E-Companion for "The Interplay Between Customer Feedback Solicitation and Innovation: A Dynamic Solution"

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EC.1. Supplementary Technical Material

EC.1.1. Detailed Derivation of the Bellman Equation (Section 3.1.4)

Consider the two independent Poisson processes: one governing the arrival of purchasing customers, and the other governing the occurrence of quality jumps. Their superposition is itself a Poisson process with rate $\Lambda := \lambda_c + \lambda_u$. Under this representation, each arrival corresponds either to a purchasing customer with probability $\frac{\lambda_c}{\Lambda}$ or to the completion of a quality improvement cycle (a quality jump) with probability $\frac{\lambda_u}{\Lambda}$.

Following Lippman (1975), this superposition simplifies the analysis by allowing us to treat decision epochs as arising from a *single* Poisson clock rather than two separate ones. As a result, the recursive structure of the firm's problem becomes transparent: $V(n_\tau, q_\tau)$ is given by the expected profit accumulated during the interval $[\mu_\tau, \mu_{\tau+1}]$ under the optimal action, plus the discounted continuation value $V(n_{\tau+1}, q_{\tau+1})$ at the next state, where the transition to $(n_{\tau+1}, q_{\tau+1})$ is governed by the event probabilities of the superposed Poisson process.

To characterize the expected profit earned over $[\mu_\tau, \mu_{\tau+1}]$, it is useful to detail the sequence of events during this interval. At time μ_τ, the firm chooses the optimal action a_τ based on the state (n_τ, q_τ). From time μ_τ to $\mu_{\tau+1}$, two types of costs are incurred continuously: a quality improvement cost rate $h_r(n_\tau, q_\tau)$ and an operational cost rate $d_r(n_\tau)$. Finally, at time $\mu_{\tau+1}$, the state transitions to $(n_{\tau+1}, q_{\tau+1})$ depending on whether the event is a customer arrival or a quality jump. In the case of a customer arrival, the firm also earns an instantaneous revenue of $p(q_\tau)$, net of cost l if the chosen action is feedback solicitation.

We now compute the expected discounted costs incurred between decision epochs. Let $h(n_\tau, q_\tau)$ denote the expected total discounted quality improvement cost from μ_τ until the next decision epoch. By the law of iterated expectations, $h(n_\tau, q_\tau) := \int_0^\infty \left[\int_0^\mu e^{-\alpha t} h_r(n_\tau, q_\tau) dt\right] \Lambda e^{-\Lambda\mu} d\mu = \frac{1}{\alpha+\Lambda} h_r(n_\tau, q_\tau)$, where the inner integration is the conditional expectation of the discounted cost, given that the next epoch occurs μ units of time after the current epoch, and $\Lambda e^{-\Lambda\mu}$ is the p.d.f. of an exponential distribution with rate Λ. For convenience, we define $h^1(n) := \frac{1}{\alpha+\Lambda} h_r^1(n)$ and $h^2(q) := \frac{1}{\alpha+\Lambda} h_r^2(q)$. Since $h_r(n,q) = h_r^1(n) + h_r^2(q)$, it follows that $h(n_\tau, q_\tau) = h^1(n_\tau) + h^2(q_\tau)$. Note that $h^1(\cdot)$ inherits all the properties of $h_r^1(\cdot)$ stated in Assumption 2, and $h^2(\cdot)$ inherits the monotone increasing property of $h_r^2(\cdot)$. Analogously, the expected discounted operational cost incurred between μ_τ and $\mu_{\tau+1}$ is $d(n_\tau) := \frac{1}{\alpha+\Lambda} d_r(n_\tau)$, which inherits the properties of $d_r(\cdot)$ from Assumption 3.

Except for the quality improvement cost and operational cost which are incurred continuously, all other revenues, costs, and state transitions occur discretely at the next event time $\mu_{\tau+1}$, which follows an exponential distribution with rate Λ. For convenience, let ρ denote the expected discount factor applied to such events: $\rho := \int_0^\infty e^{-\alpha\mu} \cdot \Lambda e^{-\Lambda\mu} d\mu = \frac{\Lambda}{\alpha+\Lambda}$.

With probability $\frac{\lambda_u}{\Lambda}$, the next event is a quality jump that increases q_τ to $q_\tau + 1$ (if $q_\tau < \bar{q}$) but provides no direct reward. With probability $\frac{\lambda_c}{\Lambda}$, the next event is a customer arrival that generates a revenue $p(q_\tau)$. Conditional on such an arrival, the quality level remains at q_τ, but the evolution of the feedback level depends on whether the firm solicits feedback: if $a_\tau = 0$, then $n_{\tau+1} = n_\tau$ and no additional cost is incurred; if $a_\tau = 1$, the firm pays the solicitation cost l, and with probability β the feedback level increases to $n_{\tau+1} = n_\tau + 1$ (if $n_\tau < \bar{n}$), while with probability $1-\beta$ it remains at n_τ.

Bringing these components together, the Bellman equation for the firm's problem can be expressed as

$$V(n,q) = -h(n,q) - d(n) + \rho\frac{\lambda_u}{\Lambda} V(n,q+1) + \rho\frac{\lambda_c}{\Lambda}\left[p(q) + \max\left\{\underbrace{V(n,q)}_{\text{if } a=0},\ \underbrace{\beta V(n+1,q) + (1-\beta)V(n,q) - l}_{\text{if } a=1}\right\}\right]$$

for the case $n < \bar{n}$ and $q < \bar{q}$. When $q = \bar{q}$, the quality improvement cost $h(n,q)$ is dropped and $q+1$ is replaced with $\bar{q}$ to reflect the upper bound on quality. When $n = \bar{n}$, we similarly replace $n+1$ with $\bar{n}$.

Remark EC.1. We address a modeling detail that arises when quality reaches the upper bound $\bar{q}$. Recall that decision epochs are triggered either by customer arrivals or by quality jumps. At $q = \bar{q}$, however, no further quality jumps are possible, which could otherwise create an inconsistency in the definition of decision epochs. To resolve this, we introduce a fictitious or "shadow" quality improvement process: it follows the same exponential timing with rate λ_u but incurs no cost and leaves the quality state unchanged, even when it "succeeds." This device ensures that the timing of decision epochs remains consistent without altering its dynamics (Lippman 1975).

EC.1.2. Decision Epoch Construction in the Joint Setting (Section 4.1.1)

In the baseline model of Section 3, decision epochs occur either at a customer arrival or at a quality jump. Since both follow exponential clocks, their superposition is Poisson with rate $\Lambda = \lambda_c + \lambda_u$. A complication arises in the joint optimization setting: if the firm chooses *not* to invest in quality improvement at a given epoch, the next epoch would be triggered only by a customer arrival. This breaks the convenient Poisson event structure of the baseline model, where epochs are always defined by the minimum of two exponential clocks.

To resolve this, we introduce a *shadow* quality improvement process that runs whenever the firm pauses quality improvement. This shadow process has an exponential completion time with rate λ_u, just like a real quality improvement cycle, but it incurs no cost and yields no actual quality improvement upon completion. Its sole purpose is to preserve the timing structure: with both customer arrivals and shadow completions, decision epochs continue to follow a Poisson process with rate $\Lambda = \lambda_c + \lambda_u$. This modeling device is a direct application of the uniformization technique introduced by Lippman (1975), which uses fictitious transitions to preserve exponential timing in systems where transition rates depend on the action. Similar techniques have been employed in many prior studies (Iravani and Duenyas 2002, Benjaafar and ElHafsi 2006).

Crucially, the shadow process does not affect the underlying stochastic dynamics or the optimal policy. Suppose the firm pauses the quality improvement effort at the τth decision epoch. If the shadow process completes before the next customer arrival, it merely defines the $(\tau+1)$th decision epoch. But since the state has not changed (by our construction of the shadow process), the firm will apply the same action as it did in the τth epoch. By the memoryless property of the exponential distribution, this artificial event has no impact on future arrivals or decisions. Thus, the shadow process is purely a modeling device that ensures analytical consistency with the baseline model while leaving system behavior and outcomes unchanged.

EC.1.3. Numerical Experiment Parameters

EC.1.3.1. Parameter values for Figure 2

$$\beta = 0.2, \quad l = 20, \quad \bar{n} = \bar{q} = 15, \quad \lambda_u = 0.5, \quad \lambda_c = 1, \quad \alpha = 0.4,$$
$$p(q) = 10q(2\bar{q} - q) + 2000, \quad d(n) = 800e^{-0.05n}, \quad h(n,q) = 600e^{-0.05n} + 5q^2.$$

EC.1.3.2. Parameter values for Figure 7

$$\beta = 0.2, \quad l = 5, \quad \bar{n} = \bar{q} = 20, \quad \lambda_u = 0.5, \quad \lambda_c = 5, \quad \alpha = 0.3,$$
$$p(q) = 5q(2\bar{q} - q) + 200, \quad d(n) = 1800e^{-0.3n} + 10, \quad h(n,q) = 600e^{-0.3n} + 2q^2 + 1000.$$

Remark EC.2. In the left panel, to make the (otherwise largely unchanged) movement of the quality threshold visible, we consider a larger increase to $l' = 50$, which falls outside the condition in Assumption 4(ii). Importantly, Assumption 4 is sufficient but not necessary for the structural results, and the numerically computed optimal policy under $l' = 50$ continues to exhibit a clean, monotone double-threshold structure.

EC.1.4. Verification of Model Assumptions

EC.1.4.1. Verification of Assumption 4(i)

Example EC.1. Consider $d(n) = \frac{\gamma_1}{1+\gamma_2 n}$ and $h(n,q) = \frac{\gamma_3}{1+\gamma_4 n} + \gamma_5 q^2$. These functions satisfy Assumption 4(i) for appropriate parameter choices. For instance, when $\bar{n} = 400$, setting $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (5000, 0.018, 100, 0.00005, 100)$ ensures that the condition holds for all $n \le \bar{n} - 1$, illustrating that the assumption is compatible with interpretable and practically relevant cost structures.

EC.1.4.2. Verification of Assumption 4(ii)

Example EC.2. The quadratic form $p(q) = \nu_1 q - \nu_2 q^2$ satisfies Assumption 4(ii) for suitable parameter values. For instance, when $\bar{q} = 20$ and $l = 10$, setting $(\nu_1, \nu_2) = (200, 5)$ ensures that the second-order differences are consistently less than or equal to $-l$, confirming that the assumption holds under simple and analytically convenient revenue models.

EC.2. Proof

EC.2.1. Proof of Lemma 1

Consider a value iteration algorithm to solve the Bellman equation (1). Let $V_k(\cdot,\cdot)$ denote the value function after k rounds of iteration. Specifically, we have the following update rule for V_{k+1} based on the value function of the previous round V_k:

$$V_{k+1}(n,q) = \max_{a \in \{0,1\}} Q^a_{k+1}(n,q), \tag{B.1}$$

where $Q^a_{k+1}(n,q)$ is the value of taking action a in the current state (n,q) and taking optimal actions in all future states, as approximated by V_k. Specifically, for any $n < \bar{n}$ and $q < \bar{q}$:

$$Q^0_{k+1}(n,q) := \rho\sigma_c p(q) + \rho\sigma_c V_k(n,q) + \rho\sigma_u V_k(n,q+1) - h(n,q) - d(n)$$

and

$$\begin{aligned} Q^1_{k+1}(n,q) &:= \rho\sigma_c p(q) + \rho\sigma_c\big[\beta V_k(n+1,q) + (1-\beta)V_k(n,q) - l\big] + \rho\sigma_u V_k(n,q+1) - h(n,q) - d(n) \\ &= \rho\sigma_c p(q) + \rho\sigma_c V_k(n,q) + \rho\sigma_u V_k(n,q+1) + \rho\sigma_c\big[\beta\Delta_n V_k(n,q) - l\big] - h(n,q) - d(n) \end{aligned}$$

where $\sigma_c := \lambda_c/\Lambda$, $\sigma_u := \lambda_u/\Lambda$, and $\Delta_n V_k(n,q) := V_k(n+1,q) - V_k(n,q)$. Note that since $\Lambda = \lambda_u + \lambda_c$, it follows that $\sigma_c + \sigma_u = (\lambda_c + \lambda_u)/\Lambda = 1$. These definitions will be used again in subsequent proofs. When $n = \bar{n}$, we simply need to replace $n+1$ with $\bar{n}$ in the above equations, i.e., $\Delta_n V_k(\bar{n},q) = 0$. When $q = \bar{q}$, we only need to replace $q+1$ with $\bar{q}$ and remove the quality improvement cost $h(n,q)$ in the above equations.

The following lemma states that the value iteration algorithm converges for our model.

Lemma EC.1. *There exists a function V such that $V_k(n,q) \to V(n,q)$ as $k \to +\infty$.*

Proof of Lemma EC.1. Our model has a finite state space and bounded rewards. Furthermore, given that the discount rate $\alpha > 0$, the expected discount factor $\rho := \frac{\Lambda}{\alpha+\Lambda}$ satisfies the condition $0 \le \rho < 1$. Hence, according to Puterman (2014), the value iteration algorithm will converge to a function V as $k \to \infty$. $\square$

We will prove Lemma 1 by induction. We start with $k = 0$ where we set $V_0(n,q) = 0$ for all the states (n,q). Note that (i)–(ii) in Lemma 1 are true for $V = V_0$. Thus, by Lemma EC.1, if we can show that properties (i)–(ii) are preserved when $V = V_k$ for any $k \ge 1$, then this proves Lemma 1. Assume that, at iteration k, properties (i)–(ii) are true for V_k. The induction proof is complete if we can show that the properties are also true for V_{k+1}.

Before proceeding with the proof, we first establish the following useful conditions.

Lemma EC.2. *The following are the conditions for the optimal feedback solicitation actions when the firm continuously improves quality:*

1. *Suppose $n < \bar{n}$. Then $V_{k+1}(n,q) = Q^1_{k+1}(n,q)$ if and only if $\beta\Delta_n V_k(n,q) - l \ge 0$. Otherwise, $V_{k+1}(n,q) = Q^0_{k+1}(n,q)$.*
2. *Suppose $n = \bar{n}$. Then $V_{k+1}(n,q) = Q^0_{k+1}(n,q)$.*

Proof of Lemma EC.2. When $n < \bar{n}$, note that $V_{k+1}(n,q) = Q^1_{k+1}(n,q)$ if and only if inequality (3) holds, which proves part 1 of the lemma. When $n = \bar{n}$, there is $\Delta_n V_k(n,q) = 0$, since $l > 0$, thus $\beta\Delta_n V_k(n,q) - l = -l < 0$, therefore $V_{k+1}(n,q) = Q^0_{k+1}(n,q)$. $\square$

Now we are ready to proceed with the proof of Lemma 1. The value of $V_{k+1}(n,q)$ depends on the optimal action that solves (B.1). Hence, we need to check that properties (i)–(ii) hold regardless of the optimal actions.

Specifically, we need to show that these properties hold for all possible combinations of optimal actions. By the induction hypothesis, some combinations can be ruled out. For example, to show that $V_{k+1}(n,q)$ increases in n, it is sufficient to show that for any $n < \bar{n}$,

$$Q^{a_1}_{k+1}(n,q) \le Q^{a_2}_{k+1}(n+1,q) \tag{B.2}$$

holds for any actions (a_1,a_2) where a_1 and a_2 are optimal for states (n,q) and $(n+1,q)$, respectively. Due to the induction hypothesis that $\Delta_n V_k(n,q)$ decreases in n, if it is optimal to not solicit feedback at state (n,q) (i.e., $a_1 = 0$), then it must also be optimal to not solicit feedback at state $(n+1,q)$ (i.e., $a_2 = 0$). Thus, it is impossible for $a_1 = 0$ and $a_2 = 1$ to be optimal for states (n,q) and $(n+1,q)$ simultaneously. Hence, we only need to check that (B.2) is true when the optimal actions (a_1,a_2) are $(0,0)$, $(1,0)$ or $(1,1)$.

To prove that $\Delta_n V_k(n,q)$ decreases in n, it is sufficient to show that for any $n < \bar{n}$,

$$Q^{a_2}_{k+1}(n+1,q) - Q^{a_1}_{k+1}(n,q) \le Q^{a_1}_{k+1}(n,q) - Q^{a_3}_{k+1}(n-1,q) \tag{B.3}$$

holds for all combinations of (a_1,a_2,a_3) where a_1 and a_2 have the same definitions as before, and a_3 is the optimal action for state $(n-1,q)$. Similarly, by the induction hypothesis, most action combinations can be ruled out from being optimal simultaneously. We only need to check that (B.3) holds when the optimal actions (a_1,a_2,a_3) are $(0,0,0)$, $(1,1,1)$, $(1,0,1)$ or $(0,0,1)$.

Besides, to prove that $\Delta_n V_k(n,q)$ decreases in q, it is sufficient to show that for any $q < \bar{q}$,

$$Q^{a_5}_{k+1}(n+1,q+1) - Q^{a_4}_{k+1}(n,q+1) \le Q^{a_2}_{k+1}(n+1,q) - Q^{a_1}_{k+1}(n,q) \tag{B.4}$$

holds for all combinations of (a_1,a_2,a_4,a_5) where a_1 and a_2 have the same definitions as before, and a_4 and a_5 are optimal actions for states $(n,q+1)$ and $(n+1,q+1)$ respectively. Similarly, we only need to check that (B.4) holds when the optimal actions (a_1,a_2,a_4,a_5) are $(0,0,0,0)$, $(1,0,0,0)$, $(1,0,1,0)$, $(1,1,0,0)$, $(1,1,1,0)$ or $(1,1,1,1)$.

Next, we provide detailed proofs for (B.2) and (B.3). The proof of (B.4) follows the same idea and is therefore omitted. Recall that $h^{1-}(n) := h^1(n) - h^1(n+1)$ and $d^-(n) := d(n) - d(n+1)$, then according to Assumption 2, $h^{1-}(n) \ge 0$ and $h^{1-}(n)$ decreases in n, and according to Assumption 3, $d^-(n) \ge 0$ and $d^-(n)$ decreases in n.

