A Proofs

Proof of Proposition 1

Proof. Given exercised price p_i, we first consider the interior quantile level q_j with $j = 2, \ldots, m-1$. Let $F_i(x) = \text{Prob}\{D(p_i) \leq x\}$ be the distribution function of $D(p_i)$, and let F_{i,N_i} be the empirical distribution function constructed from the N_i demand observations at price p_i. Let $\widehat{Q}(q_j; p_i)$ denote the empirical q_j-quantile computed from the demand observations at price p_i. We first show that $\widehat{Q}(q_j; p_i)$ converges to $Q(q_j; p_i)$ in probability.

By the uniqueness of $Q(q_j; p_i)$ and the positive continuous density condition in a neighborhood of $Q(q_j; p_i)$, for any $\epsilon > 0$,

$$F_i(Q(q_j; p_i) - \epsilon) < q_j < F_i(Q(q_j; p_i) + \epsilon).$$

We define $\delta = \frac{1}{2} \min\{q_j - F_i(Q(q_j; p_i) - \epsilon), F_i(Q(q_j; p_i) + \epsilon) - q_j\} > 0$. By the Glivenko–Cantelli theorem, $\sup_x |F_{i,N_i}(x) - F_i(x)| \to 0$ in probability. If $\sup_x |F_{i,N_i}(x) - F_i(x)| < \delta$, then the empirical distribution function and the true distribution function differ by less than δ at every point. In particular, at the point $Q(q_j; p_i) - \epsilon$, $F_{i,N_i}(Q(q_j; p_i) - \epsilon) < F_i(Q(q_j; p_i) - \epsilon) + \delta < q_j$, and at the point $Q(q_j; p_i) + \epsilon$, $F_{i,N_i}(Q(q_j; p_i) + \epsilon) > F_i(Q(q_j; p_i) + \epsilon) - \delta > q_j$. Therefore,

$$Q(q_j; p_i) - \epsilon < \widehat{Q}(q_j; p_i) \leq Q(q_j; p_i) + \epsilon.$$

Thus, the event that the empirical distribution function is within δ of F_i implies the event that the empirical quantile is within ϵ of $Q(q_j; p_i)$:

$$\left\{ \sup_x |F_{i,N_i}(x) - F_i(x)| < \delta \right\} \subseteq \left\{ |\widehat{Q}(q_j; p_i) - Q(q_j; p_i)| \leq \epsilon \right\}.$$

Equivalently, if the empirical quantile is outside the ϵ-neighborhood of $Q(q_j; p_i)$, then the empirical distribution function cannot be uniformly within δ of F_i:

$$\left\{ |\widehat{Q}(q_j; p_i) - Q(q_j; p_i)| > \epsilon \right\} \subseteq \left\{ \sup_x |F_{i,N_i}(x) - F_i(x)| \geq \delta \right\}.$$

This implies

$$\text{Prob}\left\{ |\widehat{Q}(q_j; p_i) - Q(q_j; p_i)| > \epsilon \right\} \leq \text{Prob}\left\{ \sup_x |F_{i,N_i}(x) - F_i(x)| \geq \delta \right\} \to 0.$$

Thus, $\widehat{Q}(q_j; p_i) \to Q(q_j; p_i)$ in probability.

We next show that the bootstrap bounds converge to the empirical quantile $\widehat{Q}(q_j; p_i)$. Let $\widehat{Q}^*(q_j; p_i)$ denote the empirical q_j-quantile computed from a generic bootstrap sample drawn with replacement from the original demand observations at price p_i. Theorem 5.1 of Bickel and Freedman (1981) establishes the weak convergence of the bootstrap quantile process. This theorem gives the conditional weak convergence of $\sqrt{N_i}\left\{\widehat{Q}^*(q;p_i)-\widehat{Q}(q;p_i)\right\}$ as a stochastic process indexed by interior quantile levels q. By the continuous mapping theorem, evaluating this process at the fixed quantile level q_j implies that $\sqrt{N_i}\left\{\widehat{Q}^*(q_j;p_i)-\widehat{Q}(q_j;p_i)\right\}$ converges weakly to a finite normal random variable. Given that $N_i^{-1/2}\to 0$ when $N_i\to\infty$, Slutsky's theorem implies $\widehat{Q}^*(q_j;p_i)-\widehat{Q}(q_j;p_i)\to 0$ in distribution. Because the limiting value is zero, this convergence in distribution is equivalent to convergence in probability. Thus, $\widehat{Q}^*(q_j;p_i)-\widehat{Q}(q_j;p_i)\to 0$ in probability.

The lower and upper bounds $\underline{X}_{ij}$ and $\bar{X}_{ij}$ are fixed percentiles of the bootstrap resample quantile distribution. Given that the bootstrap resample quantile $\widehat{Q}^*(q_j;p_i)$ converges to $\widehat{Q}(q_j;p_i)$ in probability, these fixed percentile estimates also converge to $\widehat{Q}(q_j;p_i)$. Therefore, $\underline{X}_{ij}-\widehat{Q}(q_j;p_i)\to 0$, and $\bar{X}_{ij}-\widehat{Q}(q_j;p_i)\to 0$ in probability. Combining this result with $\widehat{Q}(q_j;p_i)\to Q(q_j;p_i)$ in probability yields

$$\underline{X}_{ij}\to Q(q_j;p_i),\text{ and }\bar{X}_{ij}\to Q(q_j;p_i)$$

in probability for every fixed $i\in[n]$ and every interior $j=2,\dots,m-1$.

We now consider the endpoint quantiles. By construction, $\underline{X}_{i1}=\bar{X}_{i1}$ is the sample minimum of the demand observations at price p_i. Given that $Q(0;p_i)$ is the finite lower support endpoint of $D(p_i)$, we have $D(p_i)\geq Q(0;p_i)$, and $\text{Prob}\{D(p_i)\leq Q(0;p_i)+\epsilon\}>0$ for any $\epsilon>0$. Thus,

$$\text{Prob}\{\underline{X}_{i1}>Q(0;p_i)+\epsilon\}=(1-\text{Prob}\{D(p_i)\leq Q(0;p_i)+\epsilon\})^{N_i}\to 0.$$

Since $\underline{X}_{i1}\geq Q(0;p_i)$, this proves $\underline{X}_{i1}\to Q(0;p_i)$ in probability when $N_i\to\infty$.

Similarly, $\underline{X}_{im}=\bar{X}_{im}$ is the sample maximum of the demand observations at price p_i. Since $Q(1;p_i)$ is the finite upper support endpoint of $D(p_i)$, we have $D(p_i)\leq Q(1;p_i)$, and for any $\epsilon>0$, $\text{Prob}\{D(p_i)>Q(1;p_i)-\epsilon\}>0$. Hence,

$$\text{Prob}\{\bar{X}_{im}<Q(1;p_i)-\epsilon\}\leq(1-\text{Prob}\{D(p_i)>Q(1;p_i)-\epsilon\})^{N_i}\to 0.$$

Therefore, $\bar{X}_{im}\leq Q(1;p_i)$ $\bar{X}_{im}\to Q(1;p_i)$ in probability when $N_i\to\infty$.

The preceding arguments establish pointwise convergence for each fixed pair (i,j). For any $\epsilon>0$,

$$\begin{aligned}&\text{Prob}\left\{\max_{i\in[n],\,j\in[m]}\max\left\{|\underline{X}_{ij}-Q(q_j;p_i)|,\,|\bar{X}_{ij}-Q(q_j;p_i)|\right\}>\epsilon\right\}\\&\leq\sum_{i=1}^{n}\sum_{j=1}^{m}\text{Prob}\{|\underline{X}_{ij}-Q(q_j;p_i)|>\epsilon\}+\sum_{i=1}^{n}\sum_{j=1}^{m}\text{Prob}\{|\bar{X}_{ij}-Q(q_j;p_i)|>\epsilon\}.\end{aligned}$$

Each term on the right-hand side converges to zero, and only finitely many terms are included. Therefore,

$$\max_{i\in[n],\,j\in[m]}\max\left\{|\underline{X}_{ij}-Q(q_j;p_i)|,\,|\bar{X}_{ij}-Q(q_j;p_i)|\right\}\to 0$$

in probability when $N_i\to\infty$.

Finally, we suppose that $\mathcal{D}$ is nonempty and take any $\widetilde{D} \in \mathcal{D}$. By the definition of $\mathcal{D}$, $\underline{X}_{ij} \leq Q_{\widetilde{D}}(q_j; p_i) \leq \bar{X}_{ij}, i \in [n],\ j \in [m]$. Therefore, $|Q_{\widetilde{D}}(q_j; p_i) - Q(q_j; p_i)| \leq \max\left\{|\underline{X}_{ij} - Q(q_j; p_i)|,\ |\bar{X}_{ij} - Q(q_j; p_i)|\right\}$. We first take the maximum over $i \in [n]$ and $j \in [m]$, and then take the supremum over $\widetilde{D} \in \mathcal{D}$, which yields

$$\sup_{\widetilde{D} \in \mathcal{D}} \max_{i \in [n],\, j \in [m]} |Q_{\widetilde{D}}(q_j; p_i) - Q(q_j; p_i)| \leq \max_{i \in [n],\, j \in [m]} \max\left\{|\underline{X}_{ij} - Q(q_j; p_i)|,\ |\bar{X}_{ij} - Q(q_j; p_i)|\right\}.$$

The right-hand side converges to zero in probability when $N_i \to \infty$. This completes the proof. □

Proof of Proposition 2

Proof. By the definition of $\mathcal{D}^a$, as $\underline{X}^a_{ij} \geq \underline{X}_{ij}$ and $\bar{X}^a_{ij} \leq \bar{X}_{ij}$, we have $\mathcal{D}^a \subseteq \mathcal{D}$. Thus, it is sufficient to show $\mathcal{D} \subseteq \mathcal{D}^a$. For any $D(\cdot) \in \mathcal{D}$, we have $\underline{X}_{ij} \leq Q(q_j; p_i) \leq \bar{X}_{ij}$ for all $i \in [n],\ j \in [m]$. $Q(q_j; p)$ is decreasing in p for each $j \in [m]$. Therefore, it suffices to verify that $\underline{X}^a_{ij} \leq Q(q_j; p_i) \leq \bar{X}^a_{ij}$ for all i, j. For the lower bound, $\underline{X}^a_{ij} = \max_{k \geq i} \underline{X}_{kj}$. Given that $D(\cdot) \in \mathcal{D}$, we have $\underline{X}_{kj} \leq Q(q_j; p_k)$ for each $k \geq i$. Thus,

$$\underline{X}^a_{ij} = \max_{k \geq i} \underline{X}_{kj} \leq \max_{k \geq i} Q(q_j; p_k).$$

If $Q(q_j; p)$ is decreasing in p and $p_k \geq p_i$ for $k \geq i$, then $Q(q_j; p_k) \leq Q(q_j; p_i)$ for each such k. Therefore,

$$\max_{k \geq i} Q(q_j; p_k) \leq Q(q_j; p_i),$$

which implies $\underline{X}^a_{ij} \leq Q(q_j; p_i)$. For the upper bound, $\bar{X}^a_{ij} = \min_{k \leq i} \bar{X}_{kj}$. Given that $D(\cdot) \in \mathcal{D}$, we have $Q(q_j; p_k) \leq \bar{X}_{kj}$ for each $k \leq i$. Thus,

$$\bar{X}^a_{ij} = \min_{k \leq i} \bar{X}_{kj} \geq \min_{k \leq i} Q(q_j; p_k).$$

If $Q(q_j; p)$ is decreasing in p and $p_k \leq p_i$ for $k \leq i$, then $Q(q_j; p_k) \geq Q(q_j; p_i)$ for each such k. Therefore,

$$\min_{k \leq i} Q(q_j; p_k) \geq Q(q_j; p_i),$$

which implies $Q(q_j; p_i) \leq \bar{X}^a_{ij}$. The monotonicity condition on $Q(q_j; p)$ holds by assumption. Hence, $D(\cdot) \in \mathcal{D}^a$. □

We prove the theorems and propositions for the maxmin profit model and the minmax regret model by first deriving some properties of $\hat{w}(\cdot)$ and $w(\cdot)$ in the following lemma.

Lemma 1. *1. The optimal value of Problem (19) equals $\hat{w}(p, y, x)$, and $\hat{w}(p, y, x)$ is an increasing function of p and $x_j, j = 1 \cdots, m$.*

2. $x_{t(p)+1} = \arg\max_{y \geq 0} \hat{w}(p, y, x)$, where $t(p) = \min\left\{k \in [m-1] \,\middle|\, \frac{p-c}{p} \leq q_{k+1}\right\}$, as defined in (7).

3. *The optimal value of Problem (6) equals $w(p, y, x)$, and $w(p, y, x)$ is an increasing function of p and $x_j, j = 1, \cdots, m$.*

4. $x_{t(p)} = \arg\max_{y \geq 0} w(p, y, x)$.

