

E-Companion to “A Decision-Support Framework for Food Bank Operations: Overcoming Waste and Uncertainty in Food Distribution”

Xianjia Gao, Xin Chen

EC.1. Technical Proofs in Section 4

EC.1.1. Proof of Lemma 1

Notice that V^F can be reformulated as a dynamic programming problem as

$$\begin{aligned} V_t^F(y_t) = & \max_{x_t} \mathbb{E}[U_t(x_t)] + V_{t+1}^F\left(y_t - \sum_i x_{i,t} + s_{t+1}\right) \\ \text{s.t. } & \sum_{i=1}^N x_{i,j,m,t} \leq y_{j,m,t} \quad \forall m \in [M], j \in [J] \\ & x_t \geq 0 \end{aligned}$$

with end-of-horizon term $V_{T+1}^F(y) = 0$ for all state y . To prove $V^* \leq V^F$, it suffices to prove the following claim: $V_t(y) \leq V_t^F(y)$ for all $t \in [T]$ and state y .

To this end, we prove by induction that $V_t^F(\cdot)$ is concave. The result is obviously true for $V_{T+1}^F(\cdot)$. Now consider the inductive step. Since $V_{t+1}^F(\cdot)$ is concave, we obviously have $V_{t+1}^F\left(y_t - \sum_i x_{i,t} + s_{t+1}\right)$ concave in x_t , meaning that the objective function $\mathbb{E}[U_t(x_t)] + V_{t+1}^F\left(y_t - \sum_i x_{i,t} + s_{t+1}\right)$ is concave in x_t and the optimization problem is a convex optimization. To prove concavity of $V_t^F(\cdot)$, let us consider two different states y and y' , and we aim to prove that $V_t^F(\lambda y + (1-\lambda)y') \geq \lambda V_t^F(y) + (1-\lambda)V_t^F(y')$ for any $\lambda \in [0, 1]$.

We define a new state $\tilde{y}_t = \lambda y_t + (1-\lambda)y'_t$ for arbitrary two states y_t, y'_t and $\lambda \in [0, 1]$. Then we have

$$V_t^F(\tilde{y}_t) = \mathbb{E}[U_t(\tilde{x}_t^*)] + V_{t+1}^F\left(\lambda y_t + (1-\lambda)y'_t - \sum_i \tilde{x}_{i,t}^* + s_{t+1}\right)$$

where $\tilde{x}_t^*$ is optimal allocation maximizing the value function under state $\tilde{y}_t$. Let x_t^* and x'_t denote the optimal point under state y_t and y'_t , respectively. Then we can further calculate as follows:

$$\begin{aligned} V_t^F(\tilde{y}_t) &= V_t^F(\lambda y_t + (1-\lambda)y'_t) = \mathbb{E}[U_t(\tilde{x}_t^*)] + V_{t+1}^F\left(\lambda y_t + (1-\lambda)y'_t - \sum_i \tilde{x}_{i,t}^* + s_{t+1}\right) \\ &\geq \mathbb{E}[U_t(\lambda x_t^* + (1-\lambda)x'_t)] + V_{t+1}^F\left(\lambda y_t + (1-\lambda)y'_t - \lambda \sum_i x_{i,t}^* - (1-\lambda) \sum_i x'_{i,t} + s_{t+1}\right) \\ &\geq \lambda \left[\mathbb{E}[U_t(x_t^*)] + V_{t+1}^F\left(y_t - \sum_i x_{i,t}^* + s_{t+1}\right) \right] + (1-\lambda) \left[\mathbb{E}[U_t(x'_t)] + V_{t+1}^F\left(y'_t - \sum_i x'_{i,t} + s_{t+1}\right) \right] \\ &\geq \lambda V_t^F(y_t) + (1-\lambda)V_t^F(y'_t) \end{aligned}$$

where the first inequality is due to the fact that $\tilde{x}_t^*$ is the optimal solution under state $\tilde{y}_t$, and the second inequality is because $U_t(\cdot)$ and $V_{t+1}^F(\cdot)$ are concave.

Now let us prove the claim by induction. Again, for $T+1$ the claim holds trivially. Consider the inductive step, by inductive hypothesis, we have for any x_t ,

$$\begin{aligned} \mathbb{E}[U_t(x_t)] + V_{t+1}^F\left(y_t - \sum_i x_{i,t} + s_{t+1}\right) &\geq \mathbb{E}[U_t(x_t)] + \mathbb{E}_{S_{t+1}}\left[V_{t+1}^F\left(y_t - \sum_i x_{i,t} + S_{t+1}\right)\right] \\ &\geq \mathbb{E}[U_t(x_t)] + \mathbb{E}_{S_{t+1}}\left[V_{t+1}\left(y_t - \sum_i x_{i,t} + S_{t+1}\right)\right], \end{aligned}$$

where the first inequality is from Jensen's inequality with the previous claim that $V_t^F(\cdot)$ is concave for all t , and the second inequality is from inductive hypothesis. As a result, we have

$$V_t^F(y_t) \geq \mathbb{E}[U_t(x_t)] + V_{t+1}^F\left(y_t - \sum_i x_{i,t} + s_{t+1}\right) \geq V_t(y_t),$$

where x_t is the optimal solution of $V_t(y_t)$, and we are done.

EC.1.2. Proof of Theorem 1

To prove this theorem, let us first state a technical lemma below.

LEMMA EC.1. *We have $\mathbb{E}[Y_{j,m,t}] = y_{j,m,t}^*$ for all j, m, t .*

Proof. Notice that it suffices to prove this result for a fixed j . Next, we will prove the lemma by induction on t .

When $t = 1$, $\mathbb{E}[Y_{j,m,1}] = \mathbb{E}[S_{j,m,1}] + \mathbb{E}[I_{j,m,1}] = s_{j,m,1} = y_{j,m,1}^*$ for all m , the relation holds since there doesn't exist any on-hand inventory at the beginning stage. Assume the relation holds at $t \in [1, T-1]$, i.e., $\mathbb{E}[Y_{j,m,t}] = y_{j,m,t}^*$ for all m . Then

$$\begin{aligned} \mathbb{E}[Y_{j,m+1,t+1}] &= \mathbb{E}[S_{j,m+1,t+1}] + \mathbb{E}[I_{j,m+1,t+1}] = \mathbb{E}[S_{j,m+1,t+1}] + \mathbb{E}\left[Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^{SF}\right] \\ &= \mathbb{E}[S_{j,m+1,t+1}] + \mathbb{E}[Y_{j,m,t}] - \sum_{i=1}^N \frac{x_{i,j,m,t}^*}{y_{j,m,t}^*} \mathbb{E}[Y_{j,m,t}] \\ &= s_{j,m+1,t+1} + y_{j,m,t}^* - \sum_{i=1}^N \frac{x_{i,j,m,t}^*}{y_{j,m,t}^*} y_{j,m,t}^* = y_{j,m+1,t+1}^*. \end{aligned}$$

Moreover, when $m = 1$, we have $\mathbb{E}[Y_{j,1,t+1}] = \mathbb{E}[S_{j,1,t+1}] = s_{j,1,t+1} = y_{j,1,t+1}^*$ by definition. Thus we are done. $\square$

Now we are ready to prove the theorem. Since $V^* \leq V^F$ due to Lemma 1, we can rewrite the regret as

$$R^{SF} := V^* - V^{SF} \leq V^F - V^{SF} = \sum_{t=1}^T \mathbb{E}[U_t(x^*) - U_t(x^{SF})] \leq L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|x_{i,j,m,t}^* - x_{i,j,m,t}^{SF}|] \quad (\text{EC.1})$$

where the inequality is due to the Lipschitz continuity by Assumption 2. Notice that by definition of $x_{i,j,m,t}^{SF}$, for any fixed j, m, t , we have either $x_{i,j,m,t}^* \geq x_{i,j,m,t}^{SF}$ or $x_{i,j,m,t}^* \leq x_{i,j,m,t}^{SF}$ for all $i \in [N]$. As a result, we have

$$\sum_{i=1}^N \mathbb{E}[|x_{i,j,m,t}^* - x_{i,j,m,t}^{SF}|] = \mathbb{E}\left[\left|\sum_{i=1}^N x_{i,j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^{SF}\right|\right] = \mathbb{E}[|\mu_{j,m,t} y_{j,m,t}^* - \mu_{j,m,t} Y_{j,m,t}|] \quad (\text{EC.2})$$

where $\mu_{j,m,t} := \sum_{i=1}^N x_{i,j,m,t}^* / y_{j,m,t}^*$, and the second equality is by definition of $x_{i,j,m,t}^{SF}$ in Algorithm 1. Therefore, we have

$$\begin{aligned} \mathbb{E}[|\mu_{j,m,t} y_{j,m,t}^* - \mu_{j,m,t} Y_{j,m,t}|] &= \mu_{j,m,t} \mathbb{E}[|y_{j,m,t}^* - Y_{j,m,t}|] = \mu_{j,m,t} \mathbb{E}[|\mathbb{E}[Y_{j,m,t}] - Y_{j,m,t}|] \\ &\leq \mu_{j,m,t} \sqrt{\mathbb{E}[(\mathbb{E}[Y_{j,m,t}] - Y_{j,m,t})^2]} \\ &= \mu_{j,m,t} \sqrt{\text{Var}(Y_{j,m,t})} \end{aligned} \quad (\text{EC.3})$$

where the second equality is due to $\mathbb{E}[Y_{j,m,t}] = y_{j,m,t}^*$ from Lemma EC.1, and the inequality is due to Cauchy-Schwarz inequality.

