Online Appendices to “Incentive Design and Pricing under Limited Inventory”

Ruiting Zuo Tinglong Dai Jussi Keppo

OA1. Technical Proofs

In this appendix, we give justification for the specific form of the agent contract in (7). Let us first only consider incentive design under constant price as in Section 3. Let us consider the process of the agent's value function V_t^{a} that is Markov and smooth with $V_t^{\mathrm{a}} \in C^{1,2}([0,T]\times\mathbb{R})$ in the sense of functional differentiation in Dupire (2009). Then, we have

$$dV_t^{\mathrm{a}} = \partial_t V_t^{\mathrm{a}} dt + Z_t dD_t + \tfrac{1}{2}\partial_{DD} V_t^{\mathrm{a}} d\langle D\rangle_t, \tag{OA1}$$

where, by Itô's formula, the process $Z_t = \partial_D V_t^{\mathrm{a}}(t, D_t)$, and ∂_D and ∂_{DD} are the first and second partial derivatives with respect to the demand process D. Thus, Z_t represents the sensitivity of the agent's value function with respect to the demand process, and it is the key to inducing sales effort.

By the martingale optimality principle and (5), the process $V_t^{\mathrm{a}} e^{-rt} - \int_0^t e^{-rs} c(A_s)ds$ is a supermartingale for all admissible control processes, and it is a martingale for the optimal admissible control process provided that the optimizer exists. Then, by Itô's formula, we get

$$\begin{aligned}
& d\left[V_t^{\mathrm{a}} e^{-rt} - \int_0^t e^{-rs} c(A_s)ds\right] \\
&= -re^{-rt}V_t^{\mathrm{a}} dt + e^{-rt} dV_t^{\mathrm{a}} - e^{-rt}c(A_t)dt \\
&= -re^{-rt}V_t^{\mathrm{a}} dt + e^{-rt}\left[\partial_t V_t^{\mathrm{a}} dt + Z_t dD_t + \tfrac{1}{2}\partial_{DD}V_t^{\mathrm{a}} d\langle D\rangle_t\right] - e^{-rt}c(A_t)dt \\
&= e^{-rt}\left[-rV_t^{\mathrm{a}} + \partial_t V_t^{\mathrm{a}} + \tfrac{1}{2}\partial_{DD}V_t^{\mathrm{a}}\sigma^2 + Z_t\cdot(A_t+a) - c(A_t)\right]dt + Z_t \sigma dB_t.
\end{aligned}$$

Using the fact that the drift term of a martingale vanishes, we get the following path-dependent Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = -\,rV_t^{\mathrm{a}} + \partial_t V_t^{\mathrm{a}} + \tfrac{1}{2}\partial_{DD}V_t^{\mathrm{a}}\sigma^2 + H_{\mathrm{a}}(Z_t), \tag{OA2}$$

where we introduced the Hamiltonian function for the agent's problem:

$$H_{\mathrm{a}}(z) \equiv \max_{A\in\mathbb{R}} h_{\mathrm{a}}(z,A) = \frac{\eta z^2}{2} + za, \tag{OA3}$$

where $h_{\mathrm{a}}(z,A) = z\cdot(A+a) - c(A)$ for all $z\in\mathbb{R}$ and $c(A) = \frac{A^2}{2\eta}$. The maximum is attained at $\hat{A}(z) = \eta z$. Note V^{a} in (2) is the agent's value function at $t=0$. That is, $V^{\mathrm{a}} = V_0^{\mathrm{a}}$, where the value function of the agent, V_t^{a}, is the solution of HJB equation (OA2) and the optimal effort of the agent is given by $\hat{A}(Z_t) = \eta Z_t$. Substituting (OA2) into (OA1) gives

$$\begin{aligned}
dV_t^{\mathrm{a}} &= \partial_t V_t^{\mathrm{a}} dt + Z_t dD_t + \tfrac{1}{2}\partial_{DD}V_t^{\mathrm{a}} d\langle D\rangle_t \\
&= \left[rV_t^{\mathrm{a}} - H_{\mathrm{a}}(Z_t)\right]dt + Z_t dD_t.
\end{aligned} \tag{OA4}$$

By (5), the agent's contract payoff depends on the value function, $V_T^{\mathrm{a}} = \xi_T$. As such, by the definition of V_t^{a} in (5) and the direct integration of (OA4), we get

$$\xi_T = V_T^{\mathrm{a}} = M_0 + \int_0^T \left(Z_t dD_t + \left[rV_t^{\mathrm{a}} - H_{\mathrm{a}}(Z_t)\right]dt\right), \tag{OA5}$$

where $M_0 = V_0^{\mathrm{a}}$. The form of contract in (OA5) is the same as the agent's contract in (7). The constant term Y_0 in (7) is equivalent to M_0 in (OA5). The process Y_t in (7) delivers the agent's continuation utility V_t. In other words, the process Y keeps track of the agent's future utility as shown in Theorem 1, which is often referred to as the "promised utility" in the literature (see, e.g., Ljungqvist and Sargent 2004).

PROOF OF THEOREM 1.

(i) To prove the equality of the sets, we are reduced to the problem of representing any contract $\xi_T \in \Xi_{\mathrm{a}}$ as $\xi_T = Y_T^{Y_0,Z}$ for some (Y_0, Z). We define the certainty equivalent of the utility continuation process $C_t^{\mathrm{a}} = C_t^{\mathrm{a}}(\xi_T)$ given any contract $\xi_T \in \Xi_{\mathrm{a}}$

$$C_t^{\mathrm{a}}(\xi_T) := \mathbb{E}_t^{\mathbb{P}^{\tilde{A}}}\left[e^{-r(T-t)}\xi_T - \frac{1}{2\eta}\int_t^T e^{-r(s-t)}\tilde{A}_s(\xi_T)^2 ds\right], \tag{OA6}$$

where $\mathbb{E}_t^{\mathbb{P}^{\tilde{A}}}$ is the expectation with respect to information $\mathcal{F}_t$, at time t, and the expectation $\mathbb{E}^{\mathbb{P}^{\tilde{A}}}$ is taken under the dynamics (1) with any control of the effort process $\tilde{A}(\xi_T)$. Indeed, consider the certainty equivalent of the utility continuation process $C_t^{\mathrm{a}} = C_t^{\mathrm{a}}(\xi_T)$ for some ξ_T. By the standard dynamic programming principle, we see that the non-negative process $\{e^{-rt}C_t^{\mathrm{a}} - \int_0^t e^{-rs}\frac{A_s^2}{2\eta}ds, t \le T\}$ is a $\mathbb{P}-$supermartingale for all admissible control A, and a martingale for some optimal control $\hat{A}$. By using the Doob-Meyer decomposition and the martingale representation theorem, we may then identify the process C^{a} with $Y^{C_0^{\mathrm{a}},Z}$ for some Z. The integrability condition, i.e., $E\left[\sup_{t\le T}|Y_t^{Y_0,Z}|\right] < \infty$ on Y is required by Corollary 4 Briand and Hu (2008), which we omitted in the main text for simplicity.

(ii) By (7), we get

$$\begin{aligned} de^{-rt}Y_t &= -re^{-rt}Y_t dt + e^{-rt}Z_t dD_t - e^{-rt}H_{\mathrm{a}}(Z_t)dt + e^{-rt}rY_t dt \\ &= e^{-rt}Z_t dD_t - e^{-rt}H_{\mathrm{a}}(Z_t)dt. \end{aligned} \tag{OA7}$$

By (2), (7) and direct integration of (OA7), the agent's problem can be written as:

$$\begin{aligned} V^{\mathrm{a}} &= \sup_{\{A_t\}_{t\in[0,T]}} E\left[Y_0 + \int_0^T e^{-rt}Z_t dD_t - e^{-rt}H_{\mathrm{a}}(Z_t)dt - \int_0^T e^{-rt}c(A_t)dt\right] \\ &= \sup_{\{A_t\}_{t\in[0,T]}} E\left[Y_0 + \int_0^T e^{-rt}\left(h_{\mathrm{a}}(A_t, Z_t) - H_{\mathrm{a}}(Z_t)\right)dt\right] \le Y_0, \end{aligned} \tag{OA8}$$

where $h_{\mathrm{a}}(Z_t, A_t) = Z_t \cdot (A_t + a) - c(A_t)$ and the expectation is taken under the demand dynamics (1). As in (6), the maximum of the Hamiltonian function is attained at $\hat{A}(Z_t) = Z_t\eta$. As such, the equality in (OA8) is obtained at the special choice of control $A_t = \hat{A}(Z_t)$. This shows that the agent's utility value is $V^{\mathrm{a}}(\xi_{\mathrm{m}}) = Y_0$, with optimal control given by $\hat{A}(Z_t)$.