Proof. 1. By definition, Problem (19) is equivalent to

$$\begin{aligned} \max_{D(p)} \quad & \{\pi(p, y, D(p) = p \min\{y, D(p)\} - cy\} \\ \text{subject to} \quad & \text{Prob}\{D(p) \leq x_j\} = q_j, j = 1 \cdots, m, \end{aligned} \tag{25}$$

By conic duality (see Popescu (2005)) , the dual problem becomes

$$\begin{aligned} \min_{z} \quad & \sum_{j=1}^{m-1} z_j \frac{1}{m-1} - cy \\ \text{subject to} \quad & p \min\{y, x\} \leq z_j, \quad \forall x \in (x_j, x_{j+1}], j = 1 \cdots, m. \end{aligned} \tag{26}$$

Let $z_j = p \min\{y, x_{j+1}\}, j = 1, \cdots, m-1$, which is a feasible solution of the dual problem. This configuration implies $\hat{w}(p, y, x)$ is an upper bound of Problem (19).

Moreover, we define a distribution $\hat{H}_\epsilon(x)$ with $0 < \epsilon < \frac{1}{m-1}$, whose CDF is

$$F_{\hat{H}_\epsilon(x)}(s) = \begin{cases} \epsilon, & x_1 \leq s < x_2 \\ \frac{j-1}{m-1}, & x_j \leq s < x_{j+1}, j = 2, \cdots, m-1 \\ 1, & s = x_m. \end{cases} \tag{27}$$

$\hat{H}_\epsilon(x)$ can easily be checked as a feasible solution of Problem (19) and

$$\pi(p, y, \hat{H}_\epsilon(x)) = p \left\{ \epsilon \min\{y, x_1\} + (\frac{1}{m-1} - \epsilon) \min\{y, x_2\} + \sum_{j=2}^{m-1} \frac{1}{m-1} \min\{y, x_{j+1}\} \right\} - cy,$$

which implies Problem (19) is greater or equal to $\pi\left(p, y, \hat{H}_\epsilon(x)\right)$. We take $\epsilon \to 0$ and find that Problem (19) is great or equal to

$$\lim_{\epsilon \to 0} \pi\left(p, y, \hat{H}_\epsilon(x)\right) = p \sum_{j=1}^{m-1} \frac{1}{m-1} \min\{y, x_{j+1}\} = \hat{w}(p, y, x). \tag{28}$$

Therefore, $\hat{w}(p, y, x)$ is the optimal value of Problem (8). By the definition of $\hat{w}(p, y, x)$, $\hat{w}(p, y, x)$ is an increasing function of p and $x_j, j = 1, \cdots, m$.

2. By the definition of $\hat{w}(p,y,x)$, $\hat{w}(\cdot)$ is a continuous function and

$$\frac{\partial \hat{w}(p,y,x)}{\partial y} = \begin{cases} p-c, & y < x_2, \\ p\frac{m-k}{m-1} - c, & x_k < y < x_{k+1}, k = 2, \cdots, m-1, \\ -c, & x_m < y. \end{cases}$$

This outcome implies that $\hat{w}(p,y,x)$ is a concave function of y. By the definition of $t(p)$, we have

$$\frac{t(p)-1}{m-1} = q_{t(p)} < \frac{p-c}{p} \leq q_{t(p)+1} = \frac{t(p)}{m-1}.$$

Thus, when $x_{t(p)} < y < x_{t(p)+1}$, we have $\frac{\partial \hat{w}(p,y,x)}{dy} = p\frac{m-t(p)}{m-1} - c > 0$, where the last inequality follows from $\frac{t(p)-1}{m-1} < \frac{p-c}{p}$. Moreover, when $x_{t(p)+1} < y < x_{t(p)+2}$, we have $\frac{\partial \hat{w}(p,y,x)}{\partial y} = p\frac{m-t(p)-1}{m-1} - c \leq 0$, where the inequality follows from $\frac{p-c}{p} \leq \frac{t(p)}{m-1}$. Therefore, $x_{t(p)+1} = \underset{y \geq 0}{arg\,max} \hat{w}(p,y,x)$.

3. By conic duality, the dual of problem (6) is

$$\begin{aligned} \max_{z} \qquad & \sum_{j=1}^{m-1} z_j \frac{1}{m-1} - cy \\ \text{subject to} \quad & p\min\{y,x\} \geq z_j, \quad \forall x \in (x_j, x_{j+1}], j = 1, \cdots, m. \end{aligned} \tag{29}$$

Let $z_j = p\min\{y, x_j\}, j = 1, \cdots, m-1$, which is a feasible solution of the dual problem, which implies that $w(p,y,x)$ is a lower bound of problem (6). Moreover, we define a distribution $H_\epsilon(x)$, with $0 < \epsilon < \frac{1}{m-1}$, whose CDF is

$$F_{H_\epsilon(x)}(s) = \begin{cases} \frac{1}{m-1} - \epsilon, & x_1 \leq s < x_2 \\ \frac{j}{m-1} - \epsilon, & x_j \leq s < x_{j+1}, j = 2, \cdots, m-1 \\ 1, & s = x_m. \end{cases} \tag{30}$$

We can easily check that $H_\epsilon(x)$ is a feasible solution of Problem (6) and

$$\pi(p,y,H_\epsilon(x)) = p\left\{ (\frac{1}{m-1} - \epsilon)\min\{y,x_1\} + \sum_{j=2}^{m-1} \frac{1}{m-1}\min\{y,x_j\} + \epsilon \min\{y,x_m\} \right\} - cy,$$

In addition, we take $\epsilon \to 0$ and obtain

$$\lim_{\epsilon \to 0} \pi\left(p,y,H_\epsilon(x)\right) = p\sum_{j=1}^{m-1} \frac{1}{m-1}\min\{y,x_j\} - cy = w(p,y,x), \tag{31}$$

which implies that $w(p,y,x)$ is a upper bound of Problem (6). Therefore, $w(p,y,x)$ equals the optimal value of Problem (6). By the definition of $w(p,y,x)$, $w(p,y,x)$ is an increasing function of p and $x_j, j = 1, \cdots, m$.

4. By the definition of $w(p,y,x)$, $w(\cdot)$ is a continuous function and

$$\frac{\partial w(p,y,x)}{\partial y} = \begin{cases} p-c, & y < x_1, \\ p\frac{m-1-k}{m-1} - c, & x_k < y < x_{k+1}, k = 1, \cdots, m-2, \\ -c, & x_{m-1} < y. \end{cases}$$

This outcome implies that $w(p,y,x)$ is a concave function of y. By the definition of $t(p)$, we have

$$\frac{t(p)-1}{m-1} = q_{t(p)} < \frac{p-c}{p} \leq q_{t(p)+1} = \frac{t(p)}{m-1}.r$$

Then, we have $\frac{\partial \hat{w}(p,y,x)}{dy} > 0$, when $y < x_{t(p)}$ and $\frac{\partial \hat{w}(p,y,x)}{\partial y} \leq 0$, when $y > x_{t(p)}$. Therefore, $x_{t(p)} = arg\max_{y \geq 0} w(p,y,x)$.

□

Proof of proposition 3

Proof. We first consider the case $p \in (p_i, p_{i+1}), i = 1, \cdots, n-1$. We prove the results by showing that $W(p,y) \geq w(p,y,\underline{X}_{i+1})$ and $W(p,y) \leq w(p,y,\underline{X}_{i+1})$, respectively.

1. We first show $W(p,y) \geq w(p,y,\underline{X}_{i+1})$. Let $\tilde{x}$ denote the quantile of q on $D(p)$, i.e. $\tilde{x}_j = Q(q_j;p), j = 1, \cdots, m$. For any $D(\cdot) \in \mathcal{D}$ and $p_i < p < p_{i+1}$, from Lemma 1, we have $\pi(p,y,D(p)) \geq w(p,y,\tilde{x})$. Given that $Q(q_j;p)$ is a decreasing function of p and $\underline{X}_{ij} \leq Q(q_j;p_i) \leq \bar{X}_{ij}$, we have $\underline{X}_{i+1} \leq \tilde{x} \leq \bar{X}_i$. Combining with $w(p,y,\tilde{x})$, which is an increasing function of $\tilde{x}_j, j = 1, \cdots, m$ from Lemma 1, we obtain

$$\pi(p,y,D(p)) \geq w(p,y,\underline{X}_{i+1}), \forall D(\cdot) \in \mathcal{D} \text{ and } p_i < p < p_{i+1}.$$

Therefore, $W(p,y) = \min_{D(\cdot)\in\mathcal{D}} \pi(p,y,D(p)) \geq w(p,y,\underline{X}_{i+1})$.

2. Then, we show $W(p,y) \leq w(p,y,\underline{X}_{i+1})$. We consider the following demand distribution:

$$D(s|\delta) = \begin{cases} \hat{H}_\delta(\bar{X}_1), & \underline{p} \leq s < p_1, \\ \hat{H}_\delta(\bar{X}_k), & p_k \leq s < p_{k+1}, k < i, \\ \hat{H}_\delta(\bar{X}_i), & p_i \leq s < p, \\ H_\delta(\underline{X}_{i+1}), & p \leq s < p_{i+1}, \\ H_\delta(\underline{X}_{k+1}), & p_k \leq s < p_{k+1}, k > i, \\ H_\delta(\underline{X}_{\hat{n}}), & s = p_{\hat{n}}, \end{cases}$$

where $\hat{H}_\delta$ is defined in (27) and H_δ is defined in (30). We can easily check that $D(s|\delta) \in \mathcal{D}$ for any $\delta \in (0, \frac{1}{m-1})$. Thus, $W(p,y) = \min_{D(\cdot)\in\mathcal{D}} \pi(p,y,D(p)) \leq \pi(p,y,H_\delta(\underline{X}_{i+1}))$. When we take $\delta \to 0$, (31) implies $W(p,y) \leq w(p,y,\underline{X}_{i+1})$.

For the case that $p = p_i, i = 1, \cdots, n$, the proof is similar. The only difference is that the lower bound of $Q(q_j; p)$ is $\underline{X}_{ij}$ instead of $\underline{X}_{i+1,j}$. For brevity, we omit the proof here. If $\underline{p} < p_1$, for any $p \in [\underline{p}, p_1)$, the quantiles' lower bound is $\underline{X}_1$. Thus, $W(p, y) = w(p, y, \underline{X}_1)$. Moreover, if $p_n < \bar{p}$, for any $p \in (p_n, \bar{p}]$, the lower bound of quantiles are all 0. Therefore, $W(p, y) = w(p, y, 0) = -cy$.

□

Proof of Theorem 1

Proof. 1. From Lemma 1, we know that

$$\arg\max_{y \geq 0} w(p, y, x) = \underline{x}_{t(p)}.$$

Combining with the formula of $W(p, y)$ in Proposition 3, we can establish the results.

2. Lemma 1 shows that $w(p, y, x)$ is an increasing function of p and $x_j, j = 1, \cdots m$. Hence, for any $p \in (p_i, p_{i+1}), i = 1, \cdots, n-1$,

$$\begin{aligned} W^*(p) &= \max_{y \geq 0} w(p, y, \underline{X}_{i+1}), \\ &\leq \max_{y \geq 0} w(p_{i+1}, y, \underline{X}_{i+1}), \\ &= W^*(p_{i+1}), \end{aligned}$$

where the inequality follows from $p \leq p_{i+1}$. This outcome implies that

$$p_{i+1} = \arg\max_{p \in (p_i, p_{i+1}]} W^*(p).$$

Similarly, when $\underline{p} < p_1$, for any $p \in [\underline{p}, p_1)$, we have $W^*(p) \leq W^*(p_1)$. When $p_n < \bar{p}$, for any $p \in (p_n, \bar{p}]$, $W^*(p) = \max_{y \geq 0} -cy = 0 \leq W^*(p_n)$. In summary, we obtain $p^* = \arg\max_{i=1,\cdots,n} W^*(p_i)$.

□

We present Proposition 4 by first proving the following two lemmas.

Lemma 2. *Let* $\Delta = \max_{i=1,\cdots,n-1} (p_{i+1} - p_i)$. *If* $Q(q_j; p)$ *is a Lipschitz continuous function of* p *with parameter* L_q, *then*

$$|W^* - \bar{W}^*| \leq 2\Delta L_q \bar{p} m.$$

Proof. We define the set of $(\underline{x}, \bar{x})$ as follows:

$$\begin{aligned} \Theta = \{(\underline{x}, \bar{x}) | \underline{x}(p_i)_j = \underline{X}_i, \bar{x}(p_i)_j = \bar{X}_i, i = 1, \cdots, n, \\ \underline{x}(p)_j \text{ and } \bar{x}(p)_j \text{ are decreasing}, j = 1, \cdots, m\}, \end{aligned} \tag{32}$$

which represents all feasible quantile bounds functions that satisfy the given information on the exercised prices. As a consequence, the true quantile bounds $(\underline{x}^*(p), \bar{x}^*(p))$ are in Θ. Given that $Q(q_j; p)$ is Lipschitz continuous with parameter L_q, for any $(\underline{x}, \bar{x}) \in \Theta$, $\underline{x}(p)_j$ and $\bar{x}(p)_j$ are also Lipschitz continuous with parameter L_q. Then, the optimal maxmin profit can be written as

$$W^* = \max_{p \in [\underline{p}, \bar{p}], y \geq 0} \min_{(\underline{x}, \bar{x}) \in \Theta} \min_{D(\cdot) \in \bar{\mathcal{D}}(\underline{x}, \bar{x})} \pi(p, y, D(p)),$$

where $\bar{\mathcal{D}}(\underline{x}, \bar{x})$ is defined in (12).