Notice that $Y_{j,m,t} = I_{j,m,t} + S_{j,m,t}$ for $m > 1$ and $Y_{j,1,t} = S_{j,1,t}$ represents the system state, where $I_{j,m,t}$ evolves as defined in (1). Therefore, we can expand $Y_{j,m,t}$ as follows.

$$\begin{aligned} Y_{j,m,t} &= I_{j,m,t} + S_{j,m,t} = Y_{j,m-1,t-1} - \sum_i x_{i,j,m-1,t-1}^{SF} + S_{j,m,t} = (1 - \mu_{j,m-1,t-1}) Y_{j,m-1,t-1} + S_{j,m,t} \\ &= (1 - \mu_{j,m-1,t-1}) ((1 - \mu_{j,m-2,t-2}) Y_{j,m-2,t-2} + S_{j,m-1,t-1}) + S_{j,m,t} \\ &\quad \dots \\ &= \prod_{\ell=1}^{m-1} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,1,t-m+1} + \prod_{\ell=1}^{m-2} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,2,t-m+2} \\ &\quad + \dots + (1 - \mu_{j,m-1,t-1}) S_{j,m-1,t-1} + S_{j,m,t}, \end{aligned} \quad (\text{EC.4})$$

where the third equality and thereafter follow from the definition of x_t^{SF} . Consequently, we have

$$\begin{aligned} &\text{Var}(Y_{j,m,t}) \\ &= \text{Var}\left(\sum_{k=1}^m \prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k}\right) \end{aligned} \quad (\text{EC.5})$$

$$\begin{aligned}
&= \sum_{k=1}^m \text{Var} \left(\prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k} \right) \\
&\quad + 2 \sum_{k < k': k, k' \in [m]} \text{Cov} \left(\prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k}, \prod_{\ell=1}^{m-k'} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k',t-m+k'} \right) \\
&\leq \sum_{k=1}^m \text{Var} \left(\prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k} \right) \\
&\quad + 2 \sum_{k < k': k, k' \in [m]} \sqrt{\text{Var} \left(\prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k} \right) \text{Var} \left(\prod_{\ell=1}^{m-k'} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k',t-m+k'} \right)} \\
&= \left(\sum_{k=1}^m \sqrt{\text{Var} \left(\prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k} \right)} \right)^2 \\
&\leq \left(\sum_{k=1}^m (1 - \underline{\mu})^{m-k} \bar{\sigma}_S \right)^2 \leq (m \bar{\sigma}_S)^2
\end{aligned}$$

where the first inequality is by the fact that $\text{Cov}(X, Y) \leq \sqrt{\text{Var}(X)\text{Var}(Y)}$, and the second inequality is by the boundedness of variance of $S_{m,t}$ in Assumption 1 and we let $\underline{\mu} = \min_{m,t} \mu_{m,t}$.

Now combine (EC.1) to (EC.5), we get $R^{SF} \leq LJM^2 T \bar{\sigma}_S$ and we are done.

EC.1.3. Proof of Corollary 1

If $S_{j,m,t}$ are independent over t , then $\text{Cov} \left(\prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k}, \prod_{\ell=1}^{m-k'} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k',t-m+k'} \right) = 0$ for $k \neq k'$. Therefore,

$$\begin{aligned}
&\text{Var}(Y_{j,m,t}) \\
&= \text{Var} \left(\sum_{k=1}^m \prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k} \right) \\
&= \sum_{k=1}^m \text{Var} \left(\prod_{\ell=1}^{m-k} (1 - \mu_{j,m-\ell,t-\ell}) S_{j,k,t-m+k} \right) \\
&\leq \sum_{k=1}^m (1 - \underline{\mu})^{2(m-k)} \text{Var}(S_{j,k,t-m+k}) \\
&\leq \sum_{k=1}^m (1 - \underline{\mu})^{2(m-k)} \bar{\sigma}_S^2 = \frac{1 - (1 - \underline{\mu})^{2m}}{1 - (1 - \underline{\mu})^2} \bar{\sigma}_S^2 \leq \frac{1}{1 - (1 - \underline{\mu})^2} \bar{\sigma}_S^2
\end{aligned} \tag{EC.6}$$

Now replace (EC.4) by (EC.5), and we obtain the result by following the proof of Theorem 1.

EC.1.4. Proof of Proposition 1

Part (a). We start by perturbation analysis of $V_t^F(a)$ with respect to state a where a is up-to-level inventory after receiving donations. By definition, V_t^F solves a convex optimization problem with the format as

$$\begin{aligned}
&\max_x \quad f(x, a) \\
&\text{s.t.} \quad g(x, a) \leq 0
\end{aligned}$$

where $f(x, a)$ is concave function in (x, a) and it represents our objective function $\sum_{s \geq t} \mathbb{E}[U_s(x_s)]$. The only part that includes state a in the objective function is the food waste part and the impact of current state a can only last for at most M period since state a includes all food with different expiration date and after at most M all the food will be expired. As a result, we can split $f(x, a) = f_1(x) - f_2(x, a)$ where $f_2(x, a) := \theta \sum_{s=t}^{t+M-1} \sum_{j=1}^J (y_{j,M,s} - \sum_{i=1}^N x_{i,j,M,s})$ and $f_1(x)$ represents the rest of the objective function irrelevant of a . For constraints, the only relevant constraint (inequality constraint) on a are those $\sum_{i=1}^N x_{i,j,m,s} - y_{j,m,s} \leq 0$ for all $j \in [J], s = t, \dots, t + M - 1, m_s = (s - t) + 1, \dots, M$ by system dynamics of $y_{j,m,s}$. All the equality constraints represent system dynamics and can be incorporated into state transition.

Therefore, we can write constraint $g(x, a)$ as $g_{j, m_s, s}(x, a) := \sum_{i=1}^N x_{i, j, m_s, s} - y_{j, m_s, s}$ for all $j \in [J], s = t, \dots, t + M - 1, m_s = (s - t) + 1, \dots, M$. Next we define Lagrangian of the optimization $V_t^F(a)$ as $L(x, \lambda, a) = f(x, a) - \lambda^\top g(x, a)$ for $\lambda \geq 0$ on those constraints $g_{j, m_s, s}(x, a) \leq 0$ with $j \in [J], s = t, \dots, t + M - 1, m_s = (s - t) + 1, \dots, M$ relevant to a . We first prove a lemma showing that our convex optimization problem has strong duality.

LEMMA EC.2. *We have*

$$V_t^F(a) = \min_{\lambda \geq 0} \max_x L(x, \lambda, a) = \min_{\lambda \geq 0} L(x^*(\lambda), \lambda, a),$$

where $x^*(\lambda) \in \arg \max_x L(x, \lambda, a)$ is a primal optimal solution corresponding to dual variable λ .

Proof. In this proof, let us fix a . Note that $L(x, \lambda, a)$ is continuous in both x, λ , and it is linear in λ given any x . Moreover, it suffices to restrict x to the compact set

$$\mathcal{X} = \left\{ x_{i, j, m, s} \in \left[0, a_{j, m} + \sum_{s'=t}^T \max_{j', m'} s_{j', m', s'} \right] : i, j, m, s \geq t \right\},$$

since no optimal solution can allocate more than the total available and future supply. This ensures the feasible region is compact.

Since $\mathcal{X}$ is convex and compact, we can apply Danskin's theorem (see, e.g., Proposition 4.5.1 in Bertsekas et al. 2003) so that any subgradient $\partial_\lambda L(x^*(\lambda), \lambda, a)$ is given by $\text{Conv}\{-g(x, a) : x \in \mathcal{X}^*(\lambda)\}$ where Conv is the convex hull and $\mathcal{X}^*(\lambda) \subset \mathcal{X}$ is the set of optimal primal solution given λ , which must be convex given the Lagrangian maximization problem has concave objective. Since $g(x, a)$ is linear and $\mathcal{X}^*(\lambda)$ is convex, we have

$$\text{Conv}\{-g(x, a) : x \in \mathcal{X}^*(\lambda)\} = \{-g(x, a) : x \in \mathcal{X}^*(\lambda)\}.$$

In the remaining of the proof, let us prove a complementary slackness condition. That is, for any dual optimal $\lambda^* \geq 0$, there exists $x^*(\lambda^*) \in \mathcal{X}$ so that $g(x^*(\lambda^*), a) \leq 0$ and

$$(\lambda^*)^\top g(x^*(\lambda^*), a) = 0. \quad (\text{EC.7})$$

To see that, by Proposition 4.7.2 in Bertsekas et al. (2003), λ^* is optimal on $\lambda \geq 0$ if and only if there exists a subgradient in $\{-g(x, a) : x \in \mathcal{X}^*(\lambda)\}$ (denoted by $-g(x(\lambda^*), a)$) so that

$$-g(x(\lambda^*), a)(\lambda^* - \lambda) \leq 0 \quad (\text{EC.8})$$

for all $\lambda \geq 0$. We claim that this is equivalent to the complementary slackness condition we want to prove.

To see “ $\rightarrow$ ”, letting $\lambda = \lambda^* + 1 \geq 0$ (as $\lambda_j^* \geq 0$) we have $g(x(\lambda^*), a) \leq 0$, showing primal feasibility. Together with $\lambda^* \geq 0$, we also have $-g(x(\lambda^*), a)\lambda^* \geq 0$. On the other hand, let $\lambda = 0$, (EC.8) implies $-g(x(\lambda^*), a)\lambda^* \leq 0$. As a result, we immediately have $g(x(\lambda^*), a)\lambda^* = 0$.

For “ $\leftarrow$ ”, this is obvious because $-g(x(\lambda^*), a)(\lambda^* - \lambda) = g(x(\lambda^*), a)\lambda \leq 0$ by primal feasibility and $\lambda \geq 0$.

To see strong duality, we have $L(x^*(\lambda^*), \lambda^*, a) = f(x^*(\lambda^*), a) \leq V_t^F(a)$, where the equality is by complementary slackness (EC.7), and the inequality is by definition of $V_t^F(a)$ and primal feasibility of $x^*(\lambda^*)$. Thus, combining this inequality with weak duality leads to the strong duality. $\square$

As a result of EC.2, the value function $V_t^F(a)$ admits the representation $V_t^F(a) = \min_{\lambda \geq 0} L(x^*(\lambda), \lambda, a)$, where $x^*(\lambda)$ is a primal optimal solution corresponding to dual variable λ . Before proceeding to the next step of the analysis, we now establish a technical lemma that provides an upper bound on the magnitude of the dual optimal variable λ^* .

LEMMA EC.3. *For any a , any dual optimal λ^* must satisfy $\sum_{m'=m_s}^M \lambda_{j, m', s+(m'-m_s)}^* \leq L$ for all $j \in [J], s = t, \dots, t + M - 1$, and $m_s = (s - t) + 1, \dots, M$.*

Proof. The KKT conditions for optimality tell us that at optimality, the gradient of the Lagrangian with respect to the decision variables x should vanish. However, in our setting, the objective function $f(x, a)$ may not be differentiable.

Since $f(x, a)$ is concave and Lipschitz continuous, it admits a nonempty supergradient set at any point x . Formally, there exists a vector $v \in \partial_x f(x, a)$, the supergradient set of f with respect to x , such that the following stationarity condition holds: $v - (\lambda^*)^\top \partial g(x, a) / \partial x = 0$. Since $f(x, a)$ is Lipschitz continuous with constant L in Assumption 2, then we have $\|v\|_\infty \leq L$. By the definition of $g(x, a)$, the partial derivative is

$$\frac{\partial g_{j', m', s'}(x, a)}{\partial x_{i, j, m, s}} = \begin{cases} 1, & \text{if } j' = j, s \leq s', \text{ and } m + (s' - s) = m', \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have $\sum_{m'=m_s}^M \lambda_{j, m', s+(m'-m_s)}^* \leq L$ for each i, j, m, s . $\square$

Next, we will continue proof for Proposition 1(a). By Lemma EC.2, strong duality holds: $V_t^F(a) = \min_{\lambda \geq 0} L(x^*(\lambda), \lambda, a)$, where $x^*(\lambda) \in \arg \max_x L(x, \lambda, a)$.

