeds its mean $(1-\psi_0)\sum_{i=1}^n \alpha_i \mathbf{Y}_i(t)$ conditional on the census $\mathbf{Y}(t)$ with probability at least q_0 uniformly in $t \geq 0$, i.e., that $\mathbb{P}\left[B(t) \geq (1-\psi_0)\sum_{i=1}^n \alpha_i \mathbf{Y}_i(t) | \mathbf{Y}(t)\right] \geq q_0$ for all $t \geq 0$. (This follows by applying Lemma EC.3 to the number of births from age-i ewes for each i, and then taking the product of the lower-bounding probabilities over each age $i = 1, 2, \ldots, n$.) Moreover, a similar result holds for the number of surviving ram lambs and age-i ewes, for each $i = 1, 2, \ldots, n$. Our next result shows that, starting from a non-target state, there is a uniform (in the non-target state) upper bound on the expected time to return to the target state.

LEMMA EC.4. *Assume A4-A14. Define $\tau := \inf\{t \geq 1 : Y(t) = \xi\}$ as the first return time to the target state. Then, there exist nonnegative constants κ_1, κ_2 that do not depend on ℓ^* such that*

$$\sup_{y \in \mathcal{S}, y \neq \xi} \mathbb{E}[\tau | Y(0) = y] \leq \kappa_1 (\ell^*)^{\kappa_2}.$$

Proof: We use the observation that if, each year, there are sufficient births and few losses, then the population will quickly build back up to the target state ξ.

More precisely, suppose $Y(0) = y = (y_1, y_2, \ldots, y_n) \in \mathcal{S} \triangleq (y \in \mathbb{Z}_+^n : 0 \leq y \leq \xi, y \neq 0)$ with $y \neq \xi$. (Previously we used the notation y to reflect *relative* counts in the different age levels, for an overall population of size 1. For this result the interpretation differs; now the y_i variables represent unscaled counts.) Suppose first that $y_* = \min_{i=1}^n y_i = 0$. Since $y \neq 0$, there is at least one ewe in the flock. Using Lemma EC.3 we can assume a uniform lower bound, q_1 say, on the probability that the number of surviving ewes (ewes that are not "lost") exceeds its mean, for each age $1, 2, \ldots, n$. Assume this happens for each successive transition $t = 0, 1, \ldots, n-1$. Similarly, assume that the number of non-lost lambs (lambs that are born and are not lost in their first year) exceeds its conditional mean $(1-\psi_0)\sum_{i=1}^n \alpha_i Y_i(t) > 0$ for $t = 0, 1, \ldots, n-1$. The conditional probabilities of these events can be bounded below by the probability $q_0 > 0$, again using Lemma EC.3. Assuming these events happen in all of the first n transitions, it follows that after n transitions, all components of the new state, $Y(n)$, are at least equal to 1. The unconditional probability of such an event is at least $(q_1^n q_0)^n$. Recall that we also sell as necessary to ensure that the ith coordinate of $Y(n) \leq \xi_1$, though sales are unlikely to occur until the population recovers sufficiently.

Now we iterate. Starting from $Y(n) \geq (\zeta_0, \zeta_1, \ldots, \zeta_{n-1})$, assume the number of non-lost lambs and non-lost ewes are at least their respective conditional means in n successive steps. The first such step, with some algebra, yields a state

$$Y(n+1) \geq (\eta, \zeta_1, \zeta_2, \ldots, \zeta_{n-1}).$$

The kth such step (with $1 \leq k \leq n$), again with some algebra, yields a state

$$Y(n+k) \geq (\eta\zeta_0, \eta\zeta_1, \ldots, \eta\zeta_{k-1}, \zeta_k, \zeta_{k+1}, \ldots, \zeta_{n-1}).$$

Thus, after $2n$ steps, $Y(2n) \geq (\eta\zeta) \wedge \xi$, where ζ is the vector $(\zeta_0, \zeta_1, \ldots, \zeta_{n-1})$, and the componentwise minimum $\wedge$ indicates any necessary sales to avoid exceeding ξ_i ewes at any age stratum i. The probability of n such successive events is again at least $(q_1^n q_0)^n$. Now the lower bound on the population has increased by a factor $\eta > 1$, due to the stability condition, A11. With each successive sequence of n steps, the lower bound on the population increases by an additional factor of η, at least until the state hits ξ. Thus, after kn successive steps where births equal or exceed their mean, $Y(kn)$ is at least $(\eta^{k-1}\zeta) \wedge \xi$.