By the definition of $W(p, y)$ and Proposition 3, we have

$$W(p, y) = w(p, y, \underline{x}(p)),$$

where $w(\cdot)$ is defined in (5). Hence, we can rewrite W^* as

$$W^* = \max_{p \in [\underline{p}, \bar{p}], y \geq 0} \min_{(\underline{x}, \bar{x}) \in \Theta} w(p, y, \underline{x}(p)). \tag{33}$$

Then, we rewrite $\bar{W}^*$ as

$$\bar{W}^* = \max_{p \in [\underline{p}, \bar{p}], y \geq 0} w(p, y, \underline{x}^*(p)), \tag{34}$$

where $\underline{x}^*(p)$ denotes the true lower bound of the quantile function.

Let (p^*, y^*) denote the optimal solution of the maximization problem in (34), i.e.,

$$(p^*, y^*) = \arg\max_{p \in [\underline{p}, \bar{p}], y \geq 0} w(p, y, \underline{x}^*(p)).$$

Then, we have $\bar{W}^* = w(p^*, y^*, \underline{x}^*(p^*))$. By (33), for any $(\underline{x}, \bar{x}) \in \Theta$, we have

$$W^* \geq \min_{(\underline{x}', \bar{x}') \in \Theta} w(p^*, y^*, \underline{x}'(p^*)) \geq w(p^*, y^*, \underline{x}(p^*)). \tag{35}$$

Moreover, by (34), we have

$$\bar{W}^* = w(p^*, y^*, \underline{x}^*(p^*)). \tag{36}$$

Suppose $p^* \in [p_i, p_{i+1}]$, for any $\underline{x} \in \Theta$, because $\underline{x}(p)_j$ is Lipschitz continuous. We have $\underline{X}_{ij} - (p_{i+1} - p_i)L_q \leq \underline{x}(p^*)_j \leq \underline{X}_{ij} + (p_{i+1} - p_i)L_q$ and $\underline{X}_{ij} - (p_{i+1} - p_i)L_q \leq \underline{x}^*(p^*)_j \leq \underline{X}_{ij} + (p_{i+1} - p_i)L_q$. Thus, we have

$$|\underline{x}^*(p^*)_j - \underline{x}(p^*)_j| \leq 2(p_{i+1} - p_i)L_q \leq 2\Delta L_q, j = 1, \cdots, m.$$

The result holds for any $p^* \in [\underline{p}, \bar{p}]$.

By Lemma 1, $w(p, y, x)$ is an increasing function of $x_j, j = 1, \cdots, m$ with parameter $\bar{p}$. For any x, x' such that $|x_j - x'_j| \leq \delta$ for all j, we have $|w(p, y, x) - w(p, y, x')| \leq \bar{p}m\delta$. Combining with the fact that $|\underline{x}^*(p^*)_j - \underline{x}(p^*)_j| \leq 2\Delta L_q$ for all j, we have

$$w(p^*, y^*, \underline{x}^*(p^*)) - w(p^*, y^*, \underline{x}(p^*)) \leq 2\Delta L_q \bar{p} m.$$

Combining with (35) and (36), we obtain $\bar{W}^* - W^* \leq 2\Delta L_q \bar{p} m$. By definition, we have $\bar{W}^* \geq W^*$. Thus, $0 \leq \bar{W}^* - W^* \leq 2\Delta L_q \bar{p} m$, which implies $|W^* - \bar{W}^*| \leq 2\Delta L_q \bar{p} m$. $\square$

Lemma 3. *Consider the learning policy defined in (14). Let Δ_n be the length of the largest interval after the nth price has been learned. If $n \geq \lfloor 2^{\alpha k+1} \rfloor$, then $\Delta_n \leq \frac{\bar{p}-\underline{p}}{2^{\alpha k}}$.*

Proof. Without loss of generality, we set $\bar{p} - \underline{p} = 1$ in the following proof. Let $\tilde{p}_n$ denote the nth learning price, and let s_n denote the length of the largest new interval that is created by the nth learning price. Given that $\delta_n = \frac{1}{2^{\alpha k+1}}$, where $k = \max\{i \in [n] | 2^{\alpha(i-1)} \leq \frac{n+1}{2}\}$, which implies that when $2^{\alpha(k-1)+1} \leq n+1 < 2^{\alpha k+1}+1$, we have $\delta_n = \frac{1}{2^{\alpha k+1}}$. Specifically, when $2^{\alpha(k-1)+1} \leq n+1 \leq 2^{\alpha k+1}$, we have $\delta_n = \frac{1}{2^{\alpha k+1}}$.

First, we show that when $n \in [2, 2^{\alpha k+1})$, the length of the largest new interval $s_n \geq \frac{1}{2^{\alpha k+1}}$. When $n+1 \leq 2^{\alpha k+1}$, we know that $\delta_n \geq \frac{1}{2^{\alpha k+1}}$. If $\hat{\Omega}_n \neq \emptyset$, then we have $\Delta_{n-1} \geq \frac{1}{2^{\alpha k}}$ and $s_n \geq \frac{1}{2^{\alpha k+1}}$. If $\hat{\Omega}_n = \emptyset$, then the learning price $\tilde{p}_n$ is the middle point of the largest interval. After $n-1$th learning, we have at most n intervals (including boundary intervals), which implies $\Delta_{n-1} \geq \frac{1}{n} \geq \frac{1}{2^{\alpha k+1}}$. Hence, the length of the largest new interval $s_n \geq \frac{1}{2}\frac{1}{2^{\alpha k+1}} = \frac{1}{2^{\alpha k+2}} \geq \frac{1}{2^{\alpha(k+1)+1}}$, where the last inequality follows from $\alpha(k+1) \geq \alpha k + 1$, because $\alpha \geq 1$. Therefore,

$$s_n \geq \frac{1}{2^{\alpha(k+1)+1}}, \forall n \in [2, 2^{\alpha k+1}). \tag{37}$$

Let $n = \lfloor 2^{\alpha k+1} \rfloor$. We consider the following two cases. If $\Delta_{n-1} < \frac{1}{2^{\alpha k}}$, then $\Delta_n < \frac{1}{2^{\alpha k}}$. When $\Delta_{n-1} \geq \frac{1}{2^{\alpha k}}$, for any $\hat{n} \in [2^{\alpha(k-1)+1}, \lfloor 2^{\alpha k+1} \rfloor]$, if $\hat{\Omega}_{\hat{n}} \neq \emptyset$, then $\Delta_{\hat{n}-1} \geq \frac{1}{2^{\alpha k}}$ and $s_{\hat{n}} \geq \frac{1}{2^{\alpha k+1}}$. If $\hat{\Omega}_{\hat{n}} = \emptyset$, then from (37), we have $s_{\hat{n}} \geq \frac{1}{2^{\alpha k+1}}$. Furthermore, from (37), for any $n \in [2, 2^{\alpha(k-1)+1})$, we have $s_{\hat{n}} \geq \frac{1}{2^{\alpha k+1}}$. Therefore, after the nth price has been learned, we have at least $n = \lfloor 2^{\alpha k+1} \rfloor$ intervals whose length is greater or equal to $\frac{1}{2^{\alpha k+1}}$. However, we start with $n = 1$. Therefore, after n steps, we have $n+1$ price points, which create at most $n+2$ intervals (including boundary intervals). Hence,

$$\begin{aligned}
\Delta_n \leq& 1 - n\frac{1}{2^{\alpha k+1}} \\
=& 1 - \lfloor 2^{\alpha k+1} \rfloor \frac{1}{2^{\alpha k+1}} \\
\leq& 1 - 2^{\alpha k+1}\frac{1}{2^{\alpha k+1}} + \frac{1}{2^{\alpha k+1}} \\
=& \frac{1}{2^{\alpha k+1}} < \frac{1}{2^{\alpha k}}.
\end{aligned}$$

This completes the proof. $\square$

Proof of Proposition 4

Proof. From Lemma 3, we know that after $n \geq \lfloor 2^{\alpha k+1} \rfloor$ steps, the largest interval $\Delta_n \leq \frac{\bar{p}-\underline{p}}{2^{\alpha k}}$. Combining with $|W^* - \bar{W}^*| \leq 2\Delta L_q \bar{p} m$ from Lemma 2, where L_q is the Lipschitz continuous parameter of quantile function $Q(q_j; p)$. Hence, for any $\epsilon > 0$, we have $|W_n^* - \bar{W}^*| \leq \epsilon$ if $n \geq \frac{1}{\epsilon} 2^{\alpha+2}(\bar{p} - \underline{p})\bar{p} m L_q$. Therefore, $|W_n^* - \bar{W}^*| \leq \epsilon$ after no more than $O(\frac{1}{\epsilon})$ steps. $\square$

We prove Theorem 2 by using the properties derived in Lemma 1.

Proof of Theorem 2

Proof. We first consider the case that $p \in (p_i, p_{i+1}), i = 1, \cdots, \hat{n} - 1$. For notational simplicity, we let $R_l(p, y) = \max\{\max_{k<i} \hat{w}(p_{k+1}, \bar{X}_{k,t(p_{k+1})+1}, \bar{X}_k), \hat{w}(p, \bar{X}_{i,t(p)+1}, \bar{X}_i)\} - w(p, y, \underline{X}_{i+1})$ and

$$R_u(p, y) = \max_{k \geq i} \max_{\hat{y} \in \hat{Y}_{ik}} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y})).$$

Then, it is equivalent to show $R(p, y) = \max\{R_l(p, y), R_u(p, y)\}$. We prove it by showing $R(p, y) \geq \max\{R_l(p, y), R_u(p, y)\}$ and $R(p, y) \leq \max\{R_l(p, y), R_u(p, y)\}$.

1. We first show $R(p, y) \geq \max\{R_l(p, y), R_u(p, y)\}$. Consider the following demand distribution:

$$D(s|\delta) = \begin{cases} \hat{H}_\delta(\bar{X}_k), & p_k \leq s < p_{k+1}, k < i, \\ \hat{H}_\delta(\bar{X}_i), & p_i \leq s < p, \\ H_\delta(\underline{X}_{i+1}), & p \leq s < p_{i+1}, \\ H_\delta(\underline{X}_{k+1}), & p_k \leq s < p_{k+1}, k > i, \\ H_\delta(\underline{X}_{\hat{n}}), & s = p_{\hat{n}}. \end{cases}$$

We can easily check that $D(\cdot|\delta) \in \mathcal{D}$ for any $\delta \in (0, \frac{1}{m-1})$. Thus, for any $\epsilon \in (0, \min\{p - p_i, p_{i+1} - p, \min_{k=1,\cdots,\hat{n}-1} p_{k+1} - p_k\})$ and $\hat{y} \geq 0$,

$$\begin{aligned} R(p, y) &\geq \max_{\hat{p} \in [\underline{p}, \bar{p}], \hat{y} \geq 0} \pi(\hat{p}, \hat{y}, D(\hat{p}|\delta)) - \pi(p, y, D(p|\delta)) \\ &\geq \max_{\hat{y} \geq 0} \max\{\max_{k<i} \pi(\hat{p}, \hat{y}, \hat{H}_\delta(\bar{X}_k))|_{\hat{p}=p_{k+1}-\epsilon}, \pi(\hat{p}, \hat{y}, \hat{H}_\delta(\bar{X}_i))|_{\hat{p}=p-\epsilon}\} - w(p, y, H_\delta(\underline{X}_{i+1})). \end{aligned}$$