From Lemma EC.3, we can restrict the domain of dual variables to the compact and convex set:

$$\Lambda = \left\{ \lambda \geq 0 : \sum_{m'=m}^M \lambda_{j, m', s+(m'-m)} \leq L, \quad \forall j, m, s \geq t \right\}.$$

Thus, we can write $V_t^F(a) = \min_{\lambda \in \Lambda} L(x^*(\lambda), \lambda, a)$.

The function $L(x, \lambda, a)$ is affine in a and continuous in both λ, a , and $\lambda \in \Lambda$ lies in a compact set. Therefore, by Danskin's theorem, the superdifferential of $V_t^F(a)$ is given by:

$$\partial V_t^F(a) = \text{Conv} \left\{ \frac{\partial L(x^*(\lambda), \lambda, a)}{\partial a} : \lambda \in \Lambda^*(a) \right\},$$

where $\Lambda^*(a) \subseteq \Lambda$ is the set of optimal dual solutions at a , and Conv denotes the convex hull. For any $\lambda \in \Lambda^*(a)$, and for any j, m , we can compute:

$$\frac{\partial L(x^*(\lambda), \lambda, a)}{\partial a_{j, m}} = -\theta - \sum_{s=t}^{t+M-m} \lambda_{j, m+(s-t), s}.$$

To see this equation, we note that the change in $a_{j, m}$ affects multiple constraints $g_{j, m', s}$, where $m' = m + (s - t)$ for $s = t, t+1, \dots, t+M-m$. Each of these constraints has a corresponding dual variable $\lambda_{j, m+(s-t), s}$, hence the sum: $\sum_{s=t}^{t+M-m} \lambda_{j, m+(s-t), s}$. By the definition of Λ , we know $\sum_{s=t}^{t+M-m} \lambda_{j, m+(s-t), s} \leq L$, which implies

$$\left| \frac{\partial L(x^*(\lambda), \lambda, a)}{\partial a_{j, m}} \right| \leq \theta + L.$$

Thus, every supergradient $w \in \partial V_t^F(a)$ satisfies $|w_{j, m}| \leq \theta + L$. Hence, for any a, a' ,

$$|V_t^F(a) - V_t^F(a')| \leq \sup_{w \in \partial V_t^F(a)} \sum_{j, m} |w_{j, m}| \cdot |a_{j, m} - a'_{j, m}| \leq (\theta + L) \sum_{j, m} |a_{j, m} - a'_{j, m}|.$$

This completes the proof of Part (a).

Proof of Part (b). Next we will show the Jensen's gap. Let us define $f_{t+1} = f_t - x_t^t + s_{t+1}$ and $\tilde{f}_{t+1} = f_t - x_t^t + S_{t+1}$ (thus we have $f_{t+1} = \mathbb{E}[\tilde{f}_{t+1}]$). Therefore, we have

$$\begin{aligned} V_{t+1}^F(f_{t+1}) - \mathbb{E}[V_{t+1}^F(\tilde{f}_{t+1})] &= \mathbb{E}[V_{t+1}^F(f_{t+1}) - V_{t+1}^F(\tilde{f}_{t+1})] \\ &\leq \mathbb{E} \left[(L + \theta) \sum_{j=1}^J \sum_{m=1}^M |f_{j, m, t+1} - \tilde{f}_{j, m, t+1}| \right] = (L + \theta) \sum_{m=1}^M \sum_{j=1}^J \mathbb{E}[|s_{j, m, t+1} - S_{j, m, t+1}|] \\ &\leq (L + \theta) \sum_{m=1}^M \sum_{j=1}^J \sqrt{\mathbb{E}[(s_{j, m, t+1} - S_{j, m, t+1})^2]} \\ &= (L + \theta) \sum_{m=1}^M \sum_{j=1}^J \sqrt{\text{Var}(S_{j, m, t+1})} \leq (L + \theta) M J \bar{\sigma}_S \end{aligned}$$

where the first inequality is from part (a) of this proposition, the second inequality is from Jensen's inequality, and the last inequality is by invoking Assumption 1.

EC.1.5. Proof of Theorem 2

To prove this theorem, let us make the claim that for any t and state Y_t , $V_t^F(Y_t) - V_t^{\text{ReOPT}}(Y_t) \leq (T - t)(L + \theta) M J \bar{\sigma}_S$, where ReOPT represents the actual implemented re-optimized policy and $V_t^{\text{ReOPT}}(Y_t)$ denotes its cor-

responding objective value starting from time t with state Y_t . Then the result follows immediately as $R^{ReOPT} = V^* - V_1^{ReOPT}(Y_1) \leq V_1^F(Y_1) - V_1^{ReOPT}(Y_1)$ where the inequality is from Lemma 1.

To prove this claim, we use induction on t . When $t = T$, the claim holds trivially as by definition of the re-optimization policy, $V_T^F(Y) = V_T^{ReOPT}(Y)$ for any state Y . Now let us assume that the result holds for $t + 1$, we then have

$$\begin{aligned} V_t^F(Y_t) &= U_t(x_t^t) + V_{t+1}^F(Y_t - x_t^t + s_{t+1}) \\ &\leq U_t(x_t^t) + \mathbb{E}[V_{t+1}^F(Y_t - x_t^t + S_{t+1})] + (L + \theta)MJ\bar{\sigma}_S \\ &\leq U_t(x_t^t) + \mathbb{E}[V_{t+1}^{ReOPT}(Y_t - x_t^t + S_{t+1})] + (L + \theta)MJ\bar{\sigma}_S + (T - t - 1)(L + \theta)MJ\bar{\sigma}_S \\ &= V_t^{ReOPT}(Y_t) + (T - t)(L + \theta)MJ\bar{\sigma}_S, \end{aligned}$$

where the first inequality is from the Jensen's gap in Proposition 1.(b), the second inequality is from the inductive hypothesis, and the last equality is by definition of the re-optimization policy $ReOPT$.

EC.2. Further Content in Section 6

EC.2.1. Detailed Algorithms in Section 6

Algorithm 3 Budget Scaled Fluid (BSF) policy

Require: All parameters for the original problem (2) and the purchasing budget B .

- 1: **Compute fluid solution** (x^*, z^*) . At the beginning of the time horizon, solve the fluid optimization problem V^{BF} , which extends (8) with the budget constraint $\sum_{j,m,t} z_{j,m,t} c_{j,m} \leq B$ and the modified inventory relation $y_{j,m,t} = s_{j,m,t} + i_{j,m,t} + z_{j,m,t}$. Obtain the fluid optimal allocation $x_{i,j,m,t}^*$, the fluid optimal purchase quantity $z_{j,m,t}^*$, and the resulting sample path of on-hand inventory $y_{j,m,t}^*$.
- 2: **Calibrate the operator** $\psi_{j,m,t}$ **from** $z_{j,m,t}^*$. For each $j \in [J], m \in [M], t \in [T]$, choose $\psi_{j,m,t} \in [0, y_{j,m,t}^*]$ as the unique solution to

$$\mathbb{E}[\min\{(y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+, \psi_{j,m,t}\}] = z_{j,m,t}^*,$$

where the expectation is taken over $\tilde{Y}_{j,m,t} = S_{j,m,t} + I_{j,m,t}$. Existence and uniqueness of such $\psi_{j,m,t}$ follow from the intermediate value theorem applied to the continuous, monotone map $\psi \mapsto \mathbb{E}[\min\{(y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+, \psi\}]$ (see the proof of Theorem 3 in the appendix). The operator $\psi_{j,m,t}$ is preset at time 0 and does not depend on the realized sample path.

- 3: **Calculate quantity of food purchased** $Z_{j,m,t}$. For each time period $t = 1, \dots, T$, observe the on-hand inventory after donations as $\tilde{Y}_{j,m,t} = S_{j,m,t} + I_{j,m,t}$, and set

$$Z_{j,m,t} = \min\{(y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+, \psi_{j,m,t}\}.$$

By construction of $\psi_{j,m,t}$ in Step 2, $\mathbb{E}[Z_{j,m,t}] = z_{j,m,t}^*$.

- 4: **Apply the scaled fluid policy**. For each time period $t = 1, \dots, T$, observe the on-hand inventory after replenishment as $Y_{j,m,t} = \tilde{Y}_{j,m,t} + Z_{j,m,t}$, and define x_t^{BSF} as $x_{i,j,m,t}^{BSF} = (x_{i,j,m,t}^* / y_{j,m,t}^*) \cdot Y_{j,m,t}$ when $y_{j,m,t}^* > 0$; otherwise, $x_{i,j,m,t}^{BSF} = 0$. Apply the allocation policy x_t^{BSF} and update the on-hand inventory as $I_{j,m+1,t+1} = Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^{BSF}$ for $m \leq M - 1$, and $I_{j,1,t+1} = 0$.
-

Algorithm 4 Scaled Fluid with fairness (FSF) policy

Require: All parameters for the original problem V^* with parameter $\Delta \geq 0$.

- 1: **Compute minimum ratio $\underline{K}$ to guarantee fairness constraint**. At the beginning of time horizon, solve the fluid optimization problem V^F and obtain the fluid optimal solution x^* together with the sample path of on-hand inventory after donation $y_{j,m,t}^*$. Define cutting ratio $K_{j,m,t} = \frac{Y_{j,m,t}}{y_{j,m,t}^*}$ and pick the smallest $K_{j,m,t}$ over m less than one, $\underline{K}_{j,t} = \min_m \{K_{j,m,t}\}$ for all j .
- 2: **Apply the scaled fluid policy**. For each j , define $x_{j,t}^{FSF}$ as

$$x_{i,j,m,t}^{FSF} = x_{i,j,m,t}^* \max\{\underline{K}_{j,t}, 1 - \sqrt{6 \log(JMT)} \tilde{\sigma}_S / s\}$$

if $\sum_i x_{i,j,m,t}^{FSF} \leq Y_{j,m,t}$ for all m ; otherwise, define $x_{j,t}^{FSF}$ as $x_{i,j,m,t}^{FSF} = x_{i,j,m,t}^* \underline{K}_{j,t}$. Apply the allocation policy x_t^{FSF} and update the on-hand inventory as $I_{j,m+1,t+1} = Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^{FSF}$ for $m \leq M - 1$, and $I_{j,1,t+1} = 0$.