1. Proof of (B.2):

First, we check all the off-boundary states, i.e., (n,q) where $n+1 < \bar{n}$ and $q < \bar{q}$. We check all the possible combinations of optimal actions as discussed above.

(a) Case 1: $Q^0_{k+1}(n+1,q) - Q^0_{k+1}(n,q) = \rho\sigma_u\Delta_n V_k(n,q+1) + \rho\sigma_c\Delta_n V_k(n,q) + h^{1-}(n) + d^-(n) \ge 0$, where the inequality is because of the properties that $h^{1-}(n) \ge 0$ and $d^-(n) \ge 0$, and the induction hypothesis that $V_k(n,q)$ increases in n. Hence, this proves (B.2) for $(0,0)$.

(b) Case 2 $Q^0_{k+1}(n+1,q) - Q^1_{k+1}(n,q) = \rho\sigma_u\Delta_n V_k(n,q+1) + (1-\beta)\rho\sigma_c\Delta_n V_k(n,q) + \rho\sigma_c l + h^{1-}(n) + d^-(n) \ge 0$, where the inequality is because of the properties that $l > 0$, $h^{1-}(n) \ge 0$ and $d^-(n) \ge 0$, and the induction hypothesis that $V_k(n,q)$ increases in n. Hence, this proves (B.2) for $(1,0)$.

(c) Case 3: $Q^1_{k+1}(n+1,q) - Q^1_{k+1}(n,q) = \rho\sigma_u\Delta_n V_k(n,q+1) + \rho\sigma_c\left[\beta\Delta_n V_k(n+1,q) + (1-\beta)\Delta_n V_k(n,q)\right] + h^{1-}(n) + d^-(n) \ge 0$, where the inequality is because of the properties that $h^{1-}(n) \ge 0$ and $d^-(n) \ge 0$, and the induction hypothesis that $V_k(n,q)$ increases in n. Hence, this proves (B.2) for $(1,1)$.

Thus, we complete the proof for off-boundary states, i.e., $n+1<\bar{n}$ and $q<\bar{q}$. The proof when $n+1<\bar{n}$ and $q=\bar{q}$ uses the same cases as above, but we only need to replace $q+1$ with q, and there is no $h^{1-}(n)$. When $n+1=\bar{n}$, based on Lemma EC.2, $V_{k+1}(n+1,q)=Q^0_{k+1}(n+1,q)$, then only Cases 1 and 2 apply. It can be verified that the proofs still hold for boundary states. Hence, we have proved (B.2).

2. Proof of (B.3)

First, we check all the off-boundary states, i.e., (n,q) where $n+1<\bar{n}$ and $q<\bar{q}$.

(a) Case 1: $Q^0_{k+1}(n+1,q)-Q^0_{k+1}(n,q)-\left(Q^0_{k+1}(n,q)-Q^0_{k+1}(n-1,q)\right)=\rho\sigma_u\left[\Delta_n V_k(n,q+1)-\Delta_n V_k(n-1,q+1)\right]+\rho\sigma_c\left[\Delta_n V_k(n,q)-\Delta_n V_k(n-1,q)\right]+h^{1-}(n)-h^{1-}(n-1)+d^-(n)-d^-(n-1)\le 0$, where the inequality is because of the properties that $h^{1-}(n)$ and $d^-(n)$ decrease in n and the induction hypothesis that $\Delta_n V_k(n,q)$ decreases in n. Then we have proved (B.3) for $(0,0,0)$.

(b) Case 2: $Q^1_{k+1}(n+1,q)-Q^1_{k+1}(n,q)-\left(Q^1_{k+1}(n,q)-Q^1_{k+1}(n-1,q)\right)=\rho\sigma_u\left[\Delta_n V_k(n,q+1)-\Delta_n V_k(n-1,q+1)\right]+\rho\sigma_c\beta\left[\Delta_n V_k(n+1,q)-\Delta_n V_k(n,q)\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k(n,q)-\Delta_n V_k(n-1,q)\right]+h^{1-}(n)-h^{1-}(n-1)+d^-(n)-d^-(n-1)\le 0$, where the inequality is because of the properties that $h^{1-}(n)$ and $d^-(n)$ decrease in n and the induction hypothesis that $\Delta_n V_k(n,q)$ decreases in n. Then we have proved (B.3) for $(1,1,1)$.

(c) Case 3: $Q^0_{k+1}(n+1,q)-Q^1_{k+1}(n,q)-\left(Q^1_{k+1}(n,q)-Q^1_{k+1}(n-1,q)\right)=\rho\sigma_u\left[\Delta_n V_k(n,q+1)-\Delta_n V_k(n-1,q+1)\right]+\rho\sigma_c\left[-\beta\Delta_n V_k(n,q)+l\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k(n,q)-\Delta_n V_k(n-1,q)\right]+h^{1-}(n)-h^{1-}(n-1)+d^-(n)-d^-(n-1)\le 0$, where the inequality is because of the properties that $h^{1-}(n)$ and $d^-(n)$ decrease in n, the induction hypothesis that $\Delta_n V_k(n,q)$ decreases in n, and Lemma EC.2 part 1 (here we have $V_{k+1}(n,q)=Q^1_{k+1}(n,q)$). Hence we proved (B.3) for $(1,0,1)$.

(d) Case 4: $Q^0_{k+1}(n+1,q)-Q^0_{k+1}(n,q)-\left(Q^0_{k+1}(n,q)-Q^1_{k+1}(n-1,q)\right)=\rho\sigma_u\left[\Delta_n V_k(n,q+1)-\Delta_n V_k(n-1,q+1)\right]+\rho\sigma_c\left[\beta\Delta_n V_k(n,q)-l\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k(n,q)-\Delta_n V_k(n-1,q)\right]+h^{1-}(n)-h^{1-}(n-1)+d^-(n)-d^-(n-1)\le 0$, where the inequality is because of the properties that $h^{1-}(n)$ and $d^-(n)$ decrease in n, the induction hypothesis that $\Delta_n V_k(n,q)$ decreases in n, and Lemma EC.2 part 1 (here we have $V_{k+1}(n,q)=Q^0_{k+1}(n,q)$). Hence we proved (B.3) for $(0,0,1)$.

Thus, we complete the proof for off-boundary states. The proof when $n+1<\bar{n}$ and $q=\bar{q}$ uses the same cases as above, but we only need to replace $q+1$ with q, and there is no $h^{1-}(n)$. When $n+1=\bar{n}$, Case 2 is not applicable. In all boundary cases, it can be verified that the arguments still hold. Hence, we have proved (B.3). □

EC.2.2. Proof of Proposition 1

According to the optimality condition (3), it is optimal to solicit feedback at state (n,q) if and only if $\beta\Delta_n V(n,q)-l\ge 0$. Suppose that at some state (n_0,q_0), it is optimal to not solicit feedback, i.e., $\beta\Delta_n V(n_0,q_0)-l<0$. Since $\Delta_n V(n,q)$ decreases in n and $\beta>0$, then for any $n\ge n_0$, we have $\beta\Delta_n V(n,q_0)\le\beta\Delta_n V(n_0,q_0)$, so $\beta\Delta_n V(n,q_0)-l<0$, implying that it is also optimal to not solicit feedback at (n,q_0). Therefore, for each fixed q, the set of states where it is optimal to solicit feedback is of the form $\{n\le\tilde{n}(q)\}$ for some threshold $\tilde{n}(q)$. Equivalently, it is optimal to solicit feedback at (n,q) if and only if $n\le\tilde{n}(q)$. To show that $\tilde{n}(q)$ is decreasing in q, suppose $q_1<q_2$ and fix any n. If it is optimal not to solicit at (n,q_1), then $\beta\Delta_n V(n,q_1)-l<0$. Since $\Delta_n V(n,q)$ decreases in q, we have $\Delta_n V(n,q_2)\le\Delta_n V(n,q_1)$, so $\beta\Delta_n V(n,q_2)-l<0$, which implies it is also optimal not to solicit at (n,q_2). Hence, $\tilde{n}(q)$ is decreasing in q. We also characterize when the threshold attains boundary values. Since $\Delta_n V(n,q)$ decreases in n, the following condition fully determines the degenerate cases: if $\beta\Delta_n V(0,q)-l<0$, then $\beta\Delta_n V(n,q)-l<0$ for all n, and it is never optimal to solicit feedback at quality level q. In this case, we define $\tilde{n}(q)=-1$. □

EC.2.3. Proof of Proposition 2

Proof of Conclusion (i): Conclusion (i) is established through the following lemma.

Lemma EC.3. *Consider two instances of the baseline model that satisfy Assumptions 2 and 3. If the instances are identical except that $h^{1-}(n)\le h^{1'-}(n)$ for all $n\le\bar{n}-1$, then $\Delta_n V'(n,q)\ge\Delta_n V(n,q)$ holds for all $n\le\bar{n}-1$ and q.*

Proof of Lemma EC.3. Consider two instances that are identical except that $h^{1-}(n)\le h^{1'-}(n)$ for all $n\le\bar{n}-1$. We perform value iteration as in the proof of Lemma 1 for both instances simultaneously, using the prime symbol ($'$) to denote the second instance. Let $V_k(\cdot,\cdot)$ and $V'_k(\cdot,\cdot)$ denote the value functions after k rounds of iteration, and

$$V_{k+1}(n,q)=\max_{a\in\{0,1\}}Q^a_{k+1}(n,q),\ V'_{k+1}(n,q)=\max_{a\in\{0,1\}}Q'^a_{k+1}(n,q).$$

As before, $\Delta_n V_k(n,q):=V_k(n+1,q)-V_k(n,q)$ and $\Delta_n V'_k(n,q):=V'_k(n+1,q)-V'_k(n,q)$.

We prove $\Delta_n V'(n,q)\ge\Delta_n V(n,q)$ by induction. Since we know from Lemma EC.1 that, as $k\to+\infty$, $\Delta_n V_k(n,q)$ converges to $\Delta_n V(n,q)$ and $\Delta_n V'_k(n,q)$ converges to $\Delta_n V'(n,q)$, then if we can prove that $\Delta_n V'_k(n,q)\ge\Delta_n V_k(n,q)$ holds for all k, then this proves that $\Delta_n V'(n,q)\ge\Delta_n V(n,q)$. Note that based on the proofs of Lemma 1, the properties in Lemma 1 also hold for V_k and V'_k.

We start with $k=0$. Let $V_0(n,q)=0$ and $V'_0(n,q)=0$, then $\Delta_n V'_0(n,q)\ge\Delta_n V_0(n,q)$ trivially holds. Assume $\Delta_n V'_k(n,q)\ge\Delta_n V_k(n,q)$ holds for iteration k, then the induction proof is complete if we can show $\Delta_n V'_{k+1}(n,q)\ge\Delta_n V_{k+1}(n,q)$. To prove this, it is sufficient to show that

$$Q^{a_2}_{k+1}(n+1,q)-Q^{a_1}_{k+1}(n,q)\le Q'^{a_4}_{k+1}(n+1,q)-Q'^{a_3}_{k+1}(n,q) \tag{B.5}$$

holds for any actions (a_1,a_2,a_3,a_4). With a slight abuse of notation, we redefine a_1,a_2,a_3,a_4 (distinct from those in the proof of Lemma 1) such that a_1 and a_2 are optimal for states (n,q) and $(n+1,q)$ in instance 1, and a_3 and a_4 are optimal for states (n,q) and $(n+1,q)$ in instance 2, respectively. According to the proofs of Lemma 1, both $\Delta_n V_k(n,q)$ and $\Delta_n V_k'(n,q)$ decrease in n and q. Then similarly, we could rule out some combinations of (a_1,a_2,a_3,a_4) and only need to check that (B.5) holds when the optimal actions (a_1,a_2,a_3,a_4) are $(0,0,0,0),(0,0,1,0),(0,0,1,1),(1,0,1,0),(1,0,1,1)$ or $(1,1,1,1)$.

We first check all the off-boundary states.

(a) Case 1: $\left[Q'^0_{k+1}(n+1,q)-Q'^0_{k+1}(n,q)\right]-\left[Q^0_{k+1}(n+1,q)-Q^0_{k+1}(n,q)\right]=\rho\sigma_u\left[\Delta_n V_k'(n,q+1)-\Delta_n V_k(n,q+1)\right]+\rho\sigma_c\left[\Delta_n V_k'(n,q)-\Delta_n V_k(n,q)\right]+h^{1'-}(n)-h^{1-}(n)\geq 0$, where the inequality is because of the induction hypothesis that $\Delta_n V_k'(n,q)\geq\Delta_n V_k(n,q)$ and the condition that $h^{1'-}(n)\geq h^{1-}(n)$. Hence, (B.5) holds for $(0,0,0,0)$.

(b) Case 2: $\left[Q'^0_{k+1}(n+1,q)-Q'^1_{k+1}(n,q)\right]-\left[Q^0_{k+1}(n+1,q)-Q^0_{k+1}(n,q)\right]=\rho\sigma_u\left[\Delta_n V_k'(n,q+1)-\Delta_n V_k(n,q+1)\right]+\rho\sigma_c\left[l-\beta\Delta_n V_k(n,q)\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k'(n,q)-\Delta_n V_k(n,q)\right]+h^{1'-}(n)-h^{1-}(n)\geq\rho\sigma_c\left[l-\beta\Delta_n V_k(n,q)\right]\geq 0$, where the first inequality is because of the induction hypothesis and the condition that $h^{1'-}(n)\geq h^{1-}(n)$, and the second inequality is based on Lemma EC.2 part 1 (here we have $V_{k+1}(n,q)=Q^0_{k+1}(n,q)$). Hence, (B.5) holds for $(0,0,1,0)$.

(c) Case 3: $\left[Q'^1_{k+1}(n+1,q)-Q'^1_{k+1}(n,q)\right]-\left[Q^0_{k+1}(n+1,q)-Q^0_{k+1}(n,q)\right]=\rho\sigma_u\left[\Delta_n V_k'(n,q+1)-\Delta_n V_k(n,q+1)\right]+h^{1'-}(n)-h^{1-}(n)+\rho\sigma_c\beta\left[\Delta_n V_k'(n+1,q)-\Delta_n V_k(n,q)\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k'(n,q)-\Delta_n V_k(n,q)\right]\geq\rho\sigma_c\beta\left[\Delta_n V_k'(n+1,q)-\Delta_n V_k(n,q)\right]\geq\rho\sigma_c\left[l-l\right]=0$, where the first inequality is because of the induction hypothesis and $h^{1'-}(n)\geq h^{1-}(n)$, and the second inequality is based on Lemma EC.2 part 1 (here we have $V_{k+1}(n,q)=Q^0_{k+1}(n,q)$ and $V'_{k+1}(n+1,q)=Q'^1_{k+1}(n+1,q)$). Hence, (B.5) holds for $(0,0,1,1)$.

(d) Case 4: $\left[Q'^0_{k+1}(n+1,q)-Q'^1_{k+1}(n,q)\right]-\left[Q^0_{k+1}(n+1,q)-Q^1_{k+1}(n,q)\right]=\rho\sigma_u\left[\Delta_n V_k'(n,q+1)-\Delta_n V_k(n,q+1)\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k'(n,q)-\Delta_n V_k(n,q)\right]+h^{1'-}(n)-h^{1-}(n)\geq 0$, where the inequality is because of the induction hypothesis and $h^{1'-}(n)\geq h^{1-}(n)$. Hence, (B.5) holds for $(1,0,1,0)$.

(e) Case 5: $\left[Q'^1_{k+1}(n+1,q)-Q'^1_{k+1}(n,q)\right]-\left[Q^0_{k+1}(n+1,q)-Q^1_{k+1}(n,q)\right]=\rho\sigma_u\left[\Delta_n V_k'(n,q+1)-\Delta_n V_k(n,q+1)\right]+\rho\sigma_c\left[\beta\Delta_n V_k'(n+1,q)-l\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k'(n,q)-\Delta_n V_k(n,q)\right]+h^{1'-}(n)-h^{1-}(n)\geq\rho\sigma_c\left[\beta\Delta_n V_k'(n+1,q)-l\right]\geq 0$, where the first inequality is because of the induction hypothesis and $h^{1'-}(n)\geq h^{1-}(n)$, and the second inequality is based on Lemma EC.2 part 1 (here we have $V'_{k+1}(n+1,q)=Q'^1_{k+1}(n+1,q)$). Hence, (B.5) holds for $(1,0,1,1)$.

(f) Case 6: $\left[Q'^1_{k+1}(n+1,q)-Q'^1_{k+1}(n,q)\right]-\left[Q^1_{k+1}(n+1,q)-Q^1_{k+1}(n,q)\right]=\rho\sigma_u\left[\Delta_n V_k'(n,q+1)-\Delta_n V_k(n,q+1)\right]+\rho\sigma_c\beta\left[\Delta_n V_k'(n+1,q)-\Delta_n V_k(n+1,q)\right]+\rho\sigma_c(1-\beta)\left[\Delta_n V_k'(n,q)-\Delta_n V_k(n,q)\right]+h^{1'-}(n)-h^{1-}(n)\geq 0$, where the inequality is because of the induction hypothesis and $h^{1'-}(n)\geq h^{1-}(n)$. Hence, (B.5) holds for $(1,1,1,1)$.

Finally, we discuss the states at the boundary, i.e., $q=\bar{q}$ or $n+1=\bar{n}$. The proof when $n+1<\bar{n}$ and $q=\bar{q}$ uses the same cases as above, but we replace $q+1$ with $\bar{q}$ and remove the quality improvement cost. When $n+1=\bar{n}$, according to Lemma EC.2 part 2, the optimal action at $(n+1,q)$ in both instances is to not solicit feedback, which means Cases 3, 5, and 6 will not happen. It is straightforward to verify that the above arguments still hold in all boundary cases. $\square$

Therefore, we have proved $\Delta_n V'(n,q)\geq\Delta_n V(n,q)$, which implies that $\beta\Delta_n V'(n,q)-l\geq\beta\Delta_n V(n,q)-l$. This result indicates that whenever it is optimal to solicit feedback for instance 1, it should also be optimal to solicit feedback for instance 2. Then in this case, $\tilde{n}'(q)<\tilde{n}(q)$ would never happen. Therefore, we have proved that if two instances are identical except that $h^{1'-}(n)\geq h^{1-}(n)$ for all $n\leq\bar{n}-1$, then $\tilde{n}'(q)\geq\tilde{n}(q)$ for all q. $\square$

Proof of Conclusion (ii): The proof parallels that of Conclusion (i) and is therefore omitted for brevity.

