Appendix - Delay Information Sharing in Two-Sided On-Demand Platforms

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A.1. Eqs. (4) and (6) under limiting conditions

When $b_I^c = \infty$ and $b_I^p < \infty$:

$$\pi_I(0) = \frac{(1-\rho_I^c)(1-\rho_I^p)}{1-\rho_I^c\rho_I^p - (1-\rho_I^c)(\rho_I^p)^{b_I^p+1}}. \tag{A.1}$$

$$M_I = \pi_I(0)\left(\lambda_I^{c^-}\frac{1}{1-\rho_I^c} + \lambda_I^{c^+}\rho_I^p\frac{1-(\rho_I^p)^{b_I^p}}{1-\rho_I^p}\right) = \pi_I(0)\left(\lambda_I^{p^+}\frac{1-(\rho_I^p)^{b_I^p}}{1-\rho_I^p} + \lambda_I^{p^-}\frac{\rho_I^c}{1-\rho_I^c}\right), \tag{A.2}$$

When $b_I^c < \infty$ and $b_I^p = \infty$:

$$\pi_I(0) = \frac{(1-\rho_I^c)(1-\rho_I^p)}{1-\rho_I^c\rho_I^p - (1-\rho_I^p)(\rho_I^c)^{b_I^c+1}}. \tag{A.3}$$

$$M_I = \pi_I(0)\left(\lambda_I^{c^-}\frac{1-(\rho_I^c)^{b_I^c}}{1-\rho_I^c} + \lambda_I^{c^+}\frac{\rho_I^p}{1-\rho_I^p}\right) = \pi_I(0)\left(\lambda_I^{p^+}\frac{1}{1-\rho_I^p} + \lambda_I^{p^-}\rho_I^c\frac{1-(\rho_I^c)^{b_I^c}}{1-\rho_I^c}\right), \tag{A.4}$$

When $b_I^c = \infty$ and $b_I^p = \infty$:

$$\pi_I(0) = \frac{(1-\rho_I^c)(1-\rho_I^p)}{1-\rho_I^c\rho_I^p}. \tag{A.5}$$

$$M_I = \pi_I(0)\left(\lambda_I^{c^-}\frac{1}{1-\rho_I^c} + \lambda_I^{c^+}\frac{\rho_I^p}{1-\rho_I^p}\right) = \pi_I(0)\left(\lambda_I^{p^+}\frac{1}{1-\rho_I^p} + \lambda_I^{p^-}\frac{\rho_I^c}{1-\rho_I^c}\right), \tag{A.6}$$

A.2. Providers' expected delay expressions and the proof of Lemma 1

A.2.1. Providers' expected delays

When the joining strategy profile $\mathbb{P}_{\mathbb{NB}}$ leads to an unstable system under Regime I, we define $\mathbb{E}_{\mathbb{P}_{\mathbb{NB}}}[W_I^p] = \infty$. Otherwise, $\forall n \geq 0$, $\mathbb{E}_{\mathbb{P}_{\mathbb{NB}}}[W_I^p]$ follows:

$$\mathbb{E}_{\mathbb{P}_{\mathbb{NB}}}[W_{\mathbb{NB}}^p] = \frac{1}{j_{\mathbb{NB}}^c\delta^c\Lambda^c - j_{\mathbb{NB}}^p\delta^p\Lambda^p}, \tag{A.7}$$

$$\mathbb{E}_{\mathbb{P}_{\mathbb{NO}}}[W_{\mathbb{NO}}^p] = \frac{n+1}{j_{\mathbb{NO}}^c\delta^c\Lambda^c}, \tag{A.8}$$

$$\mathbb{E}_{\mathbb{P}_{\mathbb{BB}}}[W_{\mathbb{BB}}^p] = \frac{1}{\Lambda^c - j_{\mathbb{BB}}^p\delta^p\Lambda^p}, \tag{A.9}$$

$$\mathbb{E}_{\mathbb{P}_{\mathbb{BO}}}[W_{\mathbb{BO}}^p] = \frac{n+1}{\Lambda^c}, \tag{A.10}$$

$$\mathbb{E}_{\mathbb{P}_{\mathbb{OO}}}[W_{\mathbb{OO}}^p] = \frac{n+1}{\Lambda^c}. \tag{A.11}$$

A.2.2. Proof of Lemma 1

Proof for Eq. (8). Under Regime $\mathbb{NB}$, $b_{\mathbb{NB}}^c = b_{\mathbb{NB}}^p = \infty$. Therefore, based on Eq. (A.5), we have:

$$\pi_{\mathbb{NB}}(0) = \frac{(1-\rho_{\mathbb{NB}}^c)(1-\rho_{\mathbb{NB}}^p)}{1-\rho_{\mathbb{NB}}^c\rho_{\mathbb{NB}}^p}, \tag{A.12}$$

where based on the definitions for ρ_I^c and ρ_I^p (§4) and the transition rates under Regime $\mathbb{NB}$ (Fig. 1a):

$$\rho_{\mathbb{NB}}^c = \frac{\lambda_{\mathbb{NB}}^{c^-}}{\lambda_{\mathbb{NB}}^{p^-}} = \frac{j_{\mathbb{NB}}^c\delta^c\Lambda^c}{\Lambda^p}, \tag{A.13}$$

$$\rho_{\mathbb{NB}}^p = \frac{\lambda_{\mathbb{NB}}^{p^+}}{\lambda_{\mathbb{NB}}^{c^+}} = \frac{j_{\mathbb{NB}}^p\delta^p\Lambda^p}{j_{\mathbb{NB}}^c\delta^c\Lambda^c}. \tag{A.14}$$

Based on the above expressions and $\lambda_{\mathbb{NB}}^{p^-} = \Lambda^p$ and using Eq. (7), we derive $\mathbb{E}_{\mathbb{P}_{\mathbb{NB}}}[W_{\mathbb{NB}}^p]$ in Eq. (8).

Proof for Eq. (9). Under Regime $\mathbb{NO}$, $b_I^c = \infty$ and $b_I^p < \infty$. Therefore, based on Eq. (A.1):

$$\pi_{\mathbb{NO}}(0) = \frac{(1-\rho_{\mathbb{NO}}^c)(1-\rho_{\mathbb{NO}}^p)}{1-\rho_{\mathbb{NO}}^c\rho_{\mathbb{NO}}^p - (1-\rho_{\mathbb{NO}}^c)(\rho_{\mathbb{NO}}^p)^{b_{\mathbb{NO}}^p+1}}, \tag{A.15}$$

where based on the definitions for ρ_I^c and ρ_I^p (§4) and the transition rates under Regime $\mathbb{NO}$ (Fig. 1b):

$$\rho^c_{\mathbb{NO}} = \frac{\lambda^{c-}_{\mathbb{NO}}}{\lambda^{p-}_{\mathbb{NO}}} = \frac{j^c_{\mathbb{NO}}\delta^c\Lambda^c}{\Lambda^p}, \quad (A.16) \qquad \rho^p_{\mathbb{NO}} = \frac{\lambda^{p+}_{\mathbb{NO}}}{\lambda^{c+}_{\mathbb{NO}}} = \frac{\delta^p\Lambda^p}{j^c_{\mathbb{NO}}\delta^c\Lambda^c}. \quad (A.17)$$

Based on the above expressions and $\lambda^{p-}_{\mathbb{NO}} = \Lambda^p$ and using Eq. (7), we derive $\mathbb{E}_{\mathbb{P}_{\mathbb{NO}}}[W^p_{\mathbb{NO}}]$ in Eq. (9).

Proof for Eqs. (10), (A.7), and (A.9). When customers receive a binary signal under Regimes $\mathbb{BB}$ and $\mathbb{BO}$ indicating a delay upon arrival, their expected delay upon arrival is that of an $M/M/1$ with the arrival rate being the relevant joining rate of customers (i.e., λ_I^{c-}) and the service rate being the relevant joining rate of providers (i.e., λ_I^{p-}). Therefore, the customers' expected delay under these two regimes can be expressed as:

$$\mathbb{E}_{\mathbb{P}_I}[W^c_I] = \frac{1}{\lambda_I^{p-} - \lambda_I^{c-}}, \quad I \in \{\mathbb{BB}, \mathbb{BO}\}. \quad (A.18)$$

Based on the transition rates in Figs. 1c and 1d, setting $\lambda_I^{p-} = \Lambda^p$ and $\lambda_I^{c-} = j^c\delta^c\Lambda^c$, $I \in \{\mathbb{BB}, \mathbb{BO}\}$, in Eq. (10) yields the expected delay expression for Regimes $\mathbb{BB}$ and $\mathbb{BO}$ in Eq. (A.18).

The same reasoning results in the providers' expected delay expression under Regimes $\mathbb{NB}$ and $\mathbb{BB}$, given in Eqs. (A.7) and (A.9), as providers receive binary information under these regimes.

Proof for Eqs. (11), (A.7), (A.10), and (A.11). When customers receive occupancy information under Regime $\mathbb{OO}$ upon arrival at state $n < 0$ (n customers already in the system), they have to wait for the $|n|$ customers ahead of them to be matched first and then wait for their match. The time for each match is exponential with an average equivalent to the average inter-arrival time of providers at states $n < 0$, which under Regime $\mathbb{OO}$ is $1/\Lambda^p$. This results in the expected delay expression given in Eq. (11) under Regime $\mathbb{OO}$.

Applying the same reasoning and using the appropriate customers' average inter-arrival times results in the providers' expected delay expressions in Eqs. (A.8), (A.10), (A.11) under Regimes $\mathbb{NO}$, $\mathbb{BO}$, and $\mathbb{BB}$, respectively, as providers received occupancy information under those regimes.

A.3. Proof of Proposition 1

Table A.1 Equilibrium joining strategy profile $\mathbb{P}^*_{\mathbb{NB}} = (j^{c^*}_{\mathbb{NB}}, j^{p^*}_{\mathbb{NB}})$ under Regime $\mathbb{NB}$

Condition set	$j^{c^*}_{\mathbb{NB}}, j^{p^*}_{\mathbb{NB}}$
$\overline{W}^c \ge \mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^c_{\mathbb{NB}}]$, $\overline{W}^p \ge \mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^p_{\mathbb{NB}}]$	$1, 1$
$\overline{W}^c \ge \mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^c_{\mathbb{NB}}]$, $\overline{W}^p < \mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^p_{\mathbb{NB}}]$	$1, 0$
$\overline{W}^c \ge \mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^c_{\mathbb{NB}}]$, $\mathbb{E}_{(1,0)}[W^p_{\mathbb{NB}}] \le \overline{W}^p < \mathbb{E}_{(1,1)}[W^p_{\mathbb{NB}}]$	$1, \dfrac{\delta^c\Lambda^c\overline{W}^p - 1}{\delta^p\Lambda^p\overline{W}^p}$
$\overline{W}^c < \mathbb{E}_{(1,1)}[W^c_{\mathbb{NB}}]$, $\overline{W}^p \ge \mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^p_{\mathbb{NB}}]$	$\dfrac{\Lambda^p}{2\delta^c\Lambda^c} + \dfrac{\sqrt{\left(\Lambda^p(1-\delta^p)\overline{W}^c - 1\right)^2 + 4\Lambda^p\delta^p(1-\delta^p)\overline{W}^c} - 1}{2\delta^c\Lambda^c\overline{W}^c(1-\delta^p)}, 1$
$\mathbb{E}_{(0,0)}[W^c_{\mathbb{NB}}] \le \overline{W}^c < \mathbb{E}_{(1,0)}[W^c_{\mathbb{NB}}]$, $\overline{W}^p < \mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^p_{\mathbb{NB}}]$	$\dfrac{\Lambda^p\overline{W}^c - 1}{\delta^c\Lambda^c\overline{W}^c}, 0$
$\exists\ j^{c^*}_{\mathbb{NB}} \in (0,1), j^{p^*}_{\mathbb{NB}} \in (0,1)$ that solve $\mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^c_{\mathbb{NB}}] = \overline{W}^c$ and $\mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^p_{\mathbb{NB}}] = \overline{W}^p$	Solution(s) to $\mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^c_{\mathbb{NB}}] = \overline{W}^c$ and $\mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^p_{\mathbb{NB}}] = \overline{W}^p$
Otherwise	$0, 0$

Table A.2 Equilibrium joining strategy profile $\mathbb{P}^*_{\mathbb{NO}} = (j^{c^*}_{\mathbb{NO}}, b^{p^*}_{\mathbb{NO}})$ under Regime $\mathbb{NO}$

Condition set	$j^{c^*}_{\mathbb{NO}}, b^{p^*}_{\mathbb{NO}}$
$\overline{W}^c \ge \mathbb{E}_{\mathbb{P}^*_{\mathbb{NO}}}[W^c_{\mathbb{NO}}]$, $\overline{W}^p < \mathbb{E}_{\mathbb{P}^*_{\mathbb{NO}}}[W^p_{\mathbb{NO}} \mid n=0]$	$1, 0$
$\overline{W}^c \ge \mathbb{E}_{\mathbb{P}^*_{\mathbb{NO}}}[W^c_{\mathbb{NO}}]$, $\overline{W}^p \ge \mathbb{E}_{(j^{c^*}_{\mathbb{NO}},0)}[W^p_{\mathbb{NO}} \mid n=0]$	$1, \lfloor \delta^c\Lambda^c\overline{W}^p \rfloor$
$\mathbb{E}_{(0,0)}[W^c_{\mathbb{NO}}] \le \overline{W}^c < \mathbb{E}_{(1,0)}[W^c_{\mathbb{NO}}]$, $\overline{W}^p < \mathbb{E}_{(j^{c^*}_{\mathbb{NO}},0)}[W^p_{\mathbb{NO}} \mid n=0]$	$\dfrac{\Lambda^p\overline{W}^c - 1}{\delta^c\Lambda^c\overline{W}^c}, 0$
$\exists j^{c^*}_{\mathbb{NO}} \in (0,1)$ that solves $\mathbb{E}_{\mathbb{P}^*_{\mathbb{NO}}}[W^c_{\mathbb{NO}}] = \overline{W}^c$, where $b^{p^*}_{\mathbb{NO}} = \lfloor j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p \rfloor$	Solution(s) to $\mathbb{E}_{\mathbb{P}^*_{\mathbb{NO}}}[W^c_{\mathbb{NO}}] = \overline{W}^c$, $\lfloor j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p \rfloor$
Otherwise	$0, 0$