Finally, to identify Y with the certainty equivalent of the agent's dynamic value function, we apply the same argument. For all $t \in [0,T]$, by (5) and direct integration of (OA7), we have

$$\begin{aligned} V_t^{\mathrm{a}} &= \sup_{\{A_s\}_{s\in[t,T]}} E\left[e^{-r(T-t)}Y_T^{Y_0Z} - \int_t^T e^{-r(s-t)}c(A_s)ds|D_t\right] \\ &= \sup_{\{A_s\}_{s\in[t,T]}} E\left[e^{rt}e^{-rt}Y_t^{Y_0Z} + e^{rt}\int_t^T de^{-rs}Y_s^{Y_0,Z} - \int_t^T e^{-r(s-t)}c(A_s)ds|D_t\right] \\ &= \sup_{\{A_s\}_{s\in[t,T]}} E\left[Y_t^{Y_0Z} + \int_t^T e^{-r(s-t)}Z_s dD_s - e^{-r(s-t)}H_{\mathrm{a}}(Z_s)ds - \int_t^T e^{-r(s-t)}c(A_s)ds|D_t\right] \\ &= \sup_{\{A_t\}_{s\in[t,T]}} E\left[Y_t^{Y_0Z} + \int_t^T e^{-r(s-t)}\left(h_{\mathrm{a}}(A_s, Z_s) - H_{\mathrm{a}}(Z_s)\right)ds|D_t\right] \le Y_t^{Y_0,Z}, \end{aligned} \tag{OA9}$$

where the second and third equality follow from integrating (OA7) from t to T. Therefore, for any $t \in [0,T]$, $V_t^{\mathrm{a}} \le Y_t^{Y_0,Z}$ and equality (i.e., $V_t^{\mathrm{a}} = Y_t^{Y_0,Z}$) is achieved with the agent's optimal effort $\hat{A}(Z_t)$, which complete the proof that the process $Y_t^{Y_0,Z}$ in (7) delivers the agent's continuation utility V_t.

Q.E.D.

By (OA4) and (OA9), we have the following lemma about the agent's value function.

Lemma OA1. *Given the payoff process characterized in (7) with terminal payoff $\xi_T = Y_T^{Y_0,Z}$, the agent's value function evolves according to*

$$dV_t^{\mathrm{a}} = [rV_t^{\mathrm{a}} - H_{\mathrm{a}}(Z_t)]\,dt + Z_t dD_t. \tag{OA10}$$

Proof of Lemma OA1. Following (OA9) and the optimal agent's effort in Theorem 1(ii), we have $V_t^{\mathrm{a}} = Y_t$. By the process of Y in (7), the agent's continuation utility V_t^{a} evolves according to

$$dV_t^{\mathrm{a}} = [rV_t^{\mathrm{a}} - H_{\mathrm{a}}(Z_t)]\,dt + Z_t dD_t,$$

which completes the proof. *Q.E.D.*

Lemma OA1 sheds light on the agent's dynamic contract. The basic idea is that by (5), the agent's contract payoff depends on the value function, $V_T^{\mathrm{a}} = \xi_T$. *Q.E.D.*

Proof of Theorem 2. The proof follows from the arguments that precede Theorem 2. *Q.E.D.*

Proof of Lemma 2. For any process $\{Z_s\}_{s\in[t,T]}$ and overbooking parameter $\pi > 1$, by (7) and (9), we denote, the principal's value function under a given incentive process, Z, and overbooking-penalty parameter, π, is

$$U_t^{\mathrm{p}}(\{Z_s\}_{s\in[t,T]}, c) = E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s - \eta Z_s^2/2\right) ds - (1+c)e^{-r(T-t)}\left(D_T - I_0\right)^+\right], \tag{OA11}$$

where the expectation is taken under the demand dynamics (1) with the agent's optimal effort $\hat{A}_t(Z_t)$. We denote $c := \pi - 1$ and $c > 0$. Then, for any $c \ge c'$, let $\{Z_s^{*c}\}_{s\in[t,T]}$ be the optimizer for the firm's expected profit function with parameter c, $\hat{V}_t^{\mathrm{p}}(c) = \sup_{\{Z_s\}_{s\in[t,T]}} U_t^{\mathrm{p}}(\{Z_s\}_{s\in[t,T]}, c)$ and $\{Z_s^{*c'}\}_{s\in[t,T]}$ be the optimizer for $\hat{V}_t^{\mathrm{p}}(c') = \sup_{\{Z_s\}_{s\in[t,T]}} U_t^{\mathrm{p}}(\{Z_s\}_{s\in[t,T]}, c')$. Next, we have the following:

$$\begin{aligned}
&\hat{V}_t^{\mathrm{p}}(c) = U_t^{\mathrm{p}}(\{Z_s^{*c}\}_{s\in[t,T]}, c) = E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s^{*c} - \eta Z_s^{*c2}/2\right) ds - (1+c)e^{-r(T-t)}\left(D_T - I_0\right)^+\right]\\
&\le U_t^{\mathrm{p}}(\{Z_s^{*c}\}_{s\in[t,T]}, c') = E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s^{*c} - \eta Z_s^{*c2}/2\right) ds - (1+c')e^{-r(T-t)}\left(D_T - I_0\right)^+\right]\\
&\le U_t^{\mathrm{p}}(\{Z_s^{*c'}\}_{s\in[t,T]}, c') = E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s^{*c'} - \eta Z_s^{*c'2}/2\right) ds - (1+c')e^{-r(T-t)}\left(D_T - I_0\right)^+\right]\\
&= \hat{V}_t^{\mathrm{p}}(c') = \sup_{\{Z_s\}_{s\in[t,T]}} U_t^{\mathrm{p}}((Z_s)_{s\in[t,T]}, c')
\end{aligned}$$

for any $c \ge c'$. Let us denote $V_t^{\mathrm{p}}(\pi) := \hat{V}_t^{\mathrm{p}}(c)$ for all π and $\pi = 1 + c$. Therefore, it is straightforward to obtain that for any $\pi \ge \pi'$, we have $V_t^{\mathrm{p}}(\pi) \le V_t^{\mathrm{p}}(\pi')$.

The proof of part (ii) of the proposition follows an approach similar to that of part (i) and is omitted for brevity. *Q.E.D.*

Proof of Lemma 3. To avoid abuse of notation, we denote $D_t^{s,d,\boldsymbol{z}}$ as the demand quantity at time t with the initial demand d, starting from time s controlled by the incentive process $\boldsymbol{z}$. By (OA11), the firm's expected profit function at time t is given by

$$V_t^{\mathrm{p}}(d) = \sup_{\boldsymbol{z}=\{Z_s\}_{s\in[t,T]}} E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_s - \eta Z_s^2/2\right)ds - \pi e^{-r(T-t)}\left(D_T^{t,d,\boldsymbol{z}} - I_0\right)^+ |d\right].$$

For any $\lambda \in [0,1]$ and initial demand quantities d_1 and d_2 at time t, and any incentive processes $\boldsymbol{z_1} = \{Z_{1,s}\}_{s\in[t,T]}$ and $\boldsymbol{z_2} = \{Z_{2,s}\}_{s\in[t,T]}$, we have

$$\begin{aligned}
V_t^{\mathrm{p}}(\lambda d_1 + (1-\lambda)d_2) &\geq E_t\left[\int_t^T e^{-r(s-t)}\left(\eta\left[\lambda Z_{1,s} + (1-\lambda)Z_{2,s}\right] - \eta\left[\lambda Z_{1,s} + (1-\lambda)Z_{2,s}\right]^2/2\right)ds\right.\\
&\left.-\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda\boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2}} - I_0\right)^+ |d = \lambda d_1 + (1-\lambda)d_2\right]\\
&\geq E_t\left[\lambda\int_t^T e^{-r(s-t)}\left(\eta Z_{1,s} - \eta Z_{1,s}^2/2\right)ds + (1-\lambda)\int_t^T e^{-r(s-t)}\left(\eta Z_{2,s} - \eta Z_{2,s}^2/2\right)ds\right]\\
&- E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda\boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2}} - I_0\right)^+ |d = \lambda d_1 + (1-\lambda)d_2\right]\\
&\geq E_t\left[\lambda\int_t^T e^{-r(s-t)}\left(\eta Z_{1,s} - \eta Z_{1,s}^2/2\right)ds + (1-\lambda)\int_t^T e^{-r(s-t)}\left(\eta Z_{2,s} - \eta Z_{2,s}^2/2\right)ds\right]\\
&- \lambda E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1}} - I_0\right)^+ |d = d_1\right] - (1-\lambda)E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2}} - I_0\right)^+ |d = d_2\right]\\
&\geq \lambda E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_{1,s} - \eta Z_{1,s}^2/2\right)ds - \pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1}} - I_0\right)^+ |d = d_1\right]\\
&+ (1-\lambda)E_t\left[\int_t^T e^{-r(s-t)}\left(\eta Z_{2,s} - \eta Z_{2,s}^2/2\right)ds - \pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2}} - I_0\right)^+ |d = d_2\right].
\end{aligned} \tag{OA12}$$

Maximizing the right-hand side over the incentive process $\boldsymbol{z_1}$ and $\boldsymbol{z_2}$, the above inequality in (OA12) can be rewritten as

$$V_t^{\mathrm{p}}(\lambda d_1 + (1-\lambda)d_2) \geq \lambda V_t^{\mathrm{p}}(d_1) + (1-\lambda)V_t^{\mathrm{p}}(d_2). \tag{OA13}$$

The concavity is obtained by (OA13). *Q.E.D.*

Proof of Proposition 1. First, the HJB equation in (10) gives the optimal incentive variable Z_t^* by

$$\sup_{Z_t}\left(\eta Z_t - \eta Z_t^2/2\right) + \frac{\partial V_t^{\mathrm{p}}}{\partial D}\eta Z_t.$$

Then, by the first-order condition, we obtain $Z_t^* = 1 + \partial V_t^{\mathrm{p}}/\partial D_t$. Next, by taking the first order derivative with respect to D_t, we get

$$\partial Z_t^*/\partial D_t = \partial^2 V_t^{\mathrm{p}}/\partial D_t^2. \tag{OA14}$$

By Lemma 3 and (OA14), the concavity of V_t^{p} gives $\partial Z_t^*/\partial D_t \leq 0$. Therefore, the optimal incentive Z_t^* is decreasing in D_t.