EC.2.2. An Auxiliary Lemma for Theorem 3

The following lemma is a counterpart of Lemma EC.1 for the budget-constrained model. It relates the expected on-hand inventory after donation, $\mathbb{E}[\tilde{Y}_{j,m,t}]$, to the fluid quantities $y_{j,m,t}^*$ and $z_{j,m,t}^*$, and is invoked repeatedly in the proof of Theorem 3.

LEMMA EC.4. *We have $\mathbb{E}[\tilde{Y}_{j,m,t}] + z_{j,m,t}^* = y_{j,m,t}^*$ for all j, m, t .*

Proof. The proof is analogous to that of Lemma EC.1. Again, we fix one j and prove by induction together with $\mathbb{E}[Z_{j,m,t}] = z_{j,m,t}^*$. When $t = 1$, $\mathbb{E}[\tilde{Y}_{j,m,1}] = \mathbb{E}[S_{j,m,1}] + \mathbb{E}[I_{j,m,1}] = s_{j,m,1} = y_{j,m,1}^* - z_{j,m,1}^*$ for all m by definition of $y_{j,m,1}^*, z_{j,m,1}^*$. Assume the relation holds at $t \in [1, T-1]$, i.e., $\mathbb{E}[\tilde{Y}_{j,m,t}] = y_{j,m,t}^* - z_{j,m,t}^*$ (or equivalently $\mathbb{E}[Y_{j,m,t}] = y_{j,m,t}^*$ as $\mathbb{E}[Z_{j,m,t}] = z_{j,m,t}^*$) for all m . Then

$$\begin{aligned} \mathbb{E}[\tilde{Y}_{j,m+1,t+1}] &= \mathbb{E}[S_{j,m+1,t+1}] + \mathbb{E}[I_{j,m+1,t+1}] = \mathbb{E}[S_{j,m+1,t+1}] + \mathbb{E}\left[Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^{SF}\right] \\ &= \mathbb{E}[S_{j,m+1,t+1}] + \mathbb{E}\left[Y_{j,m,t} - \sum_{i=1}^N \frac{x_{i,j,m,t}^*}{y_{j,m,t}^*} Y_{j,m,t}\right] = \mathbb{E}[S_{j,m+1,t+1}] + \mathbb{E}[Y_{j,m,t}] - \sum_{i=1}^N \frac{x_{i,j,m,t}^*}{y_{j,m,t}^*} \mathbb{E}[Y_{j,m,t}] \\ &= s_{j,m+1,t+1} + y_{j,m,t}^* - \sum_{i=1}^N \frac{x_{i,j,m,t}^*}{y_{j,m,t}^*} y_{j,m,t}^* = s_{j,m+1,t+1} + y_{j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^* = y_{j,m+1,t+1}^* - z_{j,m+1,t+1}^* \end{aligned}$$

where the fifth equality is by inductive hypothesis, and the last equality is by system dynamics of $y_{j,m+1,t+1}^*$. Moreover, when $m = 1$, we have $\mathbb{E}[Y_{j,1,t+1}] = \mathbb{E}[S_{j,1,t+1}] = s_{j,1,t+1} = y_{j,1,t+1}^* - z_{j,1,t+1}^*$ by definition. Thus we are done. $\square$

EC.2.3. Proof of Theorem 3

Proof. Let us first prove that $\psi_{j,m,t}$ with $\mathbb{E}[Z_{j,m,t}] = z_{j,m,t}^*$ is obtainable. We note that $\mathbb{E}[\tilde{Y}_{j,m,t}] = y_{j,m,t}^* - z_{j,m,t}^*$ according to Lemma EC.4. When $\psi_{j,m,t}$ goes to zero, $\mathbb{E}[Z_{j,m,t}] = 0$. When $\psi_{j,m,t}$ goes to $y_{j,m,t}^*$, $\mathbb{E}[Z_{j,m,t}] = \mathbb{E}[(y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+] \geq \mathbb{E}[y_{j,m,t}^* - \tilde{Y}_{j,m,t}] = z_{j,m,t}^*$. As $\mathbb{E}[Z_{j,m,t}]$ is continuous and increasing in $\psi_{j,m,t}$, we must have $\mathbb{E}[Z_{j,m,t}] = z_{j,m,t}^*$ for some value of $\psi_{j,m,t}$ by intermediate value theorem.

Following the same procedure, we can rewrite the regret of scaled fluid policy with budget constraints as

$$R^{BSF} := V^* - V^{BSF} \leq V^F - V^{BSF} = \sum_{t=1}^T \mathbb{E}[U_t(x^*) - U_t(x^{BSF})] \leq L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|x_{i,j,m,t}^* - x_{i,j,m,t}^{BSF}|]$$

Since $Z_{j,m,t}$ is implemented in practice instead of $z_{j,m,t}^*$, there is a risk of depleting the budget prematurely, which could lead to utility loss. To better quantify this loss, we define a new variable $\tilde{x}_{i,j,m,t}$, representing the food distribution following the same allocation policy as in Algorithm 3, but without the budget constraint. Therefore, the regret of the budget-scaled fluid policy can be expressed as follows:

$$\begin{aligned} R^{BSF} &\leq L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|x_{i,j,m,t}^* - x_{i,j,m,t}^{BSF}|] = L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|x_{i,j,m,t}^* - \tilde{x}_{i,j,m,t} + \tilde{x}_{i,j,m,t} - x_{i,j,m,t}^{BSF}|] \\ &\leq L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M (\mathbb{E}[|x_{i,j,m,t}^* - \tilde{x}_{i,j,m,t}|] + \mathbb{E}[|\tilde{x}_{i,j,m,t} - x_{i,j,m,t}^{BSF}|]) \end{aligned}$$

The first part $\mathbb{E}[|x_{i,j,m,t}^* - \tilde{x}_{i,j,m,t}|]$ comes from policy difference since both of them are under enough budget constraints. While the second part $\mathbb{E}[|\tilde{x}_{i,j,m,t} - x_{i,j,m,t}^{BSF}|]$ is the loss from budget constraint. Let us start with the first part. Invoking Lemma EC.4 (the budget-constrained counterpart of Lemma EC.1, established in Section EC.2.2), analyzing this term follows the same steps as Equations (EC.2) and (EC.3) in the proof of Theorem 1.

$$L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|x_{i,j,m,t}^* - \tilde{x}_{i,j,m,t}|] \leq L \sum_{t=1}^T \sum_{j=1}^J \sum_{m=1}^M \mu_{j,m,t} \sqrt{\text{Var}(Y_{j,m,t})} \quad (\text{EC.9})$$

Observe that $Y_{j,m,t} = I_{j,m,t} + S_{j,m,t} + Z_{j,m,t}$ for $m > 1$, which can also be written as $Y_{j,m,t} = \tilde{Y}_{j,m,t} + Z_{j,m,t}$, where $\tilde{Y}_{j,m,t} = I_{j,m,t} + S_{j,m,t}$. The variable $Z_{j,m,t}$ is designed to mitigate the variance caused by fluctuations in donations, $S_{j,m,t}$. As a result, we have $\text{Var}(Y_{j,m,t}) \leq \rho \text{Var}(\tilde{Y}_{j,m,t})$ for all j, m , and t . This implies that analyzing $\text{Var}(Y_{j,m,t})$

is effectively the same as analyzing $Var(\tilde{Y}_{j,m,t})$. Following similar steps as in Equation (EC.4), $Var(\tilde{Y}_{j,m,t})$ can be expanded as follows:

$$\begin{aligned} Var(\tilde{Y}_{j,m,t}) &= Var((1 - \mu_{j,m-1,t-1})Y_{j,m-1,t-1} + S_{j,m,t}) \\ &= Var((1 - \mu_{j,m-1,t-1})Y_{j,m-1,t-1}) + Var(S_{j,m,t}) + 2Cov((1 - \mu_{j,m-1,t-1})Y_{j,m-1,t-1}, S_{j,m,t}) \end{aligned}$$

Note that $Y_{j,m-1,t-1} = S_{j,m-1,t-1} + I_{j,m-1,t-1} + Z_{j,m-1,t-1}$. We assume $S_{j,m,t}$ is independent at dimension m, t , so $Cov(S_{j,m,t}, S_{j,m-1,t-1}) = 0$. $Z_{j,m-1,t-1}$ is determined by current up-to-level inventory $Y_{j,m-1,t-1}$, so $Cov(Z_{j,m-1,t-1}, S_{j,m,t}) = 0$. Based on this argument, we can further simplify $Var(\tilde{Y}_{j,m,t})$ as

$$\begin{aligned} Var(\tilde{Y}_{j,m,t}) &= Var((1 - \mu_{j,m-1,t-1})Y_{j,m-1,t-1}) + Var(S_{j,m,t}) \\ &\leq \rho(1 - \mu_{j,m-1,t-1})^2 Var(\tilde{Y}_{j,m-1,t-1}) + Var(S_{j,m,t}) \\ &\leq \rho(1 - \mu_{j,m-1,t-1})^2 (\rho(1 - \mu_{j,m-2,t-2})^2 Var(\tilde{Y}_{j,m-2,t-2}) + Var(S_{j,m-1,t-1})) + Var(S_{j,m,t}) \\ &\quad \dots \\ &\leq \prod_{\ell=1}^{m-1} \rho^{m-1} (1 - \mu_{j,m-\ell,t-\ell})^2 Var(\tilde{Y}_{j,1,t-m+1}) + \prod_{\ell=1}^{m-2} \rho^{m-2} (1 - \mu_{j,m-\ell,t-\ell})^2 Var(S_{j,2,t-m+2}) \\ &\quad + \dots + \rho(1 - \mu_{j,m-1,t-1})^2 Var(S_{j,m-1,t-1}) + Var(S_{j,m,t}) \\ &\leq \prod_{\ell=1}^{m-1} \rho^{m-1} (1 - \mu_{j,m-\ell,t-\ell})^2 Var(S_{j,1,t-m+1}) + \prod_{\ell=1}^{m-2} \rho^{m-2} (1 - \mu_{j,m-\ell,t-\ell})^2 Var(S_{j,2,t-m+2}) \\ &\quad + \dots + \rho(1 - \mu_{j,m-1,t-1})^2 Var(S_{j,m-1,t-1}) + Var(S_{j,m,t}) \\ &= \sum_{k=1}^m \prod_{\ell=1}^{m-k} \rho^{m-k} (1 - \mu_{j,m-\ell,t-\ell})^2 Var(S_{j,k,t-m+k}) \\ &\leq \sum_{k=1}^m \rho^{m-k} (1 - \underline{\mu})^{2(m-k)} \bar{\sigma}_S^2 = \frac{1 - \rho^m (1 - \underline{\mu})^{2m}}{1 - \rho(1 - \underline{\mu})^2} \bar{\sigma}_S^2 \leq \frac{1}{1 - \rho(1 - \underline{\mu})^2} \bar{\sigma}_S^2. \end{aligned}$$

Thus we are done with the first part which lead to the regret of $O(JTM\sqrt{\rho N})$.