Table A.3 **Equilibrium joining strategy profile $\mathbb{P}^*_{\mathbb{BB}} = (j^{c^*}_{\mathbb{BB}}, j^{p^*}_{\mathbb{BB}})$ under Regime $\mathbb{BB}$**

Condition set	$\overline{W}^p \geq \mathbb{E}_{(\cdot,1)}[W^p_{\mathbb{BB}}]$	$\overline{W}^p < \mathbb{E}_{(\cdot,0)}[W^p_{\mathbb{BB}}]$	Otherwise
$\overline{W}^c \geq \mathbb{E}_{(1,\cdot)}[W^c_{\mathbb{BB}}]$	$1, 1$	$1, 0$	$1, \dfrac{\overline{W}^p\Lambda^c - 1}{\overline{W}^p\delta^p\Lambda^p}$
$\overline{W}^c < \mathbb{E}_{(0,\cdot)}[W^c_{\mathbb{BB}}]$	$0, 1$	$0, 0$	$0, \dfrac{\overline{W}^p\Lambda^c - 1}{\overline{W}^p\delta^p\Lambda^p}$
Otherwise	$\dfrac{\overline{W}^c\Lambda^p - 1}{\overline{W}^c\delta^c\Lambda^c}, 1$	$\dfrac{\overline{W}^c\Lambda^p - 1}{\overline{W}^c\delta^c\Lambda^c}, 0$	$\dfrac{\overline{W}^c\Lambda^p - 1}{\overline{W}^c\delta^c\Lambda^c}, \dfrac{\overline{W}^p\Lambda^c - 1}{\overline{W}^p\delta^p\Lambda^p}$

Table A.4 **Equilibrium joining strategy profile $\mathbb{P}^*_{\mathbb{BO}} = (j^{c^*}_{\mathbb{BO}}, b^{p^*}_{\mathbb{BO}})$ under Regime $\mathbb{BO}$**

Condition set	$\overline{W}^p \geq \mathbb{E}_{(\cdot,\cdot)}[W^p_{\mathbb{BO}} \mid n=0]$	Otherwise
$\overline{W}^c \geq \mathbb{E}_{(1,\cdot)}[W^c_{\mathbb{BO}}]$	$1, \lfloor \Lambda^c\overline{W}^p \rfloor$	$1, 0$
$\overline{W}^c < \mathbb{E}_{(0,\cdot)}[W^c_{\mathbb{BO}}]$	$0, \lfloor \Lambda^c\overline{W}^p \rfloor$	$0, 0$
Otherwise	$\dfrac{\overline{W}^c\Lambda^p - 1}{\overline{W}^c\delta^c\Lambda^c}, \lfloor \Lambda^c\overline{W}^p \rfloor$	$\dfrac{\overline{W}^c\Lambda^p - 1}{\overline{W}^c\delta^c\Lambda^c}, 0$

Table A.5 **Equilibrium joining strategy profile $\mathbb{P}^*_{\mathbb{OO}} = (b^{c^*}_{\mathbb{OO}}, b^{p^*}_{\mathbb{OO}})$ under Regime $\mathbb{OO}$**

Condition set	$\overline{W}^p \geq \mathbb{E}_{(\cdot,\cdot)}[W^p_{\mathbb{OO}} \mid n=0]$	Otherwise
$\overline{W}^c \geq \mathbb{E}_{(\cdot,\cdot)}[W^c_{\mathbb{OO}} \mid n=0]$	$\lfloor \Lambda^p\overline{W}^c \rfloor, \lfloor \Lambda^c\overline{W}^p \rfloor$	$\lfloor \Lambda^p\overline{W}^c \rfloor, 0$
Otherwise	$0, \lfloor \Lambda^c\overline{W}^p \rfloor$	$0, 0$

A.3.1. Proof for Regime $\mathbb{NB}$

Based on the transition rates under Regime $\mathbb{NB}$ (Fig. 1a) and the fact that the average match rate is equal to expected joining rate of each of the user classes (Eq. (5)), the equilibrium match rate under Regime $\mathbb{NB}$ follows $M^*_{\mathbb{NB}} = j^*_{\mathbb{NB}}\delta_c\Lambda^c$.

Now, we need to characterize $j^*_{\mathbb{NB}}$. Patient customers can employ two joining strategies that result in non-zero match rates in equilibrium: (1) All arriving patient customers join with probability $j^{c^*}_{\mathbb{NB}} = 1$ in equilibrium. This occurs when customers' expected delay $\mathbb{E}[W^c_{\mathbb{NB}}]$ is smaller than their patience level $\overline{W}^c$ even if they all join. (2) All arriving patient customers join with probability $j^{c^*}_{\mathbb{NB}} \in (0,1)$ in equilibrium, where $j^{c^*}_{\mathbb{NB}}$ is chosen such that customers are indifferent between joining and not joining, i.e., $\mathbb{E}[W^c_{\mathbb{NB}}] = \overline{W}^c$.

The above strategies combined with the providers' decisions result in six cases: (i) $j^{c^*}_{\mathbb{NB}} = j^{p^*}_{\mathbb{NB}} = 1$, (ii) $j^{c^*}_{\mathbb{NB}} = 1$ and $j^{p^*}_{\mathbb{NB}} = 0$, (iii) $j^{c^*}_{\mathbb{NB}} = 1$ and $j^{p^*}_{\mathbb{NB}} \in (0,1)$, and (iv) $j^{c^*}_{\mathbb{NB}} \in (0,1)$ and $j^{p^*}_{\mathbb{NB}} = 1$ and $j^{p^*}_{\mathbb{NB}} = 1$, (v) $j^{c^*}_{\mathbb{NB}} \in (0,1)$ and $j^{p^*}_{\mathbb{NB}} = 0$, and (vi) $j^{c^*}_{\mathbb{NB}} \in (0,1)$ and $j^{p^*}_{\mathbb{NB}} \in (0,1)$. We analyze these cases below:

Case (i) ($j^{c^*}_{\mathbb{NB}} = j^{p^*}_{\mathbb{NB}} = 1$). Based on the transitions in the CTMC in Fig. 1a and setting $j^c_{\mathbb{NB}} = j^p_{\mathbb{NB}} = 1$, we need to ensure the system's stability under this case to be able to compute the expected delay expressions. The stability condition, in this case, is given by the first condition in Eq. (A.19).

For patient customers and providers to join with probability one, we need that $\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NB}}]$ and $\overline{W}^p \geq \mathbb{E}[W^p_{\mathbb{NB}}]$. By setting $j^c_{\mathbb{NB}} = j^p_{\mathbb{NB}} = 1$ in Eq. (8), $\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NB}}]$ simplifies to the second condition in Eq. (A.19). Also, by setting $j^c_{\mathbb{NB}} = j^p_{\mathbb{NB}} = 1$ in Eq. (A.8), $\overline{W}^p \geq \mathbb{E}[W^p_{\mathbb{NB}}]$ simplifies to the third condition in Eq. (A.19).

$$\delta^p\Lambda^p < \delta^c\Lambda^c < \Lambda^p \quad \& \quad \overline{W}^c \geq \frac{1}{\Lambda^p - \delta^c\Lambda^c} - \frac{\delta^p}{\delta^c\Lambda^c(1-\delta^p)} \quad \& \quad \overline{W}^p \geq \frac{1}{\delta^c\Lambda^c - \delta^p\Lambda^p}. \tag{A.19}$$

Case (ii) ($j^{c^*}_{\mathbb{NB}} = 1$ and $j^{p^*}_{\mathbb{NB}} = 0$). Based on the transitions in the CTMC in Fig. 1a and setting $j^c_{\mathbb{NB}} = 1$ and $j^p_{\mathbb{NB}} = 0$, we need to ensure the system's stability under this case to be able to compute the expected delay expressions. In this case, the first condition in Eq. (A.20) gives the stability condition.

In this case, all patient customers join, while patient providers receiving the "delay" signal do not join. For this to occur, we need to have $\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NB}}]$ for customers. On the other hand, patient providers need to be so impatient that they do not join even if no other providers join, i.e., $j^p_{\mathbb{NB}} = 0$ leads to an expected delay $\overline{W}^p < \mathbb{E}[W^p_{\mathbb{NB}}]$. By setting $j^c_{\mathbb{NB}} = 1$ and $j^p_{\mathbb{NB}} = 0$ in Eq. (8), $\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NB}}]$

simplifies to the second condition in Eq. (A.20). Then, we obtain the relevant lower bound on providers' expected delay $\mathbb{E}[W^p_{\mathbb{NB}}]$ by setting $j^c_{\mathbb{NB}}=1$ and $j^p_{\mathbb{NB}}=0$ in Eq. (A.7), following which $\overline{W}^p<\mathbb{E}[W^p_{\mathbb{NB}}]$ simplifies to the third condition in Eq. (A.20).

$$\delta^c\Lambda^c<\Lambda^p \quad \& \quad \overline{W}^c\geq\frac{1}{\Lambda^p-\delta^c\Lambda^c} \quad \& \quad \overline{W}^p<\frac{1}{\delta^c\Lambda^c}. \tag{A.20}$$

Case (iii) ($j^{c^*}_{\mathbb{NB}}=1$ and $j^{p^*}_{\mathbb{NB}}\in(0,1)$). In this case, customers join with probability one, while providers mix. Providers will join with probability $j^{p^*}_{\mathbb{NB}}$ in equilibrium such that they are indifferent between joining and not joining. Therefore, we can derive $j^{p^*}_{\mathbb{NB}}$ in this case as follows:

$$\mathbb{E}[W^p_{\mathbb{NB}}]=\overline{W}^p \implies \frac{1}{\Lambda^p-j^{p^*}_{\mathbb{NB}}\delta^c\Lambda^c}=\overline{W}^p(\text{based on Eq. (A.7) and setting } j^c_{\mathbb{NB}}=1) \implies j^{p^*}_{\mathbb{NB}}=\frac{\delta^c\Lambda^c\overline{W}^p-1}{\delta^p\Lambda^p\overline{W}^p}.$$

Now, we derive the conditions for $j^{c^*}_{\mathbb{NB}}=1$ and $j^{p^*}_{\mathbb{NB}}$ to emerge as the equilibrium. As all patient customers join in this case, we need to make sure that the system on their side is stable (the system on the provider's side will be stable as they mix with probability $j^{p^*}_{\mathbb{NB}}$ given in the formula above). The first condition in Eq. (A.21) ensures system stability. For $\mathbb{P}^{c^*}_{\mathbb{NB}}=(1,j^{p^*}_{\mathbb{NB}})$ to be an equilibrium joining strategy profile, customers' patience level must be beyond their expected delay (i.e., $\overline{W}^c\geq\mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^c_{\mathbb{NB}}]$). Setting $j^c_{\mathbb{NB}}=1$ and $j^p_{\mathbb{NB}}=j^{p^*}_{\mathbb{NB}}$ in Eq. (8), the condition $\overline{W}^c\geq\mathbb{E}_{\mathbb{P}^*_{\mathbb{NB}}}[W^c_{\mathbb{NB}}]$ results in the second condition in Eq. (A.21).