Let $\delta \to 0$. Combining with equations (28) and (31), we obtain

$$\begin{aligned} R(p, y) &\geq \max_{\hat{y} \geq 0} \max\{\max_{k<i} \hat{w}(p_{k+1} - \epsilon, \hat{y}, \bar{X}_k), \hat{w}(p - \epsilon, \hat{y}, \bar{X}_i)\} - w(p, y, \underline{X}_{i+1}) \\ &\geq \max_{\hat{y} \geq 0} \max\{\max_{k<i} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k), \hat{w}(p, \hat{y}, \bar{X}_i)\} - w(p, y, \underline{X}_{i+1}) \\ &\geq \max\{\max_{k<i} \hat{w}(p_{k+1}, \bar{X}_{k,t(p_{k+1})+1}, \bar{X}_k), \hat{w}(p, \bar{X}_{i,t(p)+1}, \bar{X}_i)\} - w(p, y, \underline{X}_{i+1}) \\ &= R_l(p, y), \end{aligned}$$

where the last inequality follows from $\bar{X}_{i,t(p)+1} \geq 0$ and $\bar{X}_{k,t(p_{k+1})+1} \geq 0, k = 1, \cdots, i$. Next, for any $\hat{y} \geq 0, 0 < \delta < \frac{1}{m-1}$, and $k \geq i$, we consider the following demand distribution:

$$D_k(s|\delta, \hat{y}) = \begin{cases} \hat{H}_\delta\left(\bar{X}_k\right), & p_k \leq s < p_{k+1}, k < i, \\ \hat{H}_\delta\left(\bar{X}_i\right), & p_i \leq s < p, \\ H_\delta\left(\hat{X}_k^i(\hat{y})\right), & p \leq s < p_{i+1} \\ \hat{H}_\delta\left(\min\left\{\hat{X}_k^i(\hat{y}), \bar{X}_k\right\}\right), & p_k \leq s < p_{k+1}, k > i \\ \hat{H}_\delta\left(\min\left\{\hat{X}_k^i(\hat{y}), \bar{X}_{\hat{n}}\right\}\right), & s = p_{\hat{n}}. \end{cases}$$

We can easily check that $D_k(\cdot|\delta, \hat{y}) \in \mathcal{D}$. Thus, for any $\epsilon \in (0, \min\{p - p_i, p_{i+1} - p, \min_{k=1,\cdots,\hat{n}-1} p_{k+1} - p_k\}), k \geq i$ and $\hat{y} \geq 0$, we have

$$\begin{aligned} R(p, y) &\geq \max_{\hat{p} \in [\underline{p}, \bar{p}]} \pi(\hat{p}, \hat{y}, D_k(\hat{p}|\delta, \hat{y})) - \pi(p, y, D_k(p|\delta, \hat{y})) \\ &\geq \pi(\hat{p}, \hat{y}, \hat{H}_\delta(\min\{\hat{X}_k^i(\hat{y}), \bar{X}_k\}))|_{\hat{p}=p_{k+1}-\epsilon} - \pi(p, y, H_\delta(\hat{X}_k^i(\hat{y}))). \end{aligned}$$

Let $\delta \to 0$ and $\epsilon \to 0$. By (28) and (31), we obtain for any $k \geq i, \hat{y} \geq 0$,

$$\begin{aligned} R(p, y) &\geq \hat{w}(p_{k+1} - \epsilon, \hat{y}, \min\{\hat{X}_k^i(\hat{y}), \bar{X}_k\}) - w(p, y, \hat{X}_k^i(\hat{y})), \\ &\geq \hat{w}(p_{k+1}, \hat{y}, \min\{\hat{X}_k^i(\hat{y}), \bar{X}_k\}) - w(p, y, \hat{X}_k^i(\hat{y})). \end{aligned} \tag{38}$$

By the definition of $\hat{X}_k^i(\hat{y})$ from (20) and $\hat{w}(\cdot)$, we have

$$\begin{aligned} &\hat{w}\left(\hat{p}, \hat{y}, \min\left\{\hat{X}_k^i(\hat{y}), \bar{X}_k\right\}\right) \\ =&\hat{p}\frac{1}{m-1}\left(\sum_{j=2}^{m-1} \min\left\{\max\left\{\min\left\{\hat{y}, \bar{X}_{kj}\right\}, \underline{X}_{i+1,j}\right\}, \bar{X}_{kj}, \hat{y}\right\} + \min\left\{\bar{X}_{im}, \bar{X}_{km}, \hat{y}\right\}\right) - c\hat{y}. \end{aligned}$$

When $\min\left\{\hat{y}, \bar{X}_{kj}\right\} \geq \underline{X}_{i+1,j}$, we have $\min\left\{\max\left\{\min\left\{\hat{y}, \bar{X}_{kj}\right\}, \underline{X}_{i+1,j}\right\}, \bar{X}_{kj}, \hat{y}\right\} = \min\left\{\bar{X}_{kj}, \hat{y}\right\}$. When $\min\left\{\hat{y}, \bar{X}_{kj}\right\} < \underline{X}_{i+1,j}$, we also obtain $\min\left\{\max\left\{\min\left\{\hat{y}, \bar{X}_{kj}\right\}, \underline{X}_{i+1,j}\right\}, \bar{X}_{kj}, \hat{y}\right\} = \min\left\{\bar{X}_{kj}, \hat{y}\right\}$. Moreover, we have $\bar{X}_{im} \geq \bar{X}_{km}$ because $k \geq i$. Therefore, we obtain

$$\hat{w}(\hat{p}, \hat{y}, \min\{\hat{X}_k^i(\hat{y}), \bar{X}_k\}) = \hat{w}(\hat{p}, \hat{y}, \bar{X}_k), \forall \hat{p}, \hat{y}, k \geq i. \tag{39}$$

Thus, (38) implies that $R(p, y) \geq \max_{k \geq i} \max_{\hat{y} \in \hat{Y}_{ik}} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y})) = R_u(p, y)$. Therefore, we have

$$R(p, y) \geq \max\{R_l(p, y), R_u(p, y)\}.$$

2. Next we show that $R(p,y) \le \max\{R_l(p,y), R_u(p,y)\}$. Let $\hat{x}$ denote the quantile of q on $D(\hat{p})$, i.e., $\hat{x}_j = Q(q_j;\hat{p}), j = 1,\cdots,m$, and $\tilde{x}$ denote the quantile of q on $D(p)$. First, for any $D(\cdot) \in \mathcal{D}$ and $p_i < p < p_{i+1}$, we have

$$\pi(\hat{p},\hat{y},D(\hat{p})) - \pi(p,y,D(p)) \le \hat{w}(\hat{p},\hat{y},\hat{x}) - w(p,y,\tilde{x}), \tag{40}$$

where the inequality follows from Lemma 1. Given that $p_i < p < p_{i+1}$, $Q(q_j;p)$ is a decreasing function of p and $\underline{X}_{ij} \le Q(q_j;p_i) \le \bar{X}_{ij}$, we have

$$\underline{X}_{i+1} \le \tilde{x} \le \bar{X}_i, \tag{41}$$

and

$$\begin{cases} \hat{x} \le \bar{X}_k, & \hat{p} \in [p_k, p_{k+1}), k < i, \\ \hat{x} \le \bar{X}_i, & \hat{p} \in [p_i, p), \\ \hat{x} \le \tilde{x}, & \hat{p} \in [p, p_{i+1}), \\ \hat{x} \le \min\{\tilde{x}, \bar{X}_k\}, & \hat{p} \in [p_k, p_{k+1}), k > i, \\ \hat{x} \le \min\{\tilde{x}, \bar{X}_{\hat{n}-1}\}, & \hat{p} = p_{\hat{n}}. \end{cases}$$

Given that $\hat{w}(\hat{p},\hat{y},\hat{x})$ is an increasing function of $\hat{p}$ and $\hat{x}$ from Lemma 1, we obtain

$$\begin{aligned} &\max_{\hat{p}\in[\underline{p},\bar{p}]} \hat{w}(\hat{p},\hat{y},\hat{x}) - w(p,y,\tilde{x}) \\ \le &\max\{\max_{k<i} \max_{\hat{p}\in[p_k,p_{k+1})} \hat{w}(\hat{p},\hat{y},\bar{X}_k), \max_{\hat{p}\in[p_i,p)} \hat{w}(\hat{p},\hat{y},\bar{X}_i), \\ &\quad \max_{\hat{p}\in[p,p_{i+1})} \hat{w}(\hat{p},\hat{y},\tilde{x}), \max_{k>i} \max_{\hat{p}\in[p_k,p_{k+1})} \hat{w}(\hat{p},\hat{y},\min\{\bar{X}_k,\tilde{x}\}), \\ &\quad \hat{w}(p_{\hat{n}},\hat{y},\min\{\bar{X}_{\hat{n}-1},\tilde{x}\})\} - w(p,y,\tilde{x}) \\ \le &\max\{\max_{k<i} \hat{w}(p_{k+1},\hat{y},\bar{X}_k), \hat{w}(p,\hat{y},\bar{X}_i), \hat{w}(p_{i+1},\hat{y},\tilde{x}), \\ &\quad \max_{k>i} \hat{w}(p_{k+1},\hat{y},\min\{\bar{X}_k,\tilde{x}\}), \hat{w}(p_{\hat{n}},\hat{y},\min\{\bar{X}_{\hat{n}-1},\tilde{x}\})\} - w(p,y,\tilde{x}). \end{aligned}$$

When $i = \hat{n}-1$, $\hat{w}(p_{i+1},\hat{y},\tilde{x}) = \hat{w}(p_{\hat{n}},\hat{y},\tilde{x}) \ge \hat{w}(p_{\hat{n}},\hat{y},\min\{\bar{X}_{\hat{n}-1},\tilde{x}\})$. Meanwhile when $i \ne \hat{n}-1$, $\max_{k>i} \hat{w}(p_{k+1},\hat{y},\min\{\bar{X}_k,\tilde{x}\}) \ge \hat{w}(p_{\hat{n}},\hat{y},\min\{\bar{X}_{\hat{n}-1},\tilde{x}\})$, which implies that for any $D(\cdot) \in \mathcal{D}$,

$$\begin{aligned} &\max_{\hat{p}\in[\underline{p},\bar{p}]} \hat{w}(\hat{p},\hat{y},\hat{x}) - w(p,y,\tilde{x}) \\ \le &\max\{\max_{k<i} \hat{w}(p_{k+1},\hat{y},\bar{X}_k), \hat{w}(p,\hat{y},\bar{X}_i), \max_{k\ge i} \hat{w}(p_{k+1},\hat{y},\min\{\bar{X}_k,\tilde{x}\})\} - w(p,y,\tilde{x}). \end{aligned} \tag{42}$$

Given that $w(p,y,\tilde{x})$ is an increasing function of $\tilde{x}$, and $\tilde{x} \ge \underline{X}_{i+1}$, we have

$$\begin{aligned} &\max\{\max_{k<i} \hat{w}(p_{k+1},\hat{y},\bar{X}_k), \hat{w}(p,\hat{y},\bar{X}_i)\} - w(p,y,\tilde{x}) \\ \le &\max\{\max_{k<i} \hat{w}(p_{k+1},\hat{y},\bar{X}_k), \hat{w}(p,\hat{y},\bar{X}_i)\} - w(p,y,\underline{X}_{i+1}). \end{aligned} \tag{43}$$

For any $k \geq i$, by the definition of $\hat{w}(\cdot)$ and $w(\cdot)$,

$$\begin{aligned}
&\max_{k\geq i} \hat{w}(p_{k+1}, \hat{y}, \min\{\bar{X}_k, \tilde{x}\}) - w(p, y, \tilde{x}) \\
=& -\frac{p}{m-1}\min\{\tilde{x}_1, y\} \\
&+ \sum_{j=2}^{m-1}\left\{\frac{p_{k+1}}{m-1}\min\{\tilde{x}_j, \bar{X}_{kj}, \hat{y}\} - \frac{p}{m-1}\min\{\tilde{x}_j, y\}\right\} \\
&+ \frac{p_{k+1}}{m-1}\min\{\tilde{x}_m, \bar{X}_{km}, \hat{y}\} - c\hat{y} + cy.
\end{aligned}$$

Let $g_{kj}(x) = p_{k+1}\min\{x, \bar{X}_{kj}, \hat{y}\} - p\min\{x, y\}$. Then, we have

$$g_{kj}(x) = \begin{cases} p_{k+1}x - p\min\{x, y\}, & x \leq \min\{\bar{X}_{kj}, \hat{y}\}, \\ p_{k+1}\min\{\bar{X}_{kj}, \hat{y}\} - p\min\{x, y\}, & x \geq \min\{\bar{X}_{kj}, \hat{y}\}. \end{cases}$$

Given that $p_{k+1} \geq p$, $g_{kj}(x)$ is increasing when $x \leq \min\{\bar{X}_{kj}, \hat{y}\}$ and decreasing when $x \geq \min\{\bar{X}_{kj}, \hat{y}\}$. Moreover, given that $\bar{X}_{kj} \leq \bar{X}_{ij}$, we obtain

$$\max_{x\in[\underline{X}_{i+1,j}, \bar{X}_{ij}]} g_{kj}(x) = g_{kj}(\max\{\underline{X}_{i+1,j}, \min\{\bar{X}_{kj}, \hat{y}\}\}) = g_{kj}(\hat{X}_k^i(\hat{y})).$$