Next, we start analyzing the second part $L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|\tilde{x}_{i,j,m,t} - x_{i,j,m,t}^{BSF}|]$, which leads to the following equation:

$$\begin{aligned} L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|\tilde{x}_{i,j,m,t} - x_{i,j,m,t}^{BSF}|] &= L \mathbb{E} \left[\sum_{t,j,i,m} \tilde{x}_{i,j,m,t} - \sum_{t,j,i,m} x_{i,j,m,t}^{BSF} \right] \\ &\leq \frac{L}{\underline{c}} \mathbb{E} \left[\left(\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} - B \right)^+ \right] \leq \frac{L}{\underline{c}} \mathbb{E} \left[\left(\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} - \sum_{t,j,m} z_{j,m,t}^* c_{j,m} \right)^+ \right] \\ &= \frac{L}{\underline{c}} \mathbb{E} \left[\left(\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} - \mathbb{E} \left[\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} \right] \right)^+ \right] \\ &\leq \frac{L}{\underline{c}} \sqrt{\mathbb{E} \left[\left(\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} - \mathbb{E} \left[\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} \right] \right)^2 \right]} \\ &= \frac{L}{\underline{c}} \sqrt{Var \left(\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} \right)} \\ &\leq \frac{L\bar{c}}{\underline{c}} \sqrt{Var \left(\sum_{t,j,m} \tilde{Z}_{j,m,t} \right)}, \end{aligned} \tag{EC.10}$$

where $\tilde{Z}_{j,m,t}$ is the counterpart of $Z_{j,m,t}$ in the hypothetical unbounded system. In the above system of equations, the first equality stems from the fact that $\tilde{x}_{i,j,m,t} \geq x_{i,j,m,t}^{BSF}$, where $\tilde{x}_{i,j,m,t}$ represents the food allocation under the same strategy but without budget constraints. When the budget is sufficient, we have $\tilde{x}_{i,j,m,t} = x_{i,j,m,t}^{BSF}$; otherwise, $\tilde{x}_{i,j,m,t} > x_{i,j,m,t}^{BSF}$. This relationship justifies the initial inequality, as the difference between $\tilde{x}_{i,j,m,t}$ and $x_{i,j,m,t}^{BSF}$ reflects the reduction in food allocation caused by budget limitations. Although our policy satisfies the budget constraint in expectation, that is, $\mathbb{E} \left[\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} \right] \leq B$, the randomness of $\tilde{Z}_{j,m,t}$ implies that the entire budget B may be consumed prematurely (i.e., at some $t_0 < T$). This can lead to a shortfall, quantified by $\frac{L}{\underline{c}} \mathbb{E} \left[\left(\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m} - B \right)^+ \right]$,

where $\underline{c}$ represents the minimum unit cost for food. To further refine this bound, we replace B with $\sum_{t,j,m} z_{j,m,t}^* c_{j,m}$, noting that $\sum_{t,j,m} z_{j,m,t}^* c_{j,m} \leq B$, which leads to the second inequality. The third inequality follows from Jensen's inequality, which can then be simplified to: $\frac{L}{\underline{c}} \sqrt{\text{Var}\left(\sum_{t,j,m} \tilde{Z}_{j,m,t} c_{j,m}\right)}$, as shown in the final expression.

Next, we focus on analyzing $\text{Var}(\sum_{t,j,m} \tilde{Z}_{j,m,t})$. Let us partition all (j, m, t) into the following index set:

$$\mathcal{A}_{j,t} := \{(j, m', t') : (m', t') = (1, t), (2, t+1), \dots, (M, t+M-1)\} \cap \{(j, m', t') : m \in [M], t \in [T]\}$$

for each $j \in [J], t \in \{-(M-2), \dots, T\}$. Since we assume that $S_{j,m,t}$ is independent with t , it is not difficult to see that for any pair of index (j_1, m_1, t_1) and (j_2, m_2, t_2) belonging to different index set $\mathcal{A}_{j,t}$, Z_{j_1, m_1, t_1} and Z_{j_2, m_2, t_2} are independent. As a result, we can write

$$\begin{aligned} \text{Var}\left(\sum_{j,m,t} \tilde{Z}_{j,m,t}\right) &= \text{Var}\left(\sum_{j,t} \sum_{(j,m',t') \in \mathcal{A}_{j,t}} \tilde{Z}_{j,m',t'}\right) = \sum_t \text{Var}\left(\sum_j \sum_{(j,m',t') \in \mathcal{A}_{j,t}} \tilde{Z}_{j,m',t'}\right) \\ &\leq \sum_t \left(\sum_j \sum_{(j,m',t') \in \mathcal{A}_{j,t}} \sqrt{\text{Var}(\tilde{Z}_{j,m',t'})} \right)^2 \\ &\leq T J^2 M^2 \bar{\sigma}_z^2 \end{aligned} \quad (\text{EC.11})$$

where $\bar{\sigma}_z^2$ is the upper bound of all $\text{Var}(\tilde{Z}_{j,m,t})$. Thus, all left is to bound $\text{Var}(\tilde{Z}_{j,m,t})$, which can be bounded as

$$\begin{aligned} \text{Var}(\tilde{Z}_{j,m,t}) &= \text{Var}\left(\min\{(y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+, \psi_{j,m,t}\}\right) \leq \text{Var}\left((y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+\right) \\ &\leq \text{Var}\left(y_{j,m,t}^* - \tilde{Y}_{j,m,t}\right) \leq \text{Var}(\tilde{Y}_{j,m,t}). \end{aligned} \quad (\text{EC.12})$$

The first inequality follows from the fact that $\text{Var}\left(\min\{(y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+, \psi_{j,m,t}\}\right) \leq \text{Var}\left((y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+\right)$ because $\psi_{j,m,t}$ is a constant. The second inequality holds for the same reason, as $\text{Var}\left((y_{j,m,t}^* - \tilde{Y}_{j,m,t})^+\right)$ can be rewritten as $\text{Var}\left(\max\{y_{j,m,t}^* - \tilde{Y}_{j,m,t}, 0\}\right)$. The last inequality is valid since $y_{j,m,t}^*$ is a constant, contributing no variance. We already calculate $\text{Var}(\tilde{Y}_{j,m,t})$ before. Combining (EC.10) to (EC.12), we have the second part of the regret $O(JM\sqrt{TN})$. Finally, after putting the results from first part and second part together, we obtain the final upper bound for our BSF policy.

EC.2.4. Proof of Theorem 4

Proof. We first prove that the FSF policy respects the fairness constraint. To see this, by definition, we notice that for fixed j, t , $x_{i,j,m,t}^{FSF}$ is always equal to $x_{i,j,m,t}^*$ multiplied with a constant (i.e., $\max\{\underline{K}_{j,t}, 1 - \sqrt{6\log(JMT)}\tilde{\sigma}_S/\underline{s}\}$ or $\underline{K}_{j,t}$). In either case, we have

$$\frac{\sum_m x_{i,j,m,t}^{FSF}}{\sum_m \sum_i x_{i,j,m,t}^{FSF}} = \frac{\sum_m x_{i,j,m,t}^*}{\sum_m \sum_i x_{i,j,m,t}^*}.$$

Since $x_{i,j,m,t}^*$ solves the fluid approximation, it obviously satisfies the fairness constraint. As a result, the FSF policy also satisfies the fairness constraint. Second, by definition of $\underline{K}_{j,t}$, our policy FSF also satisfies the constraint $\sum_i x_{i,j,m,t}^{FSF} \leq Y_{j,m,t}$. As a result, the FSF policy is feasible in implementation.

Next we prove the regret. We begin by rewriting the regret of the scaled fluid policy with fairness constraints. Let $R^{FSF} := V^* - V^{FSF}$. Using the same procedure as before, we have $R^{FSF} = V^* - V^{FSF} \leq V^F - V^{FSF} = \sum_{t=1}^T \mathbb{E}[U_t(x^*) - U_t(x^{FSF})]$. Since each period's utility U_t is L -Lipschitz, it follows that

$$\sum_{t=1}^T \mathbb{E}[U_t(x^*) - U_t(x^{FSF})] \leq L \sum_{t=1}^T \sum_{i=1}^N \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}[|x_{i,j,m,t}^* - x_{i,j,m,t}^{FSF}|] = L \sum_{t=1}^T \sum_{j=1}^J \sum_{m=1}^M \mathbb{E}\left[\left|\sum_{i=1}^N x_{i,j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^{FSF}\right|\right]$$

where the equality is by definition of $x_{i,j,m,t}^{FSF}$. Before we proceed further, let us prove a technical lemma regarding the deviation of system dynamics to its fluid mean.

LEMMA EC.5. *Conditioned on the event $\mathcal{A} = \{S_{j,m,t} \leq s_{j,m,t} + \sqrt{2\log(1/\delta)}\tilde{\sigma}_S \text{ and } S_{j,m,t} \geq (1 - \sqrt{2\log(1/\delta)}\tilde{\sigma}_S/\underline{s})s_{j,m,t} \quad \forall j, m, t\}$ for constant δ so that $\sqrt{2\log(1/\delta)}\tilde{\sigma}_S/\underline{s} = \sqrt{6\log(JMT)}\tilde{\sigma}_S/\underline{s}$, we have*

$$Y_{j,m,t} \geq (1 - \sqrt{2\log(1/\delta)}\tilde{\sigma}_S/\underline{s})y_{j,m,t}^*$$

$$Y_{j,m,t} \leq y_{j,m,t}^* + \frac{\bar{s}}{\underline{s}} \sqrt{2 \log(1/\delta)} m^2 \tilde{\sigma}_S$$

almost surely for all j, m, t . Moreover, we always have

$$x_{i,j,m,t}^{FSF} = x_{i,j,m,t}^* \max\{\underline{K}_{j,t}, 1 - \sqrt{6 \log(JMT)} \tilde{\sigma}_S / \underline{s}\}$$

for all i, j, m, t .

Proof. Let us fix j as the proof repeats itself for different values of j . Let us first prove the two inequalities regarding $Y_{j,m,t}$ and the definition of $x_{i,j,m,t}^{FSF}$ by induction on m and t .