Now, we derive sufficient and necessary conditions for the providers to use a mixed joining strategy in equilibrium ($j^{p^*}_{\mathbb{NB}}\in(0,1)$) in response to the customers' joining strategy $j^{c^*}_{\mathbb{NB}}=1$. Providers will join with a non-zero probability (i.e., $j^{p^*}_{\mathbb{NB}}>0$) when their patience level is beyond their expected delay if none of them join (i.e., $\overline{W}^p\geq\mathbb{E}_{(1,0)}[W^p_{\mathbb{NB}}]$). On the other hand, patient providers are not willing to join with probability one in equilibrium (i.e., $j^{p^*}_{\mathbb{NB}}<1$) for either of the following reasons: (1) If their patience level is lower than their expected delay when they all join ($\overline{W}^p<\mathbb{E}_{(1,1)}[W^p_{\mathbb{NB}}]$). (2) If they all join (i.e., $j^p_{\mathbb{NB}}=1$), the system becomes unstable on their side (i.e., $\delta^c\Lambda^c\leq\delta^p\Lambda^p$ based on Fig. 1a and setting $j^c_{\mathbb{NB}}=j^p_{\mathbb{NB}}=1$). The condition for $j^{p^*}_{\mathbb{NB}}>0$ and the conditions for $j^{p^*}_{\mathbb{NB}}<1$, mentioned above, result in the third set of conditions in Eq. (A.21).

$$\begin{cases}\delta^c\Lambda^c<\Lambda^p\\ \overline{W}^c\geq\dfrac{\Lambda^p}{\delta^c\Lambda^c(\Lambda^p-\delta^c\Lambda^c)(1+\overline{W}^p(\Lambda^p-\delta^c\Lambda^c))}\\ \overline{W}^p\geq\dfrac{1}{\delta^c\Lambda^c} \text{ and } \left(\overline{W}^p<\dfrac{1}{\delta^c\Lambda^c-\delta^p\Lambda^p} \text{ or } \delta^c\Lambda^c\leq\delta^p\Lambda^p\right)\end{cases}. \tag{A.21}$$

Case (iv) ($j^{c^*}_{\mathbb{NB}}\in(0,1)$ and $j^{p^*}_{\mathbb{NB}}=1$). In this case, customers mix, while providers join with probability one. Customers will join with probability $j^{c^*}_{\mathbb{NB}}$ in equilibrium such that they are indifferent between joining and not joining. Therefore, we can derive $j^{c^*}_{\mathbb{NB}}$ in this case as follows:

$$\mathbb{E}[W^c_{\mathbb{NB}}]=\overline{W}^c \implies \frac{1}{\Lambda^p-j^c_{\mathbb{NB}}\delta^c\Lambda^c}-\frac{\delta^p}{j^c_{\mathbb{NB}}\delta^c\Lambda^c(1-\delta^p)}=\overline{W}^c(\text{based on Eq. (8) and setting } j^p_{\mathbb{NB}}=1).$$

Solving the above equation for $j^c_{\mathbb{NB}}$ results in two roots:

$$j^{c1}_{\mathbb{NB}}=\frac{\Lambda^p}{2\delta^c\Lambda^c}+\frac{\sqrt{\left(\Lambda^p(1-\delta^p)\overline{W}^c-1\right)^2+4\Lambda^p\delta^p(1-\delta^p)\overline{W}^c}-1}{2\delta^c\Lambda^c\overline{W}^c(1-\delta^p)},$$

$$j^{c2}_{\mathbb{NB}}=\frac{\Lambda^p}{2\delta^c\Lambda^c}-\frac{\sqrt{\left(\Lambda^p(1-\delta^p)\overline{W}^c-1\right)^2+4\Lambda^p\delta^p(1-\delta^p)\overline{W}^c}+1}{2\delta^c\Lambda^c\overline{W}^c(1-\delta^p)},$$

where it can be verified that the first root is positive and the second root is negative. Therefore, in this case, $j^{c^*}_{\mathbb{NB}} = j^{c1}_{\mathbb{NB}}$. As all patient providers join in this case, we need to make sure that the system on their side is stable while customers join with probability $j^{c^*}_{\mathbb{NB}} = j^{c1}_{\mathbb{NB}}$. At the joining probability $j^{c^*}_{\mathbb{NB}} = j^{c1}_{\mathbb{NB}}$, it can be verified that system stability is ensured; i.e., $\delta^p\Lambda^p < j^{c^*}\delta^c\Lambda^c$. Therefore, we do not need to impose any additional stability conditions.

Now, we derive sufficient and necessary conditions for customers to use a mixed joining strategy in equilibrium in response to the providers' joining strategy $j^{p^*}_{\mathbb{NB}} = 1$. Customers may mix due to either of the following reasons: (1) They are relatively impatient such that $\mathbb{E}[W^c_{\mathbb{NB}}] > \overline{W}^c$ when they all join (i.e., $j^c_{\mathbb{NB}} = 1$).[§] (2) If they all join (i.e., $j^c_{\mathbb{NB}} = 1$), the system becomes unstable on their side (i.e., $\delta^c\Lambda^c \geq \Lambda^p$ based on Fig. 1a and setting $j^c_{\mathbb{NB}} = 1$). These result in the second set of conditions in Eq. (A.22). For all patient providers to join in equilibrium (i.e., $j^{p^*}_{\mathbb{NB}} = 1$), they must be patient enough such that if they all join, we have $\mathbb{E}[W^p_{\mathbb{NB}}] \leq \overline{W}^p$, given customers join with probability $j^{c^*}_{\mathbb{NB}} = j^{c1}_{\mathbb{NB}}$ (given above). Setting $j^c_{\mathbb{NB}} = j^{c^*}_{\mathbb{NB}}$ and $j^p_{\mathbb{NB}} = 1$ in Eq. (A.7) results in the third condition in Eq. (A.22).

$$\overline{W}^c < \frac{1}{\Lambda^p - \delta^c\Lambda^c} - \frac{\delta^p}{\delta^c\Lambda^c(1-\delta^p)} \text{ or } \delta^c\Lambda^c \geq \Lambda^p \qquad \& \qquad \overline{W}^p \geq \frac{1}{j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c - \delta^p\Lambda^p}. \tag{A.22}$$

<u>**Case (v) ($j^{c^*}_{\mathbb{NB}} \in (0,1)$ and $j^{p^*}_{\mathbb{NB}} = 0$).**</u> In this case, customers mix, while providers do not join. Customers will join with probability $j^{c^*}_{\mathbb{NB}}$ in equilibrium such that they are indifferent between joining and not joining. Therefore, we can derive $j^{c^*}_{\mathbb{NB}}$ in this case as follows:

$$\mathbb{E}[W^c_{\mathbb{NB}}] = \overline{W}^c \implies \frac{1}{\Lambda^p - j^c_{\mathbb{NB}}\delta^c\Lambda^c} = \overline{W}^c \text{(based on Eq. (8) and setting } j^p_{\mathbb{NB}} = 0) \implies j^{c^*}_{\mathbb{NB}} = \frac{\Lambda^p\overline{W}^c - 1}{\delta^c\Lambda^c\overline{W}^c}.$$

Now, we derive sufficient and necessary conditions for customers to use a mixed joining strategy in equilibrium ($j^{c^*}_{\mathbb{NB}} \in (0,1)$) in response to the providers' joining strategy $j^{p^*}_{\mathbb{NB}} = 0$. Customers will join with a non-zero probability (i.e., $j^{c^*}_{\mathbb{NB}} > 0$) when their patience level is beyond their expected delay if none of them join (i.e., $\overline{W}^c \geq \mathbb{E}_{(0,0)}[W^c_{\mathbb{NB}}]$). On the other hand, patient customers are not willing to join with probability one in equilibrium (i.e., $j^{c^*}_{\mathbb{NB}} < 1$) for either of the following reasons: (1) If their patience level is lower than their expected delay when they all join (i.e., $\overline{W}^c < \mathbb{E}_{(1,0)}[W^c_{\mathbb{NB}}]$). (2) If they all join (i.e., $j^c_{\mathbb{NB}} = 1$), the system becomes unstable on their side (i.e., $\delta^c\Lambda^c \geq \Lambda^p$ based on Fig. 1a and setting $j^c_{\mathbb{NB}} = 1$). The condition for $j^{c^*}_{\mathbb{NB}} > 0$ and the conditions for $j^{c^*}_{\mathbb{NB}} < 1$, mentioned above, result in the first set of conditions in Eq. (A.23). For no patient providers to join in equilibrium (i.e., $j^{p^*}_{\mathbb{NB}} = 0$), their patience level must be lower than their expected delay when none of them join (i.e., $\overline{W}^p < \mathbb{E}_{(j^{c^*}_{\mathbb{NB}},0)}[W^p_{\mathbb{NB}}]$). This results in the second condition in Eq. (A.23).

$$\overline{W}^c \geq \frac{1}{\Lambda^p} \text{ and } \left(\overline{W}^c < \frac{1}{\Lambda^p - \delta^c\Lambda^c} \text{ or } \delta^c\Lambda^c \geq \Lambda^p\right) \qquad \& \qquad \overline{W}^p < \frac{\overline{W}^c}{\Lambda^p\overline{W}^c - 1}. \tag{A.23}$$

<u>**Case (vi) ($j^{c^*}_{\mathbb{NB}} \in (0,1)$ and $j^{p^*}_{\mathbb{NB}} \in (0,1)$).**</u> In this case, customers and providers mix. Customers (resp., providers) will join with probability $j^{c^*}_{\mathbb{NB}}$ (resp. $j^{p^*}_{\mathbb{NB}}$) in equilibrium such that they are indifferent between joining and not joining. Then, we derive $j^{c^*}_{\mathbb{NB}}$ and $j^{p^*}_{\mathbb{NB}}$ in terms of each other in this case by solving the following equations simultaneously:

$$\mathbb{E}[W^c_{\mathbb{NB}}] = \overline{W}^c \implies \frac{1}{\Lambda^p - j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c} - \frac{j^{p^*}_{\mathbb{NB}}\delta^p}{j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c\left(1 - j^{p^*}_{\mathbb{NB}}\delta^p\right)} = \overline{W}^c,$$

[§] This case differs subtly from Case (iii) because the customers' expected delay $\mathbb{E}[W^c_{\mathbb{NB}}]$ is lower bounded by zero (verified by substituting the lower-bound condition $j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c = \delta^p\Lambda^p$ from Eq. (A.22) into Eq. (8)). Consequently, customers can always choose a mixing probability that drives their expected delay arbitrarily close to zero, so no lower bound on $\overline{W}^c$ is required. By contrast, in Case (iii), providers receiving binary information indicating a delay must wait for at least one customer to arrive before service, yielding a strictly positive lower bound on their delay even when no other providers join.

based on Eq. (8) and setting $j^p_{\mathbb{NB}} = j^{p^*}_{\mathbb{NB}}$ and $j^c_{\mathbb{NB}} = j^{c^*}_{\mathbb{NB}}$,

$\mathbb{E}[W^p_{\mathbb{NB}}] = \overline{W}^p \implies \dfrac{1}{j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c - j^{p^*}_{\mathbb{NB}}\delta^p\Lambda^p} = \overline{W}^p$, based on Eq. (A.7) and setting $j^p_{\mathbb{NB}} = j^{p^*}_{\mathbb{NB}}$ and $j^c_{\mathbb{NB}} = j^{c^*}_{\mathbb{NB}}$.

The system will be stable if customers and providers join according to the joining behavior described above. Now, we derive sufficient and necessary conditions for customers to use a mixed joining strategy in equilibrium in response to the providers' joining strategy $j^{p^*}_{\mathbb{NB}}$. Customers may mix due to either of the following reasons: (1) They are relatively impatient such that $\mathbb{E}[W^c_{\mathbb{NB}}] > \overline{W}^c$ when they all join (i.e., $j^c_{\mathbb{NB}} = 1$).¶(2) If they all join (i.e., $j^c_{\mathbb{NB}} = 1$), the system becomes unstable on their side (i.e., $\delta^c\Lambda^c \geq \Lambda^p$ based on Fig. 1a and setting $j^c_{\mathbb{NB}} = 1$). These result in the first set of conditions in Eq. (A.24). Now, we derive sufficient and necessary conditions for the providers to use a mixed joining strategy in response to the customers' joining strategy. Providers will mix if they are willing to join at a probability higher than zero, that is, if their expected delay if no other providers join (i.e., $j^p_{\mathbb{NB}} = 0$) is such that $\mathbb{E}[W^p_{\mathbb{NB}}] \leq \overline{W}^p$. Additionally, mixing requires that providers are not willing to join with probability one, which may occur due to either of the following reasons: (1) Their patience level $\overline{W}^p$ is such that their expected delay $\mathbb{E}[W^p_{\mathbb{NB}}] > \overline{W}^p$ if they all join (i.e., $j^p_{\mathbb{NB}} = 1$). (2) If they all join (i.e., $j^p_{\mathbb{NB}} = 1$), the system becomes unstable on their side (i.e., $j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c \leq \delta^p\Lambda^p$ based on Fig. 1a and setting $j^c_{\mathbb{NB}} = j^p_{\mathbb{NB}} = 1$). These two reasons result in the second condition in Eq. (A.24).

$$\begin{cases} \overline{W}^c < \dfrac{1}{\Lambda^p - \delta^c\Lambda^c} - \dfrac{j^{p^*}_{\mathbb{NB}}\delta^p\Lambda^p}{\delta^c\Lambda^c\Lambda^p - \delta^c\Lambda^c j^{p^*}_{\mathbb{NB}}\delta^p\Lambda^p} \text{ or } \delta^c\Lambda^c \geq \Lambda^p \\ \overline{W}^p \geq \dfrac{1}{j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c}, \overline{W}^p < \dfrac{1}{j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c - \delta^p\Lambda^p} \text{ or } \delta^p\Lambda^p \geq j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c \end{cases}. \tag{A.24}$$

A.3.2. Proof for Regime $\mathbb{NO}$

The two relevant elements of the equilibrium joining strategy profile under Regime $\mathbb{NO}$ are $j^{c^*}_{\mathbb{NO}}$ and $b^{p^*}_{\mathbb{NO}}$. Accordingly, we consider four cases: (i) $j^{c^*}_{\mathbb{NO}} = 1$ and $b^p_{\mathbb{NO}} = 0$, (ii) $j^{c^*}_{\mathbb{NO}} = 1$ and $b^p_{\mathbb{NO}} > 0$, (iii) $j^{c^*}_{\mathbb{NO}} \in (0,1)$ and $b^p_{\mathbb{NO}} = 0$, and (iv) $j^{c^*}_{\mathbb{NO}} \in (0,1)$ and $b^p_{\mathbb{NO}} > 0$.