Then, by substituting $Z_t^* = 1 + \partial V_t^{\mathrm{p}}/\partial D_t$ into (10), we obtain

$$0 = \eta\left(1 + \frac{\partial V^{\mathrm{p}}}{\partial D}\right)^2/2 - rV^{\mathrm{p}} + \frac{\partial V^{\mathrm{p}}}{\partial t} + \frac{1}{2}\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}\sigma^2. \tag{OA15}$$

Next, we take the first order derivative with respect to the demand quantity D for equation (OA15), which yields

$$0=\eta\left(1+\frac{\partial V^{\mathrm{p}}}{\partial D}\right)\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}-r\frac{\partial V^{\mathrm{p}}}{\partial D}+\frac{\partial^2 V^{\mathrm{p}}}{\partial t\partial D}+\frac{1}{2}\frac{\partial^3 V^{\mathrm{p}}}{\partial D^3}\sigma^2. \tag{OA16}$$

Let $J=\frac{\partial V^{\mathrm{p}}}{\partial D}$, the above equation can be written as

$$0=\eta\left(1+J\right)\frac{\partial J}{\partial D}-rJ+\frac{\partial J}{\partial t}+\frac{1}{2}\frac{\partial^2 J}{\partial D^2}\sigma^2. \tag{OA17}$$

Next, we define $c\triangleq\pi-1$ and denote $J(c):=J(\pi)$ for any c. We take the first order derivative with respect to the parameter c for the equation (OA17), which gives:

$$0=\eta\frac{\partial J}{\partial c}\frac{\partial J}{\partial D}+\eta\left(1+J\right)\frac{\partial^2 J}{\partial D\partial c}-r\frac{\partial J}{\partial c}+\frac{\partial^2 J}{\partial t\partial c}+\frac{1}{2}\frac{\partial^3 J}{\partial D^2\partial c}\sigma^2. \tag{OA18}$$

Then, by Itô's formula, we have

$$\begin{aligned}
d\left(\frac{\partial J}{\partial c}\right)&=\frac{\partial^2 J}{\partial c\partial t}dt+\frac{\partial^2 J}{\partial c\partial D}dD+\frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}d\langle D\rangle\\
&=\frac{\partial^2 J}{\partial c\partial t}dt+\frac{\partial^2 J}{\partial c\partial D}\left[\eta Z_t^* dt+\sigma dB_t\right]+\frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}\sigma^2 dt\\
&=\frac{\partial^2 J}{\partial c\partial t}dt+\frac{\partial^2 J}{\partial c\partial D}\left[\eta(1+J)dt+\sigma dB_t\right]+\frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}\sigma^2 dt\\
&=\left[\frac{\partial^2 J}{\partial c\partial t}+\frac{\partial^2 J}{\partial c\partial D}\eta(1+J)+\frac{1}{2}\frac{\partial^3 J}{\partial c\partial D^2}\sigma^2\right]dt+\frac{\partial^2 J}{\partial c\partial D}\sigma dB_t\\
&=\frac{\partial J}{\partial c}\left[r-\eta\frac{\partial J}{\partial D}\right]dt+\frac{\partial^2 J}{\partial c\partial D}\sigma dB_t.
\end{aligned} \tag{OA19}$$

The last equality is obtained by substituting (OA18) into the above drift term. Next, by taking the expectations on both sides and conditioning on an arbitrary state value $D_{t-dt}=d$, we obtain

$$E\left[\frac{\partial J}{\partial c}(t,D_t)|D_{t-dt}=d\right]=\frac{\partial J}{\partial c}(t-dt,d)+\frac{\partial J}{\partial c}(t-dt,d)\left[r-\eta\frac{\partial J}{\partial D}(t-dt,d)\right]dt, \tag{OA20}$$

where dt is an infinitesimal time interval. Because V^{p} is concave, we have $r-\eta\frac{\partial J}{\partial D}=r-\eta\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}\geq 0$. Furthermore, note $V_T^{\mathrm{p}}=-(1+c)(D_T-I_0)^+$ and $J=\frac{\partial V^{\mathrm{p}}}{\partial D}$, $\frac{\partial J}{\partial c}(T,\cdot)$ is non-positive when it exists. By (OA20), and using $\frac{\partial J}{\partial c}(T,\cdot)\leq 0$, we obtain, for any state value d,

$$\frac{\partial J}{\partial c}(T-dt,d)+\frac{\partial J}{\partial c}(T-dt,d)\left[r-\eta\frac{\partial J}{\partial D}(T-dt,d)\right]dt=E\left[\frac{\partial J}{\partial c}(T,D_T)|D_{T-dt}=d\right]\leq 0.$$

Since $r-\eta\frac{\partial J}{\partial D}\geq 0$, it follows that $\frac{\partial J}{\partial c}(T-dt,d)\leq 0$ for all d, i.e., $\frac{\partial J}{\partial c}(T-dt,\cdot)\leq 0$. Then, iterating this argument backward in time yields

$$\frac{\partial J}{\partial c}(t,D_t)\leq 0\quad \text{for all } t\in[0,T],$$

and therefore, $\frac{\partial Z_t^*}{\partial c}=\frac{\partial^2 V^{\mathrm{p}}}{\partial D\partial c}=\frac{\partial J}{\partial c}\leq 0$. Thus, the optimal incentive Z_t^* decreases in the parameter c, which means that Z_t^* also decreases in the overbooking penalty parameter π. *Q.E.D.*

PROOF OF THEOREM 3. The proof follows from the arguments that precede Theorem 3. *Q.E.D.*

Proof of Lemma 4. For any processes $\{Z_s\}_{s\in[t,T]}$, $\{p_s\}_{s\in[t,T]}$ and overbooking parameter π, by (17), we define

$$
\begin{aligned}
&U_t^{\mathrm{p}}(\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]},\pi)\\
&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s\eta\beta^2 Z_s+\alpha p_s-\alpha p_s^2-\eta\beta^2 Z_s^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T-I_0\right)^+\right],
\end{aligned}
$$

where the expectation is taken under the dynamics (12) with the agent's optimal effort $\hat{A}_t(Z_t)$ in (14).

For any $\pi\ge\pi'$, let $\{Z_s^{*\pi}\}_{s\in[t,T]}$ and $\{p_s^{*\pi}\}_{s\in[t,T]}$ be the optimizer of the firm's expected profit function with parameter π, $V_t^{\mathrm{p}}(\pi)=\sup_{\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]}}U_t^{\mathrm{p}}\left(\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]},\pi\right)$. Further, let $\{Z_s^{*\pi'}\}_{s\in[t,T]}$ and $\{p_s^{*\pi'}\}_{s\in[t,T]}$ be the optimizer of $V_t^{\mathrm{p}}(\pi')=\sup_{\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]}}U_t^{\mathrm{p}}\left(\{Z_s\}_{s\in[t,T]},\{p_s\}_{s\in[t,T]},\pi'\right)$.