For base case when $t = 1$, notice $Y_{j,m,1} = S_{j,m,1}$ for all m . As a result, the inequalities follow immediately from the definition of $\mathcal{A}$. Moreover, for any m ,

$$\frac{Y_{j,m,1}}{y_{j,m,1}^*} \geq 1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s} = 1 - \sqrt{6 \log(JMT)} \tilde{\sigma}_S / \underline{s}.$$

Therefore, we immediately have $\underline{K}_{j,t} \geq 1 - \sqrt{6 \log(JMT)} \tilde{\sigma}_S / \underline{s}$, proving the definition of $x_{i,j,m,1}^{FSF}$.

For inductive step, let us consider $m = 1$ first. In this case, we again have $Y_{j,1,t+1} = S_{j,1,t+1}$ and the inequalities follow from $\mathcal{A}$ again. For inequality for $Y_{j,m+1,t+1}$. By system dynamics, $Y_{j,m+1,t+1} = Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^{FSF} + S_{j,m+1,t+1}$. For lower bound, we have

$$\begin{aligned} Y_{j,m+1,t+1} &= Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^{FSF} + S_{j,m+1,t+1} \\ &\geq Y_{j,m,t} + (1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s}) s_{j,m+1,t+1} - \sum_{i=1}^N x_{i,j,m,t}^{FSF} \\ &\geq Y_{j,m,t} + (1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s}) s_{j,m+1,t+1} - \sum_{i=1}^N x_{i,j,m,t}^* \max\{Y_{j,m,t} / y_{j,m,t}^*, 1 - \sqrt{6 \log(JMT)} \tilde{\sigma}_S / \underline{s}\} \\ &= (1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s}) s_{j,m+1,t+1} + \min \left\{ Y_{j,m,t} \left(1 - \sum_{i=1}^N x_{i,j,m,t}^* / y_{j,m,t}^* \right), \right. \\ &\quad \left. Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^* (1 - \sqrt{6 \log(JMT)} \tilde{\sigma}_S / \underline{s}) \right\} \\ &\geq (1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s}) \left(s_{j,m+1,t+1} \right. \\ &\quad \left. + \min \left\{ y_{j,m,t}^* \left(1 - \sum_{i=1}^N x_{i,j,m,t}^* / y_{j,m,t}^* \right), y_{j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^* \right\} \right) \\ &= (1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s}) \left(s_{j,m+1,t+1} + y_{j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^* \right) \\ &= (1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s}) y_{j,m+1,t+1}^* \end{aligned}$$

where the first inequality is by $\mathcal{A}$, the second inequality is by inductive hypothesis on definition of $x_{i,j,m,t}^{FSF}$, the subsequent equality uses $-\max\{a, b\} = \min\{-a, -b\}$, the third inequality applies the inductive hypothesis on the lower bound of $Y_{j,m,t}$ to both terms of the minimum, using $1 - \sum_{i=1}^N x_{i,j,m,t}^* / y_{j,m,t}^* \geq 0$ and $\sqrt{2 \log(1/\delta)} = \sqrt{6 \log(JMT)}$ (definition of δ), so both terms equal $y_{j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^*$, and the last equality is by the dynamics of y^* .

For upper bound,

$$\begin{aligned} Y_{j,m+1,t+1} &= Y_{j,m,t} - \sum_{i=1}^N x_{i,j,m,t}^{FSF} + S_{j,m+1,t+1} \\ &\leq y_{j,m,t}^* + \bar{s} / \underline{s} \sqrt{2 \log(1/\delta)} m^2 \tilde{\sigma}_S - \sum_{i=1}^N x_{i,j,m,t}^{FSF} + s_{j,m,t} + \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S \\ &\leq y_{j,m,t}^* + \bar{s} / \underline{s} \sqrt{2 \log(1/\delta)} m^2 \tilde{\sigma}_S - \sum_{i=1}^N x_{i,j,m,t}^* (1 - \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s}) + s_{j,m,t} + \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S \\ &\leq y_{j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^* + s_{j,m,t} + \bar{s} / \underline{s} \sqrt{2 \log(1/\delta)} m^2 \tilde{\sigma}_S + m \bar{S} \cdot \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S / \underline{s} + \sqrt{2 \log(1/\delta)} \tilde{\sigma}_S \end{aligned}$$

$$\leq y_{j,m+1,t+1} + \bar{s}/\underline{s} \sqrt{2\log(1/\delta)}(m+1)^2 \tilde{\sigma}_S$$

where the first inequality is by inductive hypothesis on upper bound of $Y_{j,m,t}$, the second inequality is by inductive hypothesis on definition of $x_{i,j,m,t}^{FSF}$, and the third inequality is because $\sum_{i=1}^N x_{i,j,m,t}^* \leq y_{j,m,t}^* \leq m\bar{s}$. Therefore, we are done with the upper bound.

In the end, for definition of $x_{i,j,m,t+1}^{FSF}$, because

$$\frac{Y_{j,m,t+1}}{y_{j,m,t+1}^*} \geq 1 - \sqrt{2\log(1/\delta)} \tilde{\sigma}_S / \underline{s} = 1 - \sqrt{6\log(JMT)} \tilde{\sigma}_S / \underline{s}$$

where the inequality is from lower bound of $Y_{j,m,t+1}$. As a result, we immediately have $\underline{K}_{j,t} \geq 1 - \sqrt{6\log(JMT)} \tilde{\sigma}_S / \underline{s}$, proving the definition of $x_{i,j,m,t}^{FSF}$. $\square$

Now we are ready to proceed with the proof of our theorem. Since all $S_{i,j,m,t}$ are sub-Gaussian, according to Assumption 3, we have with probability $1 - 2\delta$, the following two inequalities hold simultaneously by tail bound of sub-Gaussian distribution (see, e.g., Lemma 1.3 in [Rigollet 2015](#)):

$$\begin{aligned} S_{j,m,t} &\leq s_{j,m,t} + \sqrt{2\log(1/\delta)} \tilde{\sigma}_S \\ S_{j,m,t} &\geq (1 - \sqrt{2\log(1/\delta)} \tilde{\sigma}_S / \underline{s}) s_{j,m,t}. \end{aligned}$$

Therefore, by applying union bound on all j, m, t , we have with probability at least $1 - 2JMT\delta$, event $\mathcal{A}$ in Lemma [EC.5](#) holds. To bound the regret $L \sum_{t=1}^T \sum_{j=1}^J \sum_{m=1}^M \mathbb{E} \left[\left| \sum_{i=1}^N x_{i,j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^{FSF} \right| \right]$, it is thus sufficient to condition on $\mathcal{A}$ as the regret on its complement can be bounded above by $O(JM^2T) \cdot (2JMT\delta) = O(1)$ by definition of δ . Invoke Lemma [EC.5](#), we have

$$\begin{aligned} &L \sum_{t=1}^T \sum_{j=1}^J \sum_{m=1}^M \mathbb{E} \left[\left| \sum_{i=1}^N x_{i,j,m,t}^* - \sum_{i=1}^N x_{i,j,m,t}^{FSF} \right| \middle| \mathcal{A} \right] \\ &= L \sum_{t=1}^T \sum_{j=1}^J \sum_{m=1}^M \sum_{i=1}^N x_{i,j,m,t}^* \mathbb{E} [|1 - \max\{\underline{K}_{j,t}, 1 - \sqrt{6\log(JMT)} \tilde{\sigma}_S / \underline{s}\}| | \mathcal{A}] \\ &\leq O(JM^3T \sqrt{N \log(JMT)}) \end{aligned}$$

where the equality is from Lemma [EC.5](#), and the inequality is by combining Lemma [EC.5](#) and the value of parameters in Assumption 3.

EC.3. Extended Synthetic Simulation Results

This section collects the full numerical tables underlying the figures and narrative in Section 5.1. All values are averaged over $K = 50$ iterations for all policies.

Table [EC.1](#) reports the complete parameter set used in the synthetic simulation of Section 5.1.

Table EC.1 Synthetic simulation parameters.

<i>Panel A: Supply parameters by food category.</i>						<i>Panel B: Demand and other parameters.</i>	
Food	M_j (wk)	$\mathbb{E}[S_{j,m,t}]$	per age	σ_S	λ_j	Mean weekly supply	
Bread	1	35.09	5.26	4.600	35.09	Number of agencies N	18
Produce	2	40.94	6.14	1.500	81.88	Planning horizon T	26 weeks
Dairy	4	14.62	2.19	0.800	58.48	Per-capita weekly need	0.23 lbs
Meat	8	7.31	1.10	0.400	58.48	Demand split (bread / produce / dairy / meat)	15% / 35% / 25% / 25%
Total weekly mean supply:						Demand coefficient of variation	0.20
						Agency population (scaled, range)	10.9 to 262.0
						Total scaled population	941.8
						Aggregate weekly mean demand	216.61
						Aggregate supply over aggregate demand	1.08

Table EC.2 reports the full numerical data underlying Figure 2: percentage regret for all six policies under three representative scenarios at $(C_u, C_o) = (0.4, 1.0)$, as the number of agencies N scales from 18 to 450.

Table EC.2 Percentage regret (%) by system size N at $(C_u, C_o) = (0.4, 1.0)$.

Scenario	Policy	$N = 18$	$N = 90$	$N = 180$	$N = 270$	$N = 360$	$N = 450$
Non-stat (mixed freq.)	SF	5.0	1.6	1.4	1.0	1.0	1.0
	ReOPT	1.8	0.9	1.1	1.0	0.8	0.5
	MP	5.6	5.7	4.8	5.0	5.6	5.2
	PA	12.2	10.8	10.8	10.9	10.9	10.4
	MMFR	171.3	185.1	194.2	195.5	200.7	194.3
	FIFO	53.2	51.2	51.3	51.3	51.4	51.2
Holiday Drive	SF	7.5	2.9	1.2	1.7	1.7	1.2
	ReOPT	3.6	1.7	1.9	1.6	0.4	1.4
	MP	31.0	28.8	29.2	28.2	28.4	27.2
	PA	57.2	54.1	54.8	53.9	53.7	53.4
	MMFR	350.9	365.4	391.2	396.9	394.1	403.0
	FIFO	147.9	145.0	144.5	144.7	144.6	144.6
Harvest Glut	SF	6.0	2.4	1.7	1.1	1.1	1.2
	ReOPT	3.6	0.5	1.9	0.7	0.7	0.9
	MP	20.8	19.7	19.3	19.8	19.2	19.2
	PA	27.4	25.9	26.8	25.7	25.6	25.1
	MMFR	270.5	272.7	287.0	286.3	292.2	277.7
	FIFO	117.2	115.8	116.0	115.2	115.3	115.6

Table EC.3 reports the percentage regret for all six policies under all seven scenarios at $(C_u, C_o) = (0.4, 1.0)$ with $N = 18$.

Table EC.3 Percentage regret (%) across scenarios at $(C_u, C_o) = (0.4, 1.0)$.