Case (i) ($j^{c^*}_{\mathbb{NO}} = 1$ and $b^{p^*}_{\mathbb{NO}} = 0$). In this case, Fig. 1b represents the CTMC under Regime $\mathbb{NO}$ after setting $j^c_{\mathbb{NO}} = 1$ and the bounding state $b^p_{\mathbb{NO}} = 0$. As $j^c_{\mathbb{NO}} = 1$, we need to ensure the system's stability under this case to compute the expected delay expressions. The stability condition in this case is given by the first condition in Eq. (A.25). For patient customers to join with probability one (i.e., $j^c_{\mathbb{NO}} = 1$), we need that $\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NO}}]$. By setting $j^c_{\mathbb{NO}} = 1$ and $b^p_{\mathbb{NO}} = 0$ in Eq. (9), $\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NO}}]$ simplifies to the second condition in Eq. (A.25). To ensure $b^p_{\mathbb{NO}} = 0$, based on Eq. (16), we need to have $\lambda^{c+}_{\mathbb{NO}}\overline{W}^p = j^c_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p = \delta^c\Lambda^c\overline{W}^p < 1$ in this case, which results in the third condition in Eq. (A.25).

$$\delta^c\Lambda^c < \Lambda^p \quad \& \quad \overline{W}^c \geq \frac{1}{\Lambda^p - \delta^c\Lambda^c} \quad \& \quad \overline{W}^p < \frac{1}{\delta^c\Lambda^c}. \tag{A.25}$$

Case (ii) ($j^{c^*}_{\mathbb{NO}} = 1$ and $b^{p^*}_{\mathbb{NO}} > 0$). In this case, Fig. 1b represents the CTMC under Regime $\mathbb{NO}$ after setting $j^c_{\mathbb{NO}} = 1$ and the bounding state $b^p_{\mathbb{NO}} = \left\lfloor \delta^c\Lambda^c\overline{W}^p \right\rfloor$ (based on Eq. (16)). As $j^c_{\mathbb{NO}} = 1$, we need to ensure the system's stability under this case on the customers' side (the infinite side of the CTMC) to be able to compute the expected delay expressions. In this case, the first condition in Eq. (A.26) gives the stability condition. For patient customers to join with probability one (i.e., $j^c_{\mathbb{NO}} = 1$), we need that $\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NO}}]$. By setting $j^c_{\mathbb{NO}} = 1$ and $b^p_{\mathbb{NO}} = \left\lfloor \delta^c\Lambda^c\overline{W}^p \right\rfloor$ in Eq. (9),

¶ As with Case (iv), customers' expected delay $\mathbb{E}[W^c_{\mathbb{NB}}]$ is lower bounded by zero (this can be verified by setting $j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c = j^{p^*}_{\mathbb{NB}}\delta^p\Lambda^p$, its lower bound for system stability in Eq. (8)). Therefore, customers can always find a mixing probability that makes their expected wait arbitrarily small, and we do not need to impose a lower bound on $\overline{W}^c$.

$\overline{W}^c \geq \mathbb{E}[W^c_{\mathbb{NO}}]$ results in the second condition in Eq. (A.26). The condition $b^p_{\mathbb{NO}} = \left\lfloor \delta^c \Lambda^c \overline{W}^p \right\rfloor > 0$ (equivalently, $b^p_{\mathbb{NO}} = \left\lfloor \delta^c \Lambda^c \overline{W}^p \right\rfloor \geq 1$) in this case simplifies to the third condition in Eq. (A.26).

$$\begin{cases} \delta^c \Lambda^c < \Lambda^p \\ \overline{W}^c \geq \dfrac{1-\delta^p \Lambda^p/(\delta^c\Lambda^c)}{(\Lambda^p - \delta^c\Lambda^c)\left(1-\delta^p - (1-\delta^c\Lambda^c/\Lambda^p)\left(\delta^p\Lambda^p/(\delta^c\Lambda^c)\right)^{\lfloor \delta^c\Lambda^c\overline{W}^p\rfloor+1}\right)} \\ \overline{W}^p \geq \dfrac{1}{\delta^c\Lambda^c} \end{cases} . \tag{A.26}$$

<u>**Case (iii) ($j^{c^*}_{\mathbb{NO}} \in (0,1)$ and $b^{p^*}_{\mathbb{NO}} = 0$).**</u> In this case, customers mix, while providers do not join. Customers will join with probability $j^{c^*}_{\mathbb{NO}}$ in equilibrium such that they are indifferent between joining and not joining. Therefore, we can derive $j^{c^*}_{\mathbb{NO}}$ in this case as follows:

$$\mathbb{E}[W^c_{\mathbb{NO}}] = \overline{W}^c \implies \frac{1}{\Lambda^p - j^c_{\mathbb{NO}}\delta^c\Lambda^c} = \overline{W}^c \text{(based on Eq. (9) and setting } b^p_{\mathbb{NO}} = 0) \implies j^{c^*}_{\mathbb{NO}} = \frac{\Lambda^p\overline{W}^c - 1}{\delta^c\Lambda^c\overline{W}^c}.$$

Now, we derive sufficient and necessary conditions for customers to use a mixed joining strategy in equilibrium (i.e., $j^{c^*}_{\mathbb{NO}} \in (0,1)$) in response to the providers' bounding state $b^{p^*}_{\mathbb{NO}} = 0$. Customers will join with a probability greater than zero in equilibrium (i.e., $j^{c^*}_{\mathbb{NO}} > 0$) when their patience level is beyond their expected delay if none of them join, given the providers' joining strategy (i.e., $\overline{W}^c \geq \mathbb{E}_{(0,0)}[W^c_{\mathbb{NO}}]$). On the other hand, patient customers are not willing to join with probability one in equilibrium (i.e., $j^{c^*}_{\mathbb{NO}} < 1$) for either of the following reasons: (1) If their patience level is lower than their expected delay if they all join, given the providers' joining strategy (i.e., $\overline{W}^c < \mathbb{E}_{(1,0)}[W^c_{\mathbb{NO}}]$). (2) If they all join (i.e., $j^c_{\mathbb{NO}} = 1$), the system becomes unstable on their side (i.e., $\delta^c\Lambda^c \geq \Lambda^p$ based on Fig. 1b and setting $j^c_{\mathbb{NO}} = 1$). The condition for $j^{c^*}_{\mathbb{NO}} > 0$ and the conditions for $j^{c^*}_{\mathbb{NO}} < 1$, mentioned above, result in the first set of conditions in Eq. (A.27). For providers not being willing to join in equilibrium when they expect a non-zero delay (i.e., $b^{p^*}_{\mathbb{NO}} = 0$), we need to have $\lfloor \lambda^{c+}_{\mathbb{NO}} \overline{W}^p \rfloor = j^c_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p < 1$ (according to Eq. (16) and the transition rates shown in Fig. 1b), which after setting $j^c_{\mathbb{NO}} = j^{c^*}_{\mathbb{NO}}$, as derived above in this case, simplifies to the second condition in Eq. (A.27).

$$\overline{W}^c \geq \frac{1}{\Lambda^p} \text{ and } \left(\overline{W}^c < \frac{1}{\Lambda^p - \delta^c\Lambda^c} \text{ or } \delta^c\Lambda^c \geq \Lambda^p\right) \qquad \& \qquad \overline{W}^p < \frac{\overline{W}^c}{\overline{W}^c\Lambda^p - 1}. \tag{A.27}$$

<u>**Case (iv) ($j^{c^*}_{\mathbb{NO}} \in (0,1)$ and $b^{p^*}_{\mathbb{NO}} > 0$).**</u> In this case, customers mix, while providers join up to a bounding state greater than 0. Customers will join with probability $j^{c^*}_{\mathbb{NO}}$ in equilibrium such that they are indifferent between joining and not joining. Therefore, we can derive $j^{c^*}_{\mathbb{NO}}$ by the following equation, based on Eq. (9) :

$$\mathbb{E}[W^c_{\mathbb{NO}}] = \frac{1-\delta^p\Lambda^p/(j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c)}{(\Lambda^p - j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c)\left(1-\delta^p-(1-j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c/\Lambda^p)\left(\delta^p\Lambda^p/(j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c)\right)^{b^{p^*}_{\mathbb{NO}}+1}\right)} = \overline{W}^c,$$

where providers join up to state $b^{p^*}_{\mathbb{NO}}$, consistent with $j^{c^*}_{\mathbb{NO}}$, according to Eq. (16): $b^{p^*}_{\mathbb{NO}} = \lfloor j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p \rfloor$.

Now, we derive sufficient and necessary conditions for customers to use a mixed joining strategy in equilibrium (i.e., $j^{c^*}_{\mathbb{NO}} \in (0,1)$) in response to the providers' bounding state $b^{p^*}_{\mathbb{NO}} > 0$. It can be verified that customers can obtain an arbitrarily small delay by reducing $j^c_{\mathbb{NO}}$[‖]. Therefore, customers will always join with a probability greater than zero in equilibrium (i.e., $j^{c^*}_{\mathbb{NO}} > 0$). On the other hand,

[‖] Customers' expected delay is decreasing in $b^p_{\mathbb{NO}}$, and the limit of the expected delay at $b^p_{\mathbb{NO}} = 0$ as $j^{c^*}_{\mathbb{NO}} \to 0$ is zero patient customers do not join with probability one in equilibrium (i.e., $j^{c^*}_{\mathbb{NO}} < 1$), given providers' joining strategy, for either of the following reasons: (1) If their patience level is lower than their expected delay when they all join (i.e., $\overline{W}^c < \mathbb{E}_{(1,b^{p^*}_{\mathbb{NB}})}[W^c_{\mathbb{NO}}]$). (2) If they all join (i.e., $j^c_{\mathbb{NO}} = 1$), the system becomes unstable on their side (i.e., $\delta^c\Lambda^c \geq \Lambda^p$ based on Fig. 1b and setting $j^c_{\mathbb{NO}} = 1$). These result in the first set of conditions in Eq. (A.28). For providers, the condition for a positive bounding state is the second condition in Eq. (A.28), obtained by rearranging the terms in $b^{p^*}_{\mathbb{NO}} = \lfloor j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p\rfloor > 0$.

$$\begin{cases} \overline{W}^c < \dfrac{1-\delta^p\Lambda^p/(\delta^c\Lambda^c)}{(\Lambda^p-\delta^c\Lambda^c)\left(1-\delta^p-(1-\delta^c\Lambda^c/\Lambda^p)\left(\delta^p\Lambda^p/(\delta^c\Lambda^c)\right)^{b^{p^*}_{\mathbb{NO}}+1}\right)} \text{ or } \delta^c\Lambda^c \geq \Lambda^p \\ \overline{W}^p > \dfrac{1}{j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c} \end{cases}. \tag{A.28}$$

A.3.3. Proof for Regime $\mathbb{BB}$

To obtain the match rate expression in Eq. (18) in terms of the equilibrium joining strategy profile $(j^{c^*}_{\mathbb{BB}}, j^{p^*}_{\mathbb{BB}})$, we substitute $\lambda^{c^-}_I = j^{c^*}_{\mathbb{BB}}\delta^c\Lambda^c$, $\lambda^{p^+}_I = j^{p^*}_{\mathbb{BB}}\delta^p\Lambda^p$, $\lambda^{c^+}_I = \Lambda^c$, and $\lambda^{p^-}_I = \Lambda^p$ (according to the transition rates shown in Fig. 1c) in Eq. (A.6).

Next, we discuss how to obtain the necessary and sufficient conditions associated with each of the elements $j^{c^*}_{\mathbb{BB}}$ and $j^{p^*}_{\mathbb{BB}}$ of the equilibrium strategy profile. Under Regime $\mathbb{BB}$, the two elements of the strategy profile are independent of each other, as $\mathbb{E}_{\mathbb{P}_{\mathbb{BB}}}[W^u_{\mathbb{BB}}], j \in \{c,p\}$, depends only on $j^{u^*}_{\mathbb{BB}}$ and not on $j^{u'^*}_{\mathbb{BB}}, u' \neq u$ (see Eqs. (10) and (A.9)). Then, for each user class u, we consider three possible equilibrium joining strategies, leading to a total of nine possibilities considering both user classes' joining behaviors. The three cases for each user class are:

Case (i) $j^{u^*}_{\mathbb{BB}} = 1$: For patient users of class u to join at full rate in equilibrium, we need to make sure that this joining behavior does not result in an unstable system on their side. The first condition in Eq. (A.29) ensures system stability. Patient users of class u will join at the full rate if the resulting expected delay is not beyond their patience level, i.e., $\mathbb{E}_{\mathbb{P}_{\mathbb{BB}}}[W^u_{\mathbb{BB}}] \leq \overline{W}^u$ when $\mathbb{P}_{\mathbb{BB}}$ is such that $j^{u^*}_{\mathbb{BB}} = 1$. Substituting the expected delay expression from Eqs. (10) and (A.9), we obtain the second condition in Eq. (A.29).

$$\delta^u\Lambda^u < \Lambda^{u'} \quad \& \quad \overline{W}^u \geq \frac{1}{\Lambda^{u'}-\delta^u\Lambda^u}. \tag{A.29}$$

Case (ii) $j^{u^*}_{\mathbb{BB}} = 0$: This is the equilibrium strategy if the user class's expected wait when they join with the smallest possible joining probability $j^{u^*}_{\mathbb{BB}} = 0$ results in an expected delay $\mathbb{E}_{\mathbb{P}^*{}_{\mathbb{BB}}}[W^u_{\mathbb{BB}}] > \overline{W}^u$. Substituting the expected delay expression from Eqs. (10) and (A.9), we obtain the condition in (A.30).