Therefore,

$$\begin{aligned}
&\max_{\tilde{x}\in[\underline{X}_{i+1}, \bar{X}_i]} \max_{k\geq i} \hat{w}(p_{k+1}, \hat{y}, \min\{\bar{X}_k, \tilde{x}\}) - w(p, y, \tilde{x}) \\
=&\max_{k\geq i} \hat{w}(p_{k+1}, \hat{y}, \min\{\bar{X}_k, \hat{X}_k^i(\hat{y})\}) - w(p, y, \hat{X}_k^i(\hat{y})) \\
=&\max_{k\geq i} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y})),
\end{aligned}$$

where the last equality follows from (39). Combining with equations (40), (42), and (43), for any $D(\cdot) \in \mathcal{D}$, $\hat{y} \geq 0$,

$$\begin{aligned}
&\max_{\hat{p}\in[\underline{p}, \bar{p}]} \pi(\hat{p}, \hat{y}, D(\hat{p})) - \pi(p, y, D(p)) \\
&\leq \max\{\max_{k<i} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \underline{X}_{i+1}), \\
&\qquad \hat{w}(p, \hat{y}, \bar{X}_i) - w(p, y, \underline{X}_{i+1}), \\
&\qquad \max_{k\geq i} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y}))\}.
\end{aligned} \tag{44}$$

Given that $x_{t(p)+1} = \arg\max_{\hat{y}\geq 0} \hat{w}(p, y, x)$ from Lemma 1, we have

$$\begin{aligned}
&\max_{\hat{y}\geq 0} \max\{\max_{k<i} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k), \hat{w}(p, \hat{y}, \bar{X}_i)\} - w(p, y, \underline{X}_{i+1}) \\
=&\max\{\max_{k<i} \hat{w}(p_{k+1}, \bar{X}_{k,t(p_{k+1})+1}, \bar{X}_k), \hat{w}(p, \bar{X}_{i,t(p)+1}, \bar{X}_i\} - w(p, y, \underline{X}_{i+1}), \\
=&R_l(p, y).
\end{aligned}$$

In addition, by the definition of $\hat{w}(\cdot)$ and $w(\cdot)$, for any $k \geq i$, $\hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y}))$ is a piecewise linear function of $\hat{y}$ with breaking points set $\hat{Y}_{ik}$. Thus, $\max\limits_{\hat{y}\geq 0} \max\limits_{k\geq i} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y})) = \max\limits_{k\geq i} \max\limits_{\hat{y}\in\hat{Y}_{ik}} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y})) = R_u(p, y)$. Combining with (44) and the definition of $R(p, y)$, we obtain $R(p, y) \leq \max\{R_l(p, y), R_u(p, y)\}$.

For the case that $p = p_i, i = 1, \cdots, \hat{n}$, the proof is similar. The only difference is that the lower bound of $Q(q_j; p)$ is $\underline{X}_{ij}$ instead of $\underline{X}_{i+1,j}$. Additionally, the interval $[p_i, p)$ is empty. For brevity, we omit the proof here. □

Proof of proposition 6

Proof. By the definition of $w(p, y, x)$ from (5), $w(p, y, x)$ is a concave function of y, which implies that every function of the right side of equality of $R(p, y)$ in Theorem 2 is a convex function of y. Therefore, for any p, $R(p, y)$ is a convex function, because point-wise maximization preserves convexity. □

Proof of Proposition 6.

Proof. It suffices to show that $\min_{p\in(p_i,p_{i+1})} R^*(p) \geq R^*(p_{i+1})$ for any $i > i^*$. Let

$$\pi^* = \hat{w}(p_{i^*+1}, \bar{X}_{i^*,t(p_{i^*+1})+1}, \bar{X}_{i^*}).$$

For $i > i^*$ and any $p \in (p_i, p_{i+1})$, the first term in (21) satisfies $\max_{k<i} \hat{w}(p_{k+1}, \bar{X}_{k,t(p_{k+1})+1}, \bar{X}_k) - w(p, y, \underline{X}_{i+1}) = \pi^* - w(p, y, \underline{X}_{i+1})$. By Lemma 1, we have $\hat{w}(p, \cdot, \cdot)$, which is increasing in p and $x_{t(p)+1} = \arg\max_{y\geq 0} \hat{w}(p, y, x)$. Thus, given that $p \leq p_{i+1}$, we have

$$\begin{aligned}
\pi^* &= \hat{w}(p_{i^*+1}, \bar{X}_{i^*,t(p_{i^*+1})+1}, \bar{X}_{i^*}) \\
&\geq \hat{w}(p_{i+1}, \bar{X}_{i,t(p_{i+1})+1}, \bar{X}_i) \\
&= \max_{\hat{y}\geq 0} \hat{w}(p_{i+1}, \hat{y}, \bar{X}_i) \\
&\geq \max_{\hat{y}\geq 0} \hat{w}(p, \hat{y}, \bar{X}_i) \\
&= \hat{w}(p, \bar{X}_{i,t(p)+1}, \bar{X}_i),
\end{aligned}$$

which implies that $\pi^* - w(p, y, \underline{X}_{i+1}) \geq \hat{w}(p, \bar{X}_{i,t(p)+1}, \bar{X}_i) - w(p, y, \underline{X}_{i+1})$. Furthermore, for any $k \geq i$ and $\hat{y} \in \hat{Y}_{ik}$, we have

$$\begin{aligned}
\pi^* &\geq \hat{w}(p_{k+1}, \bar{X}_{k,t(p_{k+1})+1}, \bar{X}_k) \\
&= \max_{\tilde{y}\geq 0} \hat{w}(p_{k+1}, \tilde{y}, \bar{X}_k) \\
&\geq \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k).
\end{aligned}$$

By the definition of $\hat{X}_k^i(\hat{y})$, we have $\hat{X}_k^i(\hat{y}) \geq \underline{X}_{i+1}$ componentwise, which implies that $w(p, y, \underline{X}_{i+1}) \leq w(p, y, \hat{X}_k^i(\hat{y}))$. Therefore, $\pi^* - w(p, y, \underline{X}_{i+1}) \geq \max_{k \geq i} \max_{\hat{y} \in \hat{Y}_{ik}} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \hat{X}_k^i(\hat{y}))$. In summary, $R(p, y) = \pi^* - w(p, y, \underline{X}_{i+1})$, which implies that $R^*(p) = \min_{y \geq 0}\{\pi^* - w(p, y, \underline{X}_{i+1})\}$. Given that $w(p, y, \underline{X}_{i+1})$ is increasing in p, we obtain

$$\min_{p \in (p_i, p_{i+1})} R^*(p) = \min_{y \geq 0}\{\pi^* - w(p_{i+1}, y, \underline{X}_{i+1})\}. \tag{45}$$

For $p = p_{i+1}$, by a similar argument, we have $\pi^* - w(p, y, \underline{X}_{i+1}) = \max\limits_{k < i+1} \hat{w}(p_{k+1}, \bar{X}_{k, t(p_{k+1})+1}, \bar{X}_k) - w(p, y, \underline{X}_{i+1})$, and $\pi^* - w(p_{i+1}, y, \underline{X}_{i+1}) \geq \max\limits_{k \geq i+1} \max\limits_{\hat{y} \in \tilde{Y}_{i+1,k}} \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \tilde{X}_k^{i+1}(\hat{y}))$. Therefore, $R^*(p_{i+1}) = \min_{y \geq 0}\{\pi^* - w(p_{i+1}, y, \underline{X}_{i+1})\}$. Combining with (45), we have $\min_{p \in (p_i, p_{i+1})} R^*(p) = R^*(p_{i+1})$, which implies that $p_i^* = p_{i+1}$ for all $i > i^*$. $\square$

We show Proposition 8 by first proving the following two lemmas.

Lemma 4. *Let* $\Delta = \max\limits_{i=1,\cdots,\hat{n}-1}(p_{i+1} - p_i)$. *If* $Q(q_j; p)$ *is a Lipschitz continuous function of* p *with parameter* L_q, *then*

$$|R^* - \bar{R}^*| \leq 4\Delta L_q \bar{p} m.$$

Proof. We define the set of $(\underline{x}, \bar{x})$ as follows:

$$\begin{aligned} \Theta = \{(\underline{x}, \bar{x}) | \underline{x}(p_i)_j = \underline{X}_i, \bar{x}(p_i) = \bar{X}_i, i = 1, \cdots, n, \\ \underline{x}(p)_j \text{ and } \bar{x}(p)_j \text{ are decreasing}, j = 1, \cdots, m\}, \end{aligned} \tag{46}$$

which represents all feasible quantile bounds functions that satisfy the given information on the exercised prices. As a consequence, the true quantile bounds $(\underline{x}^*(p), \bar{x}^*(p))$ are in Θ. Given that $Q(q_j; p)$ is Lipschitz continuous with parameter L_q, for any $(\underline{x}_j, \bar{x}_j) \in \Theta$, $\underline{x}(p)_j$ and $\bar{x}(p)_j$ are also Lipschitz continuous with parameter L_q. Then, the optimal minmax regret can be written as

$$R^* = \min_{p \in [\underline{p}, \bar{p}], y \geq 0} \max_{(\underline{x}, \bar{x}) \in \Theta} \max_{D(\cdot) \in \bar{\mathcal{D}}(\underline{x}, \bar{x})} [\max_{\hat{p} \in [\underline{p}, \bar{p}], \hat{y} \geq 0} \pi(\hat{p}, \hat{y}, D(\hat{p})) - \pi(p, y, D(p))].$$

Let

$$r(p, y, x, \hat{p}, \hat{y}, \hat{x}) = \begin{cases} \hat{w}(\hat{p}, \hat{y}, \hat{x}) - w(p, y, x), & \hat{p} < p, \\ \hat{w}(\hat{p}, \hat{y}, \theta_1(x, \hat{x}, \hat{y})) - w(p, y, \theta_1(x, \hat{x}, \hat{y})), & \hat{p} = p, \\ \hat{w}(\hat{p}, \hat{y}, \hat{x}) - w(p, y, \theta_2(x, \hat{x}, \hat{y})), & \hat{p} > p, \end{cases}$$

where

$$\theta_1(x, \hat{x}, \hat{y})_j = \begin{cases} \max\{\min\{\hat{x}_{j+1}, \hat{y}\}, x_j\}, & j = 1, \cdots, m-1, \\ \max\{\min\{\hat{x}_m, \hat{y}\}, x_m\}, & j = m, \end{cases}$$

and

$$\theta_2(x, \hat{x}, \hat{y})_j = \begin{cases} x_1, & j = 1, \\ \max\{\min\{\hat{x}_j, \hat{y}\}, x_j\}, & j = 2, \cdots, m-1, \\ \hat{x}_m, & j = m. \end{cases}$$

Here, x represents the lower bound of $Q(q;p)$, and $\hat{x}$ represents the upper bound of $Q(q;\hat{p})$. We can show that

$$\max_{D(\cdot)\in\bar{\mathcal{D}}(\underline{x},\bar{x})} \pi(\hat{p}, \hat{y}, D(\hat{p})) - \pi(p, y, D(p)) = r(p, y, \underline{x}(p), \hat{p}, \hat{y}, \bar{x}(\hat{p})).$$

The proof is similar to the proof of $R(p, y)$ in Theorem 2. The only difference is that the lower bound of $Q(q;p)$ and upper bound of $Q(q;\hat{p})$ are given. Moreover, we do not have to optimize $\hat{p}$ and $\hat{y}$. For brevity, we omit the proof here. By the definition of $r(\cdot)$, $r(p, y, x, \hat{p}, \hat{y}, \hat{x})$ is a Lipschitz continuous function of x_j and $\hat{x}_j, j = 1, \cdots, m$ with parameter $\bar{p}$.