It is straightforward to show

$$
\begin{aligned}
&V_t^{\mathrm{p}}(\pi)=U_t^{\mathrm{p}}(\{Z_s^{*\pi}\}_{s\in[t,T]},\{p_s^{*\pi}\}_{s\in[t,T]},\pi)\\
&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s^{*\pi}\eta\beta^2 Z_s^{*\pi}+\alpha p_s^{*\pi}-\alpha p_s^{*\pi 2}-\eta\beta^2 Z_s^{*\pi 2}/2\right)ds-\pi e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\
&\le U_t^{\mathrm{p}}(\{Z_s^{*\pi}\}_{s\in[t,T]},\{p_s^{*\pi}\}_{s\in[t,T]},\pi')\\
&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s^{*\pi}\eta\beta^2 Z_s^{*\pi}+\alpha p_s^{*\pi}-\alpha p_s^{*\pi 2}-\eta\beta^2 Z_s^{*\pi 2}/2\right)ds-\pi' e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\
&\le U_t^{\mathrm{p}}(\{Z_s^{*\pi'}\}_{s\in[t,T]},(\{p_s^{*\pi'}\}_{s\in[t,T]},\pi')\\
&=E_t\left[\int_t^T e^{-r(s-t)}\left(p_s^{*\pi'}\eta\beta^2 Z_s^{*\pi'}+\alpha p_s^{*\pi'}-\alpha p_s^{*\pi'2}-\eta\beta^2 Z_s^{*\pi'2}/2\right)ds-\pi' e^{-r(T-t)}\left(D_T-I_0\right)^+\right]\\
&=V_t^{\mathrm{p}}(\pi').
\end{aligned}
$$

Therefore, we obtained that $V_t^{\mathrm{p}}(\pi)\le V_t^{\mathrm{p}}(\pi')$ for any $\pi\ge\pi'$. *Q.E.D.*

Proof of Lemma 5. Because $\alpha>\eta\beta^2$ and thereby, $\alpha>\eta\beta^2/2$, it is straightforward to show that $p_t\eta\beta^2 Z_t+\alpha p_t-\alpha p_t^2-\eta\beta^2 Z_t/2$ is jointly concave in (Z_t,p_t). To avoid abuse of notation, we denote $D_t^{s,d,\boldsymbol{z},\boldsymbol{q}}$ as the demand quantity at time t with demand d at time s and with incentive and price processes $\boldsymbol{z}$ and $\boldsymbol{p}$. The firm's expected profit function at time t is given by:

$$
\begin{aligned}
V_t^{\mathrm{p}}(d)=\sup_{\boldsymbol{z}=\{Z_s\}_{s\in[t,T]},\boldsymbol{p}=\{p_s\}_{s\in[t,T]}}&E_t\left[\int_t^T e^{-r(s-t)}\left(p_s\eta\beta^2 Z_s+\alpha p_s-\alpha p_s^2-\eta\beta^2 Z_s^2/2\right)ds\right.\\
&\left.-\pi e^{-r(T-t)}\left(D_T^{t,d,\boldsymbol{z},\boldsymbol{p}}-I_0\right)^+|d\right].
\end{aligned}
$$

For any $\lambda\in[0,1]$ and initial demand quantities d_1 and d_2, incentive processes $\boldsymbol{z_1}=\{Z_{1,s}\}_{s\in[t,T]}$ and $\boldsymbol{z_2}=\{Z_{2,s}\}_{s\in[t,T]}$, and any price processes $\boldsymbol{p_1}=\{p_{1,s}\}_{s\in[t,T]}$ and $\boldsymbol{p_2}=\{p_{2,s}\}_{s\in[t,T]}$, we have

$$
\begin{aligned}
&V_t^{\mathrm{p}}(\lambda d_1+(1-\lambda)d_2)\\
&\ge E_t\left[\int_t^T e^{-r(s-t)}\left([\lambda p_{1,s}+(1-\lambda)p_{2,s}]\,\eta\beta^2\,[\lambda Z_{1,s}+(1-\lambda)Z_{2,s}]+\alpha\,[\lambda p_{1,s}+(1-\lambda)p_{2,s}]-\alpha\,[\lambda p_{1,s}+(1-\lambda)p_{2,s}]^2\right.\right.\\
&\left.\left.-\eta\beta^2\,[\lambda Z_{1,s}+(1-\lambda)Z_{2,s}]^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda\boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2},\lambda\boldsymbol{p_1}+(1-\lambda)\boldsymbol{p_2}}-I_0\right)^+|d=\lambda d_1+(1-\lambda)d_2\right]\\
&\ge E_t\left[\lambda\int_t^T e^{-r(s-t)}\left(p_{1,s}\eta\beta^2 Z_{1,s}+\alpha p_{1,s}-\alpha p_{1,s}^2-\eta\beta^2 Z_{1,s}^2/2\right)ds\right.
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\lambda)\int_t^T e^{-r(s-t)}\left(p_{2,s}\eta\beta^2 Z_{2,s}+\alpha p_{2,s}-\alpha p_{2,s}^2-\eta\beta^2 Z_{2,s}^2/2\right)ds\Big]\\
&-E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda\boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2},\lambda\boldsymbol{p_1}+(1-\lambda)\boldsymbol{p_2}}-I_0\right)^+|d=\lambda d_1+(1-\lambda)d_2\right]\\
&\geq\lambda E_t\left[\int_t^T e^{-r(s-t)}\left(p_{1,s}\eta\beta^2 Z_{1,s}+\alpha p_{1,s}-\alpha p_{1,s}^2-\eta\beta^2 Z_{1,s}^2/2\right)ds\right]\\
&+E_t\left[(1-\lambda)\int_t^T e^{-r(s-t)}\left(p_{2,s}\eta\beta^2 Z_{2,s}+\alpha p_{2,s}-\alpha p_{2,s}^2-\eta\beta^2 Z_{2,s}^2/2\right)ds\right]\\
&-\lambda E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1},\boldsymbol{p_1}}-I_0\right)^+|d=d_1\right]-(1-\lambda)E_t\left[\pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2},\boldsymbol{p_2}}-I_0\right)^+|d=d_2\right]\\
&\geq\lambda E_t\left[\int_t^T e^{-r(s-t)}\left(p_{1,s}\eta\beta^2 Z_{1,s}+\alpha p_{1,s}-\alpha p_{1,s}^2-\eta\beta^2 Z_{1,s}^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,d_1,\boldsymbol{z_1},\boldsymbol{p_1}}-I_0\right)^+|d=d_1\right]\\
&+(1-\lambda)E_t\left[\int_t^T e^{-r(s-t)}\left(p_{2,s}\eta\beta^2 Z_{2,s}+\alpha p_{2,s}-\alpha p_{2,s}^2-\eta\beta^2 Z_{2,s}^2/2\right)ds-\pi e^{-r(T-t)}\left(D_T^{t,d_2,\boldsymbol{z_2},\boldsymbol{p_2}}-I_0\right)^+|d=d_2\right].
\end{aligned}
\tag{OA21}
$$

The second inequality follows from the joint concavity of the running payoff $f(p,Z)=p\eta\beta^2 Z+\alpha p-\alpha p^2-\eta\beta^2 Z^2/2$ in (p,Z). The third inequality follows from the convexity of $(x-I_0)^+$ and the fact that, pathwise, $D_T^{t,\lambda d_1+(1-\lambda)d_2,\lambda\boldsymbol{z_1}+(1-\lambda)\boldsymbol{z_2},\lambda\boldsymbol{p_1}+(1-\lambda)\boldsymbol{p_2}}=\lambda D_T^{t,d_1,\boldsymbol{z_1},\boldsymbol{p_1}}+(1-\lambda)D_T^{t,d_2,\boldsymbol{z_2},\boldsymbol{p_2}}$ due to the linearity of the demand dynamics. Maximizing the right-hand side over the incentive process and pricing process, $\boldsymbol{z_1},\boldsymbol{p_1}$, $\boldsymbol{z_2}$, and $\boldsymbol{p_2}$, the above inequality in (OA21) can be rewritten as

$$
V_t^{\mathrm{p}}(\lambda d_1+(1-\lambda)d_2)\geq\lambda V_t^{\mathrm{p}}(d_1)+(1-\lambda)V_t^{\mathrm{p}}(d_2),
\tag{OA22}
$$

which means $V_t^{\mathrm{p}}(d)$ is concave in d. *Q.E.D.*

PROOF OF PROPOSITION 2. The proof consists of two parts:

(i) The optimal incentive variable and price, that is, Z_t^* and p_t^*, are given by the equation in (19). Because $\alpha>\eta\beta^2$, $(p_t\eta\beta^2 Z_t+\alpha p_t-\alpha p_t^2-\eta\beta^2 Z_t^2/2)+\frac{\partial V_t^{\mathrm{p}}}{\partial D}(\eta\beta^2 Z_t-\alpha p_t)$ is jointly concave in (p_t,Z_t) and Lemma 5 gives the concavity of the principal's value function; that is, $\frac{\partial^2 V_t^{\mathrm{p}}}{\partial D^2}\leq 0$. Then, the first-order condition with respect to p_t and Z_t gives

$$
(\eta\beta^2 Z_t+\alpha-2\alpha p_t)-\frac{\partial V_t^{\mathrm{p}}}{\partial D}\alpha=0,\qquad (p_t-Z_t)+\frac{\partial V_t^{\mathrm{p}}}{\partial D}=0.
$$

These equations give the optimal price and incentive variable $p_t^*=\frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}(\eta\beta^2-\alpha)+\alpha}{2\alpha-\eta\beta^2}$ and $Z_t^*=\frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}\alpha+\alpha}{2\alpha-\eta\beta^2}$.