$$\overline{W}^u < \frac{1}{\Lambda^{u'}}. \tag{A.30}$$

Case (iii) $j^{u^*}_{\mathbb{BB}} \in (0,1)$: This case occurs when neither the Case (i) nor Case (ii) conditions hold. In this case, a class u user will employ a mixed strategy that leads to $\mathbb{E}_{\mathbb{P}^*{}_{\mathbb{BB}}}[W^u_{\mathbb{BB}}] = \overline{W}^u$. Solving this equation, we obtain:

$$j^{u^*}_{\mathbb{BB}} = \frac{\overline{W}^u\Lambda^{u'}-1}{\overline{W}^u\delta^u\Lambda^u}.$$

A.3.4. Proof for Regime $\mathbb{BO}$

To obtain the match rate expression in Eq. (19) in terms of the equilibrium joining strategy profile $(j^{c^*}_{\mathbb{BB}}, b^{p^*}_{\mathbb{BO}})$, we substitute $\lambda^{c^-}_I = j^{c^*}_{\mathbb{BO}}\delta^c\Lambda^c$, $\lambda^{c^+}_I = \Lambda^c$, $\lambda^{p^-}_I = \Lambda^p$, and $\lambda^{p^+}_I = \delta^p\Lambda^p$ (according to the transition rates shown in Fig. 1d) in Eq. (A.2). The two relevant elements of the equilibrium joining strategy profile under Regime $\mathbb{BO}$ are $j^{c^*}_{\mathbb{BO}}$ and $b^{p^*}_{\mathbb{BO}}$. Accordingly, we consider four cases: (i) $j^{c^*}_{\mathbb{BO}} = 1$ and $b^{p}_{\mathbb{BO}} > 0$, (ii) $j^{c^*}_{\mathbb{BO}} = 1$ and $b^{p}_{\mathbb{BO}} = 0$, (iii) $j^{c^*}_{\mathbb{BO}} = 0$ and $b^{p}_{\mathbb{BO}} > 0$, (iv) $j^{c^*}_{\mathbb{BO}} = 0$ and $b^{p}_{\mathbb{BO}} = 0$, (v) $j^{c^*}_{\mathbb{BO}} \in (0,1)$ and $b^{p}_{\mathbb{BO}} = 0$, and (vi) $j^{c^*}_{\mathbb{BO}} \in (0,1)$ and $b^{p}_{\mathbb{BO}} > 0$.

Case (i) ($j^{c^*}_{\mathbb{BO}} = 1$ and $b^{p^*}_{\mathbb{BO}} = 0$). In this case, Fig. 1d represents the CTMC under Regime $\mathbb{BO}$ after setting $j^{c}_{\mathbb{BO}} = 1$ and the bounding state $b^{p}_{\mathbb{BO}} = 0$. As $j^{c}_{\mathbb{BO}} = 1$, we need to ensure the system's stability on the customers' side to compute the expected delay expressions. In this case, the first condition in Eq. (A.31) gives a stability condition. For patient customers to join with probability one (i.e., $j^{c}_{\mathbb{BO}} = 1$), we need that $\overline{W}^c \geq \mathbb{E}_{(1,.)}[W^c_{\mathbb{BO}}]$. By setting $j^{c}_{\mathbb{BO}} = 1$ in Eq. (10), $\overline{W}^c \geq \mathbb{E}_{(1,.)}[W^c_{\mathbb{BO}}]$ simplifies to the second condition in Eq. (A.31). To ensure $b^{p}_{\mathbb{BO}} = 0$, based on Eq. (16), we need to have $\lambda^{c+}_{\mathbb{BO}}\overline{W}^p = \Lambda^c\overline{W}^p < 1$ in this case, which results in the third condition in Eq. (A.31).

$$\delta^c\Lambda^c < \Lambda^p \quad \& \quad \overline{W}^c \geq \frac{1}{\Lambda^p - \delta^c\Lambda^c} = \mathbb{E}_{(1,.)}[W^c_{\mathbb{BO}}] \quad \& \quad \overline{W}^p < \frac{1}{\Lambda^c} = \mathbb{E}_{(.,.)}[W^p_{\mathbb{BO}}|n=0]. \tag{A.31}$$

Case (ii) ($j^{c^*}_{\mathbb{BO}} = 1$ and $b^{p^*}_{\mathbb{BO}} > 0$). The first two conditions in this case (given in Eq. (A.32)) will be identical to the first two conditions of Case (i) as $j^{c^*}_{\mathbb{BO}} = 1$ (like Case (i)) and the joining decisions of customers and providers are independent of each other under Regime $\mathbb{BO}$. To ensure $b^{p^*}_{\mathbb{BO}} > 0$, based on Eq. (16), we need to have $\lambda^{c+}_{\mathbb{BO}}\overline{W}^p = \Lambda^c\overline{W}^p \geq 1$ in this case, which results in the third condition in Eq. (A.32).

$$\delta^c\Lambda^c < \Lambda^p \quad \& \quad \overline{W}^c \geq \frac{1}{\Lambda^p - \delta^c\Lambda^c} = \mathbb{E}_{(1,.)}[W^c_{\mathbb{BO}}] \quad \& \quad \overline{W}^p \geq \frac{1}{\Lambda^c} = \mathbb{E}_{(.,.)}[W^p_{\mathbb{BO}}|n=0]. \tag{A.32}$$

Case (iii) ($j^{c^*}_{\mathbb{BO}} = 0$ and $b^{p^*}_{\mathbb{BO}} = 0$). For $j^{c^*}_{\mathbb{BO}} = 0$, patient customers need to be so impatient that they do not join even if no other customers join, i.e., $j^{c}_{\mathbb{BO}} = 0$ leads to an expected delay $\mathbb{E}_{(0,.)}[W^c_{\mathbb{BO}}] > \overline{W}^c$. By setting $j^{c}_{\mathbb{BO}} = 0$ in Eq. (10), this condition simplifies to the first condition in Eq. (A.33). The condition for $b^{p^*}_{\mathbb{BO}} = 0$ in this case (as given in the second condition in Eq. (A.33)) is identical to the corresponding condition in Case (i).

$$\overline{W}^c < \frac{1}{\Lambda^p} = \mathbb{E}_{(0,.)}[W^c_{\mathbb{BO}}] \qquad \& \qquad \overline{W}^p < \frac{1}{\Lambda^c} = \mathbb{E}_{(.,.)}[W^p_{\mathbb{BO}}|n=0]. \tag{A.33}$$

Case (iv) ($j^{c^*}_{\mathbb{BO}} = 0$ and $b^{p^*}_{\mathbb{BO}} > 0$). The condition for $j^{c^*}_{\mathbb{BO}} = 0$ in this case (as given in the first condition in Eq. (A.34)) is identical to the corresponding condition in Case (iii). The condition for $b^{p^*}_{\mathbb{BO}} > 0$ in this case (as given in the second condition in Eq. (A.34)) is identical to the corresponding condition in Case (ii).

$$\overline{W}^c < \frac{1}{\Lambda^p} = \mathbb{E}_{(0,.)}[W^c_{\mathbb{BO}}] \quad \& \quad \overline{W}^p \geq \frac{1}{\Lambda^c} = \mathbb{E}_{(.,.)}[W^p_{\mathbb{BO}}|n=0]. \tag{A.34}$$

Case (v) ($j^{c^*}_{\mathbb{BO}} \in (0,1)$ and $b^{p^*}_{\mathbb{BO}} = 0$). In this case, customers mix, while providers do not join when they expect a delay. Customers will join with probability $j^{c^*}_{\mathbb{BO}}$ in equilibrium such that they are indifferent between joining and not joining. Therefore, we can derive $j^{c^*}_{\mathbb{BO}}$ in this case as follows:

$$\mathbb{E}[W^c_{\mathbb{BO}}] = \overline{W}^c \implies \frac{1}{\Lambda^p - j^c_{\mathbb{BO}}\delta^c\Lambda^c} = \overline{W}^c (\text{based on Eq. (10)}) \implies j^{c^*}_{\mathbb{BO}} = \frac{\Lambda^p\overline{W}^c - 1}{\delta^c\Lambda^c\overline{W}^c}.$$

Now, we derive sufficient and necessary conditions for customers to use a mixed joining strategy in equilibrium (i.e., $j^{c^*}_{\mathbb{BO}} \in (0,1)$). Customers will join with a probability greater than zero in equilibrium (i.e., $j^{c^*}_{\mathbb{BO}} > 0$) when their patience level is beyond their expected delay if none of them join (i.e., $\overline{W}^c \geq \mathbb{E}_{(0,.)}[W^c_{\mathbb{BO}}]$). On the other hand, patient customers are not willing to join with probability one in equilibrium (i.e., $j^{c^*}_{\mathbb{BO}} < 1$) for either of the following reasons: (1) If their patience level is lower than their expected delay when they all join (i.e., $\overline{W}^c < \mathbb{E}_{(1,.)}[W^c_{\mathbb{BO}}]$). (2) If they all join (i.e., $j^c_{\mathbb{BO}} = 1$), the system becomes unstable on their side (i.e., $\delta^c\Lambda^c \geq \Lambda^p$ based on Fig. 1d and setting $j^c_{\mathbb{BO}} = 1$). The condition for $j^{c^*}_{\mathbb{BO}} > 0$ and the conditions for $j^{c^*}_{\mathbb{BO}} < 1$, mentioned above, result in the first set of conditions in Eq. (A.35). The condition for $b^{p^*}_{\mathbb{BO}} = 0$ in this case (as given in the second condition in Eq. (A.35)) is identical to the corresponding condition in Case (i).

$$\begin{cases} \overline{W}^c \geq \dfrac{1}{\Lambda^p} = \mathbb{E}_{(0,.)}[W^c_{\mathbb{BO}}] \text{ and } \left(\overline{W}^c < \dfrac{1}{\Lambda^p - \delta^c\Lambda^c} = \mathbb{E}_{(1,.)}[W^c_{\mathbb{BO}}] \text{ or } \delta^c\Lambda^c \geq \Lambda^p\right) \\ \overline{W}^p < \dfrac{1}{\Lambda^c} = \mathbb{E}_{(.,.)}[W^p_{\mathbb{BO}}|n=0] \end{cases}. \tag{A.35}$$

<u>**Case (vi)** ($j^{c^*}_{\mathbb{BO}} \in (0,1)$ **and** $b^{p^*}_{\mathbb{BO}} > 0$)**.**</u> The condition for $j^{c^*}_{\mathbb{BO}} \in (0,1)$ (as given in the first condition in Eq. (A.36)) is identical to the corresponding condition in Case (v). The condition for $b^{p^*}_{\mathbb{BO}} > 0$ (as given in the second condition in Eq. (A.36)) is identical to the corresponding condition in Case (ii).

$$\begin{cases} \overline{W}^c \geq \dfrac{1}{\Lambda^p} = \mathbb{E}_{(0,.)}[W^c_{\mathbb{BO}}] \text{ and } \left(\overline{W}^c < \dfrac{1}{\Lambda^p - \delta^c\Lambda^c} = \mathbb{E}_{(1,.)}[W^c_{\mathbb{BO}}] \text{ or } \delta^c\Lambda^c \geq \Lambda^p\right) \\ \overline{W}^p \geq \dfrac{1}{\Lambda^c} = \mathbb{E}_{(.,.)}[W^p_{\mathbb{BO}}|n=0] \end{cases}. \tag{A.36}$$

A.3.5. Proof for Regime $\mathbb{OO}$

The cases follow from a straightforward application of the bounding state specifications in (16) and the observation that customers (resp., providers) join at rate Λ^c (resp., Λ^p) when $n > 0$ (resp., $n < 0$) under Regime $\mathbb{OO}$.

A.4. Proof of Proposition 2

The conditions corresponding to Regime $\mathbb{NB}$ follow from substituting the expressions for $\mathbb{E}[W^c_{\mathbb{NB}}]$ and $\mathbb{E}[W^p_{\mathbb{NB}}]$ from Eqs. (8)and (A.7) and ensuring that the resulting joining probabilities for both customers and providers are between 0 and 1. Similarly, the conditions corresponding to Regime $\mathbb{NO}$ of the proposition follow from substituting the expression for $\mathbb{E}[W^c_{\mathbb{NO}}]$ from Eqs. (9).

A.5. Proof of Proposition 3

The conditions corresponding to Regime $\mathbb{BB}$ follows from Table A.3: the match rate is zero if either $\overline{W}^c < \mathbb{E}_{(0,\cdot)}[W^c_{\mathbb{BB}}]$ and $\overline{W}^p < \mathbb{E}_{(\cdot,0)}[W^p_{\mathbb{BB}}]$, leading to an equilibrium joining strategy of $(0,0)$, or if the mixed strategy employed by both user classes leads to a joining rate of zero (which occurs if and only if $\overline{W}^c\Lambda^p - 1 = 0$ and $\overline{W}^p\Lambda^c - 1 = 0$. Similarly, the conditions corresponding to Regime $\mathbb{BO}$ follow from Table A.4, where the match rate is zero if $\overline{W}^p < \mathbb{E}_{(\cdot,\cdot)}[W^p_{\mathbb{BO}}|n=0]$) and $\overline{W}^c < \mathbb{E}_{(0,\cdot)}[W^c_{\mathbb{BO}}]$ or $\overline{W}^c\Lambda^p - 1 = 0$. The conditions corresponding to Regime $\mathbb{NO}$ follow from the condition associated with the bottom right cell of Table A.5. Finally, to compare the conditions governing the existence of a non-trivial equilibrium in regimes with no information (Regimes $\mathbb{NB}$ and $\mathbb{NO}$), and those without no information (Regimes $\mathbb{BB}$, $\mathbb{BO}$ and $\mathbb{OO}$), observe that $M^*_{\mathbb{BB}} = 0 \vee M^*_{\mathbb{BO}} = 0 \vee M^*_{\mathbb{OO}} = 0 \implies \overline{W}^c \leq 1/\Lambda^c \wedge \overline{W}^p \leq 1/\Lambda^p$.