Hence, we can rewrite R^* as

$$R^* = \min_{p\in[\underline{p},\bar{p}],y\geq 0} \max_{\hat{p}\in[\underline{p},\bar{p}],\hat{y}\geq 0} \max_{(\underline{x},\bar{x})\in\Theta} r(p, y, \underline{x}(p), \hat{p}, \hat{y}, \bar{x}(\hat{p})). \tag{47}$$

Then, we rewrite $\bar{R}^*$ as

$$\bar{R}^* = \min_{p\in[\underline{p},\bar{p}],y\geq 0} \max_{\hat{p}\in[\underline{p},\bar{p}],\hat{y}\geq 0} r(p, y, \underline{x}^*(p), \hat{p}, \hat{y}, \bar{x}^*(\hat{p})). \tag{48}$$

Let (p^*, y^*) denote the optimal solution of the outer minimization problem of (48), i.e.,

$$(p^*, y^*) = \arg\min_{p\in[\underline{p},\bar{p}],y\geq 0} \max_{\hat{p}\in[\underline{p},\bar{p}],\hat{y}\geq 0} r(p, y, \underline{x}^*(p), \hat{p}, \hat{y}, \bar{x}^*(\hat{p})).$$

Then, we have $\bar{R}^* = \max_{\hat{p}\in[\underline{p},\bar{p}],\hat{y}\geq 0} r(p^*, y^*, \underline{x}^*(p^*), \hat{p}, \hat{y}, \bar{x}^*(\hat{p}))$. Let (p', y') denote the optimal solution $(\hat{p}, \hat{y})$ of the inner maximization in (47) with $(p, y) = (p^*, y^*)$, i.e.,

$$(p', y') = \arg\max_{\hat{p}\in[\underline{p},\bar{p}],\hat{y}\geq 0} \max_{(\underline{x},\bar{x})\in\Theta} r(p^*, y^*, \underline{x}(p^*), \hat{p}, \hat{y}, \bar{x}(\hat{p})).$$

Therefore, we have

$$R^* \leq \max_{(\underline{x},\bar{x})\in\Theta} r(p^*, y^*, \underline{x}(p^*), p', y', \bar{x}(p')). \tag{49}$$

Moreover, by (48), we have

$$\bar{R}^* \geq r(p^*, y^*, \underline{x}^*(p^*), p', y', \bar{x}^*(p')). \tag{50}$$

Suppose $p^* \in [p_i, p_{i+1}]$, for any $(\underline{x}, \bar{x}) \in \Theta$. Given that $\underline{x}(p)_j$ is Lipschitz continuous, we have $\underline{X}_{ij} - (p_{i+1} - p_i)L_q \leq \underline{x}(p^*)_j \leq \underline{X}_{ij} + (p_{i+1} - p_i)L_q$ and $\underline{X}_{ij} - (p_{i+1} - p_i)L_q \leq \underline{x}^*(p^*)_j \leq \underline{X}_{ij} + (p_{i+1} - p_i)L_q$. Thus, we have

$$|\underline{x}^*(p^*)_j - \underline{x}(p^*)_j| \leq 2(p_{i+1} - p_i)L_q \leq 2\Delta L_q, j = 1, \cdots, m.$$

The result holds for any $p^* \in [\underline{p}, \bar{p}]$. Similarly, we also have

$$|\bar{x}^*(p')_j - \bar{x}(p')_j| \leq 2\Delta L_q, j = 1, \cdots, m,$$

Moreover, combining with that $r(p, y, x, \hat{p}, \hat{y}, \hat{x})$ is a Lipschitz continuous function of x_j and $\hat{x}_j, j = 1, \cdots, m$ with parameter $\bar{p}$, we have

$$\max_{(\underline{x}, \bar{x}) \in \Theta} r(p^*, y^*, \underline{x}(p^*), p', y', \bar{x}(p')) - r(p^*, y^*, \underline{x}^*(p^*), p', y', \bar{x}^*(p')) \leq 4\Delta L_q m.$$

Combining with (49) and (50), we obtain $R^* - \bar{R}^* \leq 4\Delta L_q \bar{p} m$. ☐

Lemma 5. *Consider the learning policy defined in (22). Let Δ_n be the length of the largest interval after the nth price has been learned. If $n \geq \lfloor 2^{\alpha k+1} \rfloor$, then $\Delta_n \leq \frac{\bar{p}-\underline{p}}{2^{\alpha k}}$.*

Proof. Without loss of generality, we set $\bar{p} - \underline{p} = 1$ in the following proof. Let $\tilde{p}_n$ denote the nth learning price and s_n denote the length of the largest new interval that is created by the nth learning price. Given that $\delta_n = \frac{1}{2^{\alpha k+1}}$, where $k = \max\{i \in [n] | 2^{\alpha(i-1)} \leq \frac{n}{2}\}$, which implies that when $2^{\alpha(k-1)+1} \leq n < 2^{\alpha k+1}$, we have $\delta_n = \frac{1}{2^{\alpha k+1}}$.

First, we show that when $n \in [3, 2^{\alpha k+1})$, the length of the largest new interval $s_n \geq \frac{1}{2^{\alpha k+1}}$. When $n < 2^{\alpha k+1}$, we know that $\delta_n \geq \frac{1}{2^{\alpha k+1}}$. If $\Omega_n \neq \emptyset$, then we have $\Delta_{n-1} \geq \frac{1}{2^{\alpha k}}$ and $s_n \geq \frac{1}{2^{\alpha k+1}}$. If $\Omega_n = \emptyset$, then the learning price $\tilde{p}_n$ is the middle point of the largest interval. After the $n-1$th learning, we have $n-1$ intervals, which implies that $\Delta_{n-1} \geq \frac{1}{n-1} \geq \frac{1}{2^{\alpha k+1}-1}$. Hence, the length of the largest new interval $s_n \geq \frac{1}{2}\frac{1}{2^{\alpha k+1}-1} \geq \frac{1}{2^{\alpha(k+1)+1}}$, where the last inequality follows from $\alpha(k+1) \geq \alpha k + 1$, because $\alpha \geq 1$. Therefore,

$$s_n \geq \frac{1}{2^{\alpha(k+1)+1}}, \forall n \in [3, 2^{\alpha k+1}). \tag{51}$$

Let $n = \lfloor 2^{\alpha k+1} \rfloor$. We consider the following two cases. If $\Delta_{n-1} < \frac{1}{2^{\alpha k}}$, then $\Delta_n < \frac{1}{2^{\alpha k}}$. When $\Delta_{n-1} \geq \frac{1}{2^{\alpha k}}$, for any $\hat{n} \in [2^{\alpha(k-1)+1}, \lfloor 2^{\alpha k+1} \rfloor]$, $\Delta_{\hat{n}} \geq 2^{\alpha k}$. Hence, $s_{\hat{n}} \geq \frac{1}{2^{\alpha k+1}}$, because $\Delta_{\hat{n}} = \frac{1}{2^{\alpha k+1}}$. Furthermore, from (51), for any $n \in [3, 2^{\alpha(k-1)+1})$, we have $s_{\hat{n}} \geq \frac{1}{2^{\alpha k+1}}$. Therefore, after the nth price has been learned, we have at least $n - 1 = \lfloor 2^{\alpha k+1} \rfloor - 1$ intervals whose length is greater or equal to $\frac{1}{2^{\alpha k+1}}$. Hence,

$$\begin{aligned}
\Delta_n \leq & 1 - (n-1-1)\frac{1}{2^{\alpha k+1}} \\
= & 1 - (\lfloor 2^{\alpha k+1} \rfloor - 2)\frac{1}{2^{\alpha k+1}} \\
\leq & 1 - (2^{\alpha k+1} - 2)\frac{1}{2^{\alpha k+1}} \\
= & \frac{1}{2^{\alpha k}}.
\end{aligned}$$

This complete the proof. ☐

Proof of Proposition 8

Proof. From Lemma 5, we know that after $n \geq \lfloor 2^{\alpha k+1} \rfloor$ steps, the largest interval $\Delta_n \leq \frac{\bar{p}-p}{2^{\alpha k}}$. Combining with $|R^* - \bar{R}^*| \leq 4\Delta L_q \bar{p} m$ from Lemma 4, where L_q is the Lipschitz continuous parameter of quantile function $Q(q_j; p)$. Hence, for any $\epsilon > 0$, we have $|R_n^* - \bar{R}^*| \leq \frac{1}{\epsilon}$ if $n \geq \frac{1}{\epsilon} 2^{\alpha+3}(\bar{p} - \underline{p})\bar{p} m L_q$. Therefore, $|R_n^* - \bar{R}^*| \leq \epsilon$ after no more than $O(\frac{1}{\epsilon})$ steps. □

B Online Appendix

In this section, we provide the details of the algorithm and proof of Proposition 7. Our algorithm handles two cases: $p = p_i, i = 1, \cdots, \hat{n}-1$ and $p \in (p_i, p_{i+1}), i = 1, \cdots, \hat{n}-1$. For the first case, we can use golden search to find the optimal $y(p)$ easily. For the second case, Proposition 6 implies that for any $i > i^*$ where $i^* = \arg\max_{i=1,\cdots,\hat{n}-1} \hat{w}(p_{i+1}, \bar{X}_{i,t(p_{i+1})+1}, \bar{X}_i)$, the optimal price over interval $(p_i, p_{i+1}]$ is p_{i+1}. Therefore, we only need to solve the optimization problem for intervals with $i \leq i^*$. $\min_{p \in (p_i, p_{i+1}), y \geq 0} R(p, y)$ is equivalent to

$$
\begin{aligned}
&\min_{p \in (p_i, p_{i+1})} \min_{y \geq 0} \min z \\
&\quad \text{s.t.} \hat{w}(p_{k+1}, \bar{X}_{k,t(p_{k+1})+1}, \bar{X}_k) - w(p, y, \underline{X}_{i+1}) \leq z, k = 1, \cdots, i-1 \\
&\quad \hat{w}(p, \bar{X}_{ij}, \bar{X}_i) - w(p, y, \underline{X}_{i+1}) \leq z, j = 1, \cdots, m, \\
&\quad \hat{w}(p_{k+1}, y, \bar{X}_k) - w(p, y, \max\{\bar{X}_k, \underline{X}_{i+1}\}) \leq z, k = i, \cdots, \hat{n}-1 \\
&\quad \hat{w}(p_{k+1}, \hat{y}, \bar{X}_k) - w(p, y, \max\{\min\{\hat{y}, \bar{X}_k\}, \underline{X}_{i+1}\}) \leq z, k = i, \cdots, \hat{n}-1, \hat{y} \in \bar{Y}_{ik},
\end{aligned}
\tag{52}
$$

where $\bar{Y}_{ik} = \{0\} \cup \{\bar{X}_{kj}\}_{j=1}^m \cup \{\underline{X}_{i+1,j}\}_{j=1}^m$. The above equivalence is implied by Theorem 2, which provides an explicit expression for $R(p, y)$ when $p \in (p_i, p_{i+1})$. In Theorem 2, $R(p, y)$ is expressed as the maximum of three terms by (21). An auxiliary variable z is introduced in (52) to bound each of the three terms. The four groups of constraints in Problem (52) correspond to the three terms of (21) as follows. The first group enforces the first term of (21). The second group refines the second term of (21) by splitting $\bar{X}_{i,t(p)+1}$ into the m quantile-specific terms $\bar{X}_{ij}$. The third and fourth groups encode the third term of (21), where the third group handles $\hat{y} = y$ with $\hat{X}_k^i(y) = \max\{\bar{X}_k, \underline{X}_{i+1}\}$, while the fourth group handles $\hat{y} \in \hat{Y}_{ik} \setminus \{y\}$ with $\hat{X}_k^i(\hat{y}) = \max\{\min\{\hat{y}, \bar{X}_k\}, \underline{X}_{i+1}\}$. At optimality, z equals the maximum of the three terms in (21), which implies that $z = R(p, y)$.

Each constraint in Problem (52) is a piece-wise linear function of y with breakpoints from $\cup_{k=i}^{\hat{n}-1} \bar{Y}_{ik}$. Hence, we can divide the feasible regions $y \geq 0$ into subintervals where the constraints are linear in y. Then, we can solve each subproblem over each $[y_l, y_u] \subset [0, +\infty)$.

In each subinterval $y \in [y_l, y_u]$, each constraint is of the form

$$apy + bp + dy + e \le z,$$

where a, b, d, e are constants. By the definition of $w(\cdot)$ and the expression in (52), we have $a \le 0$. Therefore, it is reduced to solve the following bilinear optimization problem:

$$\begin{aligned} &\min_{p \in (p_l, p_u)} \min_{y \in [y_l, y_u]} \min z \\ &\text{s.t.} a_i py + b_i p + d_i y + e_i \le z, i = 1, \cdots, t. \end{aligned} \tag{53}$$

Suppose that $z \in [z_l, z_u]$ and $a_i \le 0, i = 1, \cdots t$. One natural lower bound z_l for z is 0, because the minmax regret is always nonnegative. z_u is set to be the largest profit under the current information, i.e., $\max_{i=1,\cdots,\hat{n}-1} \hat{w}(p_{i+1}, \bar{X}_{i,t(p_{i+1})+1}, \bar{X}_i)$. Given that the objective is to find the minimal z, we can use binary search over $[z_l, z_u]$ to find the optimal solution by checking whether a fixed value of z is feasible for all the constraints. We achieve this process, by rewriting each constraint as

$$\begin{cases} p \le \frac{z - d_i y - e_i}{a_i y + b_i}, & \text{if } a_i y + b_i > 0, \\ p \ge \frac{z - d_i y - e_i}{a_i y + b_i}, & \text{if } a_i y + b_i < 0, \\ d_i y + e_i \le z, & \text{if } a_i y + b_i = 0. \end{cases}$$

The constraints can be categorized into three groups. The index sets are defined as $J(y) = \{j \mid a_j y + b_j < 0\}$, $I(y) = \{i \mid a_i y + b_i > 0\}$, and $K(y) = \{k \mid a_k y + b_k = 0\}$. We define $w_i(y) = \frac{z - d_i y - e_i}{a_i y + b_i}$. Then, the upper bound of p is $\min_{i \in I(y)} w_i(y)$ and the lower bound of p is $\max_{j \in J(y)} w_j(y)$. For $p \in (p_l, p_u)$ to be feasible, we need $\max_{j \in J(y)} w_j(y) \le \min_{i \in I(y)} w_i(y)$, $\max_{j \in J(y)} w_j(y) < p_u$, and $\min_{i \in I(y)} w_i(y) > p_l$.