Scenario	SF	ReOPT	MP	PA	MMFR	FIFO
Stationary	6.0	2.9	8.8	3.4	150.9	55.6
Non-stat (mixed)	5.9	2.7	6.3	11.9	171.6	52.9
Non-stat (same)	6.0	2.4	7.1	9.7	169.3	53.7
Surge (3×)	6.2	3.0	19.5	33.0	264.6	112.3
Holiday Drive	8.0	4.0	30.5	55.9	350.8	146.9
Harvest Glut	5.9	2.3	21.5	27.7	271.1	118.2
Retail Clearance	6.9	2.8	21.0	25.3	238.3	90.9

Figure EC.1 and Table EC.4 demonstrate the robustness check of all policies with different combinations of (C_u, C_o) .

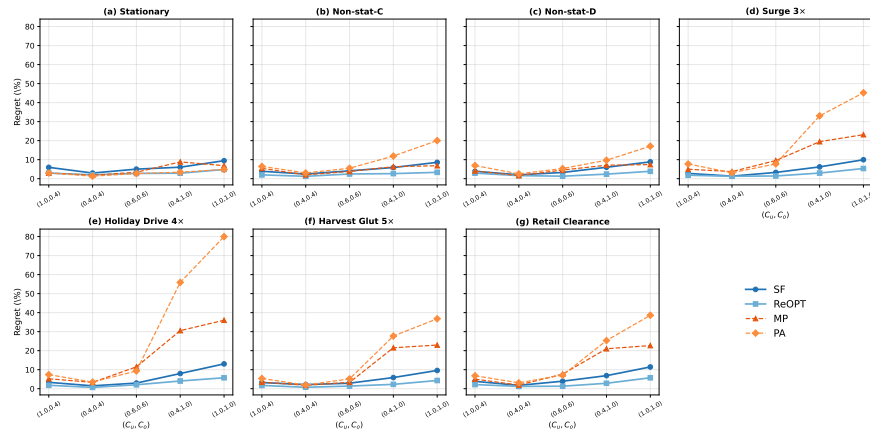


Figure EC.1 Percentage regret versus (C_u, C_o) in the seven supply scenarios. SF and ReOPT: solid lines; MP and PA: dashed. MMFR and FIFO (above 50% in every cell) are omitted; see Table EC.4.

Table EC.4 Percentage regret (%) by supply scenario and cost pair (C_u, C_o) . **Bold: smallest regret for each scenario and cost pair.**

(C_u, C_o)	Policy	Stationary	Non-stat-C	Non-stat-D	Surge-C-3x	Holiday-Drive	Harvest-Glut	Retail-Clearance
(1.0, 0.4)	SF	6.0	3.9	3.9	2.8	3.4	3.2	3.9
	ReOPT	2.9	2.0	2.9	1.8	1.8	1.7	2.2
	MP	3.0	5.4	4.1	5.0	5.2	3.4	5.1
	PA	3.0	6.5	6.9	7.7	7.4	5.4	6.8
	MMFR	145.9	171.9	177.5	202.9	218.3	196.9	194.7
	FIFO	60.1	65.7	67.0	111.5	127.4	112.1	100.6
(0.4, 0.4)	SF	3.0	2.4	2.0	1.3	1.5	2.2	1.7
	ReOPT	2.0	1.2	1.6	1.3	0.7	0.7	1.2
	MP	1.8	2.0	1.6	3.7	3.3	1.8	1.9
	PA	1.3	3.0	2.5	3.1	3.5	2.0	3.1
	MMFR	82.4	95.9	97.7	132.9	147.3	134.5	122.5
	FIFO	53.2	55.4	55.0	99.4	113.4	102.2	87.3
(0.6, 0.6)	SF	5.0	4.1	3.3	3.3	3.0	2.9	3.9
	ReOPT	2.8	2.5	1.3	1.4	2.1	1.4	1.3
	MP	3.4	3.8	4.5	9.5	11.5	3.3	7.7
	PA	2.7	5.5	5.3	7.8	9.3	5.1	7.1
	MMFR	124.7	144.0	145.8	192.7	219.0	191.8	179.0
	FIFO	56.5	58.2	58.1	107.5	126.3	109.6	92.5
(0.4, 1.0)	SF	6.0	5.9	6.0	6.2	8.0	5.9	6.9
	ReOPT	2.9	2.7	2.4	3.0	4.0	2.3	2.8
	MP	8.8	6.3	7.1	19.5	30.5	21.5	21.0
	PA	3.4	11.9	9.7	33.0	55.9	27.7	25.3
	MMFR	150.9	171.6	169.3	264.6	350.8	271.1	238.3
	FIFO	55.6	52.9	53.7	112.3	146.9	118.2	90.9
(1.0, 1.0)	SF	9.5	8.6	8.8	10.0	13.0	9.6	11.4
	ReOPT	4.8	3.3	3.9	5.3	5.7	4.3	5.7
	MP	6.9	6.8	7.4	23.1	36.0	23.0	22.7
	PA	4.8	20.0	17.1	45.2	80.0	36.8	38.6
	MMFR	229.7	283.5	289.7	413.4	582.5	410.7	389.4
	FIFO	65.1	66.5	68.6	137.2	189.8	138.6	116.2

EC.3.1. Effectiveness, Efficiency, and Equity Metrics

For any allocation $x_{i,j,m,t}$ and demand $D_{i,j,t}$ we define the following metrics used throughout Sections 5.1.4 and 5.2.4.

Effectiveness.

- *Average fill rate:* $\text{FillRate} = \frac{1}{NT} \sum_t \sum_i \frac{\sum_{j,m} x_{i,j,m,t}}{\sum_j D_{i,j,t}}$.
- *Aggregate fill rate:* $\text{AggFillRate} = \frac{\sum_{i,j,m,t} x_{i,j,m,t}}{\sum_{i,j,t} D_{i,j,t}}$.

Efficiency.

- *Waste rate:* fraction of received supply that expires before being distributed, $\text{WasteRate} = \frac{\sum_{j,m,t} S_{j,m,t} - \sum_{i,j,m,t} x_{i,j,m,t}}{\sum_{j,m,t} S_{j,m,t}}$.

- *Overage rate:* $\text{OverageRate} = \frac{\sum_{i,j,t} (\sum_m x_{i,j,m,t} - D_{i,j,t})^+}{\sum_{i,j,m,t} x_{i,j,m,t}}$.

Equity. Per-agency fill rate $f_i = \frac{\sum_{j,m,t} x_{i,j,m,t}}{\sum_{j,t} D_{i,j,t}}$. Metrics:

- *Jain index:* $\text{Jain} = (\sum_i f_i)^2 / (N \sum_i f_i^2)$; 1 indicates perfect equity ([Jain 1984](#)).
- *Coefficient of variation* $\text{CV}(f)$: lower is more equitable.
- *Average proportional deviation:* $\text{APD} = \frac{1}{|\mathcal{P}|N} \sum_{(j,t) \in \mathcal{P}} \sum_i \left| \frac{\sum_m x_{i,j,m,t}}{\sum_{i',m} x_{i',j,m,t}} - \frac{D_{i,j,t}}{\sum_{i'} D_{i',j,t}} \right|$, where $\mathcal{P}$ is the set of food-period pairs (j, t) with positive total allocation and positive total demand. That is, APD is the absolute gap between an agency's share of the allocation of food j in period t and its share of the demand for food j in period t , averaged over agencies, food categories, and periods; lower is more equitable.

EC.4. Extra Real Data Simulation Results

EC.4.1. Robustness to Near-Expiry Donations

Food bank donations often arrive close to end-of-life rather than fresh. The main results of Section 5.2.2 draw incoming supply independently for each age level $m \in \{0, \dots, M_j - 1\}$, producing an approximately *uniform* age distribution. To probe sharper near-expiry concentration, we replace this with a bimodal distribution: a fraction α of each food

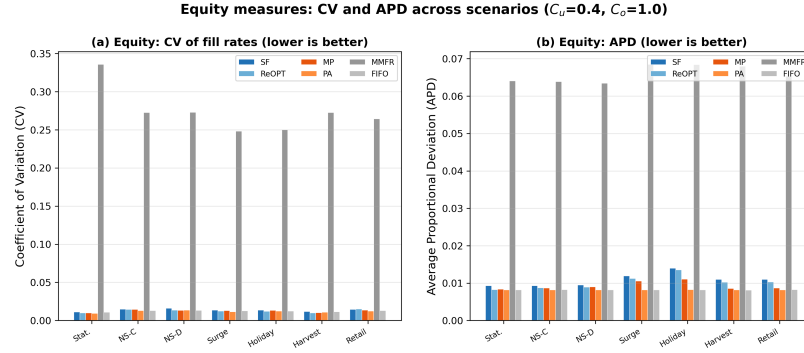


Figure EC.2 Sensitive equity metrics across seven scenarios at $(C_u, C_o) = (0.4, 1.0)$: coefficient of variation of per-agency fill rates (left) and average proportional deviation (right); lower is more equitable.

Table EC.5 Effectiveness, efficiency, and equity metrics across seven scenarios at $(C_u, C_o) = (0.4, 1.0)$. $\uparrow/\downarrow$: higher/lower is better; bold: best policy for each metric and scenario.