1. $M^*_{\mathbb{NB}}$: Observe from (A.7) that providers, receiving binary information, face an expected delay of $\frac{1}{j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c - j^{p^*}_{\mathbb{NB}}\delta^p\Lambda^p} \geq \frac{1}{\delta^c\Lambda^c - j^{p^*}_{\mathbb{NB}}\delta^p\Lambda^p} \geq \frac{1}{\delta^c\Lambda^c} > \frac{1}{\Lambda^c}$. Given that $\overline{W}^p \leq \dfrac{1}{\Lambda^c}$, providers find it optimal to join with probability $j^{p^*}_{\mathbb{NB}} = 0$. Accordingly, customers who receive no information face an expected delay of $\frac{1}{\Lambda^p - j^{c^*}_{\mathbb{NB}}\delta^c\Lambda^c} \geq \frac{1}{\Lambda^p - \delta^c\Lambda^c} > \frac{1}{\Lambda^p}$. Given that $\overline{W}^c \leq \frac{1}{\Lambda^p}$, customers find it optimal to join at rate $j^{c^*}_{\mathbb{NB}} = 0$. Accordingly, $M^*_{\mathbb{NB}} = 0$.
2. $M^*_{\mathbb{NO}}$: Observe from (16) that $b^{p^*}_{\mathbb{NO}} = \lfloor j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p\rfloor$. But, we have that $\lfloor j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c\overline{W}^p\rfloor < \lfloor\Lambda^c\overline{W}^p\rfloor \leq 1$, leading to $b^{p^*}_{\mathbb{NO}} = 0$. Then, as argued above for the case of $M^*_{\mathbb{NB}}$, customers who receive no information face an expected delay of $\frac{1}{\Lambda^p - j^{c^*}_{\mathbb{NO}}\delta^c\Lambda^c} \geq \frac{1}{\Lambda^p - \delta^c\Lambda^c} > \frac{1}{\Lambda^p}$. Given that $\overline{W}^c \leq \frac{1}{\Lambda^p}$, customers find it optimal to join at rate $j^{c^*}_{\mathbb{NO}} = 0$. Accordingly, $M^*_{\mathbb{NB}} = 0$. This completes the proof.

A.6. Proof of Proposition 4

Under Regimes $\mathbb{NB}$ and $\mathbb{NO}$, the equilibrium match rates follows $M_I^* = j_I^{c^*}\delta^c\Lambda^c$ (according to Proposition 1). Therefore to compare $M^*_{\mathbb{NB}}$ and $M^*_{\mathbb{NO}}$, it suffices to compare the equilibrium joining probabilities $j^{c^*}_{\mathbb{NB}}$ and $j^{c^*}_{\mathbb{NO}}$. Under Cases (i)-(iii) of Regime $\mathbb{NB}$ (conditions laid out in Table A.1 and expanded in Eqs. (A.19)–(A.21)), customers join at full rate (i.e., $j^{c^*}_{\mathbb{NB}} = 1$), and therefore, the equilibrium match rates will be weakly higher under Regime $\mathbb{NB}$ compared to Regime $\mathbb{NO}$ ($M^*_{\mathbb{NB}} \geq M^*_{\mathbb{NO}}$).

Now, we compare $M^*_{\mathbb{NB}}$ and $M^*_{\mathbb{NO}}$ when the conditions in Case (iv) of Regime $\mathbb{NB}$ holds. In this case, customers mix in equilibrium (at the rate $j^{c^*}_{\mathbb{NB}}$ provided in Table A.1), while providers join at full rate (i.e., $0 < j^{c^*}_{\mathbb{NB}} < 1$ and $j^{p^*}_{\mathbb{NB}} = 1$). Comparing the BD processes for Regimes $\mathbb{NB}$ and $\mathbb{NO}$ shown respectively in Figs. 1a and 1b when $j^P_{\mathbb{NB}} = 1$, we observe that the providers' joining rates are identical under the two regimes, except that they join up to the bounding state $b^p_{\mathbb{NO}}$ under Regime $\mathbb{NO}$. To prove $M^*_{\mathbb{NB}} \geq M^*_{\mathbb{NO}}$ under Case (iv), we show the following:

1. Customers' expected delay $\mathbb{E}_{\mathbb{P}_{\mathbb{NO}}}[W^c_{\mathbb{NO}}]$ under Regime $\mathbb{NO}$ is decreasing in the providers' bounding state $b^p_{\mathbb{NO}}$.
2. Customers' expected delay $\mathbb{E}_{\mathbb{P}_{\mathbb{NO}}}[W^c_{\mathbb{NO}}]$ under Regime $\mathbb{NO}$ is increasing in their joining rate $j^c_{\mathbb{NO}}$.
3. Regime $\mathbb{NB}$ is equivalent to Regime $\mathbb{NO}$ with $b^p_{\mathbb{NO}} = \infty$.

We can show the first point, by writing $\mathbb{E}_{\mathbb{P}_{\mathbb{NO}}}[W^c_{\mathbb{NO}}]$ as follows:

$$\mathbb{E}[W^c_{\mathbb{NO}}] = \mathbb{E}[W^c_{\mathbb{NO}}|n \leq 0]\Pr(n \leq 0) + \mathbb{E}[W^c_{\mathbb{NO}}|n > 0]\Pr(n > 0) = \frac{1}{\Lambda^p - j^c_{\mathbb{NO}}\delta^c\Lambda^c}\Pr(n \leq 0) + 0 \times \Pr(n > 0),$$

where the first term is increasing in $j^c_{\mathbb{NO}}$. Therefore, to establish the first point, we now need to show that $\Pr(n \leq 0)$ under Regime $\mathbb{NO}$ is increasing in $j^c_{\mathbb{NO}}$. This follows directly from Theorem 5 in Smith and Whitt (1981), whose implication is that the state of a CTMC with higher $j^c_{\mathbb{NO}}$ is stochastically smaller (see Fig. 1b).

To show the second point, we use the $\mathbb{E}[W^c_{\mathbb{NO}}]$ expression given in Eq. (9) and take its partial derivative with respect to $b^p_{\mathbb{NO}}$ as follows:

$$\frac{\partial\,\mathbb{E}[W^c_{\mathbb{NO}}]}{\partial b^p_{\mathbb{NO}}} = -\frac{\delta^p(\delta^p\Lambda^p - j^c_{\mathbb{NO}}\delta^c\Lambda^c)\left(\dfrac{\delta^p\Lambda^p}{j^c_{\mathbb{NO}}\delta^c\Lambda^c}\right)^{b^p_{\mathbb{NO}}}\log\left(\dfrac{\delta^p\Lambda^p}{j^c_{\mathbb{NO}}\delta^c\Lambda^c}\right)}{\left(\delta^p(j^c_{\mathbb{NO}}\delta^c\Lambda^c - \Lambda^p)\left(\dfrac{\delta^p\Lambda^p}{j^c_{\mathbb{NO}}\delta^c\Lambda^c}\right)^{b^p_{\mathbb{NO}}} + j^c_{\mathbb{NO}}\delta^c\Lambda^c - j^c_{\mathbb{NO}}\delta^c\Lambda^c\delta^p\right)^2},$$

which is negative both when $\delta^p\Lambda^p < j^c_{\mathbb{NO}}\delta^c\Lambda^c$ and $\delta^p\Lambda^p > j^c_{\mathbb{NO}}\delta^c\Lambda^c$ as both cases results in $(\delta^p\Lambda^p - j^c_{\mathbb{NO}}\delta^c\Lambda^c) \times \log\left(\dfrac{\delta^p\Lambda^p}{j^c_{\mathbb{NO}}\delta^c\Lambda^c}\right)$ being positive. This establishes the second fact. The third fact follows from a visual inspection of the CTMCs. This completes the proof.

A.7. Proof of Proposition 5

Proof for $M^*_{\mathbb{BN}} \geq M^*_{\mathbb{BB}}$: When providers are sufficiently patient under Regime $\mathbb{BB}$ such that $\overline{W}^p \geq \mathbb{E}_{(.,1)}[W^p_{\mathbb{BB}}]$, they join with full probability in equilibrium (i.e., $j^{p^*}_{\mathbb{BB}} = 1$, according to Table A.3). As the match rate is equivalent to the average joining rate of either user class (Eq. (5)), setting $j^{p^*}_{\mathbb{BB}} = 1$ in the CTMC of Regime $\mathbb{BB}$ (as shown in Fig. 1c) implies that the average joining rate of providers will be between $\delta^p\Lambda^p$ and Λ^p. Therefore, we will have $\delta^p\Lambda^p \leq M^*_{\mathbb{BB}} \leq \Lambda^p$ when $\overline{W}^p \geq \mathbb{E}_{(.,1)}[W^p_{\mathbb{BB}}]$. On the other hand, under Regime $\mathbb{BN}$, providers receive no information, and therefore, impatient providers do not join. As a result, we will have $M^*_{\mathbb{BN}} \leq \delta^p\Lambda^p$. Therefore, the match rate under Regime $\mathbb{NB}$ will be weakly less than under Regime $\mathbb{BB}$.

Proof for $M^*_{\mathbb{ON}} \geq M^*_{\mathbb{OB}}$: When providers are sufficiently patient such that $\overline{W}^p \geq \mathbb{E}_{(\cdot,1)}[W^p_{\mathbb{OB}}]$ (the providers' analog to the first row of Table A.4), they join at rate $\delta^p\Lambda^p$ in all states $n \geq 0$ and at rate Λ^p in all states $n < 0$ under Regime $\mathbb{OB}$. Therefore, the resulting match rate is in $[\delta^p\Lambda^p, \Lambda^p)$. On the other hand, the largest possible match rate under Regime $\mathbb{ON}$ is $\delta^p\Lambda^p$, which is weakly smaller than the match rate under Regime $\mathbb{BN}$.

A.8. Proof of Lemma 2

We need to prove that $M^*_{\mathbb{BO}} \geq M^*_{\mathbb{BB}} \Leftrightarrow M^*_{\mathbb{OO}} \geq M^*_{\mathbb{OB}}$. Clearly, we can see from Tables A.3–A.4 that $j^{c^*}_{\mathbb{BO}} = j^{c^*}_{\mathbb{BB}}$, leading to identical customers' joining rates $\lambda^c = j^c_{\mathbb{BO}}\delta^c\Lambda^c = j^c_{\mathbb{BB}}\delta^c\Lambda^c$ in states $n \leq 0$ under Regimes $\mathbb{BB}$ and $\mathbb{BO}$. Similarly, $b^c_{\mathbb{OO}} = b^c_{\mathbb{OB}}$ (Tables A.5 and A.4). For identical reasons, the provider bounding state b^p_I when $I \in \{\mathbb{BO}, \mathbb{OO}\}$ and the providers' joining rate λ^p when facing the "delay" signal is identical under $I \in \{\mathbb{BO}, \mathbb{BB}\}$.

For ease of exposition, we define the following notation:

$$p_{\mathbb{O}} = \sum_{k=1}^{b^p} \left(\frac{\delta^p\Lambda^p}{\Lambda^c}\right)^k, \quad p_{\mathbb{B}} = \sum_{k=1}^{\infty} \left(\frac{\lambda^p}{\Lambda^c}\right)^k, \quad c_{\mathbb{O}} = \sum_{k=1}^{b^c} \left(\frac{\delta^c\Lambda^c}{\Lambda^p}\right)^k, \quad c_{\mathbb{B}} = \sum_{k=1}^{\infty} \left(\frac{\lambda^c}{\Lambda^p}\right)^k.$$

Then, we can write:

$$M^*_{\mathbb{BO}} = \lambda^c\left(1 - \pi_{\mathbb{BO}}(0)p_{\mathbb{O}}\right) + \Lambda^c\pi_{\mathbb{BO}}(0)p_{\mathbb{O}}, \quad M^*_{\mathbb{BB}} = \lambda^c\left(1 - \pi_{\mathbb{BB}}(0)p_{\mathbb{B}}\right) + \Lambda^c\pi_{\mathbb{BB}}(0)p_{\mathbb{B}},$$
$$M^*_{\mathbb{OO}} = \delta^c\Lambda^c\left(1 - \pi_{\mathbb{OO}}(0)p_{\mathbb{O}}\right) + \Lambda^c\pi_{\mathbb{OO}}(0)p_{\mathbb{O}}, \quad M^*_{\mathbb{OB}} = \delta^c\Lambda^c\left(1 - \pi_{\mathbb{OB}}(0)p_{\mathbb{B}}\right) + \Lambda^c\pi_{\mathbb{OB}}(0)p_{\mathbb{B}}.$$

Following some basic algebra and noting that $\lambda^c \leq \delta^c\Lambda^c < \Lambda^c$, it then emerges that

$$M^*_{\mathbb{BO}} \geq M^*_{\mathbb{BB}} \iff \frac{p_{\mathbb{O}}}{p_{\mathbb{B}}} \geq \frac{\pi_{\mathbb{BB}}(0)}{\pi_{\mathbb{BO}}(0)} \qquad \text{(A.37)} \qquad M^*_{\mathbb{OO}} \geq M^*_{\mathbb{OB}} \iff \frac{p_{\mathbb{O}}}{p_{\mathbb{B}}} \geq \frac{\pi_{\mathbb{OB}}(0)}{\pi_{\mathbb{OO}}(0)} \qquad \text{(A.38)}$$