Based on this information, the set of feasible inventory decisions y for a fixed z, which is denoted as S_z, is defined as follows:

$$S_z = \left\{ y \in [y_l, y_u] \;\middle|\; \begin{array}{l} \max_{j \in J(y)} w_j(y) \le \min_{i \in I(y)} w_i(y) \\ \text{and} \quad \max_{j \in J(y)} w_j(y) < p_u \\ \text{and} \quad \min_{i \in I(y)} w_i(y) > p_l \\ \text{and} \quad \max_{k \in K(y)} (d_k y + e_k) \le z \end{array} \right\}. \tag{54}$$

We introduce the following lemma, which simplifies the bilinear feasibility check on (p, y) for a fixed z to a one-dimensional feasibility check over y.

Lemma 6. *z is feasible for Problem (53) if and only if $S_z \neq \emptyset$. Moreover, if $y \in S_z$ and we set $\tilde{p} = \frac{\max\{p_l, \max_{j \in J(y)} w_j(y)\} + \min\{p_u, \min_{i \in I(y)} w_i(y)\}}{2}$, then $\tilde{p} \in (p_l, p_u)$ and $(\tilde{p}, y, z)$ is a feasible solution to Problem (53).*

Proof. We prove both directions of the equivalence.

($\Rightarrow$) Suppose that z is feasible for Problem (53). Then, there exists (p, y) with $p \in (p_l, p_u)$ and $y \in [y_l, y_u]$ such that all constraints are satisfied:

$$a_i p y + b_i p + d_i y + e_i \leq z, \quad \forall i = 1, \cdots, t. \tag{55}$$

For this fixed y, we rewrite each constraint according to the sign of $a_i y + b_i$:

- If $a_i y + b_i > 0$ (i.e., $i \in I(y)$), then $p \leq \frac{z - d_i y - e_i}{a_i y + b_i} = w_i(y)$.
- If $a_i y + b_i < 0$ (i.e., $i \in J(y)$), then $p \geq \frac{z - d_i y - e_i}{a_i y + b_i} = w_j(y)$.
- If $a_i y + b_i = 0$ (i.e., $i \in K(y)$), then $d_i y + e_i \leq z$.

Given that $p \in (p_l, p_u)$ satisfies all constraints, we must have:

$$\max_{j \in J(y)} w_j(y) \leq p \leq \min_{i \in I(y)} w_i(y), \tag{56}$$

$$\max_{j \in J(y)} w_j(y) < p_u, \tag{57}$$

$$\min_{i \in I(y)} w_i(y) > p_l, \tag{58}$$

$$\max_{k \in K(y)} (d_k y + e_k) \leq z. \tag{59}$$

This outcome implies that $y \in S_z$ by the definition of S_z in (53). Hence, $S_z \neq \emptyset$.

($\Leftarrow$) Suppose that $S_z \neq \emptyset$. Then, there exists $y \in S_z$ such that

$$\max_{j \in J(y)} w_j(y) \leq \min_{i \in I(y)} w_i(y), \tag{60}$$

$$\max_{j \in J(y)} w_j(y) < p_u, \tag{61}$$

$$\min_{i \in I(y)} w_i(y) > p_l, \tag{62}$$

$$\max_{k \in K(y)} (d_k y + e_k) \leq z. \tag{63}$$

Given that $\tilde{p} = \frac{\max\left\{p_l, \max_{j \in J(y)} w_j(y)\right\} + \min\left\{p_u, \min_{i \in I(y)} w_i(y)\right\}}{2}$, we have $\tilde{p} \in (p_l, p_u)$. The following results hold.

- For $i \in I(y)$: $\tilde{p} \leq \min\left\{p_u, \min_{i \in I(y)} w_i(y)\right\} \leq w_i(y)$, which implies that $a_i \tilde{p} y + b_i \tilde{p} + d_i y + e_i \leq z$.
- For $j \in J(y)$: $\tilde{p} \geq \max\left\{p_l, \max_{j \in J(y)} w_j(y)\right\} \geq w_j(y)$, which implies that $a_j \tilde{p} y + b_j \tilde{p} + d_j y + e_j \leq z$.
- For $k \in K(y)$: $d_k y + e_k \leq \max_{k \in K(y)} (d_k y + e_k) \leq z$, which implies that $a_k \tilde{p} y + b_k \tilde{p} + d_k y + e_k = d_k y + e_k \leq z$ (given that $a_k y + b_k = 0$).

Therefore, $(\tilde{p}, y)$ satisfies all constraints in Problem (53). Hence, z is feasible. $\square$

Before applying the complementary approach, we first handle degenerate constraints where $a_i = b_i = 0$. In this case, the constraint $a_i py + b_i p + d_i y + e_i \le z$ simplifies to $d_i y + e_i \le z$. We consider the following three cases.

- If $d_i = 0$, then the constraint becomes $e_i \le z$. Hence, if $e_i > z$, then z is infeasible. If $e_i \le z$, then the constraint is satisfied for any y.
- If $d_i > 0$, then we have $y \le \frac{z-e_i}{d_i}$. This outcome means that $\frac{z-e_i}{d_i}$ is an upper bound on y if z is feasible. Therefore, we update the upper bound $y_u = \min\{y_u, \frac{z-e_i}{d_i}\}$.
- If $d_i < 0$, then we have $y \ge \frac{z-e_i}{d_i}$. This outcome means that $\frac{z-e_i}{d_i}$ is a lower bound on y if z is feasible. Therefore, we update the lower bound $y_l = \max\{y_l, \frac{z-e_i}{d_i}\}$.

After updating y_l and y_u, if $y_l > y_u$, then z is infeasible. Otherwise, we remove these degenerate constraints from further consideration because they have been incorporated into the domain bounds.

We check the nonemptiness of S_z for the remaining constraints by adopting a complementary interval-covering approach that builds the violation set directly. We test whether it exhausts the domain $[y_l, y_u]$. For these constraints, we consider Problem (53) with $a_i \neq 0$ or $b_i \neq 0$. Given that $a_i \le 0$, we have a strict geometric structure on the denominators $a_i y + b_i$. Let $\hat{y}_i = -\frac{b_i}{a_i}$ (assuming $a_i \neq 0$) be the singular points for the ith constraint.

The sign of the denominator $a_i y + b_i = a_i(y - \hat{y}_i)$ depends solely on the position of y relative to $\hat{y}_i$. Given that $a_i \le 0$, we have

$$\begin{aligned} &\text{If } y < \hat{y}_i, \text{ then } a_i(y - \hat{y}_i) > 0, \text{ which implies } i \in I(y). \\ &\text{If } y > \hat{y}_i, \text{ then } a_i(y - \hat{y}_i) < 0, \text{ which implies } i \in J(y). \end{aligned} \tag{64}$$

Consequently, a constraint i acts as a lower bound on price (I) to the left of its singularity and as an upper bound (J) to the right. This configuration precisely defines the intervals where constraint conflicts may occur.

If $a_i = 0$, then the denominator $a_i y + b_i = b_i$ is constant. Additionally, its sign depends solely on b_i. We maintain consistency with the case $a_i \neq 0$ by extending the definition of $\hat{y}_i$ by setting $\hat{y}_i = +\infty$ if $b_i > 0$, and $\hat{y}_i = -\infty$ if $b_i < 0$. With this extended definition, the classification rule in (64) remains valid. When $b_i > 0$ (so $\hat{y}_i = +\infty$), we have $a_i y + b_i = b_i > 0$ for all finite y, which implies that $i \in I(y)$ for all y. When $b_i < 0$ (so $\hat{y}_i = -\infty$), we have $a_i y + b_i = b_i < 0$ for all finite y, which implies that $i \in J(y)$ for all y.

We can recall that

$$S_z = \left\{ y \in [y_l, y_u] \;\middle|\; \begin{array}{l} \max\limits_{j \in J(y)} w_j(y) \leq \min\limits_{i \in I(y)} w_i(y) \\ \text{and} \quad \max\limits_{j \in J(y)} w_j(y) < p_u \\ \text{and} \quad \min\limits_{i \in I(y)} w_i(y) > p_l \\ \text{and} \quad \max\limits_{k \in K(y)} (d_k y + e_k) \leq z \end{array} \right\}.$$

We define the violation set (or bad set) $B_z = [y_l, y_u] \setminus S_z$ and construct B_z by collecting intervals where a part of the feasibility conditions of S_z fails. Let $\mathcal{U}$ be this interval collection after sorting $\hat{y}_i$ in an ascending order.

1. **Pairwise Constraint Conflict (Type A1)**

   For any $y \in S_z$, $\max_{j \in J(y)} w_j(y) \leq \min_{i \in I(y)} w_i(y)$. A conflict between $i \in I(y)$ (upper price bound) and $j \in J(y)$ (lower price bound) arises when $w_j(y) > w_i(y)$, which can only occur for $y \in (\hat{y}_j, \hat{y}_i)$, because $y > \hat{y}_j$ implies that $j \in J(y)$ and $y < \hat{y}_i$ implies that $i \in I(y)$ from (64). For any pair with $\hat{y}_j < \hat{y}_i$,

$$U_{\text{A1}} = \bigcup_{j,i:\hat{y}_j < \hat{y}_i} \left(\{ y \mid w_j(y) > w_i(y) \} \cap (\hat{y}_j, \hat{y}_i) \right). \tag{65}$$

2. **Boundary Violations (Type A2 and A3)**

   For any $y \in S_z$, the price bounds require $\max_{j \in J(y)} w_j(y) < p_u$ and $\min_{i \in I(y)} w_i(y) > p_l$ to ensure $p \in (p_l, p_u)$. Type A2 violates the upper price bound of S_z when a lower bound from $j \in J(y)$ satisfies $w_j(y) \geq p_u$. This outcome can only occur for $y > \hat{y}_j$, i.e., within $(\hat{y}_j, +\infty)$,

$$U_{\text{A2}} = \bigcup_{j} \left(\{ y \mid w_j(y) \geq p_u \} \cap (\hat{y}_j, +\infty) \right). \tag{66}$$

   Type A3 violates the lower price bound of S_z when an upper bound from $i \in I(y)$ satisfies $w_i(y) \leq p_l$. This outcome can only occur for $y < \hat{y}_i$, i.e., within $(-\infty, \hat{y}_i)$,

$$U_{\text{A3}} = \bigcup_{i} \left(\{ y \mid w_i(y) \leq p_l \} \cap (-\infty, \hat{y}_i) \right). \tag{67}$$

3. **Discrete Singular Violation (Type B)**

   For $y \in S_z$, the $K(y)$ condition requires $\max_{k \in K(y)} (d_k y + e_k) \leq z$. At singular points $\hat{y}_k$ (where $a_k y + b_k = 0$), feasibility reduces to $d_k \hat{y}_k + e_k \leq z$. If this scenario fails, the single point is infeasible and is treated as a degenerate interval:

$$U_{\text{B}} = \bigcup_{k:d_k \hat{y}_k + e_k > z} [\hat{y}_k, \hat{y}_k]. \tag{68}$$

The final collection is $\mathcal{U} = U_{\mathrm{A1}} \cup U_{\mathrm{A2}} \cup U_{\mathrm{A3}} \cup U_{\mathrm{B}}$. Checking whether $\mathcal{U}$ covers $[y_l, y_u]$ is equivalent to testing whether S_z is empty. We employ a sweep-line algorithm by de Berg et al. (2008) to determine whether the union of bad intervals $\mathcal{U}$ completely covers the domain $[y_l, y_u]$. The sweep-line algorithm is a classical computational geometry technique that can be used to solve one-dimensional interval coverage problems. The core idea is to traverse the domain from left to right using a virtual "sweep line," which processes events (interval start and end points) in a sorted order. At each event point, we maintain a counter that tracks the number of active intervals that are covering the current position. If at any point the counter becomes zero, we have found a gap in the coverage, thus indicating that the union does not cover the entire domain. This approach efficiently transforms the geometric problem of checking interval union coverage into a one-dimensional counting problem. Given the $N = |\mathcal{U}|$ intervals, the sweep-line algorithm requires $O(N \log N)$ times.

The feasibility check of Problem (53) is summarized in Algorithm 1. Algorithm 1 follows three logical steps. Step 1 incorporates degenerate constraints with $a_i = b_i = 0$ into the bounds of y and immediately declares infeasibility if they conflict. Step 2 enumerates violation intervals of Types A1–A3 and B to build $\mathcal{U}$, thus capturing every way that the defining inequalities of S_z can fail. Step 3 runs the sweep-line coverage check. If a gap remains, the midpoint of the feasible price interval at that y yields a feasible (p, y).