(ii) By the optimal price and incentive variable in (i) and (18), we get:

$$
0=\frac{\frac{1}{2}\alpha^2\left(1+\frac{\partial V^{\mathrm{p}}}{\partial D}\right)^2}{2\alpha-\eta\beta^2}-rV^{\mathrm{p}}+\frac{\partial V^{\mathrm{p}}}{\partial t}+\frac{1}{2}\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}\sigma^2.
\tag{OA23}
$$

Then, we take the first order derivative with respect to the demand quantity D, which gives

$$
0=\frac{\alpha^2\left(1+\frac{\partial V^{\mathrm{p}}}{\partial D}\right)\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2}}{2\alpha-\eta\beta^2}-r\frac{\partial V^{\mathrm{p}}}{\partial D}+\frac{\partial^2 V^{\mathrm{p}}}{\partial t\partial D}+\frac{1}{2}\frac{\partial^3 V^{\mathrm{p}}}{\partial D^3}\sigma^2.
\tag{OA24}
$$

Let $J=\frac{\partial V^{\mathrm{p}}}{\partial D}$, the equation (OA24) can be rewritten as

$$
0=\frac{\alpha^2(1+J)\frac{\partial J}{\partial D}}{2\alpha-\eta\beta^2}-rJ+\frac{\partial J}{\partial t}+\frac{1}{2}\frac{\partial^2 J}{\partial D^2}\sigma^2.
\tag{OA25}
$$

Next, we denote $c := \pi - 1$ and define $J(c) := J(\pi)$ for any c. We take the first order derivative with respect to the parameter c for the equation (OA25). The reformulated equation is as follows:

$$0 = \frac{\alpha^2 \frac{\partial J}{\partial c}\frac{\partial J}{\partial D}}{2\alpha - \eta\beta^2} + \frac{\alpha^2 (1+J)\frac{\partial^2 J}{\partial D \partial c}}{2\alpha - \eta\beta^2} - r\frac{\partial J}{\partial c} + \frac{\partial^2 J}{\partial t \partial c} + \frac{1}{2}\frac{\partial^3 J}{\partial D^2 \partial c}\sigma^2. \tag{OA26}$$

Then, by the Itô's formula, we have

$$\begin{aligned}
d\left(\frac{\partial J}{\partial c}\right) &= \frac{\partial^2 J}{\partial c \partial t}dt + \frac{\partial^2 J}{\partial c \partial D}dD + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}d\langle D\rangle \\
&= \frac{\partial^2 J}{\partial c \partial t}dt + \frac{\partial^2 J}{\partial c \partial D}\left[\left(\eta\beta^2 Z_t^* + \alpha(1-p_t^*)\right)dt + \sigma dB_t\right] + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}\sigma^2 dt \\
&= \frac{\partial^2 J}{\partial c \partial t}dt + \frac{\partial^2 J}{\partial c \partial D}\left[\frac{(1+J)\alpha^2}{2\alpha - \eta\beta^2}dt + \sigma dB_t\right] + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}\sigma^2 dt \\
&= \left[\frac{\partial^2 J}{\partial c \partial t} + \frac{\partial^2 J}{\partial c \partial D}\frac{(1+J)\alpha^2}{2\alpha - \eta\beta^2} + \frac{1}{2}\frac{\partial^3 J}{\partial c \partial D^2}\sigma^2\right]dt + \frac{\partial^2 J}{\partial c \partial D}\sigma dB_t \\
&= \frac{\partial J}{\partial c}\left[r - \frac{\alpha^2}{2\alpha - \eta\beta^2}\frac{\partial J}{\partial D}\right]dt + \frac{\partial^2 J}{\partial c \partial D}\sigma dB_t.
\end{aligned} \tag{OA27}$$

The last equality is obtained by substituting (OA26) into the above drift term. Next, by taking the expectation on both sides and conditioning on an arbitrary state value $D_{t-dt} = d$, we obtain

$$E\left[\frac{\partial J}{\partial c}(t, D_t)|D_{t-dt} = d\right] = \frac{\partial J}{\partial c}(t-dt, d) + \frac{\partial J}{\partial c}(t-dt, d)\left[r - \frac{\alpha^2}{2\alpha - \eta\beta^2}\frac{\partial J}{\partial D}(t-dt, d)\right]dt, \tag{OA28}$$

where dt is an infinitesimal time interval. Because V^{p} is concave and $\alpha > \eta\beta^2$, we have $r - \frac{\alpha^2}{2\alpha-\eta\beta^2}\frac{\partial J}{\partial D} = r - \frac{\alpha^2}{2\alpha-\eta\beta^2}\frac{\partial^2 V^{\mathrm{p}}}{\partial D^2} \geq 0$. Furthermore, note $V^{\mathrm{p}} = -(1+c)(D_T - I_0)^+$ and $J = \frac{\partial V^{\mathrm{p}}}{\partial D}$, $\frac{\partial J}{\partial c}(T, \cdot)$ is non-positive when it exists. By (OA28) and $\frac{\partial J}{\partial c}(T, \cdot) \leq 0$, we obtain, for any state value d

$$\frac{\partial J}{\partial c}(T-dt, d) + \frac{\partial J}{\partial c}(T-dt, d)\left[r - \frac{\alpha^2}{2\alpha - \eta\beta^2}\frac{\partial J}{\partial D}(T-dt, d)\right]dt = E\left[\frac{\partial J}{\partial c}(T, D_T)|D_{T-dt} = d\right] \leq 0$$

Since $r - \frac{\alpha^2}{2\alpha-\eta\beta^2}\frac{\partial J}{\partial D} \geq 0$, it follows that $\frac{\partial J}{\partial c}(T-dt, d) \leq 0$ for all d, i.e., $\frac{\partial J}{\partial c}(T-dt, \cdot) \leq 0$. Then, iterating this argument backward in time yields

$$\frac{\partial J}{\partial c}(t, D_t) \leq 0 \quad \text{for all } t \in [0, T],$$

and therefore, $\frac{\partial Z_t^*}{\partial c} = \frac{\partial^2 V^{\mathrm{p}}}{\partial D \partial c} = \frac{\partial J}{\partial c} \leq 0$. Because $p_t^* = \frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}(\eta\beta^2-\alpha)+\alpha}{2\alpha-\eta\beta^2} = \frac{J(\eta\beta^2-\alpha)+\alpha}{2\alpha-\eta\beta^2}$, $\alpha > \eta\beta^2$ and $Z_t^* = \frac{\frac{\partial V_t^{\mathrm{p}}}{\partial D}\alpha+\alpha}{2\alpha-\eta\beta^2} = \frac{J\alpha+\alpha}{2\alpha-\eta\beta^2}$, we have $\frac{\partial p_t^*}{\partial c} = \frac{\frac{\partial J}{\partial c}(\eta\beta^2-\alpha)}{2\alpha-\eta\beta^2} \geq 0$ and $\frac{\partial Z_t^*}{\partial c} = \frac{\alpha\frac{\partial J}{\partial c}}{2\alpha-\eta\beta^2} \leq 0$. As such, we obtain that the optimal incentive Z_t^* falls in c and the optimal price p_t^* increases in c, which means Z_t^* also falls in the overbooking penalty parameter π and p_t^* increases in π. *Q.E.D.*

PROOF OF PROPOSITION 3. By (16) and (12), the firm's expected profit under the fully-dynamic ($V_{\mathrm{p}}^{\mathrm{fd}}$), dynamic-pricing-only ($V_{\mathrm{p}}^{\mathrm{dp}}$), dynamic-contracting-only ($V_{\mathrm{p}}^{\mathrm{dc}}$), and fully-static strategies ($V_{\mathrm{p}}^{\mathrm{fs}}$), respectively, are given as follows:

$$V_{\mathrm{p}}^{\mathrm{fd}} = \sup_{\hat{p}=\{p_t\}_{t\in[0,T]}, Z=\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p_t\eta\beta^2 Z_t + \alpha p_t - \alpha p_t^2 - \eta\beta^2 Z_t^2/2\right)dt - \pi e^{-rT}\left(D_T^{\hat{p},Z} - I_0\right)^+\right] \tag{OA29}$$