From the balance equation governing each of the regimes, we note that:

$$\pi_{\mathbb{BB}}(0) = \frac{1}{1 + c_{\mathbb{B}} + p_{\mathbb{B}}}, \quad \pi_{\mathbb{BO}}(0) = \frac{1}{1 + c_{\mathbb{B}} + p_{\mathbb{O}}}, \quad \pi_{\mathbb{OB}}(0) = \frac{1}{1 + c_{\mathbb{O}} + p_{\mathbb{B}}}, \quad \pi_{\mathbb{OO}}(0) = \frac{1}{1 + c_{\mathbb{O}} + p_{\mathbb{O}}}.$$

Now, observe from Eqs. (A.37) and (A.38) the following, which show these two equations are equivalent statements:

$$\frac{p_{\mathbb{O}}}{p_{\mathbb{B}}} \geq \frac{\pi_{\mathbb{BB}}(0)}{\pi_{\mathbb{BO}}(0)} \iff \frac{p_{\mathbb{O}}}{p_{\mathbb{B}}} \geq \frac{1 + c_{\mathbb{B}} + p_{\mathbb{O}}}{1 + c_{\mathbb{B}} + p_{\mathbb{B}}} \iff p_{\mathbb{O}}(1 + c_{\mathbb{B}}) \geq p_{\mathbb{B}}(1 + c_{\mathbb{B}}) \iff p_{\mathbb{O}} \geq p_{\mathbb{B}},$$
$$\frac{p_{\mathbb{O}}}{p_{\mathbb{B}}} \geq \frac{\pi_{\mathbb{OB}}(0)}{\pi_{\mathbb{OO}}(0)} \iff \frac{p_{\mathbb{O}}}{p_{\mathbb{B}}} \geq \frac{1 + c_{\mathbb{O}} + p_{\mathbb{O}}}{1 + c_{\mathbb{O}} + p_{\mathbb{B}}} \iff p_{\mathbb{O}}(1 + c_{\mathbb{O}}) \geq p_{\mathbb{B}}(1 + c_{\mathbb{O}}) \iff p_{\mathbb{O}} \geq p_{\mathbb{B}}.$$

To show that this is identical to the information disclosure decision in a one-sided setting, consider a one-sided setting where customers' service rate is Λ^c, and we need to determine whether to disclose binary or occupancy information to providers. Then, disclosing binary information to providers outperforms disclosing occupancy information to them if and only if:

$$\Lambda^c(1 - \pi_{\mathbb{B}}(0)) \geq \Lambda^c(1 - \pi_{\mathbb{O}}(0)) \iff \pi_{\mathbb{O}}(0) \geq \pi_{\mathbb{B}}(0) \iff \frac{1}{1 + p_{\mathbb{O}}} \geq \frac{1}{1 + p_{\mathbb{B}}} \iff p_{\mathbb{O}} \geq p_{\mathbb{B}},$$

which is identical to the conditions in Eqs. (A.37) and (A.38). This completes the proof.

A.9. Proof of Proposition 6

We compare $M^*_{\mathbb{BB}}$ and $M^*_{\mathbb{BO}}$. According to Lemma 2, the same set of conditions should govern the comparison between $M^*_{\mathbb{OB}}$ and $M^*_{\mathbb{OO}}$.

Proof of Part (i). According to Table A.3, providers join with probability one in equilibrium (i.e., $j^{p^*}_{\mathbb{BB}} = 1$) when $\overline{W}^p \geq \mathbb{E}_{(\cdot,1)}[W^p_{\mathbb{BB}}]$. Furthermore, under both Regimes $\mathbb{BB}$ and $\mathbb{BO}$, customers receiving the "delay" signal face an expected delay $1/(\Lambda^p - j^c_I\delta^c\Lambda^c), I \in \{\mathbb{BB}, \mathbb{BO}\}$ (check Eq. (10)). Accordingly, $j^{c^*}_{\mathbb{BB}} = j^{c^*}_{\mathbb{BO}}$ in equilibrium. As a result, the CTMCs under Regimes $\mathbb{BO}$ and $\mathbb{BB}$ (as shown in Figs. 1c and 1d) are almost identical, except that the CTMC for Regime $\mathbb{BO}$ has a finite bounding state of $b^p_{\mathbb{BO}}$. So, it suffices to prove that the match rate is increasing in the bounding state, which we do below in Lemma 3.

LEMMA 3. *$M_{\mathbb{BO}}$ under is increasing in $b^p_{\mathbb{BO}}$, all other parameters remaining constant.*

Proof of Lemma 3. When $\delta^p\Lambda^p < \Lambda^c$, the match rate under Regime $\mathbb{BO}$ follows Eq. (19). Since the match rate cannot exceed Λ^c, the second term in Eq. (19) needs to be positive. The numerator is positive as $j^c_{\mathbb{BO}}\delta^c < 1$ and $\delta^p\Lambda^p < \Lambda^c$ (the condition for this lemma to hold, which is also implied by the condition for Proposition 6). Therefore, it suffices to establish that the denominator increases in $b^p_{\mathbb{BO}}$. The derivative of the denominator with respect to $b^p_{\mathbb{BO}}$ is $\delta^p(j^c_{\mathbb{BO}}\delta^c\Lambda^c - \Lambda^p)(\rho^p_{\mathbb{BO}})^{b^p_{\mathbb{BO}}}\ln(\rho^p_{\mathbb{BO}})$, which is positive because $\delta^p\Lambda^p < \Lambda^c$ and $j^c_{\mathbb{BO}}\delta^c\Lambda^c < \Lambda^p$ (as customers' joining strategy must result in a stable system).

This completes the proof for Part (i) of Proposition 6.

Proof of Part (ii). Now, we treat the case when providers are not sufficiently patient, leading to providers mixing when they receive the "delay" signal under Regime $\mathbb{BB}$. Then, we observe from Table A.3 that the equilibrium joining probability $j^{p^*}_{\mathbb{BB}} = \frac{\overline{W}^p\Lambda^c - 1}{\overline{W}^p\delta^p\Lambda^p}$. Substituting this into Eq. (18) yields the following match rate, which does not depend on δ^p, as $j^{c^*}_{\mathbb{BB}}$ does not depend on δ^p (see Table A.3):

$$M^*_{\mathbb{BB}} = \frac{j^{c^*}_{\mathbb{BB}}\delta^c(\Lambda^c - \overline{W}^p(\Lambda^c)^2 + \Lambda^p) + \Lambda^p(\overline{W}^p\Lambda^c - 1)}{\overline{W}^p\Lambda^p - j^{c^*}_{\mathbb{BB}}(\overline{W}^p\delta^c\Lambda^c - \delta^c)},$$

Now, it suffices to show that $M^*_{\mathbb{BO}}$ increases in δ^p. This will establish that $M^*_{\mathbb{OO}} \geq M^*_{\mathbb{BO}}$ only when δ^p is large enough. From Eq. (19), we have:

$$\frac{\partial M^*_{\mathbb{BO}}}{\partial \delta^p} = \frac{(\Lambda^c - j^{c^*}_{\mathbb{BB}}\delta^c\Lambda^c)(j^{c^*}_{\mathbb{BB}}\delta^c\Lambda^c - \Lambda^p)\left((\rho^p)^{b^p_{\mathbb{BO}}}(b^p_{\mathbb{BO}}\Lambda^c - b^p_{\mathbb{BO}}\delta^p\Lambda^p + \Lambda^c) - \Lambda^c\right)}{\left(\delta^p(j^{c^*}_{\mathbb{BB}}\delta^c\Lambda^c - \Lambda^p)(\rho^p)^{b^p_{\mathbb{BO}}} + \Lambda^c - \delta^p j^{c^*}_{\mathbb{BB}}\delta^c\Lambda^c\right)^2}.$$

For $\frac{\partial M^*_{\mathbb{BO}}}{\partial \delta^p} \geq 0$, it suffices to prove that the last term in the numerator is negative (as $j^{c^*}_{\mathbb{BB}}\delta^c\Lambda^c < \min\{\Lambda^c, \Lambda^p\}$), which simplifies to the following inequality:

$$\underbrace{(\rho^p)^{b^p_{\mathbb{BO}}}(1 + b^p_{\mathbb{BO}}(1-\rho^p))}_{\text{LHS}} \leq 1 \tag{A.39}$$

To prove inequality (A.39), we show that the derivative of its left-hand side with respect to ρ^p is positive for any $b^p_{\mathbb{BO}}$ (and therefore we can ignore the dependency of $b^p_{\mathbb{BO}}$ on ρ^p). This derivative is $b^p_{\mathbb{BO}}(1 + b^p_{\mathbb{BO}})(1-\rho^p)(\rho^p)^{b^p_{\mathbb{BO}}-1}$, which is clearly non-negative for all values of $b^p_{\mathbb{BO}}$ when $\rho^p < 1$ and non-positive for all values of $b^p_{\mathbb{BO}}$ when $\rho^p \geq 1$. So, the LHS of Eq. (A.39) achieves its supremum when $\rho^p = 1$; it is easy to see that this supremum is 1. This completes the proof.

A.10. Analogs to Propositions 4–6 and Lemma 2, fixing information disclosed to providers

PROPOSITION 8 **(Analog to Proposition 4)**. *$M^*_{\mathbb{BN}} \geq M^*_{\mathbb{ON}}$ when providers and customers have patience levels such that at least one user class joins with probability one under Regime $\mathbb{BN}$, i.e., one of the first four condition sets in the analog to Table A.1 holds.*

PROPOSITION 9 **(Analog to Proposition 5)**. *$M^*_{\mathbb{NB}} \leq M^*_{\mathbb{BB}}$ and $M^*_{\mathbb{NO}} \leq M^*_{\mathbb{BO}}$ when customers are sufficiently patient such that $\overline{W}^c \geq \dfrac{1}{\Lambda^p - \delta^c\Lambda^c}$ and $\delta^c\Lambda^c < \Lambda^p$.*

LEMMA 4 **(Analog to Lemma 2)**. *$M^*_{\mathbb{BB}} \geq M^*_{\mathbb{OB}} \iff M^*_{\mathbb{BO}} \geq M^*_{\mathbb{OO}}$; i.e., when providers receive binary information, disclosing binary information to customers is preferable over occupancy if and only if the same preference holds when providers receive occupancy information.*

PROPOSITION 10 **(Analog to Proposition 6)**. *$M^*_{\mathbb{BB}} \geq M^*_{\mathbb{OB}}$ and $M^*_{\mathbb{BO}} \geq M^*_{\mathbb{OO}}$ when: (i) customers are sufficiently patient such that $\overline{W}^c \geq 1/(\Lambda^p - \delta^c\Lambda^c)$ and $\delta^c\Lambda^c < \Lambda^p$, or (ii) the proportion δ^c of patient customers is below a certain threshold (possibly 1).*

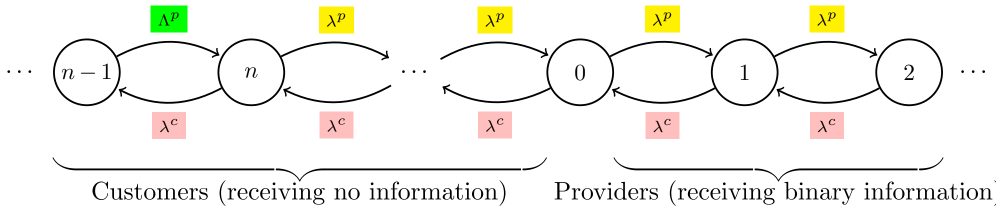

Figure A.1 **Regime $\mathbb{NB}'$**

A.11. Proof of Proposition 7

Part (i): Suppose $\overline{W}^p \geq \dfrac{1}{\Lambda^c - \delta^p \Lambda^p}$. We first show the optimality of our chosen binary style signal over no information and occupancy information. From Proposition 5, we have that $M^*_{\mathbb{BB}} \geq M^*_{\mathbb{BN}}$ and $M^*_{\mathbb{OB}} \geq M^*_{\mathbb{ON}}$. From Proposition 6, we have that $M^*_{\mathbb{BB}} \geq M^*_{\mathbb{BO}}$ and $M^*_{\mathbb{OB}} \geq M^*_{\mathbb{OO}}$. Furthermore, if $\overline{W}^p \geq \dfrac{1}{\Lambda^c - \delta^p \Lambda^p}$, then either the first or fourth set of conditions in Table A.1 must hold, as the conditions on $\overline{W}^p$ are weaker than $\overline{W}^p \geq \dfrac{1}{\Lambda^c - \delta^p \Lambda^p}$ and the conditions on $\overline{W}^c$ are perfect complements. This implies that $M^*_{\mathbb{NB}} \geq M^*_{\mathbb{NO}}$ (Proposition 4). Therefore, it is optimal to give providers binary information. The condition for customers follows from the same logic as above, with the symmetric analogs to Propositions 5, 6 and 4 (Propositions 9, 10 and 8 respectively).

Next, we show that no binary-style threshold-type signal can outperform this one.