Proof of Conclusion (iii): First, we present the following lemma.

Lemma EC.4. *Consider two instances of the baseline model that satisfy Assumptions 2 and 3, which are identical except for $l<l'$. For any policy π under which the firm does not solicit feedback at $(n,q),(n,q+1),\ldots,(n,\bar{q})$, the resulting value functions satisfy $V^{\pi}(n,q)=V'^{\pi}(n,q)$.*

The statement in Lemma EC.4 is straightforward, since differences in feedback solicitation cost have no effect when the firm chooses not to solicit. Now consider two instances of the baseline model that satisfy Assumptions 2 and 3 and are identical except for $l<l'$. Assume it is optimal to not solicit feedback at (n,q) for instance 1. By the policy structure established in Proposition 1, it is also optimal to not solicit feedback at $(n,q),(n,q+1),\ldots,(n,\bar{q})$ in instance 1. Define a policy $\hat{\pi}$ under which the firm does not solicit feedback at $(n,q),(n,q+1),\ldots,(n,\bar{q})$; then $V(n,q)=V^{\hat{\pi}}(n,q)$.

Now assume, for contradiction, that it is optimal to solicit feedback at (n,q) for instance 2. This implies $V'(n,q)\geq V'^{\hat{\pi}}(n,q)=V^{\hat{\pi}}(n,q)=V(n,q)$, where the first equality follows from Lemma EC.4. However, applying the optimal policy of instance 2, denoted π_2, in instance 1 yields $V^{\pi_2}(n,q)>V'^{\pi_2}(n,q)=V'(n,q)$ because $l<l'$ is the only difference between the two instances. Since $V(n,q)$ is the supremum over all feasible policies in instance 1, we have $V(n,q)\geq V^{\pi_2}(n,q)>V'(n,q)$, which contradicts the earlier inequality $V'(n,q)\geq V(n,q)$.

Therefore, if it is optimal to not solicit feedback at (n,q) for instance 1, it cannot be optimal to solicit feedback at (n,q) for instance 2. Equivalently, $\tilde{n}'(q)>\tilde{n}(q)$ can never occur, and thus $\tilde{n}'(q)\leq\tilde{n}(q)$ for all q. $\square$

Proof of Conclusion (iv): Recall that $\rho = \frac{\Lambda}{\alpha+\Lambda}$, $h^1(n) = \frac{1}{\alpha+\Lambda} h_r^1(n)$, and $d(n) = \frac{1}{\alpha+\Lambda} d_r(n)$. Therefore, when $\alpha < \alpha'$, we have $\rho > \rho'$, and we can also derive that $h^{1-}(n) = h^1(n) - h^1(n+1) = \frac{1}{\alpha+\Lambda}[h_r^1(n) - h_r^1(n+1)] > \frac{1}{\alpha'+\Lambda}[h_r^1(n) - h_r^1(n+1)] = h^{1'-}(n)$. The proof then proceeds by applying value iteration under these parameter changes, following the same steps as in the proof of Conclusion (i). We omit the repetitive details. $\square$

Proof of Conclusion (v): To prove conclusion (v), we first present the following lemma:

LEMMA EC.5. *Under Assumptions 2 and 3, consider a value iteration algorithm to solve the Bellman equation* (1), *and let $V_k(n,q)$ denote the value function after k rounds of iteration, with $V_0(n,q) = 0$. Then, for any $k \geq 1$ and any state (n,q) with $n \leq \bar{n} - 1$,*

$$\Delta_n V_k(n,q) \leq \frac{D(n)\left(1-\rho^k\right)}{1-\rho}.$$

Proof of Lemma EC.5. We finish this proof by induction.

1. We first prove that $\Delta_n V_k(n,q) \leq \frac{D(n)\left(1-\rho^k\right)}{1-\rho}$ holds for $k=1$, that is, $\Delta_n V_1(n,q) \leq D(n)$. Given $V_0(n,q) = 0$, there is $\beta \Delta_n V_0(n,q) - l = -l < 0$, then based on Lemma EC.2, $V_1(n,q) = Q_1^0(n,q)$, where $Q_1^0(n,q) = \rho\sigma_c p(q) - h(n,q) - d(n)$ for $q < \bar{q}$, and $Q_1^0(n,q) = \rho\sigma_c p(q) - d(n)$ for $q = \bar{q}$. Therefore, for any $q < \bar{q}$, $\Delta_n V_1(n,q) = Q_1^0(n+1,q) - Q_1^0(n,q) = h(n,q) + d(n) - h(n+1,q) - d(n+1) = h^{1-}(n) + d^-(n) = D(n)$, and for $q = \bar{q}$, $\Delta_n V_1(n,q) = Q_1^0(n+1,q) - Q_1^0(n,q) = d(n) - d(n+1) = d^-(n) < D(n)$. Thus, we have proved $\Delta_n V_1(n,q) \leq D(n)$.
2. Assume $\Delta_n V_k(n,q) \leq \frac{D(n)\left(1-\rho^k\right)}{1-\rho}$ is true for V_k.
3. We next prove that $\Delta_n V_{k+1}(n,q) \leq \frac{D(n)\left(1-\rho^{k+1}\right)}{1-\rho}$. To prove this, it is sufficient to show that

$$Q_{k+1}^{a_2}(n+1,q) - Q_{k+1}^{a_1}(n,q) \leq \frac{D(n)\left(1-\rho^{k+1}\right)}{1-\rho} \tag{B.6}$$

holds for any actions (a_1, a_2), where a_1 and a_2 are optimal for states (n,q) and $(n+1,q)$, respectively. According to the proof of Lemma 1, for this value iteration process, the properties in Lemma 1 are true for $V = V_k$ for any $k > 1$, thus it is impossible for $a_1 = 0$ and $a_2 = 1$ to be optimal for states (n,q) and $(n+1,q)$ simultaneously. Hence, we only need to check that (B.6) is true when the optimal actions (a_1, a_2) are $(0,0)$, $(1,0)$ or $(1,1)$.

First, we check all the off-boundary states, i.e., (n,q) where $n+1 < \bar{n}$ and $q < \bar{q}$. We check all the possible combinations of optimal actions as discussed above.

(a) Case 1: $Q_{k+1}^0(n+1,q) - Q_{k+1}^0(n,q) = \rho\sigma_u \Delta_n V_k(n,q+1) + \rho\sigma_c \Delta_n V_k(n,q) + h^{1-}(n) + d^-(n) \leq \rho(\sigma_u + \sigma_c)\frac{D(n)\left(1-\rho^k\right)}{1-\rho} + D(n) = \rho\frac{D(n)\left(1-\rho^k\right)}{1-\rho} + D(n) = \frac{D(n)\left(1-\rho^{k+1}\right)}{1-\rho}$, where the inequality is because of the induction hypothesis that $\Delta_n V_k(n,q) \leq \frac{D(n)\left(1-\rho^k\right)}{1-\rho}$. Hence, this proves (B.6) for $(0,0)$.

(b) Case 2: $Q_{k+1}^0(n+1,q) - Q_{k+1}^1(n,q) = \rho\sigma_u \Delta_n V_k(n,q+1) + (1-\beta)\rho\sigma_c \Delta_n V_k(n,q) + \rho\sigma_c l + h^{1-}(n) + d^-(n) \leq \rho\sigma_u \Delta_n V_k(n,q+1) + (1-\beta)\rho\sigma_c \Delta_n V_k(n,q) + \rho\sigma_c \beta \Delta_n V_k(n,q) + D(n) \leq \rho\frac{D(n)\left(1-\rho^k\right)}{1-\rho} + D(n) = \frac{D(n)\left(1-\rho^{k+1}\right)}{1-\rho}$, where the first inequality is based on Lemma EC.2 part 1 (here we have $V_{k+1}(n,q) = Q_{k+1}^1(n,q)$), and the second inequality is because of the induction hypothesis. Hence, this proves (B.6) for $(1,0)$.

(c) Case 3: $Q_{k+1}^1(n+1,q) - Q_{k+1}^1(n,q) = \rho\sigma_u \Delta_n V_k(n,q+1) + \rho\sigma_c\left[\beta \Delta_n V_k(n+1,q) + (1-\beta)\Delta_n V_k(n,q)\right] + h^{1-}(n) + d^-(n) \leq \rho\sigma_u \frac{D(n)\left(1-\rho^k\right)}{1-\rho} + \rho\sigma_c\beta\frac{D(n+1)\left(1-\rho^k\right)}{1-\rho} + \rho\sigma_c(1-\beta)\frac{D(n)\left(1-\rho^k\right)}{1-\rho} + D(n) \leq \rho\sigma_u\frac{D(n)\left(1-\rho^k\right)}{1-\rho} + \rho\sigma_c\beta\frac{D(n)\left(1-\rho^k\right)}{1-\rho} + \rho\sigma_c(1-\beta)\frac{D(n)\left(1-\rho^k\right)}{1-\rho} + D(n) = \frac{D(n)\left(1-\rho^{k+1}\right)}{1-\rho}$, where the first inequality follows from the induction hypothesis, and the second follows from the properties that $D(n)$ is a decreasing function (based on Assumptions 2 and 3). Hence, this proves (B.6) for $(1,1)$.

Thus, we complete the proof for off-boundary states, i.e., $n+1 < \bar{n}$ and $q < \bar{q}$. The proof when $n+1 < \bar{n}$ and $q = \bar{q}$ uses the same cases as above, but we only need to replace $q+1$ with q, and there is no $h^{1-}(n)$. When $n+1 = \bar{n}$, based on Lemma EC.2, $V_{k+1}(n+1,q) = Q_{k+1}^0(n+1,q)$, then only Cases 1 and 2 apply. It can be verified that the proofs still hold for boundary states.

Therefore, we have proved that $\Delta_n V_k(n,q) \leq \frac{D(n)\left(1-\rho^k\right)}{1-\rho}$ holds for $\forall k \geq 1$. $\square$

Based on Lemma EC.5, we obtain the following corollary:

COROLLARY EC.1. *Under the conditions of Lemma EC.5, for any $k \geq 1$ and any state (n,q) with $n \leq \bar{n} - 1$, $\Delta_n V_k(n,q) \leq \frac{D(n)}{1-\rho}$.*

This follows directly from Lemma EC.5 given that $1 - \rho^k \leq 1$ for all $k \geq 1$, which yields a k-independent upper bound.

Next, we present the key lemma to derive conclusion (v) in Proposition 2.

LEMMA EC.6. *Consider two instances of the baseline model that satisfy Assumptions 2 and 3. If the instances are identical except that $\beta < \beta'$, and $d^-(n) + \rho\frac{\lambda_c}{\Lambda}\left[l - \beta\frac{D(n)}{1-\rho}\right] \geq 0$ for all $n \leq \bar{n} - 1$, then $\beta' \Delta_n V'(n,q) \geq \beta \Delta_n V(n,q)$ holds for all $n \leq \bar{n} - 1$ and q.*

Proof of Lemma EC.6. The proof follows the same parallel value iteration analysis structure as the proof of Lemma EC.3 and therefore we omit repeated arguments.

We start with $k=0$. Let $V_0(n,q)=0$ and $V_0'(n,q)=0$, then $\beta'\Delta_n V_0'(n,q)\geq\beta\Delta_n V_0(n,q)$ trivially holds. Assume $\beta'\Delta_n V_k'(n,q)\geq\beta\Delta_n V_k(n,q)$ holds for iteration k, then the induction proof is complete if we can show $\beta'\Delta_n V_{k+1}'(n,q)\geq\beta\Delta_n V_{k+1}(n,q)$. To prove this, it is sufficient to show that

$$\beta\left[Q_{k+1}^{a_2}(n+1,q)-Q_{k+1}^{a_1}(n,q)\right]\leq\beta'\left[Q_{k+1}'^{a_4}(n+1,q)-Q_{k+1}'^{a_3}(n,q)\right] \tag{B.7}$$

holds for any actions (a_1,a_2,a_3,a_4), where a_1 and a_2 are optimal for states (n,q) and $(n+1,q)$ in instance 1, and a_3 and a_4 are optimal for states (n,q) and $(n+1,q)$ in instance 2, respectively. Similarly, we could rule out some combinations and only need to check that (B.7) holds when the optimal actions (a_1,a_2,a_3,a_4) are $(0,0,0,0),(0,0,1,0),(0,0,1,1),(1,0,1,0),(1,0,1,1)$ or $(1,1,1,1)$.

First, we check all the off-boundary states, i.e., (n,q) where $n+1<\bar{n}$ and $q<\bar{q}$. We check all the possible combinations of optimal actions as discussed above.

(a) Case 1: $\beta'\left[Q_{k+1}'^{0}(n+1,q)-Q_{k+1}'^{0}(n,q)\right]-\beta\left[Q_{k+1}^{0}(n+1,q)-Q_{k+1}^{0}(n,q)\right]=\rho\sigma_u[\beta'\Delta_n V_k'(n,q+1)-\beta\Delta_n V_k(n,q+1)]+\rho\sigma_c\left[\beta'\Delta_n V_k'(n,q)-\beta\Delta_n V_k(n,q)\right]+(\beta'-\beta)D(n)\geq 0$.

(b) Case 2: $\beta'\left[Q_{k+1}'^{0}(n+1,q)-Q_{k+1}'^{1}(n,q)\right]-\beta\left[Q_{k+1}^{0}(n+1,q)-Q_{k+1}^{0}(n,q)\right]=\rho\sigma_u[\beta'\Delta_n V_k'(n,q+1)-\beta\Delta_n V_k(n,q+1)]+\rho\sigma_c[\beta' l-\beta\Delta_n V_k(n,q)]+\rho\sigma_c(1-\beta')\beta'\Delta_n V_k'(n,q)+(\beta'-\beta)D(n)\geq\rho\sigma_c\left[\beta' l-\beta\Delta_n V_k(n,q)+(1-\beta')l\right]=\rho\sigma_c\left[l-\beta\Delta_n V_k(n,q)\right]\geq 0$.

(c) Case 3: $\beta'\left[Q_{k+1}'^{1}(n+1,q)-Q_{k+1}'^{1}(n,q)\right]-\beta\left[Q_{k+1}^{0}(n+1,q)-Q_{k+1}^{0}(n,q)\right]=\rho\sigma_u[\beta'\Delta_n V_k'(n,q+1)-\beta\Delta_n V_k(n,q+1)]+\rho\sigma_c\beta'[\beta'\Delta_n V_k'(n+1,q)+(1-\beta')\Delta_n V_k'(n,q)]-\rho\sigma_c\beta\Delta_n V_k(n,q)+(\beta'-\beta)D(n)\geq\rho\sigma_c[\beta' l+(1-\beta')l-\beta\Delta_n V_k(n,q)]\geq 0$.

(d) Case 4: $\beta'\left[Q_{k+1}'^{0}(n+1,q)-Q_{k+1}'^{1}(n,q)\right]-\beta\left[Q_{k+1}^{0}(n+1,q)-Q_{k+1}^{1}(n,q)\right]=\rho\sigma_u[\beta'\Delta_n V_k'(n,q+1)-\beta\Delta_n V_k(n,q+1)]+(\beta'-\beta)D(n)+\rho\sigma_c\left[\beta'(1-\beta')\Delta_n V_k'(n,q)-\beta(1-\beta)\Delta_n V_k(n,q)+(\beta'-\beta)l\right]\geq\rho\sigma_c[\beta(1-\beta')\Delta_n V_k(n,q)-\beta(1-\beta)\Delta_n V_k(n,q)+(\beta'-\beta)l]+(\beta'-\beta)D(n)=\rho\sigma_c(\beta'-\beta)\left[l-\beta\Delta_n V_k(n,q)\right]+(\beta'-\beta)D(n)\geq(\beta'-\beta)\left[\rho\sigma_c\left(l-\beta\frac{D(n)}{1-\rho}\right)+D(n)\right]\geq 0$, where the first inequality is due to the induction hypothesis, the second inequality is because of Corollary EC.1, and the last inequality is because of $\rho\sigma_c\left(l-\beta\frac{D(n)}{1-\rho}\right)+D(n)\geq\rho\sigma_c\left(l-\beta\frac{D(n)}{1-\rho}\right)+d^-(n)\geq 0$.

(e) Case 5: $\beta'\left[Q_{k+1}'^{1}(n+1,q)-Q_{k+1}'^{1}(n,q)\right]-\beta\left[Q_{k+1}^{0}(n+1,q)-Q_{k+1}^{1}(n,q)\right]=\rho\sigma_u[\beta'\Delta_n V_k'(n,q+1)-\beta\Delta_n V_k(n,q+1)]+\rho\sigma_c\beta'[\beta'\Delta_n V_k'(n+1,q)+(1-\beta')\Delta_n V_k'(n,q)]-\rho\sigma_c\beta[(1-\beta)\Delta_n V_k(n,q)+l]+(\beta'-\beta)D(n)\geq\rho\sigma_c\beta'[l+(1-\beta')\Delta_n V_k'(n,q)]-\rho\sigma_c\beta\left[(1-\beta)\Delta_n V_k(n,q)+l\right]+(\beta'-\beta)D(n)\geq\rho\sigma_c[\beta(1-\beta')\Delta_n V_k(n,q)-\beta(1-\beta)\Delta_n V_k(n,q)+(\beta'-\beta)l]+(\beta'-\beta)D(n)\geq 0$, where the last inequality is based on the same reasoning in case 4.