$$\begin{aligned}
V_{\mathrm{p}}^{\mathrm{dp}} = & \sup_{\hat{p}=\{p_t\}_{t\in[0,T]},b_0,a_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt}p_t dD_t^{\hat{p},b_0} - e^{-rT}\left(a_0 + b_0\int_0^T e^{-r(t-T)}(dD_t^{\hat{p},b_0} - \alpha(1-p_t)dt)\right)\right. \\
& \left. -\pi e^{-rT}\left(D_T^{\hat{p},b_0} - I_0\right)^+\right] \\
= & \sup_{\hat{p}=\{p_t\}_{t\in[0,T]},b_0} E\left[\int_0^T e^{-rt}\left(p_t\eta\beta^2 b_0 + \alpha p_t - \alpha p_t^2 - \eta\beta^2 b_0^2\right)dt - \pi e^{-rT}\left(D_T^{\hat{p},b_0} - I_0\right)^+\right] \qquad \text{(OA30)} \\
V_{\mathrm{p}}^{\mathrm{dc}} = & \sup_{p,\xi_T} E\left[\int_0^T e^{-rt}p dD_t^{p,Z} - e^{-rT}\xi_T - \pi e^{-rT}\left(D_T^{p,Z} - I_0\right)^+\right] \\
= & \sup_{p,Z=\{Z_t\}_{t\in[0,T]},Y_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt}p dD_t^{p,Z} - e^{-rT}Y_T^{Y_0,Z} - \pi e^{-rT}\left(D_T^{p,Z} - I_0\right)^+\right] \\
= & \sup_{p,Z=\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2\right)dt - \pi e^{-rT}\left(D_T^{p,Z} - I_0\right)^+\right] \qquad \text{(OA31)} \\
V_{\mathrm{p}}^{\mathrm{fs}} = & \sup_{p,b_0,a_0\geq\rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt}p dD_t^{p,b_0} - e^{-rT}(a_0 + b_0\int_0^T e^{-r(t-T)}(dD_t^{p,b_0} - \alpha(1-p)dt)) - \pi e^{-rT}\left(D_T^{p,b_0} - I_0\right)^+\right] \\
= & \sup_{p,b_0} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 b_0 + \alpha p - \alpha p^2 - \eta\beta^2 b_0^2\right)dt - \pi e^{-rT}\left(D_T^{p,b_0} - I_0\right)^+\right], \qquad \text{(OA32)}
\end{aligned}$$

where we denote by $D_T^{\hat{p},Z}$, $D_T^{\hat{p},b_0}$, $D_T^{p,Z}$, and D_T^{p,b_0} the controlled demand processes under the fully-dynamic, dynamic-pricing-only, dynamic-contracting-only, and fully-static strategies, respectively. By (12), we have the expectation in (OA29) is taken under the demand dynamics with a dynamic pricing policy $\hat{p} = \{p_t\}_{t\in[0,T]}$ and the agent's optimal effort $\hat{A}(Z_t) = \beta\eta Z_t$ given dynamic contract $\xi_T = Y_T^{Y_0,Z}$, as follows

$$dD_t^{\hat{p},Z} = \left(\alpha(1-p_t) + \eta\beta^2 Z_t\right)dt + \sigma dB_t.$$

Similarly, the expectation in (OA30) is taken under the demand dynamics with a dynamic pricing policy $\hat{p} = \{p_t\}_{t\in[0,T]}$ and the agent's optimal effort $\hat{a}(b_0) = \eta\beta b_0$ given a static contract $\Gamma = a_0 + \int_0^T e^{-r(t-T)}b_0(dD_t - \alpha(1-p_t)dt)$ characterized by (a_0, b_0), as follows

$$dD_t^{\hat{p},b_0} = \left(\alpha(1-p_t) + \eta\beta^2 b_0\right)dt + \sigma dB_t.$$

Note that the static contract compensates the agent only for the effort-driven demand, not the price-driven portion $\alpha(1-p_t)$. This is because the price-driven demand $\alpha(1-p_t)$ is perfectly observable and determined by the firm's pricing decision. Moreover, this demand component is not generated by the agent's effort. Then the firm does not pay commission on demand the agent does not contribute to.

The expectation in (OA31) is taken under the demand dynamics with static pricing policy p and the agent's optimal effort $\hat{A}(Z_t) = \beta\eta Z_t$ given a dynamic contract $\xi_T = Y_T^{Y_0,Z}$, as follows

$$dD_t^{p,Z} = \left(\alpha(1-p) + \eta\beta^2 Z_t\right)dt + \sigma dB_t.$$

The expectation in (OA32) is taken under the demand dynamics with a static pricing policy p and the agent's optimal effort $\hat{a}(b_0) = \eta\beta b_0$ given a static contract Γ characterized by (a_0, b_0), as follows

$$dD_t^{p,b_0} = \left(\alpha(1-p) + \eta\beta^2 b_0\right)dt + \sigma dB_t.$$

Here the agent's participation level is normalized to zero (i.e., $\rho_{\mathrm{a}} = 0$), and thereby, the optimal base compensations Y_0 and a_0 are equal to the normalized participation level, that is, $Y_0 = 0$ and $a_0 = 0$.

The instantaneous incentive cost is the instantaneous promised payment to induce the desired effort. The results are directly derived from (20) to (23). For example, to examine the difference between static contracting and dynamic contracting, let us first compare the firm's value functions under the dynamic-contracting-only strategy ($V_{\mathrm{p}}^{\mathrm{dc}}$) and under the fully-static strategy ($V_{\mathrm{p}}^{\mathrm{fs}}$). Under the dynamic-contracting-only strategy, by (16), the firm's value function is given by:

$$\begin{aligned} V_{\mathrm{p}}^{\mathrm{dc}} &= \sup_{p, Z=\{Z_t\}_{t\in[0,T]}, Y_0 \geq \rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} p dD_t^{p,Z} - e^{-rT} Y_T^{Y_0,Z} - \pi e^{-rT}\left(D_T^{p,Z} - I_0\right)^+\right] \\ &= \sup_{p, Z=\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2\right)dt - \pi e^{-rT}\left(D_T^{p,Z} - I_0\right)^+\right], \quad \text{(OA33)} \end{aligned}$$

where the agent's participation level ρ_{a} can be normalized to zero and the optimal Y_0 is just equal to the normalized participation level, that is, $Y_0 = 0$. The last equality is obtained by substituting the optimal $Y_0 = 0$, the demand dynamics in (8), and payoff process in (15) .

Under the fully-static strategy, by (16), the firm's value function, $V_{\mathrm{p}}^{\mathrm{fs}}$, is given by:

$$\begin{aligned} V_{\mathrm{p}}^{\mathrm{fs}} &= \sup_{p, b_0, a_0 \geq \rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} p dD_t^{p,b_0} - e^{-rT}\Gamma - \pi e^{-rT}\left(D_T^{p,b_0} - I_0\right)^+\right] \\ &= \sup_{p, b_0, a_0 \geq \rho_{\mathrm{a}}} E\left[\int_0^T e^{-rt} p dD_t^{p,b_0} - e^{-rT}(a_0 + b_0\int_0^T e^{-r(t-T)}(dD_t^{p,b_0} - \alpha(1-p)dt)) - \pi e^{-rT}\left(D_T^{p,b_0} - I_0\right)^+\right] \\ &= \sup_{p, b_0} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 b_0 + \alpha p - \alpha p^2 - \eta\beta^2 b_0^2\right)dt - \pi e^{-rT}\left(D_T^{p,b_0} - I_0\right)^+\right]. \quad \text{(OA34)} \end{aligned}$$

Here, the static contract is given by $\Gamma = a_0 + \int_0^T e^{-r(t-T)} b_0(dD_t^{p,b_0} - \alpha(1-p)dt)$. The base payment a_0 and incentive bonus b_0 are chosen at the initial time by the firm. The last equality is obtained by substituting the demand dynamics in (12).

Next, through (OA33) and (OA34), we can compare the firm's value under the dynamic contract and the static contract. Under the static contract, the corresponding incentive parameter b_0 is chosen by the firm at the initial time $t = 0$. For the dynamic contract, the incentive variable Z_t is adapted and dependent on the current demand level or the remaining inventory level. Dynamic contracting provides the firm with more flexibility to control the demand process. For example, when the current cumulative demand is very high, the firm can provide the agent with a lower incentive, which is intended to decrease the incentive variable Z_t and to refrain from motivating the agent to generate excessive demand. Moreover, at any time t, given the same level of incentive (i.e., $Z_t = b_0$), we compare the term $(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2)$ in (OA33) under the dynamic-contracting-only strategy and $(p\eta\beta^2 b_0 + \alpha p - \alpha p^2 - \eta\beta^2 b_0^2)$ in (OA34) under the fully-static strategy. The differences are $\eta\beta^2 Z_t^2/2$ and $\eta\beta^2 b_0^2$. Therefore, even under the same incentive level, the instantaneous incentive cost for the firm under the dynamic-contracting-only strategy is lower than that under the fully-static strategy. *Q.E.D.*

PROOF OF LEMMA 6. We provide the proof part by part:

(i) First, we aim to show that as α approaches zero or infinity, the optimal price p approaches one. The case $\alpha \to 0$ is immediate: the demand drift $\alpha(1-p_t)$ vanishes, so any price discount (i.e. $p_t < 1$) strictly reduces instantaneous revenue without generating a compensating increase in demand; thus $p_t = 1$ is optimal.