With each successive block of n transitions where the number of non-lost lambs and non-lost ewes are at least their respective conditional means, the lower bound on the census increases by another power of η, at least until sales are required to prevent exceeding the target state ξ. Thus, in at most $K = n\lceil(\ln \xi_1)/\ln(\eta) + 1\rceil$ successive transitions where the number of non-lost lambs equals or exceeds its conditional mean and the number of ewe losses is less than or equal to their conditional means, the state returns to the target ξ. Let $q_2 \triangleq q_1^n q_0$. Each such transition has probability at least $q_2 > 0$, so that the probability of that sequence is at least

$$q_2^K \geq q_2^{n((\ln \xi_1)/\ln(\eta)+2)} = q_2^{2n} \xi_1^{n(\ln q_2)/\ln(\eta)}.$$

Using a geometric trials argument, it follows that the expected value of τ, the hitting time to state ξ is at most

$$K/q_2^K \leq \frac{n\lceil(\ln \xi_1)/\ln(\eta) + 1\rceil}{q_2^{2n}} \xi_1^{-n(\ln q_2)/\ln(\eta)} \leq \kappa_0 \xi_1^{\kappa_2}$$

for appropriately defined constants $\kappa_0, \kappa_2 > 0$ that do not depend on ℓ^*. (Note that $\ln(\xi_1) \leq \xi_1$, so the impact of this term can be absorbed into κ_2, the power of ξ_1.) The observation that $\xi_1 \leq \ell^*(y_1^* + b) + 1 \leq \ell^*(1 + b) + 1$, together with the fact that $b = b(\ell^*) < 1$ means we can take the constant $\kappa_1 = \kappa_0 2^{\kappa_2}$. ■

We now build on Lemma EC.2 on the revenue when moving from the target state to the target state to show that the long-run revenue from our policy is near optimal.

LEMMA EC.5. *Under A4-A14, the mean steady-state revenue from our stated policy is at least $\ell^*\rho - O((\ell^*)^{1-r_2})$, i.e., the revenue of the optimal LP solution minus a small amount.*

Proof: The mean steady-state revenue from our policy is at least equal to the mean one-step revenue starting from the target state multiplied by the steady-state probability of being in the target state. Lemma EC.2 showed that the mean one-step revenue starting from the target state is at least $\ell^*\rho - O({\ell^*}^{1-r_2})$.

Now we use the fact that the steady-state probability of being in the target state is the reciprocal of $\mathbb{E}[\tau|Y(0)=\xi]$, the mean return time to the target state starting from the target state, e.g., see Resnick (2013, Proposition 2.12.3). Using first step analysis, and the bounds from Lemma EC.1 and Lemma EC.4, we have that

$$\begin{aligned}\mathbb{E}[\tau|Y(0)=\xi] &\le 1 + P(Y(1)\ne\xi|Y(0)=\xi)\sup_{y\in\mathcal{S},y\ne\xi}\mathbb{E}[\tau|Y(0)=y]\\ &\le 1+\theta_1 e^{-\theta_2 {\ell^*}^{\theta_3}}\kappa_1 {\ell^*}^{\kappa_2}.\end{aligned}$$

It immediately follows that the mean steady-state revenue is at least

$$\frac{\ell^*\rho - O({\ell^*}^{1-r_2})}{1+\theta_1 e^{-\theta_2 {\ell^*}^{\theta_3}}\kappa_1 {\ell^*}^{\kappa_2}} \ge \left[\ell^*\rho - O({\ell^*}^{1-r_2})\right]\left[1-\theta_1 e^{-\theta_2 {\ell^*}^{\theta_3}}\kappa_1 {\ell^*}^{\kappa_2}\right] = \ell^*\rho - O({\ell^*}^{1-r_2}).$$

■

Lemma EC.6, below, shows that the flock size constraint is met, asymptotically.

LEMMA EC.6. *Under A4-A14, the mean steady-state flock size, $\nu=\nu(\ell^*)$ say, satisfies $|\nu(\ell^*)-\ell^*| = O((\ell^*)^{1-r_2})$.*