(f) Case 6: $\beta'\left[Q_{k+1}'^{1}(n+1,q)-Q_{k+1}'^{1}(n,q)\right]-\beta\left[Q_{k+1}^{1}(n+1,q)-Q_{k+1}^{1}(n,q)\right]=\rho\sigma_u[\beta'\Delta_n V_k'(n,q+1)-\beta\Delta_n V_k(n,q+1)]+\rho\sigma_c\beta'\left[\beta'\Delta_n V_k'(n+1,q)+(1-\beta')\Delta_n V_k'(n,q)\right]-\rho\sigma_c\beta\left[\beta\Delta_n V_k(n+1,q)+(1-\beta)\Delta_n V_k(n,q)\right]+(\beta'-\beta)D(n)\geq\rho\sigma_c\left[\beta\beta'\Delta_n V_k(n+1,q)-\beta^2\Delta_n V_k(n+1,q)\right]+\rho\sigma_c\left[\beta(1-\beta')\Delta_n V_k(n,q)-\beta(1-\beta)\Delta_n V_k(n,q)\right]+(\beta'-\beta)D(n)\geq\rho\sigma_c(\beta'-\beta)\left[\beta\Delta_n V_k(n+1,q)-\beta\Delta_n V_k(n,q)\right]+(\beta'-\beta)D(n)\geq\rho\sigma_c(\beta'-\beta)\left[l-\beta\frac{D(n)}{1-\rho}\right]+(\beta'-\beta)D(n)\geq 0$.

Finally, we discuss the states at the boundary, i.e., $q=\bar{q}$ or $n+1=\bar{n}$. The proof when $n+1<\bar{n}$ and $q=\bar{q}$ uses the same cases as above, but we replace $q+1$ with $\bar{q}$ and remove the quality improvement cost. Cases 1 - 3 still hold, while in cases 4 - 6, the equation is lower bounded by $(\beta'-\beta)\left[\rho\sigma_c\left(l-\beta\frac{D(n)}{1-\rho}\right)+d^-(n)\right]$ instead, which remains nonnegative under the condition specified in Lemma EC.6. When $n+1=\bar{n}$, cases 3, 5, and 6 will not happen, and it is straightforward to verify that the above arguments still hold. □

Therefore, we have proved $\beta'\Delta_n V'(n,q)\geq\beta\Delta_n V(n,q)$, which implies that $\beta'\Delta_n V'(n,q)-l\geq\beta\Delta_n V(n,q)-l$, i.e., whenever it is optimal to solicit feedback for instance 1, it should also be optimal to solicit feedback for instance 2. Then in this case, $\tilde{n}'(q)<\tilde{n}(q)$ would never happen. Therefore, we have proved that $\tilde{n}'(q)\geq\tilde{n}(q)$ for all q. □

EC.2.4. Proof of Proposition 3

To prove Proposition 3, we first present the following lemma:

Lemma EC.7. *Consider two instances of the baseline model that satisfy Assumptions 2 and 3. Let (β,l) and (β',l') denote the solicitation effectiveness and cost in the two instances, respectively, and suppose the two instances are identical except that $l\leq l'$ and $\beta\leq\beta'$.*

(i) *If $\frac{l'-l}{\beta'-\beta}\leq\min\left\{d^-(n),\ \frac{d^-(n)+\rho\frac{\lambda_c}{\Lambda}\left(l-\beta\frac{D(n)}{1-\rho}\right)}{1-\rho}\right\}$ for all $n\leq\bar{n}-1$, then $V'(n,q)\geq V(n,q)$ for all n and q, and $\beta'\Delta V'(n,q)-l'\geq\beta\Delta V(n,q)-l$ for all $n\leq\bar{n}-1$ and q.*

(ii) *If $\frac{l'-l}{\beta'-\beta}\geq\frac{D(n)}{1-\rho}$ for all $n\leq\bar{n}-1$, then $V'(n,q)\leq V(n,q)$ for all n and q, and $\beta'\Delta V'(n,q)-l'\leq\beta\Delta V(n,q)-l$ for all $n\leq\bar{n}-1$ and q.*

Proof of Lemma EC.7. Proof of Conclusion (i): The proof follows a similar value-iteration and case-analysis structure as that of Lemma EC.3, and we therefore omit the repeated arguments.

We prove property (i) by induction. Let $V_0(n,q)=0$ and $V_0'(n,q)=0$.

1. We first prove that $V_k'(n,q) \ge V_k(n,q)$ and $\beta'\Delta V_k'(n,q) - l' \ge \beta\Delta V_k(n,q) - l$ hold for $k=1$.

According to the proof of Lemma EC.5, starting from $V_0(n,q)=0$, there is $V_1(n,q)=Q_1^0(n,q)$, where $Q_1^0(n,q) = \rho\sigma_c p(q) - h(n,q) - d(n)$ for $q<\bar{q}$, and $Q_1^0(n,q) = \rho\sigma_c p(q) - d(n)$ for $q=\bar{q}$. Since $Q_1^0(n,q)$ is independent of β or l, thus $V_1'(n,q) = V_1(n,q)$, therefore $V_1'(n,q) \ge V_1(n,q)$ trivially holds.

Given $V_1'(n,q)$ and $V_1(n,q)$, we know for any $q<\bar{q}$, $\Delta_n V_1(n,q) = \Delta_n V_1'(n,q) = h^{1-}(n) + d^-(n) = D(n)$, and for $q=\bar{q}$, $\Delta_n V_1(n,q) = \Delta_n V_1'(n,q) = d^-(n)$. Therefore, when $q<\bar{q}$, $\beta'\Delta V_1'(n,q) - l' - \left[\beta\Delta V_1(n,q) - l\right] = (\beta'-\beta)D(n) - (l'-l)$, and when $q=\bar{q}$, $\beta'\Delta V_1'(n,q) - l' - \left[\beta\Delta V_1(n,q) - l\right] = (\beta'-\beta)d^-(n) - (l'-l)$. Under the condition stated in Conclusion (i), we have $\frac{l'-l}{\beta'-\beta} \le d^-(n)$, which implies $(\beta'-\beta)d^-(n) \ge l'-l$. Since $D(n) > d^-(n)$, it follows that $(\beta'-\beta)D(n) \ge l'-l$. Therefore, we conclude that $\beta'\Delta V_1'(n,q) - l' \ge \beta\Delta V_1(n,q) - l$.

2. Assume $V_k'(n,q) \ge V_k(n,q)$ and $\beta'\Delta_n V_k'(n,q) - l' \ge \beta\Delta_n V_k(n,q) - l$ holds for iteration k.

3. We next prove that $V_{k+1}'(n,q) \ge V_{k+1}(n,q)$ and $\beta'\Delta_n V_{k+1}'(n,q) - l' \ge \beta\Delta_n V_{k+1}(n,q) - l$.

To prove $V_{k+1}'(n,q) \ge V_{k+1}(n,q)$, it is sufficient to show that

$$Q_{k+1}^{a_1}(n,q) \le Q_{k+1}'^{a_3}(n,q) \tag{B.8}$$

holds for any optimal actions (a_1,a_3). Due to the induction hypothesis that $\beta'\Delta_n V_k'(n,q) - l' \ge \beta\Delta_n V_k(n,q) - l$, if it is optimal to solicit feedback at (n,q) in instance 1, it must also be optimal to solicit at (n,q) in instance 2, thus we only need to check that (B.8) holds when the optimal actions (a_1,a_3) are $(0,0)$, $(0,1)$, or $(1,1)$.

To prove $\beta'\Delta_n V_{k+1}'(n,q) - l' \ge \beta\Delta_n V_{k+1}(n,q) - l$, it is sufficient to show that

$$\beta\left[Q_{k+1}^{a_2}(n+1,q) - Q_{k+1}^{a_1}(n,q)\right] - l \le \beta'\left[Q_{k+1}'^{a_4}(n+1,q) - Q_{k+1}'^{a_3}(n,q)\right] - l' \tag{B.9}$$

holds for any optimal actions (a_1,a_2,a_3,a_4). Similarly, we could rule out some combinations and only need to check that (B.9) holds when the optimal actions (a_1,a_2,a_3,a_4) are $(0,0,0,0),(0,0,1,0),(0,0,1,1),(1,0,1,0),(1,0,1,1)$ or $(1,1,1,1)$.

I. Proof of (B.8): First, we check all the off-boundary states, i.e., (n,q) where $n<\bar{n}$ and $q<\bar{q}$. We check all the possible combinations of optimal actions as discussed above.

(a) Case 1: $Q_{k+1}'^0(n,q) - Q_{k+1}^0(n,q) = \rho\sigma_u\left[V_k'(n,q+1) - V_k(n,q+1)\right] + \rho\sigma_c\left[V_k'(n,q) - V_k(n,q)\right] \ge 0$.

(b) Case 2: $Q_{k+1}'^1(n,q) - Q_{k+1}^0(n,q) = \rho\sigma_u\left[V_k'(n,q+1) - V_k(n,q+1)\right] + \rho\sigma_c\left[V_k'(n,q) - V_k(n,q)\right] + \rho\sigma_c[\beta'\Delta_n V_k'(n,q) - l'] \ge 0$.

(c) Case 3: $Q_{k+1}'^1(n,q) - Q_{k+1}^1(n,q) = \rho\sigma_u\left[V_k'(n,q+1) - V_k(n,q+1)\right] + \rho\sigma_c\left[V_k'(n,q) - V_k(n,q)\right] + \rho\sigma_c[\beta'\Delta_n V_k'(n,q) - l' - (\beta\Delta_n V_k(n,q) - l)] \ge 0$.

Thus, we complete the proof for off-boundary states. The proof when $n<\bar{n}$ and $q=\bar{q}$ uses the same cases as above, but we only need to replace $q+1$ with q. When $n=\bar{n}$, only case 1 applies. It can be verified that the proofs still hold for boundary states. Hence, we have proved (B.8).

II. Proof of (B.9): First, we check all the off-boundary states, i.e., (n,q) where $n+1<\bar{n}$ and $q<\bar{q}$. We check all the possible combinations of optimal actions as discussed above.

(a) Case 1: $\beta'\left[Q_{k+1}'^0(n+1,q) - Q_{k+1}'^0(n,q)\right] - l' - \beta\left[Q_{k+1}^0(n+1,q) - Q_{k+1}^0(n,q)\right] + l = \rho\sigma_u[\beta'\Delta_n V_k'(n,q+1) - \beta\Delta_n V_k(n,q+1)] + \rho\sigma_c\left[\beta'\Delta_n V_k'(n,q) - \beta\Delta_n V_k(n,q)\right] + (\beta'-\beta)D(n) + l - l' \ge \rho\sigma_u(l'-l) + \rho\sigma_c(l'-l) + (\beta'-\beta)D(n) + l - l' = (\rho-1)(l'-l) + (\beta'-\beta)D(n) \ge (\beta'-\beta)D(n) - (l'-l) \ge 0$.

(b) Case 2: $\beta'\left[Q_{k+1}'^0(n+1,q) - Q_{k+1}'^1(n,q)\right] - l' - \beta\left[Q_{k+1}^0(n+1,q) - Q_{k+1}^0(n,q)\right] + l = \rho\sigma_u[\beta'\Delta_n V_k'(n,q+1) - \beta\Delta_n V_k(n,q+1)] + \rho\sigma_c[\beta' l' - \beta\Delta_n V_k(n,q)] + \rho\sigma_c(1-\beta')\beta'\Delta_n V_k'(n,q) + (\beta'-\beta)D(n) + l - l' \ge \rho\sigma_u(l'-l) + \rho\sigma_c\left[\beta' l' - l + (1-\beta')l'\right] + (\beta'-\beta)D(n) + l - l' = (\rho-1)(l'-l) + (\beta'-\beta)D(n) \ge 0$.

(c) Case 3: $\beta'\left[Q_{k+1}'^1(n+1,q) - Q_{k+1}'^1(n,q)\right] - \beta\left[Q_{k+1}^0(n+1,q) - Q_{k+1}^0(n,q)\right] = \rho\sigma_u[\beta'\Delta_n V_k'(n,q+1) - \beta\Delta_n V_k(n,q+1)] + \rho\sigma_c\beta'\left[\beta'\Delta_n V_k'(n+1,q) + (1-\beta')\Delta_n V_k'(n,q)\right] - \rho\sigma_c\beta\Delta_n V_k(n,q) + (\beta'-\beta)D(n) + l - l' \ge \rho\sigma_u(l'-l) + \rho\sigma_c\left[\beta' l' + (1-\beta')l' - l\right] + (\beta'-\beta)D(n) + l - l' = (\rho-1)(l'-l) + (\beta'-\beta)D(n) \ge 0$.

(d) Case 4: $\beta'\left[Q_{k+1}'^0(n+1,q) - Q_{k+1}'^1(n,q)\right] - l' - \beta\left[Q_{k+1}^0(n+1,q) - Q_{k+1}^1(n,q)\right] + l = \rho\sigma_u[\beta'\Delta_n V_k'(n,q+1) - \beta\Delta_n V_k(n,q+1)] + (\beta'-\beta)D(n) + \rho\sigma_c\left[\beta'(1-\beta')\Delta_n V_k'(n,q) - \beta(1-\beta)\Delta_n V_k(n,q) + \beta' l' - \beta l\right] + l - l' \ge \rho\sigma_u(l'-l) + \rho\sigma_c\left[(1-\beta')(\beta\Delta_n V_k(n,q) + l' - l) - \beta(1-\beta)\Delta_n V_k(n,q) + \beta' l' - \beta l\right] + (\beta'-\beta)D(n) + l - l' = \rho\sigma_u(l'-l) + \rho\sigma_c(l'-l) + \rho\sigma_c l(\beta'-\beta) - \rho\sigma_c(\beta'-\beta)\beta\Delta_n V_k(n,q) + (\beta'-\beta)D(n) + l - l' \ge (\rho-1)(l'-l) + (\beta'-\beta)\left[D(n) + \rho\sigma_c\left(l - \beta\frac{D(n)}{1-\rho}\right)\right] \ge (\rho-1)(l'-l) + (\beta'-\beta)\left[d^-(n) + \rho\sigma_c\left(l - \beta\frac{D(n)}{1-\rho}\right)\right] \ge 0$, where the second inequality is because of Corollary EC.1, and the last inequality follows from the condition stated in Conclusion (i) of Lemma EC.7, namely, $\frac{l'-l}{\beta'-\beta} \le \min\left\{d^-(n),\ \frac{d^-(n)+\rho\sigma_c\left(l-\beta\frac{D(n)}{1-\rho}\right)}{1-\rho}\right\} \le \frac{d^-(n)+\rho\sigma_c\left(l-\beta\frac{D(n)}{1-\rho}\right)}{1-\rho}$.

(e) Case 5: $\beta'\left[Q_{k+1}'^1(n+1,q) - Q_{k+1}'^1(n,q)\right] - l' - \beta\left[Q_{k+1}^0(n+1,q) - Q_{k+1}^1(n,q)\right] + l = \rho\sigma_u[\beta'\Delta_n V_k'(n,q+1) - \beta\Delta_n V_k(n,q+1)] + \rho\sigma_c\beta'\left[\beta'\Delta_n V_k'(n+1,q) + (1-\beta')\Delta_n V_k'(n,q)\right] - \rho\sigma_c\beta\left[(1-\beta)\Delta_n V_k(n,q) + l\right] + (\beta'-\beta)D(n) + l - l' \ge (\rho\sigma_u - 1)(l'-l) + \rho\sigma_c\beta'\left[l' + (1-\beta')\Delta_n V_k'(n,q)\right] - \rho\sigma_c\beta\left[(1-\beta)\Delta_n V_k(n,q) + l\right] + (\beta'-\beta)D(n) \ge (\rho\sigma_u -$

$1)(l'-l)+\rho\sigma_c\left[(1-\beta')(\beta\Delta_n V_k(n,q)+l'-l)-\beta(1-\beta)\Delta_n V_k(n,q)+\beta' l'-\beta l\right]+(\beta'-\beta)D(n)\geq(\rho-1)(l'-l)+(\beta'-\beta)\left[D(n)+\rho\sigma_c\big(l-\beta\frac{D(n)}{1-\rho}\big)\right]\geq 0.$

(f) Case 6: $\beta'\left[Q'^1_{k+1}(n+1,q)-Q'^1_{k+1}(n,q)\right]-l'-\beta[Q^1_{k+1}(n+1,q)-Q^1_{k+1}(n,q)]+l=\rho\sigma_u[\beta'\Delta_n V'_k(n,q+1)-\beta\Delta_n V_k(n,q+1)]+\rho\sigma_c\beta'[\beta'\Delta_n V'_k(n+1,q)+(1-\beta')\Delta_n V'_k(n,q)]-\rho\sigma_c\beta[\beta\Delta_n V_k(n+1,q)+(1-\beta)\Delta_n V_k(n,q)]+(\beta'-\beta)D(n)+l-l'\geq\rho\sigma_u(l'-l)+\rho\sigma_c[\beta'(\beta\Delta_n V_k(n+1,q)+l'-l)-\beta^2\Delta_n V_k(n+1,q)]+\rho\sigma_c[(1-\beta')(\beta\Delta_n V_k(n,q)+l'-l)-\beta(1-\beta)\Delta_n V_k(n,q)]+(\beta'-\beta)D(n)+l-l'\geq(\rho-1)(l'-l)+\rho\sigma_c(\beta'-\beta)[\beta\Delta_n V_k(n+1,q)-\beta\Delta_n V_k(n,q)]+(\beta'-\beta)D(n)\geq(\rho-1)(l'-l)+\rho\sigma_c(\beta'-\beta)\left[l-\beta\frac{D(n)}{1-\rho}\right]+(\beta'-\beta)D(n)\geq 0.$

Finally, we discuss the states at the boundary, i.e., $q=\bar{q}$ or $n+1=\bar{n}$. The proof when $n+1<\bar{n}$ and $q=\bar{q}$ uses the same cases as above, but we replace $q+1$ with $\bar{q}$ and remove the quality improvement cost. In cases 1 - 3, the equation is instead lower bounded by $(\rho-1)(l'-l)+(\beta'-\beta)d^-(n)$, while in cases 4 - 6, the equation is lower bounded by $(\rho-1)(l'-l)+(\beta'-\beta)\left[d^-(n)+\rho\sigma_c\big(l-\beta\frac{D(n)}{1-\rho}\big)\right]$. Both lower bounds remain nonnegative under the condition stated in Conclusion (i). When $n+1=\bar{n}$, cases 3, 5, and 6 will not happen, and it is straightforward to verify that the above arguments still hold.