Metric	Policy	Stationary	Non-stat-C	Non-stat-D	Surge (3 $\times$)	Holiday Drive	Harvest Glut	Retail Clearance
(a) Average Fill Rate ($\uparrow$, Effectiveness)	SF	0.929	0.883	0.871	0.907	0.903	0.922	0.893
	ReOPT	0.939	0.892	0.887	0.919	0.911	0.932	0.900
	MP	0.935	0.887	0.883	0.912	0.906	0.929	0.901
	PA	0.945	0.879	0.869	0.893	0.883	0.915	0.884
(b) Aggregate Fill Rate ($\uparrow$, Effectiveness)	SF	0.930	0.886	0.873	0.911	0.904	0.922	0.895
	ReOPT	0.940	0.894	0.884	0.921	0.913	0.932	0.904
	MP	0.935	0.891	0.882	0.914	0.907	0.930	0.904
	PA	0.945	0.879	0.870	0.893	0.882	0.914	0.884
(c) Waste Rate ($\downarrow$, Efficiency)	SF	0.000	0.076	0.064	0.186	0.224	0.218	0.166
	ReOPT	0.003	0.078	0.066	0.188	0.227	0.217	0.167
	MP	0.003	0.079	0.067	0.216	0.299	0.266	0.218
	PA	0.000	0.000	0.000	0.000	0.000	0.000	0.000
(d) Overage Rate ($\downarrow$, Efficiency)	SF	0.131	0.128	0.130	0.210	0.266	0.212	0.196
	ReOPT	0.118	0.119	0.120	0.204	0.259	0.206	0.186
	MP	0.113	0.115	0.116	0.180	0.187	0.150	0.131
	PA	0.126	0.212	0.195	0.385	0.463	0.396	0.347
(e) Jain Fairness Index ($\uparrow$, Equity)	SF	0.9999	0.9998	0.9997	0.9998	0.9998	0.9999	0.9998
	ReOPT	0.9999	0.9998	0.9998	0.9998	0.9999	0.9999	0.9998
	MP	0.9999	0.9998	0.9998	0.9998	0.9998	0.9999	0.9998
	PA	0.9999	0.9998	0.9998	0.9999	0.9998	0.9999	0.9998
(f) Coefficient of Variation ($\downarrow$, Equity)	SF	0.0110	0.0149	0.0161	0.0135	0.0135	0.0117	0.0145
	ReOPT	0.0099	0.0144	0.0135	0.0122	0.0119	0.0100	0.0152
	MP	0.0098	0.0145	0.0134	0.0130	0.0134	0.0102	0.0136
	PA	0.0091	0.0129	0.0134	0.0115	0.0124	0.0109	0.0125
(g) Average Proportional Deviation ($\downarrow$, Equity)	SF	0.0093	0.0093	0.0095	0.0119	0.0140	0.0110	0.0110
	ReOPT	0.0082	0.0087	0.0089	0.0113	0.0135	0.0102	0.0103
	MP	0.0084	0.0087	0.0090	0.0106	0.0110	0.0085	0.0087
	PA	0.0082	0.0082	0.0082	0.0082	0.0082	0.0081	0.0082

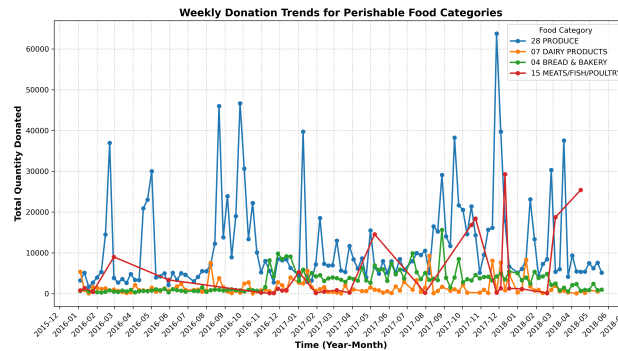


Figure EC.3 Weekly donations by food category at FBST, January 2016 to May 2018.

Table EC.6 Parameters calibrated from FBST operational data.

Category	Parameter	Value	Source
Structural	Agencies (N)	80	Agency Report
	Horizon (T)	26 weeks	Design (6 months)
	Food types (J)	4	Donation Records
	Simulation iterations (K)	50	Design
	Need per capita (PPW)	0.12 lbs/person/week	Derived from Agency Report
Shelf life M_j	Bread/Bakery	1 week	Real shelf life 1–3 days
	Produce	2 weeks	Fresh produce spoilage
	Dairy	4 weeks	Refrigerated shelf life
	Meat/Fish/Poultry	8 weeks	Frozen storage
Supply (Gamma)	Produce	mean 7,760, CV 0.36	FBST weekly records
	Dairy	mean 864, CV 0.46	FBST weekly records
	Bread/Bakery	mean 2,431, CV 0.53	FBST weekly records
	Meat/Fish/Poultry	mean 636, CV 0.35	FBST weekly records
Seasonal amplitude	Produce	0.56 (peak: Summer/Fall)	FBST data
	Dairy	0.42 (peak: Winter)	FBST data
	Bread/Bakery	0.30 (year-round)	FBST data
	Meat/Fish/Poultry	0.27 (holiday-driven)	FBST data
Surge multiplier	Produce, Dairy, Bread	5× mean	Actual max/mean ratios 6.4–11.3× Exact FBST observed peak
	Meat/Fish/Poultry	69× mean	
Demand	Base ratios (P/D/B/M)	64% / 8% / 22% / 6%	Supply composition
	Child weights	Dairy +2, Meat +1.5	Agency demographics
	Elderly weights	Produce +1.5	Agency demographics
	Seasonal amplitude	0.30 (peak: Winter)	Design, as in Table EC.1
	Coefficient of variation	0.15	Design

Table EC.7 Percentage regret (%) on FBST data under four cost pairs: mean (standard deviation) over $K = 50$ paths.

Policy	(1.0, 0.4)	(0.6, 0.6)	(0.4, 1.0)	(1.0, 1.0)
ReOPT	0.19 (6.19)	0.15 (6.47)	0.17 (7.24)	0.31 (7.51)
SF	1.30 (6.19)	1.19 (6.46)	1.68 (7.23)	2.50 (7.51)
MP	1.34 (6.20)	3.60 (6.45)	3.12 (7.24)	3.35 (7.51)
PA	18.28 (5.28)	19.97 (5.49)	28.12 (6.21)	29.01 (6.46)
MMFR	109.79 (2.50)	109.28 (2.68)	110.42 (3.24)	113.41 (3.42)
FIFO	114.11 (2.05)	114.02 (2.17)	116.21 (2.60)	119.31 (2.80)

type’s weekly supply arrives at age $m = M_j - 1$ (one period before expiry), and the remaining $1 - \alpha$ at age $m = 0$, with total supply per category and period held fixed. Bread ($M = 1$) is unaffected. We compare three settings at $(C_u, C_o) = (0.6, 0.6)$ over $K = 50$ CRN paths: **Uniform** (main results), **NE50** ($\alpha = 0.50$), and **NE70** ($\alpha = 0.70$). Table EC.8 reports percentage regret.

Table EC.8 Percentage regret (%) on FBST data at $(C_u, C_o) = (0.6, 0.6)$ under three supply-age profiles: Uniform (main results) and NE50/NE70, with 50%/70% of supply arriving at age $M_j - 1$.

Policy	Uniform	NE50	NE70
ReOPT	0.15	0.17	0.07
SF	1.19	0.56	0.64
MP	3.60	0.17	0.07
PA	19.97	19.64	21.37
MMFR	109.28	105.94	116.37
FIFO	114.02	109.74	120.57

Three findings follow. *First*, SF and ReOPT remain robust to the age shift: SF regret falls from 1.19% under Uniform to 0.56% under NE50 and 0.64% under NE70, while ReOPT stays below 0.2% throughout. SF’s decrease reflects that the bimodal redistribution makes the weekly age composition deterministic, so SF’s time-invariant ratios track the fluid upper bound more tightly. *Second*, MP converges to ReOPT under near-expiry: MP regret falls to 0.17% and 0.07% under NE50 and NE70, matching ReOPT to within tie-breaking noise. When most supply must be allocated in the current period there is little inter-temporal structure to exploit, so the advantage of our policies over

MP is conditional on meaningful carry-over potential. This extends the mechanism of Section 5.2.3: the value our policies capture depends not only on shelf life and supply variability but also on the remaining shelf life at arrival. *Third*, PA’s regret is essentially unchanged (19.64% to 21.37%) since its proportional rule is age-agnostic; MMFR and FIFO degrade modestly under NE70 (+7.1 and +6.6 pp) as their supply utilization drops from 28% to 24%.

EC.5. Numerical Validation of the Fairness Extension

We test the fairness extension on FBST data at $(C_u, C_o) = (1.0, 0.4)$, sweeping the fairness tolerance $\Delta \in \{0, 0.005, 0.02, 0.05, 0.20\}$ ($\Delta = 0$ enforces perfectly proportional allocation) with $K = 50$ common-random-number paths; FC- X denotes policy X under the fairness constraint. Figure EC.4 plots utility and average fill rate against Δ , and Table EC.9 summarizes policies at $\Delta = 0.005$. Three findings emerge. (1) Tightening from $\Delta = 0.20$ to $\Delta = 0$ costs FC-SF only about 9% of utility (from 4.65M to 4.25M); even at $\Delta = 0$, FC-SF’s 4.25M still exceeds the unconstrained PA baseline (3.88M) by 10%, because PA enforces proportionality period-by-period while FC-SF maintains the same constraint while planning across the shelf-life horizon. (2) Without the constraint, ReOPT outperforms SF and SF outperforms MP; under tight fairness, the three FC policies finish within 1.7% of one another in utility, and FC-ReOPT’s lead over FC-SF stays within 0.1–1.1% across the full Δ grid, because the constraint pins allocations near demand proportions and leaves little room for adaptive re-optimization. (3) Counterintuitively, tightening fairness can raise the average fill rate: at $\Delta = 0$, FC-SF’s average is about 5 percentage points above unconstrained SF’s, because unconstrained SF concentrates supply on large-demand agencies and leaves small ones near-zero fill, dragging the cross-agency mean down, while the constraint spreads supply more evenly and lifts the average.

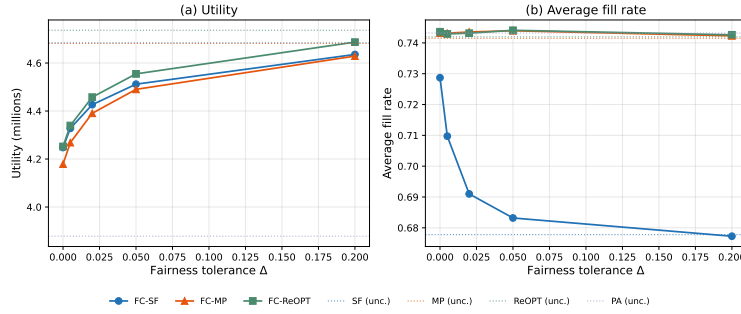


Figure EC.4 Fairness–utility trade-off on FBST data at $(C_u, C_o) = (1.0, 0.4)$. Dotted lines: unconstrained reference for each policy.

Table EC.9 Policy comparison at $(C_u, C_o) = (1.0, 0.4)$ and $\Delta = 0.005$ (FC: with fairness constraint; SF and PA unconstrained).

Policy	Utility	Avg Fill	Min/Mean Fill	Avg Prop Dev
FC-SF	4.33M	0.710	0.819	0.005
FC-MP	4.27M	0.743	0.868	0.004
FC-ReOPT	4.34M	0.743	0.865	0.004
SF (unconstrained)	4.68M	0.678	0.842	0.012
PA (unconstrained)	3.88M	0.743	0.981	0.001

References

- Bertsekas D, Nedic A, Ozdaglar A (2003) *Convex analysis and optimization*, volume 1 (Athena Scientific).
- Jain R (1984) A quantitative measure of fairness and discrimination for resource allocation in shared systems. *DEC, Sep. 1984*.
- Rigollet P (2015) 18. s997: High dimensional statistics. *Lecture Notes*, Cambridge, MA, USA: MIT OpenCourseWare.