Next we consider the case $\alpha \to \infty$. For each $\alpha > 0$, a given b_0, and an admissible pricing policy $\tilde{p} = \{\tilde{p}_t\}_{0 \leq t \leq T}$ (progressively measurable, with $0 \leq \tilde{p}_t \leq 1$ a.s.), consider the controlled demand process

$$D_t^{\tilde{p},b_0} = \int_0^t \left(\alpha(1-\tilde{p}_s) + \beta\hat{a}\right)ds + \sigma B_t, \qquad 0 \leq t \leq T,$$

where $\hat{a} = \eta\beta b_0$ is the optimal effort of the agent under the static contract Γ. By (21), the firm's expected profit under dynamic-pricing-only policy can be written as follows

$$J_\alpha^{dp}(\tilde{p}) := E\left[\int_0^T e^{-rs}\Big(\tilde{p}_s\eta\beta^2 b_0 + \alpha\tilde{p}_s - \alpha\tilde{p}_s^2 - \eta\beta^2 b_0^2\Big)ds - \pi e^{-rT}\big(D_T^{\tilde{p},b_0} - I_0\big)^+\right].$$

Let p^* denote the benchmark policy $p_t^* \equiv 1$. For any admissible $\tilde{p}$ and $\tilde{p} \neq p^*$, any sample path $\omega \in \Omega$ and the induced control path $\tilde{p}_t(\omega)$, we define the (pathwise) price deviation

$$\delta_t(\omega) := 1 - \tilde{p}_t(\omega) \in [0,1], \qquad W(\omega) := \{t \in [0,T] : \delta_t(\omega) > 0\}.$$

For any given b_0 and $\alpha > 0$, we write the performance difference under any admissible policy $\tilde{p}$ and p^* as follows

$$\Delta_\alpha^{dp}(\tilde{p}) := J_\alpha^{dp}(\tilde{p}) - J_\alpha^{dp}(p^*).$$

Compute, for each ω and almost every t,

$$\begin{aligned} f_\alpha(\tilde{p}_t) - f_\alpha(1) &= \big(\tilde{p}_t\eta\beta^2 b_0 + \alpha\tilde{p}_t - \alpha\tilde{p}_t^2 - \eta\beta^2 b_0^2\big) - \big(\eta\beta^2 b_0 - \eta\beta^2 b_0^2\big) \\ &= -\eta\beta^2 b_0\delta_t + \alpha(\delta_t - \delta_t^2), \end{aligned}$$

with the obvious shorthand $f_\alpha(\cdot)$ for the running integrand. Moreover

$$D_T^{\tilde{p},b_0} - D_T^{p^*,b_0} = \int_0^T \alpha(1-\tilde{p}_s)ds = \alpha\int_0^T \delta_s ds =: \alpha\Lambda, \qquad \Lambda := \int_0^T \delta_s ds.$$

Thus pathwise

$$\Delta_\alpha^{dp}(\tilde{p}) = E\Bigg[\underbrace{\int_0^T e^{-rs}\big(-\eta\beta^2 b_0\delta_s + \alpha(\delta_s - \delta_s^2)\big)ds}_{(A)} - \underbrace{\pi e^{-rT}\Big((D_T^{p^*,b_0} + \alpha\Lambda - I_0)^+ - (D_T^{p^*,b_0} - I_0)^+\Big)}_{(B)}\Bigg].$$

Suppose for some sample path ω, $W(\omega)$ has zero measure, which implies $(A)-(B)$ goes to zero. Suppose for some sample path ω, $W(\omega)$ has positive measure. For large α, the quantity $(D_T^{p^*,b_0}(\omega) + \alpha\Lambda(\omega) - I_0)^+$ grows at least linearly in α (once $\alpha\Lambda(\omega)$ exceeds $|D_T^{p^*,b_0}(\omega) - I_0|$). Consequently, for such ω,

$$\pi e^{-rT}\alpha\Lambda(\omega) \geq \text{(B)} \geq \pi e^{-rT}\left(\alpha\Lambda(\omega) - C(\omega)\right)$$

for some finite (path-dependent) nonnegative constant $C(\omega)$ independent of α. Also,

$$-\int_0^T e^{-rs}\eta\beta^2 b_0 ds \leq \text{(A)} \leq \alpha\int_0^T \delta_s(\omega)ds = \alpha\Lambda(\omega)$$

Hence for such ω, for all sufficiently large α,

$$\text{(A) - (B)} \leq \alpha\Lambda(\omega) - \pi e^{-rT}(\alpha\Lambda(\omega) - C(\omega)) = -\alpha\Lambda(\omega)(\pi e^{-rT} - 1) + \pi e^{-rT}C(\omega) \leq \pi e^{-rT}C(\omega). \tag{OA35}$$

Since $\pi e^{-rT} > 1$ and $0 < \Lambda(\omega) \leq T$, we have $(A)-(B) < 0$ for some sufficiently large α. $(A)-(B)$ tends to $-\infty$ as $\alpha \to \infty$. Moreover, $(A)-(B)$ is nonincreasing in α when α is sufficiently large. By Fatou's lemma, this pathwise divergence downward implies the expectation $\Delta_\alpha(\tilde{p}) \to -\infty$ if $P(\Lambda > 0) > 0$ and $\Delta_\alpha(\tilde{p}) = 0$ if $P(\Lambda > 0) = 0$, which implies $J_\alpha^{dp}(\tilde{p}) \leq J_\alpha^{dp}(p^*)$ for any admissible $\tilde{p}$. Then the optimal price policy is $p_t^* \equiv 1$ as $\alpha \to \infty$. Define the firm's value under dynamic-contracting-only policy is given by

$$\begin{aligned}
V_{\mathrm{p}}^{\mathrm{dc}} &= \sup_{p,\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 Z_t + \alpha p - \alpha p^2 - \eta\beta^2 Z_t^2/2dt\right) - \pi e^{-rT}\left(D_T^{p,Z} - I_0\right)^+\right] \\
&\geq \sup_{\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(\eta\beta^2 Z_t - \eta\beta^2 Z_t^2/2dt\right) - \pi e^{-rT}\left(D_T^{1,Z} - I_0\right)^+\right] \\
&\geq \sup_{\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(\eta\beta^2 Z_t - \eta\beta^2 Z_t^2 dt\right) - \pi e^{-rT}\left(D_T^{1,Z} - I_0\right)^+\right] \\
&\geq \sup_{b_0} J_\alpha^{dp}(p^*), \qquad \text{(OA36)}
\end{aligned}$$

where the controlled demand process under dynamic-contracting only strategy follows

$$dD_t^{p,Z} = \left(\alpha(1-p) + \eta\beta^2 Z_t\right)dt + \sigma dB_t, \qquad 0 \leq t \leq T.$$

The right hand side in last inequality of (OA36) is exactly the firm's value under the dynamic-pricing-only strategy as $\alpha \to \infty$. Hence, the dynamic-contracting-only strategy outperforms the dynamic-pricing-only strategy as $\alpha \to \infty$.

(ii) By a similar argument as in (i), as β approaches zero or infinity, the optimal incentive Z approaches zero. Hence, the dynamic-pricing-only strategy outperforms the dynamic-contracting-only strategy.

(iii) As I_0 approaches infinity, the overbooking penalty vanishes and our problem converges to a deterministic setting. In this case, both the price and incentive become time deterministic. The firm's problem simplifies to a deterministic one, rendering both the dynamic-contracting-only and dynamic-pricing-only strategies equivalent to each other.

(iv) As σ approaches zero or infinity, the optimal incentive Z approaches zero and the price p approaches one. As σ approaches zero, our problem simplifies to a deterministic problem. Hence, both the dynamic-contracting-only and dynamic-pricing-only strategies become equivalent to each other. *Q.E.D.*

PROOF OF PROPOSITION 4. By Lemma 6 and (OA35), there exists some sufficiently large $\bar{\alpha}$ such that $(A) - (B) < 0$ when $W(w)$ has positive measure, and $(A) - (B) = 0$ when $W(w)$ has zero measure. Consequently, $J_{\bar{\alpha}}^{dp}(\tilde{p}) \leq J_{\bar{\alpha}}^{dp}(p^*)$, which implies that there exists a sufficiently large $\bar{\alpha}$ for which the optimal price under the dynamic-pricing-only strategy is $p_t^* \equiv 1$; that is, $V_{\mathrm{p}}^{dp}(\bar{\alpha}) = \sup_{b_0} J_{\bar{\alpha}}^{dp}(p^*)$.

Therefore,

$$\begin{aligned}
V_{\mathrm{p}}^{\mathrm{dc}}(\bar{\alpha}) &= \sup_{p,\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(p\eta\beta^2 Z_t + \bar{\alpha} p - \bar{\alpha} p^2 - \eta\beta^2 Z_t^2/2dt\right) - \pi e^{-rT}\left(D_T^{p,Z} - I_0\right)^+\right] \\
&\geq \sup_{\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\left(\eta\beta^2 Z_t - \eta\beta^2 Z_t^2/2dt\right) - \pi e^{-rT}\left(D_T^{1,Z} - I_0\right)^+\right] \\
&\geq \sup_{b_0} J_{\bar{\alpha}}^{dp}(p^*) = V_{\mathrm{p}}^{dp}(\bar{\alpha}), \qquad \text{(OA37)}
\end{aligned}$$

where the controlled demand process under dynamic-contracting only strategy follows

$$dD_t^{p,Z} = \left(\bar{\alpha}(1-p) + \eta\beta^2 Z_t\right)dt + \sigma dB_t, \qquad 0 \leq t \leq T.$$

Since there exists a sufficiently large $\bar{\alpha}$ such that $V_{\mathrm{p}}^{dp}(\alpha) \leq V_{\mathrm{p}}^{dc}(\alpha)$, i.e., $\tilde{V}(\bar{\alpha}) \leq 0$, the claim in Proposition 4 follows.