Hence, we have proved Conclusion (i) of Lemma EC.7.

Proof of Conclusion (ii): Similarly, we also prove property (ii) by induction. Let $V_0(n,q)=0$ and $V'_0(n,q)=0$.

1. We first prove that $V'_k(n,q)\leq V_k(n,q)$ and $\beta'\Delta V'_k(n,q)-l'\leq\beta\Delta V_k(n,q)-l$ hold for $k=1$.

As in the proof of Conclusion (i), for any $q<\bar{q}$, we have $\Delta_n V_1(n,q)=\Delta_n V'_1(n,q)=h^{1-}(n)+d^-(n)=D(n)$, and for $q=\bar{q}$, $\Delta_n V_1(n,q)=\Delta_n V'_1(n,q)=d^-(n)$. Therefore, when $q<\bar{q}$, $\beta'\Delta V'_1(n,q)-l'-\left[\beta\Delta V_1(n,q)-l\right]=(\beta'-\beta)D(n)-(l'-l)$, and when $q=\bar{q}$, $\beta'\Delta V'_1(n,q)-l'-\left[\beta\Delta V_1(n,q)-l\right]=(\beta'-\beta)d^-(n)-(l'-l)$.

Under the condition stated in Conclusion (ii), we have $\frac{l'-l}{\beta'-\beta}\geq\frac{D(n)}{1-\rho}\geq D(n)\geq d^-(n)$, which implies that $(\beta'-\beta)D(n)\leq l'-l$ and $(\beta'-\beta)d^-(n)\leq l'-l$. Therefore, we conclude that $\beta'\Delta V'_1(n,q)-l'\leq\beta\Delta V_1(n,q)-l$.

2. Assume $V'_k(n,q)\leq V_k(n,q)$ and $\beta'\Delta_n V'_k(n,q)-l'\leq\beta\Delta_n V_k(n,q)-l$ holds for iteration k.

3. We next prove $V'_{k+1}(n,q)\leq V_{k+1}(n,q)$ and $\beta'\Delta_n V'_{k+1}(n,q)-l'\leq\beta\Delta_n V_{k+1}(n,q)-l$.

To prove $V'_{k+1}(n,q)\leq V_{k+1}(n,q)$, it is sufficient to show that

$$Q^{a_1}_{k+1}(n,q)\geq Q'^{a_3}_{k+1}(n,q) \tag{B.10}$$

holds for any optimal actions (a_1,a_3). Due to the induction hypothesis that $\beta'\Delta_n V'_k(n,q)-l'\leq\beta\Delta_n V_k(n,q)-l$, if it is optimal to solicit feedback at (n,q) in instance 2, it must also be optimal to solicit at (n,q) in instance 1, so we only need to check that (B.8) holds when (a_1,a_3) are $(0,0)$, $(1,0)$, or $(1,1)$. The proof of (B.10) is similar to that of (B.8). We therefore omit the detailed case-by-case verification of all possible optimal action combinations.

To prove $\beta'\Delta_n V'_{k+1}(n,q)-l'\leq\beta\Delta_n V_{k+1}(n,q)-l$, it is sufficient to show that

$$\beta\left[Q^{a_2}_{k+1}(n+1,q)-Q^{a_1}_{k+1}(n,q)\right]-l\geq\beta'\left[Q'^{a_4}_{k+1}(n+1,q)-Q'^{a_3}_{k+1}(n,q)\right]-l' \tag{B.11}$$

holds for any optimal actions (a_1,a_2,a_3,a_4). Similarly, we could rule out some combinations and only need to check that (B.11) holds when the optimal actions (a_1,a_2,a_3,a_4) are $(0,0,0,0),(1,0,0,0),(1,1,0,0),(1,0,1,0),(1,1,1,0)$ or $(1,1,1,1)$.

The proof of (B.11) is similar to that of (B.9), so we also omit the detailed case-by-case verification of all optimal action combinations. In particular, such verification shows that for all off-boundary states, the difference $\beta\left[Q^{a_2}_{k+1}(n+1,q)-Q^{a_1}_{k+1}(n,q)\right]-l-\Big(\beta'\left[Q'^{a_4}_{k+1}(n+1,q)-Q'^{a_3}_{k+1}(n,q)\right]-l'\Big)$ is lower bounded by $(1-\rho)(l'-l)+(\beta-\beta')D(n)$, which is nonnegative under the condition stated in Conclusion (ii). For boundary states, the difference is instead lower bounded by $(1-\rho)(l'-l)+(\beta-\beta')d^-(n)$, which is likewise nonnegative. Therefore, we have proved Conclusion (ii) of Lemma EC.7. □

Based on Conclusion (i) of Lemma EC.7, if $\frac{l'-l}{\beta'-\beta}\leq\min\left\{d^-(n),\ \frac{d^-(n)+\rho\frac{\lambda_c}{\Lambda}\left(l-\beta\frac{D(n)}{1-\rho}\right)}{1-\rho}\right\}$ for all $n\leq\bar{n}-1$, then $V'(n,q)\geq V(n,q)$ for all n and q and $\beta'\Delta V'(n,q)-l'\geq\beta\Delta V(n,q)-l$ for all $n\leq\bar{n}-1$ and q. Hence, whenever it is optimal to solicit feedback under instance 1, it must also be optimal under instance 2, so $\tilde{n}'(q)<\tilde{n}(q)$ cannot occur. Therefore, $\tilde{n}'(q)\geq\tilde{n}(q)$ for all q, establishing Conclusion (i) of Proposition 3. Similarly, by Conclusion (ii) of Lemma EC.7, if $\frac{l'-l}{\beta'-\beta}\geq\frac{D(n)}{1-\rho}$ for all $n\leq\bar{n}-1$, then $V'(n,q)\leq V(n,q)$ and $\beta'\Delta V'(n,q)-l'\leq\beta\Delta V(n,q)-l$ for all relevant states. Likewise, $\tilde{n}'(q)>\tilde{n}(q)$ cannot occur, implying $\tilde{n}'(q)\leq\tilde{n}(q)$ for all q, which proves Conclusion (ii). □

EC.2.5. Proof of Proposition 4

To prove Proposition 4, we first present the following lemma.

Lemma EC.8. *Under Assumptions 1 and 2, in the joint model, assuming it is optimal to solicit feedback at all states (n,q) with $n<\bar{n}$, then:*

(i) $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)$ *decreases in q;*

(ii) $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q) - h(n,q)$ *increases in* n.

Proof of Lemma EC.8. In this case, the Bellman equation for states when $n < \bar{n}$ and $q < \bar{q}$ can be written as:

$$J(n,q) = \max\Big\{\rho\sigma_u J(n,q+1) - h(n,q),\ \rho\sigma_u J(n,q)\Big\} - d(n) + \rho\sigma_c p(q) + \rho\sigma_c\big[\beta J(n+1,q) + (1-\beta)J(n,q) - l\big], \tag{B.12}$$

where again $\sigma_u := \lambda_u/\Lambda$ and $\sigma_c := \lambda_c/\Lambda$. When $q = \bar{q}$, then $q+1$ is replaced with $\bar{q}$. When $n = \bar{n}$, then the term $\rho\sigma_c\big[\beta J(n+1,q) + (1-\beta)J(n,q) - l\big]$ is replaced with $\rho\sigma_c J(n,q)$. For this proof only, we use a restricted form of $J(n,q)$ that optimizes only the quality improvement decision, while later proofs will involve the full joint optimization.

As in the proof of Lemma 1, consider a value iteration algorithm to solve the Bellman equation (B.12). Let $J_k(\cdot,\cdot)$ denote the value function after k rounds of iteration. Specifically, we have the following update rule for J_{k+1} based on the value function of the previous round J_k:

$$J_{k+1}(n,q) = \max_{a_u \in \{0,1\}} G_{k+1}^{a_u}(n,q),$$

where $G_{k+1}^{a_u}(n,q)$ is the value of taking action a_u in the current state (n,q) and taking optimal actions in all future states, as approximated by J_k. Here $a_u = 1$ ($a_u = 0$) represents improving (not improving) quality. Specifically, for any $n < \bar{n}$ and $q < \bar{q}$:

$$G_{k+1}^0(n,q) := \rho\sigma_u J_k(n,q) + \rho\sigma_c p(q) + \rho\sigma_c\big[\beta J_k(n+1,q) + (1-\beta)J_k(n,q) - l\big] - d(n),$$

and

$$G_{k+1}^1(n,q) := \rho\sigma_u J_k(n,q+1) - h(n,q) + \rho\sigma_c p(q) + \rho\sigma_c\big[\beta J_k(n+1,q) + (1-\beta)J_k(n,q) - l\big] - d(n).$$

When $q = \bar{q}$, we only need to replace $q+1$ with $\bar{q}$ in the above equations; and when $n = \bar{n}$, the term $\rho\sigma_c\big[\beta J_k(n+1,q) + (1-\beta)J_k(n,q) - l\big]$ is replaced with $\rho\sigma_c J_k(n,q)$.

We will prove Lemma EC.8 by induction. We start with $k = 0$, where we set $J_0(n,q) = 0$ for all the states (n,q). Note that (i)–(ii) in Lemma EC.8 are true for $J = J_0$. Thus, by Lemma EC.1, if we can show that properties (i)–(ii) are preserved when $J = J_k$ for any $k \geq 1$, then this proves Lemma EC.8. Assume that, at iteration k, properties (i)–(ii) are true for J_k. The induction proof is complete if we can show that the properties are also true for J_{k+1}.

Before proceeding with the proof, we first establish some useful conditions.

Lemma EC.9. *The following are the conditions for the optimal quality improvement actions when it is optimal to solicit feedback at all states* (n,q) *with* $n < \bar{n}$*:*

1. *Suppose* $q < \bar{q}$*. Then* $J_{k+1}(n,q) = G_{k+1}^1(n,q)$ *if and only if* $\rho\sigma_u\Delta_q J_k(n,q) \geq h(n,q)$*. Otherwise,* $J_{k+1}(n,q) = G_{k+1}^0(n,q)$*.*
2. *Suppose* $q = \bar{q}$*. Then* $J_{k+1}(n,q) = G_{k+1}^0(n,q)$*.*

Proof of Lemma EC.9. When $q < \bar{q}$, $J_{k+1}(n,q) = G_{k+1}^1(n,q)$ if and only if inequality (6) holds, which proves part 1. When $q = \bar{q}$, $\Delta_q J_k(n,\bar{q}) = 0$, with the fact that $h(n,q) > 0$, we have $\rho\sigma_u\Delta_q J_k(n,\bar{q}) < h(n,q)$, i.e., it is optimal to not invest in quality improvement, which proves part 2. $\square$

Now we are ready to proceed with the proof of Lemma EC.8. Here we present the detailed proof of conclusion (i). The proof of conclusion (ii) is similar. To show that $\rho\sigma_u\Delta_q J(n,q) - h(n,q)$ decreases in q, it is sufficient to show that for any $q < \bar{q}$,

$$G_{k+1}^{a_{u_3}}(n,q+1) - G_{k+1}^{a_{u_2}}(n,q) - \big[G_{k+1}^{a_{u_2}}(n,q) - G_{k+1}^{a_{u_1}}(n,q-1)\big] \leq \frac{h(n,q)}{\rho\sigma_u} - \frac{h(n,q-1)}{\rho\sigma_u} \tag{B.13}$$

holds for all combinations of $(a_{u_1}, a_{u_2}, a_{u_3})$, where a_{u_1}, a_{u_2} and a_{u_3} are the optimal actions for states $(n,q-1)$, (n,q) and $(n,q+1)$ respectively. However, the induction hypothesis can easily show that some combinations cannot be optimal actions simultaneously. We only need to check that (B.13) holds when the optimal actions $(a_{u_1}, a_{u_2}, a_{u_3})$ are $(0,0,0)$, $(1,1,1)$, $(1,1,0)$ or $(1,0,0)$.

Recall that $p^+(q) := p(q+1) - p(q)$. According to Assumption 1, $p(q)$ is an increasing and concave function, hence $p^+(q) \geq 0$ and $p^+(q) - p^+(q-1) \leq 0$. Another group of properties we will use later is that, $h(n,q) - h(n,q-1) = h^2(q) - h^2(q-1) \geq 0$, and $h(n,q) - h(n,q-1) = h^2(q) - h^2(q-1) = h(n+1,q) - h(n+1,q-1)$.

(a) Case 1: $\big[G_{k+1}^0(n,q+1) - G_{k+1}^0(n,q)\big] - \big[G_{k+1}^0(n,q) - G_{k+1}^0(n,q-1)\big] = \rho\sigma_u[\Delta_q J_k(n,q) - \Delta_q J_k(n,q-1)] + \rho\sigma_c[p^+(q) - p^+(q-1)] + \rho\sigma_c\beta[\Delta_q J_k(n+1,q) - \Delta_q J_k(n+1,q-1)] + \rho\sigma_c(1-\beta)[\Delta_q J_k(n,q) - \Delta_q J_k(n,q-1)] \leq \rho\sigma_u\big[\frac{h(n,q)}{\rho\sigma_u} - \frac{h(n,q-1)}{\rho\sigma_u}\big] + \rho\sigma_c\big[\frac{h(n,q)}{\rho\sigma_u} - \frac{h(n,q-1)}{\rho\sigma_u}\big] = \frac{1}{\sigma_u}\big[h(n,q) - h(n,q-1)\big] \leq \frac{1}{\rho\sigma_u}\big[h(n,q) - h(n,q-1)\big]$, where the first inequality is because of the induction hypothesis, plus the properties $p^+(q) - p^+(q-1) \leq 0$ and $h(n,q) - h(n,q-1) = h(n+1,q) - h(n+1,q-1)$, and the last inequality is because of the properties $h(n,q) - h(n,q-1) \geq 0$ and $\rho \in (0,1)$. Hence, (B.13) holds for $(0,0,0)$.

(b) Case 2: $\left[G^1_{k+1}(n,q+1)-G^1_{k+1}(n,q)\right]-\left[G^1_{k+1}(n,q)-G^1_{k+1}(n,q-1)\right]=\rho\sigma_u\left[\Delta_q J_k(n,q+1)-\Delta_q J_k(n,q)\right]+\rho\sigma_c\left[p^+(q)-p^+(q-1)\right]+\rho\sigma_c\beta\left[\Delta_q J_k(n+1,q)-\Delta_q J_k(n+1,q-1)\right]+\rho\sigma_c(1-\beta)\left[\Delta_q J_k(n,q)-\Delta_q J_k(n,q-1)\right]+\left[h(n,q)-h(n,q+1)\right]-\left[h(n,q-1)-h(n,q)\right]\le\frac{1}{\sigma_u}\left[h(n,q)-h(n,q-1)\right]\le\frac{1}{\rho\sigma_u}\left[h(n,q)-h(n,q-1)\right]$, where the first inequality is derived from the induction hypothesis, plus the properties that $p^+(q)-p^+(q-1)\le 0$, $h(n,q)-h(n,q-1)=h(n+1,q)-h(n+1,q-1)$, and $\sigma_c+\sigma_u=1$. Hence, (B.13) holds for $(1,1,1)$.

(c) Case 3: $\left[G^0_{k+1}(n,q+1)-G^1_{k+1}(n,q)\right]-\left[G^1_{k+1}(n,q)-G^1_{k+1}(n,q-1)\right]=\rho\sigma_c\left[p^+(q)-p^+(q-1)\right]-\rho\sigma_u\Delta_q J_k(n,q)+h(n,q)+h(n,q)-h(n,q-1)+\rho\sigma_c\beta\left[\Delta_q J_k(n+1,q)-\Delta_q J_k(n+1,q-1)\right]+\rho\sigma_c(1-\beta)\left[\Delta_q J_k(n,q)-\Delta_q J_k(n,q-1)\right]\le\frac{1}{\sigma_u}\left[h(n,q)-h(n,q-1)\right]\le\frac{1}{\rho\sigma_u}\left[h(n,q)-h(n,q-1)\right]$, where the first inequality is derived from the induction hypothesis, the properties that $p^+(q)-p^+(q-1)\le 0$, $h(n,q)-h(n,q-1)=h(n+1,q)-h(n+1,q-1)$, $\sigma_c+\sigma_u=1$, and Lemma EC.9 part 1 (here we have $J_{k+1}(n,q)=G^1_{k+1}(n,q)$). Hence, (B.13) holds for $(1,1,0)$.

(d) Case 4: $\left[G^0_{k+1}(n,q+1)-G^0_{k+1}(n,q)\right]-\left[G^0_{k+1}(n,q)-G^1_{k+1}(n,q-1)\right]=\rho\sigma_c\left[p^+(q)-p^+(q-1)\right]+\rho\sigma_u\Delta_q J_k(n,q)-h(n,q-1)+\rho\sigma_c\beta\left[\Delta_q J_k(n+1,q)-\Delta_q J_k(n+1,q-1)\right]+\rho\sigma_c(1-\beta)\left[\Delta_q J_k(n,q)-\Delta_q J_k(n,q-1)\right]\le\frac{\sigma_c}{\sigma_u}\left[h(n,q)-h(n,q-1)\right]+h(n,q)-h(n,q-1)=\frac{1}{\sigma_u}\left[h(n,q)-h(n,q-1)\right]\le\frac{1}{\rho\sigma_u}\left[h(n,q)-h(n,q-1)\right]$, where the first inequality is derived from the induction hypothesis, the properties that $p^+(q)-p^+(q-1)\le 0$, $h(n,q)-h(n,q-1)=h(n+1,q)-h(n+1,q-1)$, and Lemma EC.9 part 1 (here we have $J_{k+1}(n,q)=G^0_{k+1}(n,q)$). Hence, (B.13) holds for $(1,0,0)$.