Proof: When $Y(0)=\xi$, the flock size equals $\sum_{i=1}^n \xi_i$ which lies between $\ell^*(1-n(n-1)\epsilon/2)$ and $\ell^*(1+bR_n)+1$. When $Y(0)\ne\xi$, $0\le\sum_{i=1}^n Y_i(0)\le \ell^*(1+nb)+1$. If q is the steady-state probability of the target state ξ, then

$$q\ell^*(1-O(\epsilon)) \le \nu(\ell^*) \le q\ell^*(1+bR_n) + (1-q)\ell^*(1+nb)+1,$$

and since $R_n\le n$,

$$q\ell^*(1-O(\epsilon)) \le \nu(\ell^*) \le \ell^*(1+bn)+1.$$

The proof of Lemma EC.5 shows that $q \geq 1 - \theta_4 e^{-\theta_5 \ell^{*\theta_6}}$ for some positive constants θ_4, θ_5 and $\theta_6 \in (0,1)$ that do not depend on ℓ^* and $\theta_4 e^{-\theta_5 \ell^{*\theta_6}} = o(\ell^{*-r_1})$. Thus,

$$-O(\ell^* \epsilon) \leq \nu(\ell^*) - \ell^* \leq O(\ell^* b).$$

■

Proof of Theorem 3: This follows immediately from Lemmas EC.5 and EC.6.

EC.4. The Model-Predictive Control Formulation

The full Model-Predictive Control formulation is

$$\begin{array}{llll}
\max & \sum_{i=0}^{n} p_i w_i + \sum_{t=0}^{\tilde{T}-1}\left[\sum_{i=0}^{n} p_i \mathbf{z}_i(t) + p^R(1-\psi_0)\sum\limits_{i=1}^{n}\alpha_i \mathbf{y}_i(t) - c^H \sum_{i=1}^{n} \mathbf{y}_i(t)\right] & & \\
\text{s/t} & (1-\psi_0)\sum_{j=1}^{n}\alpha_j \mathbf{y}_j(t-1) - \mathbf{z}_0(t-1) - \mathbf{y}_1(t) & = 0 & t = 1,2,\ldots,\tilde{T} \\
& (1-\psi_{i-1})\mathbf{y}_{i-1}(t-1) - \mathbf{z}_{i-1}(t-1) - \mathbf{y}_i(t) & = 0 & t = 1,2,\ldots,\tilde{T}; i = 2,3,\ldots,n \\
& (1-\psi_n)\mathbf{y}_n(t) - \mathbf{z}_n(t) & = 0 & t = 0,1,\ldots,\tilde{T}-1 \\
& \sum_{i=1}^{n} \mathbf{y}_i(t) & \leq \ell^* & t = 0,1,\ldots,\tilde{T}-1 \\
& \mathbf{y}_i(0) + w_{i-1} & = \mathbf{Y}_{i-1} & i = 1,2,\ldots,n \\
& w_n & = \mathbf{Y}_n & \\
& w_i \geq 0 & & i = 0,1,\ldots,n \\
& \mathbf{y}_i(t) \geq 0 & & t = 0,1,\ldots,\tilde{T}; i = 1,2,\ldots,n \\
& \mathbf{z}_i(t) \geq 0 & & t = 0,1,\ldots,\tilde{T}-1; i = 0,1,\ldots,n.
\end{array}$$

EC.5. Counterexamples When the Assumptions of Theorem 1 Fail

EXAMPLE EC.1. Suppose $n = 2$, $\psi_0 = \psi_1 = \psi_2 = 0$, and $p^R = 0$. The following two solutions are feasible:

$$z_0 = (\alpha_1 + \alpha_2 - 1)\frac{\ell^*}{2} - (1 + \alpha_2 - \alpha_1)\frac{\epsilon}{2}, \quad z_1 = \epsilon, \quad z_2 = y_2 = \frac{\ell^*}{2} - \frac{\epsilon}{2}, \quad y_1 = \frac{\ell^*}{2} + \frac{\epsilon}{2};$$
$$z_0' = (\alpha_1 + \alpha_2 - 1)\frac{\ell^*}{2}, \quad z_1' = 0, \quad z_2' = y_1' = y_2' = \frac{\ell^*}{2},$$

(assuming $\alpha_1 + \alpha_2 > 1$ and ϵ is sufficiently small, to ensure all y_i, z_i are non-negative). It follows that

$$\sum_{j=0}^{2} p_j z_j' - \sum_{j=0}^{2} p_j z_j = \left[p_0(1 + \alpha_2 - \alpha_1) + p_2 - 2p_1\right]\frac{\epsilon}{2}, \tag{EC.9}$$

which can be either positive or negative, depending on the relation between α_1 and α_2. For instance, when $p_0 = p_1 = p_2$, the sign of (EC.9) equals the sign of $\alpha_2 - \alpha_1$. In particular, when $\alpha_2 < \alpha_1$, the conclusion of Theorem 1 no longer holds. This is intuitive because, for instance, when $\alpha_2 = 0$ (i.e., when the reproduction rate for ewes in their final year is zero), it becomes more beneficial to sell more ewes a year earlier to reduce the number of ewes in their final year.