1. *When customers receive no information:* First, as argued above, the first or fourth set of conditions in Table A.1 must hold under the assumed conditions, and therefore, patient providers receiving the binary "delay" signal under Regime $\mathbb{NB}$ join with probability one. Therefore, the match rate exceeds $\delta^p\Lambda^p$ under Regime $\mathbb{NB}$. We seek to show that Regime $\mathbb{NB}$ outperforms an alternative Regime $\mathbb{NB}'$, where providers are provided one signal in states $i < n$ and another signal in states $i \geq n$, where $n \neq 0$. Under such an alternative, observe that if $n > 0$, both binary signals indicate non-zero delay. Therefore, the maximum possible match rate is $\delta^p\Lambda^p$. So it suffices to consider the case of $n < 0$. The CTMC for this case is shown in Fig. A.1. Here, providers join at rate Λ^p when they receive the "no delay" signal and at rate $\lambda^p \leq \delta^p\Lambda^p$ when they receive the delay signal. Customers join at rate $\lambda^c \leq \delta^c\Lambda^c$. First, we show that the match rate λ^c that results from this joining behavior is maximized when $n = 0$, for any given value of λ^p. To do this, we show that the expected customer waiting time, $\mathbb{E}[W^c]$, is decreasing in λ^p for a fixed value of λ^c. To show this, we write:

$$\mathbb{E}[W^c] = \Pr(i \leq 0)\, \mathbb{E}[W^c | i \leq 0].$$

   Now, consider System 1 with $\lambda^p < \delta^p\Lambda^p$ and System 2 with $\lambda^p = \delta^p\Lambda^p$. Following Theorem 5 in Smith and Whitt (1981), System 2's state is stochastically to the right of System 1. As a result, $\Pr(i \leq 0)$ is higher for System 1. Furthermore, $\mathbb{E}[W^c|i \leq 0]$ is higher for System 1, as the customer service rates (provider arrival rates) are higher in System 2 than System 1 for states $i \in [n, -1]$. It follows that $\mathbb{E}[W^c]$ is the lowest, and consequently, that the match rate is highest when λ^p is at its highest possible value, i.e., $\lambda^p = \delta^p\Lambda^p$. Then, we set $\lambda^p = \delta^p\Lambda^p$, and invoke Theorem 5 in Smith and Whitt (1981) again to assert that $\mathbb{E}[W^c]$ is decreasing in $|n|$. As a final step, observe that setting $\lambda^p = \delta^p\Lambda^p$ and $n = 0$, we recover the CTMC corresponding to Regime $\mathbb{NB}$. So, Regime $\mathbb{NB}$ has a higher match rate than Regime $\mathbb{NB}'$ as defined above.
2. *When customers receive binary information:* Observe again that under the stated conditions, the match rate of Regime $\mathbb{BB}$ is a convex combination of $\delta^p\Lambda^p$ and Λ^p, and therefore exceeds $\delta^p\Lambda^p$. We seek to show that Regime $\mathbb{BB}$ outperforms an alternative Regime $\mathbb{BB}'$, where providers are provided one signal in states $i < n$ and another signal in states $i \geq n$, where $n \neq 0$. Again, if $n > 0$, both binary signals indicate non-zero delay. Therefore, the maximum possible match rate is $\delta^p\Lambda^p$. So it suffices to consider the case of $n < 0$. The CTMC for this

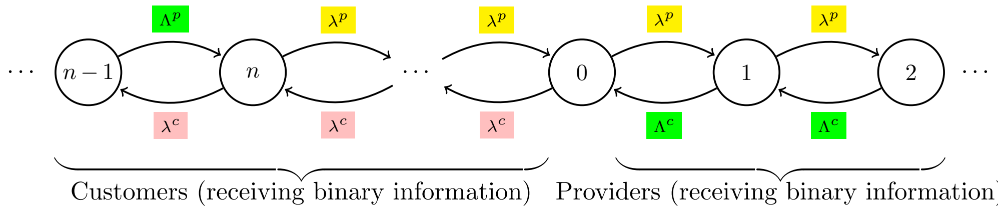

Figure A.2 **Regime $\mathbb{BB}'$**

case is shown in Fig. A.2. Here, providers join at rate Λ^p when they receive the "no delay" signal and at rate $\lambda^p \leq \delta^p \Lambda^p$ when they receive the delay signal. Observe here that the match rate can be written as:

$$\Lambda^c \Pr(i > 0) + \lambda^c \Pr(i \leq 0)$$

Now, consider System 1 with $\lambda^p < \delta^p \Lambda^p$ and System 2 with $\lambda^p = \delta^p \Lambda^p$, for a fixed value of λ^c. Following Theorem 5 in Smith and Whitt (1981), System 2's state is stochastically to the right of System 1. As a result, $\Pr(i > 0)$ is higher for System 2, leading to a higher match rate for System 2. Then, we set $\lambda^p = \delta^p \Lambda^p$, and invoke Theorem 5 in Smith and Whitt (1981) again to assert that the match rate is decreasing in $|n|$ for a fixed value of λ^c. Therefore, setting $n = 0$ increases the match rate if λ^c is unchanged. Finally, following the same logic as the proof for Regime $\mathbb{NB}$ above, $\mathbb{E}[W^c]$ is increasing in $|n|$ and therefore, the equilibrium λ^c will be higher when $n = 0$; following the same arguments as above, this will result in a further higher match rate. As a final step, observe that setting $\lambda^p = \delta^p \Lambda^p$ and $n = 0$ (and the λ^c that is consistent with these), we recover the CTMC corresponding to Regime $\mathbb{BB}$. This proves that Regime $\mathbb{BB}$ has a higher match rate than Regime $\mathbb{BB}'$ as defined above.

3. *When customers receive occupancy information:* Observe again that under the stated conditions, the match rate of Regime $\mathbb{OB}$ is a convex combination of $\delta^p \Lambda^p$ and Λ^p, and therefore exceeds $\delta^p \Lambda^p$. We seek to show that Regime $\mathbb{OB}$ outperforms an alternative Regime $\mathbb{OB}'$, where providers are provided one signal in states $i < n$ and another signal in states $i \geq n$, where $n \neq 0$. Again, if $n > 0$, both binary signals indicate non-zero delay. Therefore, the maximum possible match rate is $\delta^p \Lambda^p$. So it suffices to consider the case of $n < 0$. The CTMC for this case is shown in Fig. A.3. Providers join at rate Λ^p when they receive the "no delay" signal and at rate $\lambda^p \leq \delta^p \Lambda^p$ otherwise. The match rate can be written as:

$$\Lambda^c \Pr(i > 0) + \delta^c \Lambda^c \Pr(i \leq 0)$$

Now, consider System 1 with $\lambda^p < \delta^p \Lambda^p$ and System 2 with $\lambda^p = \delta^p \Lambda^p$, for a fixed value of λ^c. Following Theorem 5 in Smith and Whitt (1981), System 2's state is stochastically to the right of System 1. As a result, $\Pr(i > 0)$ is higher for System 2, leading to a higher match rate for System 2. Then, we set $\lambda^p = \delta^p \Lambda^p$, and invoke Theorem 5 in Smith and Whitt (1981) again to assert that the match rate is decreasing in $|n|$. Therefore, setting $n = 0$ increases the match rate. Finally, observe that setting $\lambda^p = \delta^p \Lambda^p$ and $n = 0$, we recover the CTMC corresponding to Regime $\mathbb{OB}$. So, Regime $\mathbb{OB}$ has a higher match rate than Regime $\mathbb{OB}'$ as defined above.

Part (ii): Putting together the observations above, Regime $\mathbb{BB}$ outperforms the other basic regime types we have considered. Now, we prove that no alternative binary signaling mechanism—that can potentially use non-zero thresholds for *both* customers and providers—can outperform Regime $\mathbb{BB}$. Observe that under the aforementioned conditions, the match rate under Regime $\mathbb{BB}$, $M^*_{\mathbb{BB}} > \max\{\delta^p \Lambda^p, \delta^c \Lambda^c\}$, as the provider arrival rate at every state is at least $\delta^p \Lambda^p$ and the customer arrival rate at every state is at least $\delta^c \Lambda^c$. Now, consider an alternative binary signal for customers, so that customers receive one signal for states $i < m$ and another signal for states $i \geq m + 1$. If $m < 0$, both signals indicate a delay; therefore, impatient customers do not join under either signal, and the maximum match rate is $\delta^c \Lambda^c < M^*_{\mathbb{BB}}$. Therefore, no alternative customer signal with $p < 0$ can

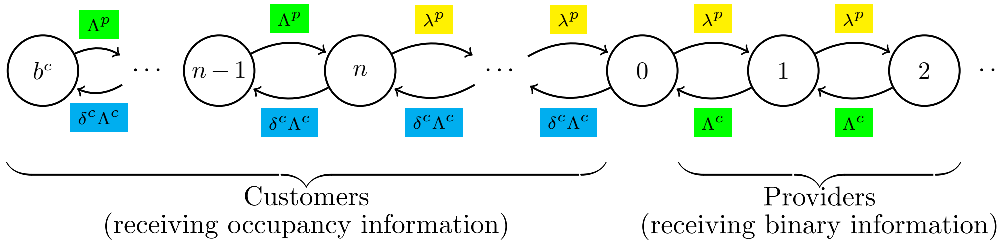

Figure A.3 **Regime $\mathbb{O}\mathbb{B}'$**

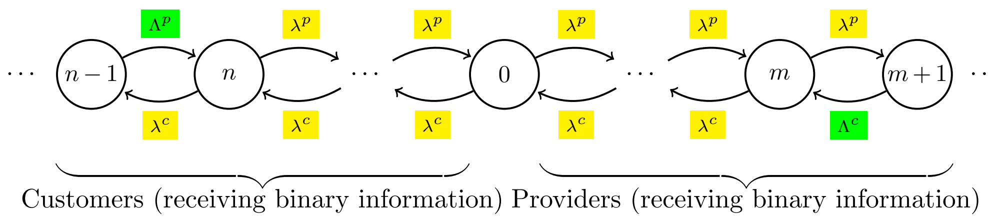

Figure A.4 **Alternative to Regime $\mathbb{B}\mathbb{B}$, with non-zero thresholds for both customers and providers.**

outperform Regime $\mathbb{B}\mathbb{B}$. Similarly, if providers receive one signal for states $i<n$ and another signal for states $i \geq n$, any signal with $n>0$ is strictly inferior to Regime $\mathbb{B}\mathbb{B}$. Therefore, it remains to consider alternative binary signaling mechanisms with $n<0$ and $m>0$.

For ease of exposition, we draw the CTMC associated with this alternative signaling mechanism in Figure A.4. Here, when customers receive a signal indicating that they are in states $m+1$ or larger, they expect no delay and join at rate Λ^c. However, when they arrive to states m or smaller, they may encounter a delay, and therefore, join at some rate $\lambda^c \leq \delta^c\Lambda^c$. An analogous argument holds for providers, who join at rate $\lambda^p \leq \delta^p\Lambda^p$ when their signal indicates that they arrive to states n or larger. In what follows, we prove that the match rate obtained using such a signaling mechanism is inferior to that of Regime $\mathbb{B}\mathbb{B}$.

To show this, we first show that the match rate corresponding to this alternative signaling mechanism is increasing in λ^p (and by symmetry, λ^c). This again follows from Theorem 5 of Smith and Whitt (1981). Therefore, the match rate of the CTMC in Fig. A.4 is upper bounded by the match rate of a CTMC with $\lambda^c = \delta^c\Lambda^c$ and $\lambda^p = \delta^p\Lambda^p$.

Now, we show that the match rate is decreasing in m (and by symmetry, $|n|$). The match rate of the CTMC in Fig. A.4 can be calculated as:

$$M = \frac{(\lambda^c-\Lambda^p)(\Lambda^c-\lambda^p)\left(\frac{(\lambda^c)^2(\lambda^p-\Lambda^p)\left(\frac{\lambda^c}{\lambda^p}\right)^n}{\lambda^c-\Lambda^p} + \frac{(\lambda^p)^2(\lambda^c-\Lambda^c)\left(\frac{\lambda^p}{\lambda^c}\right)^m}{\Lambda^c-\lambda^p}\right)}{\lambda^c(\Lambda^c-\lambda^p)(\lambda^p-\Lambda^p)\left(\frac{\lambda^c}{\lambda^p}\right)^n + \lambda^p(\lambda^c-\Lambda^c)(\lambda^c-\Lambda^p)\left(\frac{\lambda^p}{\lambda^c}\right)^m}.$$

Then,

$$\frac{dM}{dm} = \frac{-\lambda^c\lambda^p(\Lambda^c-\lambda^c)(\Lambda^c-\lambda^p)(\Lambda^p-\lambda^c)(\Lambda^p-\lambda^p)\left(\frac{\lambda^c}{\lambda^p}\right)^n\left(\frac{\lambda^p}{\lambda^c}\right)^m(\lambda^c-\lambda^p)\log\left(\frac{\lambda^c}{\lambda^p}\right)}{\left(\lambda^c(\Lambda^c-\lambda^p)(\lambda^p-\Lambda^p)\left(\frac{\lambda^c}{\lambda^p}\right)^n + \lambda^p(\lambda^c-\Lambda^c)(\lambda^c-\Lambda^p)\left(\frac{\lambda^p}{\lambda^c}\right)^p\right)^2}. \tag{A.40}$$

The denominator is clearly positive. Observing that $\lambda^c<\Lambda^c$, $\lambda^c<\Lambda^p$, $\lambda^p<\Lambda^p$, $\lambda^p<\Lambda^c$, it follows that the numerator is negative.

By symmetry, the match rate is decreasing in n. Therefore, the match rate of the CTMC in Fig A.4 is smaller than the match rate of the CTMC in Fig A.4 with $\lambda^c=\delta^c\Lambda^c$, which in turn, is smaller than the match rate of a CTMC with $m=n=0$, which is identical to the CTMC of the original Regime $\mathbb{B}\mathbb{B}$. This completes the proof.