Thus, we complete the proof for off-boundary states $q+1<\bar{q}$ and $n<\bar{n}$. When $q+1=\bar{q}$, according to Lemma EC.9 part 2, we only need to discuss cases 1, 3, and 4, and all the proofs still hold. When $n=\bar{n}$, we need to replace $n+1$ with n and remove l, and it can be verified that all the proofs still hold. Therefore, we have proved that $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)$ decreases in q. The proof of conclusion (ii) in Lemma EC.8 follows the same reasoning and is omitted for brevity. Notably, the proof of conclusion (ii) additionally relies on Assumption 2. □

Now we are ready to prove Proposition 4. The optimality condition (6) implies that it is optimal to invest in quality improvement at (n,q) if and only if $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)\ge 0$. By Lemma EC.6, $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)$ decreases in q and increases in n. We first prove part (i) of Proposition 4 for a fixed n. If it is optimal to *not* invest in quality improvement at (n,q), i.e., $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)<0$, then for any $q'>q$, we also have $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q')-h(n,q')<0$. Thus, for each fixed n, the set of states where it is optimal to invest in quality improvement is of the form $\{q\le\tilde{q}(n)\}$ for some threshold $\tilde{q}(n)$. For part (ii), now fix q. If it is optimal to invest in quality improvement at (n,q), then $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)\ge 0$. For any $n'>n$, we have $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n',q)-h(n',q)\ge\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)\ge 0$, so it remains optimal to invest when more feedback is collected. Hence, $\tilde{q}(n)$ is increasing in n.

We also characterize when the quality threshold attains its boundary values. Since $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)$ decreases in q, if $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,0)-h(n,0)<0$, then $\rho\frac{\lambda_u}{\Lambda}\Delta_q J(n,q)-h(n,q)<0$ for all q, and it is never optimal to invest in quality improvement at feedback level n. In this case, we define $\tilde{q}(n)=-1$.

Therefore, for each fixed n, there exists a quality threshold $\tilde{q}(n)$ such that it is optimal to invest in quality improvement if and only if $q\le\tilde{q}(n)$, and $\tilde{q}(n)$ is monotone increasing in n. □

EC.2.6. Proof of Lemma 2

Consider a value iteration algorithm to solve the Bellman equation (4) for the joint optimization problem. To shorten the following analysis, define $J_c(n,q):=\rho\sigma_c p(q)+\rho\sigma_c\max\{\beta J(n+1,q)+(1-\beta)J(n,q)-l,\ J(n,q)\}$ and $J_u(n,q):=\max\{\rho\sigma_u J(n,q+1)-h(n,q),\ \rho\sigma_u J(n,q)\}-d(n)$, where again $\sigma_u:=\lambda_u/\Lambda$ and $\sigma_c:=\lambda_c/\Lambda$. Then the Bellman equation (4) can be written as $J(n,q)=J_c(n,q)+J_u(n,q)$. Define the action pair (a_c,a_u), where $a_c\in\{0,1\}$ indicates whether to solicit customer feedback ($a_c=1$) or not ($a_c=0$), and $a_u\in\{0,1\}$ indicates whether to invest in quality improvement ($a_u=1$) or not ($a_u=0$).

Let $J_k(\cdot,\cdot)$ denote the value function after k rounds of iteration. Specifically, we have the following update rule for J_{k+1} based on the value function of the previous round J_k:

$$J_{k+1}(n,q)=J_{c_{k+1}}(n,q)+J_{u_{k+1}}(n,q)=\max_{a_c\in\{0,1\}}Q^{a_c}_{c_{k+1}}(n,q)+\max_{a_u\in\{0,1\}}Q^{a_u}_{u_{k+1}}(n,q),$$

where $Q^{a_c}_{c_{k+1}}(n,q)$ and $Q^{a_u}_{u_{k+1}}(n,q)$ are the values of taking action a_c and a_u in the current state (n,q) and taking optimal actions in all future states respectively, as approximated by J_k. Specifically, for any $n<\bar{n}$ and $q<\bar{q}$:

$$Q^0_{c_{k+1}}(n,q):=\rho\sigma_c\left[p(q)+J_k(n,q)\right],\ Q^1_{c_{k+1}}(n,q):=\rho\sigma_c\left[p(q)+\beta J_k(n+1,q)+(1-\beta)J_k(n,q)-l\right],$$

and

$$Q^0_{u_{k+1}}(n,q):=\rho\sigma_u J_k(n,q)-d(n),\ Q^1_{u_{k+1}}(n,q):=\rho\sigma_u J_k(n,q+1)-h(n,q)-d(n).$$

When $n=\bar{n}$, we simply need to replace $n+1$ with $\bar{n}$ in the above equations, and when $q=\bar{q}$, we only need to replace $q+1$ with $\bar{q}$ in the above equations.

Recall from Lemma EC.1 that the value iteration algorithm converges in the baseline model. This result extends naturally to the joint model. Therefore, we will also prove Lemma 2 by induction. We start with $k=0$ where we set $J_0(n,q)=0$ for all the states (n,q). Note that properties (i)–(iii) in Lemma 2 are true for $J=J_0$. Thus, if we can show that properties (i)–(iii) in Lemma 2 are preserved when $J=J_k$ for any $k\ge 1$, then this proves Lemma 2. Assume that, at iteration k, properties (i)–(iii) in Lemma 2 are true for J_k. The induction proof is complete if we can show that the properties are also true for J_{k+1}.

Before proceeding with the proof, we first establish some useful conditions:

LEMMA EC.10. *The following are the optimal conditions for the quality improvement action a_u in the joint optimization problem:*

1. *Suppose $q < \bar{q}$. Then $J_{u_{k+1}}(n,q) = Q^1_{u_{k+1}}(n,q)$ if and only if $\rho\sigma_u \Delta_q J_k(n,q) - h(n,q) \geq 0$. Otherwise, $J_{u_{k+1}}(n,q) = Q^0_{u_{k+1}}(n,q)$.*
2. *Suppose $q = \bar{q}$. Then $J_{u_{k+1}}(n,q) = Q^0_{u_{k+1}}(n,q)$.*

Proof of Lemma EC.10. When $q < \bar{q}$, $J_{u_{k+1}}(n,q) = Q^1_{u_{k+1}}(n,q)$ if and only if inequality (6) holds, which proves part 1 of the lemma. When $q = \bar{q}$, $\Delta_q J_k(n,\bar{q}) = 0$, then with the fact that $h(n,q) > 0$, there is $\Delta_q J_k(n,\bar{q}) - h(n,q) < 0$, hence $J_{u_{k+1}}(n,q) = Q^0_{c_{k+1}}(n,q)$, which proves part 2. □

LEMMA EC.11. *The following are the optimal conditions for the feedback solicitation action a_c in the joint optimization problem:*

1. *Suppose $n < \bar{n}$. Then $J_{c_{k+1}}(n,q) = Q^1_{c_{k+1}}(n,q)$ if and only if $\beta\Delta_n J_k(n,q) - l \geq 0$. Otherwise, $J_{c_{k+1}}(n,q) = Q^0_{c_{k+1}}(n,q)$.*
2. *Suppose $n = \bar{n}$. Then $J_{c_{k+1}}(n,q) = Q^0_{c_{k+1}}(n,q)$.*

Proof of Lemma EC.11. When $n < \bar{n}$, note that $J_{c_{k+1}}(n,q) = Q^1_{c_{k+1}}(n,q)$ if and only if inequality (7) holds, which proves part 1 of the lemma. When $n = \bar{n}$, $\Delta_n J_k(\bar{n},q) = 0$, then with the fact that $l > 0$, there is $\beta\Delta_n J_k(\bar{n},q) - l = -l < 0$, hence $J_{c_{k+1}}(n,q) = Q^0_{c_{k+1}}(n,q)$, which proves part 2. □

Note that analyzing the joint policy is significantly more complex, as it requires comparing far more possible combinations of optimal actions than in the proofs of Lemma 1 and Lemma EC.8, which address only single-action policies. To manage this complexity, we establish that properties (i)–(iii) hold for J_{k+1} by verifying the corresponding sufficient conditions.

Given that $J_{k+1}(n,q) = J_{c_{k+1}}(n,q) + J_{u_{k+1}}(n,q)$, it is straightforward to derive that $\Delta_n J_{k+1}(n,q) = \Delta_n J_{c_{k+1}}(n,q) + \Delta_n J_{u_{k+1}}(n,q)$ and $\rho\sigma_u\Delta_q J_{k+1}(n,q) - h(n,q) = \left[\rho\sigma_u\Delta_q J_{c_{k+1}}(n,q) - \sigma_c h(n,q)\right] + \left[\rho\sigma_u\Delta_q J_{u_{k+1}}(n,q) - \sigma_u h(n,q)\right]$. Therefore, we will decompose the proof of each property into the following steps, where we use the symbols $\uparrow$ and $\downarrow$ as abbreviations for "nondecreasing" and "nonincreasing," respectively:

(i) To prove $J_{k+1}(n,q) \uparrow q, \uparrow n$, it is sufficient to show that $J_{c_{k+1}}(n,q) \uparrow q, \uparrow n$ and $J_{u_{k+1}}(n,q) \uparrow q, \uparrow n$ respectively;
(ii) To prove $\Delta_n J_{k+1}(n,q) \downarrow q, \downarrow n$, it is sufficient to show that $\Delta_n J_{c_{k+1}}(n,q) \downarrow q, \downarrow n$ and $\Delta_n J_{u_{k+1}}(n,q) \downarrow q, \downarrow n$ respectively;
(iii) To prove $\rho\sigma_u\Delta_q J_{k+1}(n,q) - h(n,q) \downarrow q, \uparrow n$, it is sufficient to show that $\rho\sigma_u\Delta_q J_{c_{k+1}}(n,q) - \sigma_c h(n,q) \downarrow q, \uparrow n$ and $\rho\sigma_u\Delta_q J_{u_{k+1}}(n,q) - \sigma_u h(n,q) \downarrow q, \uparrow n$ respectively.

Now we are ready to proceed with the proof of Lemma 2. For brevity, we present detailed proofs only for $\Delta_n J_{k+1}(n,q) \downarrow n$ and $\rho\sigma_u\Delta_q J_{k+1}(n,q) - h(n,q) \downarrow q$, as these two properties ensure the core double-threshold structure. The proofs of the remaining properties are analogous and are omitted.

(I) $\underline{\Delta_n J_{k+1}(n,q) \downarrow n}$: To prove this, it is sufficient to show that $\Delta_n J_{c_{k+1}}(n,q) \downarrow n$ and $\Delta_n J_{u_{k+1}}(n,q) \downarrow n$, respectively. This further reduces to

$$Q^{a_{c_2}}_{c_{k+1}}(n,q) - Q^{a_{c_1}}_{c_{k+1}}(n-1,q) \geq Q^{a_{c_3}}_{c_{k+1}}(n+1,q) - Q^{a_{c_2}}_{c_{k+1}}(n,q), \tag{B.14}$$

$$Q^{a_{u_2}}_{u_{k+1}}(n,q) - Q^{a_{u_1}}_{u_{k+1}}(n-1,q) \geq Q^{a_{u_3}}_{u_{k+1}}(n+1,q) - Q^{a_{u_2}}_{u_{k+1}}(n,q) \tag{B.15}$$

hold for all possible combinations of the optimal actions $(a_{c_1}, a_{c_2}, a_{c_3})$ and $(a_{u_1}, a_{u_2}, a_{u_3})$ at the states $(n-1,q)$, (n,q), and $(n+1,q)$, respectively. Similarly, due to the induction hypothesis, we only need to check that (B.14) holds when the optimal feedback solicitation actions $(a_{c_1}, a_{c_2}, a_{c_3})$ are $(0,0,0)$, $(1,1,1)$, $(1,1,0)$ or $(1,0,0)$, and (B.15) holds when the optimal quality investment actions $(a_{u_1}, a_{u_2}, a_{u_3})$ are $(0,0,0)$, $(1,1,1)$, $(0,1,1)$ or $(0,0,1)$.

Step 1: We first prove $\Delta_n J_{c_{k+1}}(n,q) \downarrow n$ by showing that

$$Q^{a_{c_2}}_{c_{k+1}}(n,q) - Q^{a_{c_1}}_{c_{k+1}}(n-1,q) - \left[Q^{a_{c_3}}_{c_{k+1}}(n+1,q) - Q^{a_{c_2}}_{c_{k+1}}(n,q)\right] \tag{B.16}$$

is greater than or equal to zero when the optimal actions $(a_{c_1}, a_{c_2}, a_{c_3})$ are $(0,0,0)$, $(1,1,1)$, $(1,1,0)$ or $(1,0,0)$:

(a) $(a_{c_1}, a_{c_2}, a_{c_3}) = (0,0,0)$: (B.16) $= \rho\sigma_c\left[\Delta_n J_k(n-1,q) - \Delta_n J_k(n,q)\right] \geq 0$.

(b) $(a_{c_1}, a_{c_2}, a_{c_3}) = (1,1,1)$: (B.16) $= \rho\sigma_c\beta\left[\Delta_n J_k(n,q) - \Delta_n J_k(n+1,q)\right] + \rho\sigma_c(1-\beta)\left[\Delta_n J_k(n-1,q) - \Delta_n J_k(n,q)\right] \geq 0$.

(c) $(a_{c_1}, a_{c_2}, a_{c_3}) = (1,1,0)$: (B.16) $= \rho\sigma_c\left[\beta\Delta_n J_k(n,q) - l + (1-\beta)\left(\Delta_n J_k(n-1,q) - \Delta_n J_k(n,q)\right)\right] \geq 0$, where the inequality is because of Lemma EC.11 part 1 (here we have $J_{c_{k+1}}(n,q) = Q^1_{c_{k+1}}(n,q)$) and the induction hypothesis.

(d) $(a_{c_1}, a_{c_2}, a_{c_3}) = (1,0,0)$: (B.16) $= \rho\sigma_c\left[l - \beta\Delta_n J_k(n,q) + (1-\beta)\left(\Delta_n J_k(n-1,q) - \Delta_n J_k(n,q)\right)\right] \geq 0$, where the inequality is because of Lemma EC.11 part 1 (here we have $J_{c_{k+1}}(n,q) = Q^0_{c_{k+1}}(n,q)$) and the induction hypothesis.

Step 2: Next we prove $\Delta_n J_{u_{k+1}}(n,q) \downarrow n$ by showing that

$$Q^{a_{u_2}}_{u_{k+1}}(n,q) - Q^{a_{u_1}}_{u_{k+1}}(n-1,q) - \left[Q^{a_{u_3}}_{u_{k+1}}(n+1,q) - Q^{a_{u_2}}_{u_{k+1}}(n,q)\right] \tag{B.17}$$

is greater than or equal to zero when the optimal actions $(a_{u_1}, a_{u_2}, a_{u_3})$ are $(0,0,0)$, $(1,1,1)$, $(0,1,1)$ or $(0,0,1)$:

(a) $(a_{u_1}, a_{u_2}, a_{u_3}) = (0,0,0)$: (B.17) $= \rho\sigma_u\big[\Delta_n J_k(n-1,q) - \Delta_n J_k(n,q)\big] + d^-(n-1) - d^-(n) \geq 0$, where the inequality is because of the induction hypothesis and Assumption 3.

(b) $(a_{u_1}, a_{u_2}, a_{u_3}) = (1,1,1)$: (B.17) $= \rho\sigma_u\big[\Delta_n J_k(n-1,q+1) - \Delta_n J_k(n,q+1)\big] + h^{1-}(n-1) - h^{1-}(n) + d^-(n-1) - d^-(n) \geq 0$, where the inequality is because of the induction hypothesis and Assumptions 2 and 3.

(c) $(a_{u_1}, a_{u_2}, a_{u_3}) = (0,1,1)$: (B.17) $= -\rho\sigma_u\Delta_n J_k(n,q+1) - h^{1-}(n) - d^-(n) - \big[\rho\sigma_u J_k(n-1,q) - \rho\sigma_u J_k(n,q+1) + h(n,q) - d^-(n-1)\big] = \rho\sigma_u\big[\Delta_n J_k(n-1,q+1) - \Delta_n J_k(n,q+1)\big] + \rho\sigma_u\Delta_q J_k(n-1,q) - h(n,q) - h^{1-}(n) + d^-(n-1) - d^-(n) \geq \rho\sigma_u\Delta_q J_k(n-1,q) - h(n,q) - h^{1-}(n) + d^-(n-1) - d^-(n)$, where the inequality is because of the induction hypothesis $\Delta_n J_k(n,q)\downarrow n$. Given the induction hypothesis that $\Delta_n J_k(n,q)\downarrow q$, there is also $\Delta_q J_k(n,q)\downarrow n$, thus $\rho\sigma_u\Delta_q J_k(n-1,q) \geq \rho\sigma_u\Delta_q J_k(n,q)$. According to Lemma EC.10, since $J_{u_{k+1}}(n,q) = Q^1_{u_{k+1}}(n,q)$, thus $\rho\sigma_u\Delta_q J_k(n,q) \geq h(n,q)$, therefore, there is $\rho\sigma_u\Delta_q J_k(n-1,q) \geq \rho\sigma_u\Delta_q J_k(n,q) \geq h(n,q)$. Hence, (B.17) $\geq h(n,q) - h(n,q) - h^{1-}(n) + d^-(n-1) - d^-(n) = d^-(n-1) - d^-(n) - h^{1-}(n) \geq 0$, where the last inequality is because of Assumption 4. Therefore, we have proved (B.17) ≥ 0.

(d) $(a_{u_1}, a_{u_2}, a_{u_3}) = (0,0,1)$: (B.17) $= \rho\sigma_u J_k(n,q) - \rho\sigma_u J_k(n+1,q+1) + h(n+1,q) + d^-(n-1) - d^-(n) + \rho\sigma_u\Delta_n J_k(n-1,q) = \rho\sigma_u\big[\Delta_n J_k(n-1,q) - \Delta_n J_k(n,q)\big] - \rho\sigma_u\Delta_q J_k(n+1,q) + h(n+1,q) + d^-(n-1) - d^-(n) \geq -\rho\sigma_u\Delta_q J_k(n+1,q) + h(n+1,q) + d^-(n-1) - d^-(n)$. Since $\Delta_q J_k(n,q)\downarrow n$, then $\rho\sigma_u\Delta_q J_k(n,q) \geq \rho\sigma_u\Delta_q J_k(n+1,q)$. Besides, according to Lemma EC.10, since $J_{u_{k+1}}(n,q) = Q^0_{u_{k+1}}$, there is $\rho\sigma_u\Delta_q J_k(n,q) \leq h(n,q)$. Therefore, here we have $\rho\sigma_u\Delta_q J_k(n+1,q) \leq \rho\sigma_u\Delta_q J_k(n,q) \leq h(n,q)$, which implies that $-\rho\sigma_u\Delta_q J_k(n+1,q) \geq -h(n,q)$. Hence (B.17) $\geq -h(n,q) + h(n+1,q) + d^-(n-1) - d^-(n) = d^-(n-1) - d^-(n) - h^{1-}(n) \geq 0$. Thus, we have proved (B.17) ≥ 0.