Proposition 9 (Feasibility Check)**.** *Algorithm 1 determines the feasibility of Problem* (53)*. Specifically:*

1. *If the algorithm returns* True*, the constructed set of bad intervals $\mathcal{U}$ does not cover the entire domain $[y_l, y_u]$, and the returned (p, y) satisfies all constraints in Problem* (53)*.*

2. *If the algorithm returns* False*, then no pair (p, y) satisfies the constraints for the given z, i.e., the feasible set $\mathcal{S}_z$ is empty.*

Proof. 1. Suppose that Algorithm 1 returns True with output $(\tilde{p}, \tilde{y})$. By the algorithm's logic, this outcome occurs only if Step 3 finds a point $\tilde{y} \in [y_l, y_u]$ such that $\tilde{y} \notin \mathcal{U}$. It immediately implies that $\mathcal{U}$ does not cover the entire domain $[y_l, y_u]$ because at least one point $\tilde{y}$ is not covered by any bad interval.

We show that $(\tilde{p}, \tilde{y})$ satisfies all constraints in Problem (53). Given that $\tilde{y} \notin \mathcal{U}$, by the definition of the violation set $\mathcal{U}$, the point $\tilde{y}$ is not in any of the bad intervals constructed in Step 2. This outcome implies the following results.

- Type A violations (pairwise and boundary conflicts). $\tilde{y}$ is not in any interval where $w_j(\tilde{y}) > w_i(\tilde{y})$ for conflicting constraints, and is not in intervals where boundary violations occur.

- Type B violations (discrete singular violations). For all singular points $\hat{y}_k$ with $d_k\hat{y}_k + e_k > z$, we have $\tilde{y} \neq \hat{y}_k$, or if $\tilde{y} = \hat{y}_k$ for some k, then $d_k\hat{y}_k + e_k \leq z$.

Given that $\tilde{y} \notin \mathcal{U}$, we have $\tilde{y} \in S_z$ by the complementary definition of $S_z = [y_l, y_u] \setminus \mathcal{U}$ (where $\mathcal{U}$ represents the bad set B_z). By Lemma 6, given that $\tilde{y} \in S_z \neq \emptyset$, there exists a price p such that $(p, \tilde{y})$ satisfies all constraints. The algorithm computes $\tilde{p}$ as the midpoint between the lower and upper price bounds determined by the constraints at $\tilde{y}$:

$$\tilde{p} = \frac{\max\left\{p_l, \max_{j\in J(\tilde{y})} w_j(\tilde{y})\right\} + \min\left\{p_u, \min_{i\in I(\tilde{y})} w_i(\tilde{y})\right\}}{2}. \tag{69}$$

Given that $\tilde{y} \in S_z$, we have

$$\max_{j\in J(\tilde{y})} w_j(\tilde{y}) \leq \min_{i\in I(\tilde{y})} w_i(\tilde{y}), \tag{70}$$

$$\max_{j\in J(\tilde{y})} w_j(\tilde{y}) < p_u, \tag{71}$$

$$\min_{i\in I(\tilde{y})} w_i(\tilde{y}) > p_l, \tag{72}$$

$$\max_{k\in K(\tilde{y})} (d_k\tilde{y} + e_k) \leq z. \tag{73}$$

Therefore, $\tilde{p}$ must satisfy

$$\max\left\{p_l, \max_{j\in J(\tilde{y})} w_j(\tilde{y})\right\} < \tilde{p} < \min\left\{p_u, \min_{i\in I(\tilde{y})} w_i(\tilde{y})\right\}, \tag{74}$$

which ensures that $\tilde{p} \in (p_l, p_u)$. Hence, $(\tilde{p}, \tilde{y})$ satisfies all constraints in Problem (53).

2. Suppose that Algorithm 1 returns False. This outcome occurs in one of two following cases.

   - Case 2a: The algorithm returns False in Step 1, either because $e_i > z$ for some constraint with $a_i = b_i = 0$ and $d_i = 0$, or because $y_l > y_u$ after updating bounds. In the first case, the constraint $d_iy + e_i = e_i > z$ cannot be satisfied for any y, which makes z infeasible. In the second case, the domain becomes empty such that no feasible y exists.

   - Case 2b: The algorithm reaches Step 3, and the sweep-line traversal finds that $\mathcal{U}$ completely covers $[y_l, y_u]$, which implies that no point $\tilde{y} \in [y_l, y_u]$ exists such that $\tilde{y} \notin \mathcal{U}$. Therefore, $S_z = [y_l, y_u] \setminus \mathcal{U} = \emptyset$. By Lemma 6, if $S_z = \emptyset$, then z is infeasible for Problem (53), i.e., no pair (p, y) can satisfy all constraints.

   In both cases, the algorithm correctly determines that z is infeasible. Hence, $\mathcal{S}_z = \emptyset$.

□

Algorithm 1 Feasibility Check via Complementary Interval Covering

Require: Candidate value z, Domain $[y_l, y_u]$, Constraints data $\{a_i, b_i, d_i, e_i\}_{i=1}^{t}$, Price bounds $[p_l, p_u]$.

Ensure: Boolean feasibility, and feasible (p, y) if true.

Step 1: Preprocess Degenerate Constraints
for each constraint i with $a_i = b_i = 0$ **do**
  if $d_i = 0$ and $e_i > z$ **then**
    return False, $\emptyset$
  else if $d_i > 0$ **then**
    $y_u \leftarrow \min\{y_u, \frac{z-e_i}{d_i}\}$
  else if $d_i < 0$ **then**
    $y_l \leftarrow \max\{y_l, \frac{z-e_i}{d_i}\}$
  end if
  Remove constraint i from consideration
end for
if $y_l > y_u$ **then**
  return False, $\emptyset$
end if
Step 2: Construct Violation Set $\mathcal{U} \leftarrow \emptyset$
for each pair j, i with $\hat{y}_j < \hat{y}_i$ **do**
  if $(\hat{y}_j, \hat{y}_i) \cap [y_l, y_u] \neq \emptyset$ **then**
    Solve $w_j(y) > w_i(y)$ on $(\hat{y}_j, \hat{y}_i) \cap [y_l, y_u]$, add intervals to $\mathcal{U}$
  end if
end for
for each constraint j (lower bound provider) **do**
  if $(\hat{y}_j, +\infty) \cap [y_l, y_u] \neq \emptyset$ **then**
    Solve $w_j(y) \geq p_u$ on $(\hat{y}_j, +\infty) \cap [y_l, y_u]$, add intervals to $\mathcal{U}$
  end if
end for
for each constraint i (upper bound provider) **do**
  if $(-\infty, \hat{y}_i) \cap [y_l, y_u] \neq \emptyset$ **then**
    Solve $w_i(y) \leq p_l$ on $(-\infty, \hat{y}_i) \cap [y_l, y_u]$, add intervals to $\mathcal{U}$
  end if
end for
for each $\hat{y}_k$ **do**
  if $\hat{y}_k \in [y_l, y_u]$ and $d_k \hat{y}_k + e_k > z$ **then**
    $\mathcal{U} \leftarrow \mathcal{U} \cup \{[\hat{y}_k, \hat{y}_k]\}$
  end if
end for
Step 3: Sweep-Line Coverage Check
Perform sweep-line traversal over $[y_l, y_u]$
if $\exists \tilde{y} \in [y_l, y_u]$ with $\tilde{y} \notin \mathcal{U}$ **then**
  $\tilde{p} \leftarrow \frac{\max\{p_l, \max_{j \in J(\tilde{y})} w_j(\tilde{y})\} + \min\{p_u, \min_{i \in I(\tilde{y})} w_i(\tilde{y})\}}{2}$
  return True, $(\tilde{p}, \tilde{y})$
else
  return False, $\emptyset$
end if

In the rest of this section, we present the complete algorithm for solving the minmax regret optimization problem $\min_{p\in[\underline{p},\bar{p}],y\geq 0} R(p,y)$. The algorithm consists of three steps.

Step 1: Discrete Price Points Optimization. For each discrete price point $p_i, i = 1,\cdots,\hat{n}$, we solve the optimization problem $\min_{y\geq 0} R(p_i,y)$. Given that $R(p,y)$ is convex in y for any fixed p, we can efficiently use golden search to find the optimal inventory level $y^*(p_i)$ and the corresponding regret value $R^*(p_i)$. Each price point costs $O(nm^2\log(1/\epsilon))$ for precision ϵ. Hence, the total $\hat{n} = O(n)$ points require $O(n^2m^2\log(1/\epsilon))$ computational time and yield $\hat{n}$ candidates $\{(p_i, y^*(p_i)), R^*(p_i)\}$.

Step 2: Continuous Price Intervals Optimization. Proposition 6 implies that for any $i > i^*$, the optimal price over interval $(p_i, p_{i+1}]$ is p_{i+1}, which has already been considered in Step 1. Therefore, we only need to solve the bilinear optimization problem $\min_{p\in(p_i,p_{i+1})}\min_{y\geq 0} R(p,y)$ for intervals with $i \leq i^*$. The complexity analysis for Step 2 consists of three key components.

First, given that $i^* = \underset{i=1,\cdots,\hat{n}-1}{\arg\max}\, \hat{w}(p_{i+1}, \bar{X}_{i,t(p_{i+1})+1}, \bar{X}_i)$ and $i^* \leq \hat{n}-1$, the number of intervals that require continuous optimization is at most $i^* \leq \hat{n}-1 = O(n)$. Within each interval (p_i, p_{i+1}) with $i \leq i^*$, we transform the problem into Problem (52), which is then decomposed into subintervals where constraints become linear. The piecewise linear constraints generate $O(nm)$ breakpoints for y, thereby decomposing the feasible region into $O(n\cdot nm) = O(n^2m)$ subproblems that are characterized by bilinear constraints.

Second, the complexity of solving subproblem arises from performing a binary search on the regret value z and constructing the violation set $\mathcal{U}$ for feasibility verification. For a precision of ϵ, the binary search takes $O(\log(1/\epsilon))$ iterations. The computational effort for each feasibility check is primarily dictated by the construction of the violation set $\mathcal{U}$ using Algorithm 1. We bound the size of $\mathcal{U}$ by analyzing the structure of constraints in Problem (52). The total number of constraints is $O(nm)$, which includes constraints from all four groups in Problem (52). The key observation is that, at most, m constraints can belong to $I(y)$ (i.e., have positive slope $a_iy + b_i > 0$). This outcome follows from analyzing which constraint groups can potentially have positive slopes. Constraints from the first, third, and fourth groups in Problem (52) involve terms that are nonincreasing with respect to price. Consequently, their gradients satisfy $a_iy + b_i \leq 0$ for all $y \geq 0$. Therefore, these constraints can never belong to $I(y)$ and exclusively populate the set $J(y)$. Only constraints from the second constraint group, which are of the form $\hat{w}(p, \bar{X}_{ij}, \bar{X}_i) - w(p, y, \underline{X}_{i+1}) \leq z$ for $j = 1,\cdots,m$, may potentially have positive slopes $a_iy + b_i > 0$ for some values of y. Given that the second constraint group contains exactly m constraints (one for each quantile level $j = 1,\cdots,m$), we have $|I(y)| \leq m = O(m)$ and $|J(y)| \leq O(nm)$. Meanwhile, given that pairwise conflicts occur between constraints in $I(y)$ and $J(y)$, the total number of pairwise inequalities is bounded by $|I(y)|\times|J(y)| \leq O(m)\times O(nm) = O(nm^2)$. The sweep-line algorithm in Algorithm 1 requires $O(N\log N)$ time for N intervals with $N = O(nm^2)$, which implies that the feasibility verification for a single subproblem requires $O(nm^2 \log(nm))$ computational time.

In summary, for each subproblem, we perform $O(\log(1/\epsilon))$ feasibility checks, each of which costs $O(nm^2 \log(nm))$ computational time. Therefore, each subproblem costs $O(nm^2 \log(nm) \log(1/\epsilon))$ computational time. Given that we have, at most, $O(n^2m)$ subproblems, the total computational complexity for Step 2 is:

$$O(n^3m^3 \log(nm) \log(1/\epsilon))$$

yielding at most n candidates $\{(\bar{p}_i, \bar{y}_i), \bar{R}_i\}$.

Step 3: Global Optimal Solution Selection. We compare all candidate solutions from Steps 1 and 2. Then, we select the solution with the minimum regret value. Therefore, Step 3 scans the $O(n)$ candidates and costs $O(n)$ computational time.

Total Complexity. If we combine the complexity analysis of all three steps, the total computational time is

$$\begin{aligned}
T_{\text{total}} &= \underbrace{O(n^2m^2 \log(1/\epsilon))}_{\text{Step 1}} + \underbrace{O(n) \times O(nm) \times O(\log(1/\epsilon)) \times O(nm^2 \log(nm))}_{\text{Step 2}} + \underbrace{O(n)}_{\text{Step 3}} \\
&= O(n^2m^2 \log(1/\epsilon)) + O(n^3m^3 \log(nm) \log(1/\epsilon)) + O(n) \\
&= O(n^3m^3 \log(nm) \log(1/\epsilon)).
\end{aligned}$$

This notation completes the proof of Proposition 7.