EXAMPLE EC.2. Suppose $n = 2, \psi_0 = 0, \alpha_1 = \alpha_2 = 1$, and $p^R = 0$. The following two solutions are feasible:

$$\begin{aligned}
&z_0 = \frac{1-\psi_1}{2-\psi_1}\ell^* - \frac{1}{2-\psi_1}\epsilon, \quad z_1 = \epsilon, \quad z_2 = \frac{(1-\psi_1)(1-\psi_2)}{2-\psi_1}\ell^* - \frac{1-\psi_2}{2-\psi_1}\epsilon,\\
&y_1 = \frac{1}{2-\psi_1}\ell^* + \frac{1}{2-\psi_1}\epsilon, \quad y_2 = \frac{1-\psi_1}{2-\psi_1}\ell^* - \frac{1}{2-\psi_1}\epsilon;\\
&z_0' = \frac{1-\psi_1}{2-\psi_1}\ell^*, \quad z_1' = 0, \quad z_2' = \frac{(1-\psi_1)(1-\psi_2)}{2-\psi_1}\ell^*, \quad y_1' = \frac{1}{2-\psi_1}\ell^*, \quad y_2' = \frac{1-\psi_1}{2-\psi_1}\ell^*.
\end{aligned}$$

It follows that

$$\sum_{j=0}^{2} p_j z_j' - \sum_{j=0}^{2} p_j z_j = \left[\frac{1}{2-\psi_1}p_0 + \frac{1-\psi_2}{2-\psi_1}p_2 - p_1\right]\epsilon, \tag{EC.10}$$

which can be either positive or negative, depending on the relation between ψ_1 and ψ_2. For instance, when $p_0 = p_1 = p_2$, the sign of (EC.10) equals the sign of $\psi_1 - \psi_2$. In particular, when $\psi_1 < \psi_2$, the conclusion of Theorem 1 no longer holds. This is also intuitive because, for instance, when $\psi_2 = 1$ (i.e., a 100% loss of ewes in their final year), it becomes more beneficial to sell more ewes a year earlier to reduce the number of ewes in their final year.

EC.6. Price-Independent Worst-Case Ratio for the ML Policy

PROPOSITION EC.1. *Assume that $p^R = p_0 \geq p_i$ for all $i = 1\ldots, n$. The performance ratio of the ML policy value relative to the optimal LP value is at least*

$$\frac{\left(2(1-\psi_0)\sum_{j=1}^{n}\alpha_j\zeta_{j-1} - 1\right)R_n^{-1}}{\max\limits_{j=1,\ldots,n}\{2(1-\psi_0)\alpha_j - \psi_j\}}. \tag{EC.11}$$

The denominator of (EC.11) provides an upper bound on a scalar multiple of the optimal LP value. The intuition behind this upper bound is as follows: we start with the full breeding population (ℓ^*), assume that all sheep are sold at the highest possible price p_0, and take the most optimistic values for reproduction and loss — specifically, the highest reproduction rate $\max_{j=1}^n \alpha_j$ and the smallest loss rate $\min_{j=1}^n \psi_j$, which yields the largest population gain rate $\max_{j=1,\ldots,n}\{2(1-\psi_0)\alpha_j - \psi_j\}$.

The bound is tight when (1) the optimal LP solution retains only first-year ewes; (2) $p_0 = p_1$; and (3) first-year ewes exhibit both the highest reproduction rate and lowest loss rate across all age groups. This aligns with the intuition discussed above.

One example of such a parameter setting arises when $p_0 = p_1 = \ldots = p_{n-1} >> p_n$, with $\{\alpha_j\}_{j=1}^n$ decreasing in j and $\{\psi_j\}_{j=1}^n$ increasing in j. In this case, the price structure is substantially altered, and the monotonicity of both $\{\alpha_j\}$ and $\{\psi_j\}$ is the exact opposite of the assumptions that guarantee the optimality of the ML policy. Specifically, when $p_0 = p_1 = \ldots = p_{n-1} = 250$, $p_n = 0$, $\alpha = (1.4, 1.2, 1.0, 0.8, 0.6)$ and $\psi = (0.01, 0.02, 0.03, 0.04, 0.05)$, the performance ratio is 0.6414, exactly matching the lower bound established in Proposition EC.1.