It is straightforward to verify that the above arguments still hold in boundary cases. Now since we have proved $\Delta_n J_{c_{k+1}}(n,q)\downarrow n$ and $\Delta_n J_{u_{k+1}}(n,q)\downarrow n$, thus, we have proved that $\Delta_n J_{k+1}(n,q)\downarrow n$.

(II) $\underline{\rho\sigma_u\Delta_q J_{k+1}(n,q) - h(n,q)\downarrow q}$: To prove this, it is sufficient to show that $\rho\sigma_u\Delta_q J_{c_{k+1}}(n,q) - \sigma_c h(n,q)\downarrow q$ and $\rho\sigma_u\Delta_q J_{u_{k+1}}(n,q) - \sigma_u h(n,q)\downarrow q$, respectively. With a slight abuse of notation $(a_{c_1}, a_{c_2}, a_{c_3})$ and $(a_{u_1}, a_{u_2}, a_{u_3})$, this further reduces to verifying that for any $q < \bar{q}$,

$$Q^{a_{c_2}}_{c_{k+1}}(n,q) - Q^{a_{c_1}}_{c_{k+1}}(n,q-1) - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} \geq Q^{a_{c_3}}_{c_{k+1}}(n,q+1) - Q^{a_{c_2}}_{c_{k+1}}(n,q) - \sigma_c\frac{h(n,q)}{\rho\sigma_u}, \tag{B.18}$$

$$Q^{a_{u_2}}_{u_{k+1}}(n,q) - Q^{a_{u_1}}_{u_{k+1}}(n,q-1) - \sigma_u\frac{h(n,q-1)}{\rho\sigma_u} \geq Q^{a_{u_3}}_{u_{k+1}}(n,q+1) - Q^{a_{u_2}}_{u_{k+1}}(n,q) - \sigma_u\frac{h(n,q)}{\rho\sigma_u} \tag{B.19}$$

hold for all possible combinations of the optimal actions $(a_{c_1}, a_{c_2}, a_{c_3})$ and $(a_{u_1}, a_{u_2}, a_{u_3})$ at the states $(n,q-1)$, (n,q), and $(n,q+1)$, respectively. Similarly, due to the induction hypothesis, we only need to check that (B.18) holds when the optimal feedback solicitation actions $(a_{c_1}, a_{c_2}, a_{c_3})$ are $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ or $(1,1,1)$, and (B.19) holds when the optimal quality investment actions $(a_{u_1}, a_{u_2}, a_{u_3})$ are $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ or $(1,1,1)$.

Step 1: We first prove $\rho\sigma_u\Delta_q J_{c_{k+1}}(n,q) - \sigma_c h(n,q)\downarrow q$ by showing that

$$Q^{a_{c_2}}_{c_{k+1}}(n,q) - Q^{a_{c_1}}_{c_{k+1}}(n,q-1) - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} - \left[Q^{a_{c_3}}_{c_{k+1}}(n,q+1) - Q^{a_{c_2}}_{c_{k+1}}(n,q) - \sigma_c\frac{h(n,q)}{\rho\sigma_u}\right] \tag{B.20}$$

is greater than or equal to zero when the optimal actions $(a_{c_1}, a_{c_2}, a_{c_3})$ are $(0,0,0)$, $(1,1,1)$, $(1,1,0)$ or $(1,0,0)$:

(a) $(a_{c_1}, a_{c_2}, a_{c_3}) = (0,0,0)$: (B.20) $= \rho\sigma_c\big[p^+(q-1) - p^+(q) + \Delta_q J_k(n,q-1) - \Delta_q J_k(n,q)\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} \geq \rho\sigma_c\big[\Delta_q J_k(n,q-1) - \Delta_q J_k(n,q)\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} \geq \rho\sigma_c\big[\frac{h(n,q-1)}{\rho\sigma_u} - \frac{h(n,q)}{\rho\sigma_u}\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} = \frac{\sigma_c}{\rho\sigma_u}(\rho - 1)\big[h(n,q-1) - h(n,q)\big] \geq 0$, where the first inequality is because of Assumption 1, the second inequality is because of the induction hypothesis that $\rho\sigma_u\Delta_q J_k(n,q) - h(n,q)\downarrow q$, and the third inequality is because of the properties that $h(n,q)$ increases in q and $\rho\in[0,1)$.

(b) $(a_{c_1}, a_{c_2}, a_{c_3}) = (1,0,0)$: (B.20) $= \rho\sigma_c\big[p^+(q-1) - p^+(q) - \beta J_k(n+1,q-1) - (1-\beta)J_k(n,q-1) + J_k(n,q) - \Delta_q J_k(n,q) + l\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} = \rho\sigma_c\big[p^+(q-1) - p^+(q) - l - \beta J_k(n+1,q-1) - (1-\beta)J_k(n,q-1) + J_k(n,q) - \Delta_q J_k(n,q) + 2l\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} \geq \rho\sigma_c\big[p^+(q-1) - p^+(q) - l - \beta J_k(n+1,q-1) - (1-\beta)J_k(n,q-1) + J_k(n,q) - J_k(n,q+1) + J_k(n,q) + 2\beta\Delta_n J_k(n,q)\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} = \rho\sigma_c\big[p^+(q-1) - p^+(q) - l + \beta\Delta_q J_k(n+1,q-1) + \beta J_k(n+1,q) - \beta J_k(n,q+1) + (1-\beta)(\Delta_q J_k(n,q-1) - \Delta_q J_k(n,q))\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} \geq \rho\sigma_c\big[p^+(q-1) - p^+(q) - l + \beta\Delta_q J_k(n+1,q-1) + \beta J_k(n+1,q) - \beta J_k(n+1,q+1) + (1-\beta)(\Delta_q J_k(n,q-1) - \Delta_q J_k(n,q))\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} = \rho\sigma_c\big[p^+(q-1) - p^+(q) - l + \beta\Delta_q J_k(n+1,q-1) - \beta\Delta_q J_k(n+1,q) + (1-\beta)(\Delta_q J_k(n,q-1) - \Delta_q J_k(n,q))\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} \geq \rho\sigma_c\big[p^+(q-1) - p^+(q) - l\big] + \rho\sigma_c\big[\frac{h(n,q-1)}{\rho\sigma_u} - \frac{h(n,q)}{\rho\sigma_u}\big] - \sigma_c\frac{h(n,q-1)}{\rho\sigma_u} + \sigma_c\frac{h(n,q)}{\rho\sigma_u} \geq \rho\sigma_c\big[p^+(q-1) - p^+(q) - l\big] \geq 0$, where the first inequality is because of Lemma EC.11 and $J_{c_{k+1}}(n,q) = Q^0_{c_{k+1}}(n,q)$, the second inequality is because of the induction hypothesis that $J_k(n,q)\uparrow n$, the third inequality is because of the induction hypothesis that $\rho\sigma_u\Delta_q J_k(n,q) - h(n,q)\downarrow q$ and the property that $h(n+1,q-1) - h(n+1,q) = h^2(q-1) - h^2(q) = h(n,q-1) - h(n,q)$, the fourth inequality is because of the properties that $h(n,q)$ increases in q and $\rho\in[0,1)$, and the last inequality is because of Assumption 4. Thus we have proved (B.20) ≥ 0.

(c) $(a_{c_1},a_{c_2},a_{c_3})=(1,1,0)$: (B.20) $=\rho\sigma_c\big[p^+(q-1)-p^+(q)+\beta\Delta_q J_k(n+1,q-1)+(1-\beta)\Delta_q J_k(n,q-1)-J_k(n,q+1)+\beta J_k(n+1,q)+(1-\beta)J_k(n,q)-l\big]-\sigma_c\frac{h(n,q-1)}{\rho\sigma_u}+\sigma_c\frac{h(n,q)}{\rho\sigma_u}=\rho\sigma_c\big[p^+(q-1)-p^+(q)-l+\beta\Delta_q J_k(n+1,q-1)+(1-\beta)\Delta_q J_k(n,q-1)-\beta\Delta_q J_k(n+1,q)+\beta J_k(n+1,q+1)-J_k(n,q+1)+(1-\beta)J_k(n,q)\big]-\sigma_c\frac{h(n,q-1)}{\rho\sigma_u}+\sigma_c\frac{h(n,q)}{\rho\sigma_u}\geq\rho\sigma_c\big[p^+(q-1)-p^+(q)-l+\beta\Delta_q J_k(n+1,q-1)-\beta\Delta_q J_k(n+1,q)+(1-\beta)\Delta_q J_k(n,q-1)-(1-\beta)\Delta_q J_k(n,q)\big]-\sigma_c\frac{h(n,q-1)}{\rho\sigma_u}+\sigma_c\frac{h(n,q)}{\rho\sigma_u}\geq\rho\sigma_c\big[p^+(q-1)-p^+(q)-l\big]+\rho\sigma_c\big[\frac{h(n,q-1)}{\rho\sigma_u}-\frac{h(n,q)}{\rho\sigma_u}\big]-\sigma_c\frac{h(n,q-1)}{\rho\sigma_u}+\sigma_c\frac{h(n,q)}{\rho\sigma_u}\geq\rho\sigma_c\big[p^+(q-1)-p^+(q)-l\big]\geq 0$, where the first inequality is because of the induction hypothesis that $J_k(n,q)\uparrow n$, the second inequality is because of the induction hypothesis that $\rho\sigma_u\Delta_q J_k(n,q)-h(n,q)\downarrow q$ and the property that $h(n+1,q-1)-h(n+1,q)=h(n,q-1)-h(n,q)$, the third inequality is because of the properties that $h(n,q)$ increases in q and $\rho\in[0,1)$, and the last inequality is because of Assumption 4. Thus we have proved (B.20) ≥ 0.

(d) $(a_{c_1},a_{c_2},a_{c_3})=(1,1,1)$: (B.20) $=\rho\sigma_c\big[p^+(q-1)-p^+(q)+\beta\Delta_q J_k(n+1,q-1)-\beta\Delta_q J_k(n+1,q)+(1-\beta)\Delta_q J_k(n,q-1)-(1-\beta)\Delta_q J_k(n,q)\big]-\sigma_c\frac{h(n,q-1)}{\rho\sigma_u}+\sigma_c\frac{h(n,q)}{\rho\sigma_u}\geq\rho\sigma_c\big[p^+(q-1)-p^+(q)+\frac{h(n,q-1)}{\rho\sigma_u}-\frac{h(n,q)}{\rho\sigma_u}\big]-\sigma_c\frac{h(n,q-1)}{\rho\sigma_u}+\sigma_c\frac{h(n,q)}{\rho\sigma_u}\geq\rho\sigma_c\big[\frac{h(n,q-1)}{\rho\sigma_u}-\frac{h(n,q)}{\rho\sigma_u}\big]-\sigma_c\frac{h(n,q-1)}{\rho\sigma_u}+\sigma_c\frac{h(n,q)}{\rho\sigma_u}\geq 0$, where the first inequality is because of the induction hypothesis and the property that $h(n+1,q-1)-h(n+1,q)=h(n,q-1)-h(n,q)$, the second inequality is because of Assumption 1, and the last inequality is because of the properties that $h(n,q)$ increases in q and $\rho\in[0,1)$.

Step 2: We first prove $\rho\sigma_u\Delta_q J_{u_{k+1}}(n,q)-\sigma_u h(n,q)\downarrow q$ by showing that

$$Q^{a_{u_2}}_{u_{k+1}}(n,q)-Q^{a_{u_1}}_{u_{k+1}}(n,q-1)-\sigma_u\frac{h(n,q-1)}{\rho\sigma_u}-\left[Q^{a_{u_3}}_{u_{k+1}}(n,q+1)-Q^{a_{u_2}}_{u_{k+1}}(n,q)-\sigma_u\frac{h(n,q)}{\rho\sigma_u}\right]\tag{B.21}$$

is greater than or equal to zero when the optimal actions $(a_{u_1},a_{u_2},a_{u_3})$ are $(0,0,0)$, $(1,1,1)$, $(1,1,0)$ or $(1,0,0)$:

(a) $(a_{u_1},a_{u_2},a_{u_3})=(0,0,0)$: (B.21) $=\rho\sigma_u\big[\Delta_q J_k(n,q-1)-\Delta_q J_k(n,q)\big]-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}\geq h(n,q-1)-h(n,q)-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}=(1-\frac{1}{\rho})\big[h(n,q-1)-h(n,q)\big]\geq 0$, where the first inequality is because of the induction hypothesis that $\rho\sigma_u\Delta_q J_k(n,q)-h(n,q)\downarrow q$.

(b) $(a_{u_1},a_{u_2},a_{u_3})=(1,0,0)$: (B.21) $=h(n,q-1)-\rho\sigma_u\Delta_q J_k(n,q)-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}\geq h(n,q-1)-h(n,q)-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}\geq 0$, where the first inequality is because of Lemma EC.10 and $J_{u_{k+1}}(n,q)=Q^0_{u_{k+1}}(n,q)$.

(c) $(a_{u_1},a_{u_2},a_{u_3})=(1,1,0)$: (B.21) $=\rho\sigma_u\Delta_q J_k(n,q)-h(n,q)+h(n,q-1)-h(n,q)-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}\geq h(n,q-1)-h(n,q)-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}\geq 0$, where the first inequality is because of Lemma EC.10 and $J_{u_{k+1}}(n,q)=Q^1_{u_{k+1}}(n,q)$.

(d) $(a_{u_1},a_{u_2},a_{u_3})=(1,1,1)$: (B.21) $=\rho\sigma_u\Delta_q J_k(n,q)-h(n,q)+h(n,q-1)-\rho\sigma_u\Delta_q J_k(n,q+1)+h(n,q+1)-h(n,q)-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}=\rho\sigma_u\Delta_q J_k(n,q)-h(n,q)-\big[\rho\sigma_u\Delta_q J_k(n,q+1)-h(n,q+1)\big]+h(n,q-1)-h(n,q)-\frac{h(n,q-1)}{\rho}+\frac{h(n,q)}{\rho}\geq 0$, where the inequality is because of the induction hypothesis that $\rho\sigma_u\Delta_q J_k(n,q)-h(n,q)\downarrow q$.

It is straightforward to verify that the above arguments still hold in boundary cases. Thus, we have proved $\rho\sigma_u\Delta_q J_{k+1}(n,q)-h(n,q)\downarrow q$.

Therefore, we have established that $\Delta_n J_{k+1}(n,q)$ decreases in n and that $\rho\sigma_u\Delta_q J_{k+1}(n,q)-h(n,q)$ decreases in q. The proofs of the remaining properties follow the same reasoning and are omitted for brevity. □

EC.2.7. Proof of Proposition 5

Under Lemma 2, the value function $J(n,q)$ in the joint optimization model retains the same monotonicity properties as in the single-action settings: $\Delta_n J(n,q)$ decreases in both n and q, while $\rho\sigma_u\Delta_q J(n,q)-h(n,q)$ increases in n and decreases in q. These are exactly the properties we used in the proofs of Propositions 1 and 4 to establish the existence and monotonicity of the solicitation and quality thresholds. Therefore, by applying the same threshold arguments as in those proofs, we obtain the two monotone thresholds $\tilde{n}(q)$ and $\tilde{q}(n)$ that jointly characterize the optimal joint policy. □

EC.2.8. Proof of Proposition 6

Below we provide the detailed proof for the policy's sensitivity with respect to $p(q)$, as stated in Conclusion (ii) of Proposition 6. The proofs of the remaining conclusions in Proposition 6 follow similar arguments and are omitted for brevity. Note that the proof of Conclusion (v) also follows the approach used to establish Conclusion (v) in Proposition 2, where an upper bound on the marginal value of feedback is required. In particular, Lemma EC.5 and Corollary EC.1 can be extended to the joint setting, which facilitates the proof of Conclusion (v). Due to space limitations, we do not repeat the analysis here.

Specifically, Conclusion (ii) is established through the following lemma:

LEMMA EC.12. *Consider two instances of the joint model that satisfy Assumptions 1, 2, 3 and 4. Suppose the two instances are identical except that $p'^+(q)-p^+(q)\geq l$ for all $q\leq\bar{q}-1$, then*

(i) *$\Delta_n J(n,q)\leq\Delta_n J'(n,q)$ for all $n\leq\bar{n}-1$ and q;*
(ii) *$\Delta_q J(n,q)\leq\Delta_q J'(n,q)$ for all n and $q\leq\bar{q}-1$.*

Proof of Lemma EC.12. Consider two identical instances of the joint model except that $p'^{+}(q) - p^{+}(q) \geq l$ for all $q \leq \bar{q}-1$. Similarly, we conduct value iterations as in the proof of Lemma 2 for the two instances simultaneously. We use the prime ($'$) symbol for the second instance. Let $J_k(\cdot,\cdot)$ and $J'_k(\cdot,\cdot)$ denote the value functions after k rounds of iteration for the two instances, respectively, then

$$J_{k+1}(n,q) = \max_{a_c\in\{0,1\}} Q^{a_c}_{c_{k+1}}(n,q) + \max_{a_u\in\{0,1\}} Q^{a_u}_{u_{k+1}}(n,q),\ J'_{k+1}(n,q) = \max_{a_c\in\{0,1\}} Q'^{a_c}_{c_{k+1}}(n,q) + \max_{a_u\in\{0,1\}} Q'^{a_u}_{u_{k+1}}(n,q).$$

We prove the lemma by induction. Since we know from Lemma EC.1 that, as $k\to+\infty$, $J_k(n,q)$ converges to $J(n,q)$ and $J'_k(n,q)$ converges to $J'(n,q)$, then if we can prove that $\Delta_n J_k(n,q) \leq \Delta_n J'_k(n,q)$ and $\Delta_q J_k(n,q) \leq \Delta_q J'_k(n,q)$ hold for all k, then this proves that $\Delta_n J(n,q) \leq \Delta_n J'(n,q)$ and $\Delta_q J(n,q) \leq \Delta_q J'(n,q)$. Note that from the proofs of Lemma 2, the properties in Lemma 2 also hold for both J_k and J'_k.

We start with $k=0$ where we set $J_0(n,q)=0$ and $J'_0(n,q)=0$ for all the states (n,q), then $\Delta_n J_k(n,q) \leq \Delta_n J'_k(n,q)$ and $\Delta_q J_k(n,q) \leq \Delta_q J'_k(n,q)$ hold trivially for $k=0$. Assume $\Delta_n J_k(n,q) \leq \Delta_n J'_k(n,q)$ and $\Delta_q J_k(n,q) \leq \Delta_q J'_k(n,q)$ hold for iteration k, then the induction proof is complete if we can show $\Delta_n J_{k+1}(n,q) \leq \Delta_n J'_{k+1}(n,q)$ and $\Delta_q J_{k+1}(n,q) \leq \Delta_q J'_{k+1}(n,q)$ hold for iteration $k+1$.

Recall that $J_{k+1}(n,q) = J_{c_{k+1}}(n,q) + J_{u_{k+1}}(n,q)$, thus to prove $\Delta_n J_{k+1}(n,q) \leq \Delta_n J'_{k+1}(n,q)$, it is sufficient to show that $\Delta_n J_{c_{k+1}}(n,q) \leq \Delta_n J'_{c_{k+1}}(n,q)$ and $\Delta_n J_{u_k+1}(n,q) \leq \Delta_n J'_{u_k+1}(n,q)$. Similarly, to prove $\Delta_q J_{k+1}(n,q) \leq \Delta_q J'_{k+1}(n,q)$, it is sufficient to show that $\Delta_q J_{c_{k+1}}(n,q) \leq \Delta_q J'_{c_{k+1}}(n,q)$ and $\Delta_q J_{u_{k+1}}(n,q) \leq \Delta_q J'_{u_{k+1}}(n,q)$.

(I) $\underline{\Delta_n J_{k+1}(n,q) \leq \Delta_n J'_{k+1}(n,q)}$

To prove this, it is sufficient to show that

$$Q^{a_{c_2}}_{c_{k+1}}(n+1,q) - Q^{a_{c_1}}_{c_{k+1}}(n,q) \leq Q'^{a_{c_4}}_{c_{k+1}}(n+1,q) - Q'^{a_{c_3}}_{c_{k+1}}(n,q) \quad \text{(B.22)}$$

$$Q^{a_{u_2}}_{u_{k+1}}(n+1,q) - Q^{a_{u_1}}_{u_{k+1}}(n,q) \leq Q'^{a_{u_4}}_{u_{k+1}}(n+1,q) - Q'^{a_{u_3}}_{u_{k+1}}(n,q) \quad \text{(B.23)}$$

hold for all possible combinations of the optimal actions, where (a_{c_1}, a_{u_1}) and (a_{c_2}, a_{u_2}) are the optimal actions at states (n,q) and $(n+1,q)$ in instance 1, and (a_{c_3}, a_{u_3}) and (a_{c_4}, a_{u_4}) are optimal for the same states in instance 2. Due to the induction hypothesis and the proof of Lemma 2, we only need to check that (B.22) holds when the optimal feedback solicitation actions $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4})$ are $(0,0,0,0)$, $(0,0,1,0)$, $(0,0,1,1)$, $(1,0,1,0)$, $(1,0,1,1)$ or $(1,1,1,1)$, and (B.23) holds when the optimal quality investment actions $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4})$ are $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$ or $(1,1,1,1)$.

Step 1: We first prove $\Delta_n J_{c_{k+1}}(n,q) \leq \Delta_n J'_{c_{k+1}}(n,q)$ by showing that

$$Q'^{a_{c_4}}_{c_{k+1}}(n+1,q) - Q'^{a_{c_3}}_{c_{k+1}}(n,q) - \left[Q^{a_{c_2}}_{c_{k+1}}(n+1,q) - Q^{a_{c_1}}_{c_{k+1}}(n,q)\right] \quad \text{(B.24)}$$

is greater than or equal to zero when the optimal actions $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4})$ are $(0,0,0,0)$, $(0,0,1,0)$, $(0,0,1,1)$, $(1,0,1,0)$, $(1,0,1,1)$ or $(1,1,1,1)$:

(a) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,0,0,0)$: (B.24) $= \rho\sigma_c\left[\Delta_n J'_k(n,q) - \Delta_n J_k(n,q)\right] \geq 0$.

(b) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,0,1,0)$: (B.24) $= \rho\sigma_c\left[l - \beta\Delta_n J_k(n,q) + (1-\beta)(\Delta_n J'_k(n,q) - \Delta_n J_k(n,q))\right] \geq 0$.

(c) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,0,1,1)$: (B.24) $= \rho\sigma_c[\beta\Delta_n J'_k(n+1,q) + (1-\beta)\Delta_n J'_k(n,q) - \Delta_n J_k(n,q)] \geq \rho\sigma_c(l-l) = 0$.

(d) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (1,0,1,0)$: (B.24) $= \rho\sigma_c[l - l + (1-\beta)(\Delta_n J'_k(n,q) - \Delta_n J_k(n,q))] \geq 0$.

(e) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (1,0,1,1)$: (B.24) $= \rho\sigma_c[\beta\Delta_n J'_k(n+1,q) - l + (1-\beta)(\Delta_n J'_k(n,q) - \Delta_n J_k(n,q))] \geq 0$.

(f) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (1,1,1,1)$: (B.24) $= \rho\sigma_c[\beta(\Delta_n J'_k(n+1,q) - \Delta_n J_k(n+1,q)) + (1-\beta)(\Delta_n J'_k(n,q) - \Delta_n J_k(n,q))] \geq 0$.

Step 2: Next we prove $\Delta_n J_{u_{k+1}}(n,q) \leq \Delta_n J'_{u_{k+1}}(n,q)$ by showing that

$$Q'^{a_{u_4}}_{u_{k+1}}(n+1,q) - Q'^{a_{u_3}}_{u_{k+1}}(n,q) - \left[Q^{a_{u_2}}_{u_{k+1}}(n+1,q) - Q^{a_{u_1}}_{u_{k+1}}(n,q)\right] \quad \text{(B.25)}$$

is greater than or equal to zero when the optimal actions $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4})$ are $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$ or $(1,1,1,1)$.

(a) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,0,0,0)$: (B.25) $= \rho\sigma_u\Delta_n J'_k(n,q) - \rho\sigma_u\Delta_n J_k(n,q) \geq 0$.

(b) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,0,0,1)$: (B.25) $= \rho\sigma_u\Delta_n J'_k(n,q) - \rho\sigma_u\Delta_n J_k(n,q) + \rho\sigma_u\Delta_q J'_k(n+1,q) - h(n+1,q) \geq 0$.

(c) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,0,1,1)$: (B.25) $= \rho\sigma_u\Delta_n J'_k(n,q+1) - \rho\sigma_u\Delta_n J_k(n,q) + h(n,q) - h(n+1,q) \geq \rho\sigma_u\Delta_n J'_k(n,q+1) - \rho\sigma_u\Delta_n J'_k(n,q) + h(n,q) - h(n+1,q) = \rho\sigma_u\Delta_q J'_k(n+1,q) - h'(n+1,q) - \left[\rho\sigma_u\Delta_q J'_k(n,q) - h'(n,q)\right] \geq 0$.

(d) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,1,0,1)$: (B.25) $= \rho\sigma_u\Delta_n J'_k(n,q) - \rho\sigma_u\Delta_n J_k(n,q) + \rho\sigma_u\Delta_q J'_k(n+1,q) - \rho\sigma_u\Delta_q J_k(n+1,q) \geq 0$.

(e) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,1,1,1)$: (B.25) $= \rho\sigma_u J_k(n,q) - \rho\sigma_u J_k(n+1,q+1) + h(n+1,q) + \rho\sigma_u\Delta_n J'_k(n,q+1) + h(n,q) - h(n+1,q) = -\rho\sigma_u\Delta_n J_k(n,q+1) + \rho\sigma_u\Delta_n J'_k(n,q+1) - \rho\sigma_u\Delta_q J_k(n,q) + h(n,q) \geq 0$.

(f) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (1,1,1,1)$: (B.25) $= \rho\sigma_u \Delta_n J'_k(n,q+1) - \rho\sigma_u \Delta_n J_k(n,q+1) \geq 0$.

(II) $\underline{\Delta_q J(n,q) \leq \Delta_q J'(n,q)}$

To prove this, it is sufficient to show that

$$Q^{a_{c_1}}_{c_{k+1}}(n,q+1) - Q^{a_{c_2}}_{c_{k+1}}(n,q) \leq Q'^{a_{c_3}}_{c_{k+1}}(n,q+1) - Q'^{a_{c_4}}_{c_{k+1}}(n,q) \tag{B.26}$$

$$Q^{a_{u_1}}_{u_{k+1}}(n,q+1) - Q^{a_{u_2}}_{u_{k+1}}(n,q) \leq Q'^{a_{u_3}}_{u_{k+1}}(n,q+1) - Q'^{a_{u_4}}_{u_{k+1}}(n,q) \tag{B.27}$$

hold for all possible combinations of the optimal actions. For ease of exposition, with a slight abuse of notation, we redefine (a_{c_1}, a_{u_1}) and (a_{c_2}, a_{u_2}) as the optimal actions at states $(n,q+1)$ and (n,q) in instance 1, and (a_{c_3}, a_{u_3}) and (a_{c_4}, a_{u_4}) for the same states in instance 2. Due to the induction hypothesis and the proof of Lemma 2, we only need to check that (B.26) holds when the optimal feedback solicitation actions $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4})$ are $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$ or $(1,1,1,1)$, and (B.27) holds when the optimal quality investment actions $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4})$ are $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$ or $(1,1,1,1)$.

Step 1: We first prove $\Delta_q J_{c_{k+1}}(n,q) \leq \Delta_q J'_{c_{k+1}}(n,q)$ by showing that

$$Q'^{a_{c_3}}_{c_{k+1}}(n,q+1) - Q'^{a_{c_4}}_{c_{k+1}}(n,q) - \big[Q^{a_{c_1}}_{c_{k+1}}(n,q+1) - Q^{a_{c_2}}_{c_{k+1}}(n,q)\big] \tag{B.28}$$

is greater than or equal to zero when the optimal actions $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4})$ are $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$ or $(1,1,1,1)$:

(a) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,0,0,0)$: (B.28) $= \rho\sigma_c\big[p'^+(q) + \Delta_q J'_k(n,q)\big] - \rho\sigma_c\big[p^+(q) + \Delta_q J_k(n,q)\big] \geq 0$

(b) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,0,0,1)$: (B.28) $= \rho\sigma_c\big[p'^+(q) + \beta\Delta_q J'_k(n+1,q) - \beta\Delta_n J'_k(n,q+1) + (1-\beta)\Delta_q J'_k(n,q) + l\big] - \rho\sigma_c\big[p^+(q) + \Delta_q J_k(n,q)\big] \geq \rho\sigma_c\big[p'^+(q) - p^+(q) + \beta\Delta_q J'_k(n+1,q) - \beta\Delta_q J_k(n,q)\big] \geq \rho\sigma_c\big[p'^+(q) - p^+(q) + \beta\Delta_q J_k(n+1,q) - \beta\Delta_q J_k(n,q)\big] = \rho\sigma_c\big[p'^+(q) - p^+(q) - \beta\Delta_n J_k(n,q) + \beta\Delta_n J_k(n,q+1)\big] \geq \rho\sigma_c\big[p'^+(q) - p^+(q) - l\big]$, where the last inequality is because of Lemma EC.11 part 1 (here we have $J_{c_{k+1}}(n,q) = Q^0_{c_{k+1}}(n,q)$) and that $J_k(n,q)$ increases in n. Then we can prove (B.28) ≥ 0 via the condition that $p'^+(q) - p^+(q) \geq l$.

(c) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,0,1,1)$: (B.28) $= \rho\sigma_c\big[p'^+(q) + \beta\Delta_q J'_k(n+1,q) + (1-\beta)\Delta_q J'_k(n,q)\big] - \rho\sigma_c\big[p^+(q) + \Delta_q J_k(n,q)\big] \geq \rho\sigma_c\big[p'^+(q) - p^+(q)\big] + \rho\sigma_c\beta\Delta_q J_k(n+1,q) - \rho\sigma_c\beta\Delta_q J_k(n,q) = \rho\sigma_c\big[p'^+(q) - p^+(q)\big] - \rho\sigma_c\beta\Delta_n J_k(n,q) + \rho\sigma_c\beta\Delta_n J_k(n,q+1) \geq \rho\sigma_c\big[p'^+(q) - p^+(q) - l\big] \geq 0$.

(d) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,1,0,1)$: (B.28) $= \rho\sigma_c\big[p'^+(q) + \beta\Delta_q J'_k(n+1,q) - \beta\Delta_n J'_k(n,q+1) + (1-\beta)\Delta_q J'_k(n,q) + l\big] - \rho\sigma_c\big[p^+(q) + \beta\Delta_q J_k(n+1,q) - \beta\Delta_n J_k(n,q+1) + (1-\beta)\Delta_q J_k(n,q) + l\big] \geq \rho\sigma_c\big[p'^+(q) - p^+(q) - \beta\Delta_n J'_k(n,q+1) + \beta\Delta_n J_k(n,q+1)\big] \geq \rho\sigma_c\big[p'^+(q) - p^+(q) - l\big] \geq 0$.

(e) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (0,1,1,1)$: (B.28) $= \rho\sigma_c\big[p'^+(q) + \beta\Delta_q J'_k(n+1,q) + (1-\beta)\Delta_q J'_k(n,q)\big] - \rho\sigma_c\big[p^+(q) + \beta\Delta_q J_k(n+1,q) - \beta\Delta_n J_k(n,q+1) + (1-\beta)\Delta_q J_k(n,q) + l\big] \geq \rho\sigma_c\big[p'^+(q) - p^+(q) - l\big] \geq 0$.

(f) $(a_{c_1}, a_{c_2}, a_{c_3}, a_{c_4}) = (1,1,1,1)$: (B.28) $= \rho\sigma_c\big[p'^+(q) + \beta\Delta_q J'_k(n+1,q) + (1-\beta)\Delta_q J'_k(n,q)\big] - \rho\sigma_c\big[p^+(q) + \beta\Delta_q J_k(n+1,q) + (1-\beta)\Delta_q J_k(n,q)\big] \geq 0$.

Step 2: Next we prove $\Delta_q J_{u_{k+1}}(n,q) \leq \Delta_q J'_{u_{k+1}}(n,q)$ by showing that

$$Q'^{a_{u_3}}_{u_{k+1}}(n,q+1) - Q'^{a_{u_4}}_{u_{k+1}}(n,q) - \big[Q^{a_{u_1}}_{u_{k+1}}(n,q+1) - Q^{a_{u_2}}_{u_{k+1}}(n,q)\big] \tag{B.29}$$

is greater than or equal to zero when the optimal actions $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4})$ are $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,1,0,1)$, $(0,1,1,1)$ or $(1,1,1,1)$.

(a) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,0,0,0)$: (B.29) $= \rho\sigma_u \Delta_q J'_k(n,q) - \rho\sigma_u \Delta_q J_k(n,q) \geq 0$.

(b) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,0,0,1)$: (B.29) $= h(n,q) - \rho\sigma_u \Delta_q J_k(n,q) \geq 0$.

(c) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,0,1,1)$: (B.29) $= \rho\sigma_u \Delta_q J'_k(n,q+1) - h(n,q+1) + h(n,q) - \rho\sigma_u \Delta_q J_k(n,q) \geq 0$.

(d) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,1,0,1)$: (B.29) $= h(n,q) - h(n,q) = 0$.

(e) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (0,1,1,1)$: (B.29) $= \rho\sigma_u \Delta_q J'_k(n,q+1) - h(n,q+1) + h(n,q) - h(n,q) \geq 0$.

(f) $(a_{u_1}, a_{u_2}, a_{u_3}, a_{u_4}) = (1,1,1,1)$: (B.29) $= \rho\sigma_u \Delta_q J'_k(n,q+1) - \rho\sigma_u \Delta_q J_k(n,q+1) \geq 0$.

It is straightforward to verify that the above arguments still hold in boundary cases. Therefore, Lemma EC.12 has been proved. □

Now we are ready to prove the policy sensitivity with respect to $p^+(q)$. For the solicitation threshold, recall that feedback is solicited when $\beta\Delta_n J(n,q) - l \geq 0$. If a state (n,q) in instance 1 satisfies this inequality, then because $\Delta_n J'(n,q) \geq \Delta_n J(n,q)$ we also have $\beta\Delta_n J'(n,q) - l \geq \beta\Delta_n J(n,q) - l \geq 0$. Hence, every state where solicitation is optimal in instance 1 remains optimal in instance 2, implying $\tilde{n}'(q) \geq \tilde{n}(q)$ for all q. Similarly, for the quality threshold, quality improvement is chosen when $\rho\sigma_u \Delta_q J(n,q) - h(n,q) \geq 0$. If (n,q) satisfies this inequality in instance 1, then because $\Delta_q J'(n,q) \geq \Delta_q J(n,q)$ we also have $\rho\sigma_u \Delta_q J'(n,q) - h(n,q) \geq \rho\sigma_u \Delta_q J(n,q) - h(n,q) \geq 0$. Thus, every state where quality investment is optimal in instance 1 remains optimal in instance 2, implying $\tilde{q}'(n) \geq \tilde{q}(n)$ for all n. Therefore, when the marginal revenue from quality increases, both thresholds $\tilde{n}(q)$ and $\tilde{q}(n)$ increase, establishing Conclusion (ii). □