Proof of Proposition 5. Recall $k=\eta\beta^2$. Throughout this proof, we assume that the value function $U^{b^{\mathrm{dp}}}$ and the auxiliary value function $W^{(1)}$ introduced below are classical $C^{1,2}$ solutions of their respective HJB equations on $[0,T)\times\mathbb{R}$ with continuous extensions to $t=T$, as follows from standard Schauder regularity for uniformly parabolic equations (the parabolicity coefficient $\sigma^2/2$ is strictly positive); see, e.g., Touzi (2012). The monotonicity $U_d^{b^{\mathrm{dp}}}\le 0$ follows from the concavity of the principal's value function in cumulative demand established in Lemma 5.

Subsolution and comparison. We use the HJB equations corresponding to (24) and (25). For a fixed static incentive $b\ge 0$ (consistent with the sales-bonus interpretation of the agent's compensation throughout the paper), the dynamic-pricing-only continuation value satisfies

$$rU^b=U_t^b+\frac{\sigma^2}{2}U_{dd}^b+\sup_{p\in(0,1]}\left\{pkb+\alpha p-\alpha p^2+\big(\alpha(1-p)+kb\big)U_d^b-kb^2\right\}. \tag{OA38}$$

The supremum is over the model's admissible price set $p\in(0,1]$; equivalently, one may take the closure $[0,1]$, since the unconstrained maximizer below lies in $(0,1]$ whenever $q\le 0$ and $b\ge 0$. Let $q=U_d^b(t,d)$, which coincides with $U_d^{b^{\mathrm{dp}}}(t,d)\le 0$ when we specialize to $b=b^{\mathrm{dp}}$ below. Writing $p=1-\delta$, with $\delta\in[0,1)$, the maximized pricing term can be decomposed as

$$\begin{aligned}&\sup_{\delta\in[0,1)}\left\{kb(1+q)-kb^2+\delta\big(\alpha(1+q)-kb\big)-\alpha\delta^2\right\}\\&\qquad=kb(1+q)-kb^2+\frac{\big(\alpha(1+q)-kb\big)_+^2}{4\alpha}.\end{aligned} \tag{OA39}$$

The last equality follows because $q\le 0$ and $b\ge 0$, so the unconstrained maximizer $\delta^*=(\alpha(1+q)-kb)_+/(2\alpha)$ lies in $[0,1/2]\subset[0,1)$.

Now consider the dynamic-contracting-only problem with the static price fixed at $p\equiv 1$, which is admissible in (25) since the price ranges over $(0,1]$. Let $W^{(1)}(t,d)$ denote the corresponding continuation value, with terminal condition $W^{(1)}(T,d)=-\pi(d-I_0)^+$. The associated HJB equation is

$$rW^{(1)}=W_t^{(1)}+\frac{\sigma^2}{2}W_{dd}^{(1)}+\sup_{Z\ge 0}\left\{kZ-\frac{kZ^2}{2}+kZW_d^{(1)}\right\}. \tag{OA40}$$

Specializing to $b=b^{\mathrm{dp}}$ and evaluating the Hamiltonian in (OA40) at the test function $U^{b^{\mathrm{dp}}}$ gives

$$\sup_{Z\ge 0}\left\{kZ(1+q)-\frac{kZ^2}{2}\right\}=\frac{k}{2}(1+q)_+^2. \tag{OA41}$$

The dynamic-pricing-only Hamiltonian evaluated at $U^{b^{\mathrm{dp}}}$ is the bracketed expression in (OA39) with $b=b^{\mathrm{dp}}$; it is exceeded by the lower bound in (OA41) by an amount

$$\frac{k}{2}(1+q)_+^2-\left[kb^{\mathrm{dp}}(1+q)-k(b^{\mathrm{dp}})^2+\frac{\big(\alpha(1+q)-kb^{\mathrm{dp}}\big)_+^2}{4\alpha}\right],$$

which is nonnegative by (26). In the backward-HJB sign convention, this is equivalent to the subsolution inequality

$$rU^{b^{\mathrm{dp}}}-U_t^{b^{\mathrm{dp}}}-\frac{\sigma^2}{2}U_{dd}^{b^{\mathrm{dp}}}-\sup_{Z\ge 0}\left\{kZ-\frac{kZ^2}{2}+kZU_d^{b^{\mathrm{dp}}}\right\}\le 0,$$

so $U^{b^{\mathrm{dp}}}$ is a subsolution of (OA40) with the same terminal condition $-\pi(d-I_0)^+$. The comparison principle for finite-horizon parabolic HJBs in the linear-growth class (see, e.g., Touzi 2012) gives

$$U^{b^{\mathrm{dp}}}(0,0)\le W^{(1)}(0,0).$$

Since $p \equiv 1$ is feasible for the dynamic-contracting-only problem in (25), we also have $W^{(1)}(0,0) \le V_{\mathrm{p}}^{\mathrm{dc}}(\alpha)$. Therefore $U^{b^{\mathrm{dp}}}(0,0) \le W^{(1)}(0,0) \le V_{\mathrm{p}}^{\mathrm{dc}}(\alpha)$. Since b^{dp} is optimal for the dynamic-pricing-only problem, $U^{b^{\mathrm{dp}}}(0,0) = V_{\mathrm{p}}^{\mathrm{dp}}(\alpha)$, and the result follows.

Verification of cases (i) *and* (ii). Write $b = b^{\mathrm{dp}}$ and let $s = (1+q)_+ \ge 0$. If $1+q<0$, then $s=0$, the right-hand side of (26) is zero, and the left-hand side is $-kb^{\mathrm{dp}}(1+q) + k(b^{\mathrm{dp}})^2 \ge 0$; the inequality holds trivially. We therefore focus on $1+q \ge 0$, so that $s = 1+q$. The left-hand side of (26) satisfies

$$\frac{k}{2}s^2 - kbs + kb^2 = k\left(b - \frac{s}{2}\right)^2 + \frac{k}{4}s^2 \ge \frac{k}{4}s^2,$$

whereas the right-hand side satisfies $(\alpha s - kb)_+^2/(4\alpha) \le \alpha s^2/4$. Thus case ($i$), $k \ge \alpha$, implies (26). Alternatively, if $\alpha(1+q) \le kb^{\mathrm{dp}}$, then the local value of price flexibility is zero. The left-hand side of (26) is nonnegative by the preceding quadratic bound, so (26) holds automatically. This is precisely the price-cap condition (ii); under it, the optimizer in (OA39) has $\delta^* = 0$, and hence the dynamic-pricing-only policy is at $p_t = 1$.

Primitive sufficient condition. A primitive condition that implies the price-cap condition (ii) is

$$\pi e^{-r(T-t)}\bar{\Phi}\left(\frac{I_0 - d}{\sigma\sqrt{T-t}}\right) \ge 1 - \frac{\eta\beta^2 b^{\mathrm{dp}}}{\alpha}, \qquad t<T,$$

where $\bar{\Phi}(x) = 1 - \Phi(x)$. To see this, fix $b = b^{\mathrm{dp}}$ and $h>0$. Let $\hat{p}$ denote an optimal price process for $U^b(t, d+h)$; if an optimizer does not exist, the same argument applies to an ε-optimal process and the bound is preserved as $\varepsilon \downarrow 0$, since the final lower bound below depends on $\hat{p}$ only through the nonnegative-drift property and not through its optimality. Apply the same sample-path price process $\hat{p}$ from state (t,d), which is feasible because the admissible set for price processes imposes only measurability and the static range $p_t \in (0,1]$, independent of the initial demand state. Under the common Brownian path,

$$D_T^{d+h,\hat{p}} = D_T^{d,\hat{p}} + h.$$

The flow payoff is the same under the two coupled systems, so by the suboptimality of $\hat{p}$ at state (t,d),

$$U^b(t,d+h) - U^b(t,d) \le -\pi e^{-r(T-t)}\mathbb{E}\left[\left(D_T^{d,\hat{p}} + h - I_0\right)^+ - \left(D_T^{d,\hat{p}} - I_0\right)^+\right].$$

Since $(x+h-I_0)^+ - (x-I_0)^+ \ge h\,\mathbf{1}_{\{x>I_0\}}$, we obtain

$$\frac{U^b(t,d+h) - U^b(t,d)}{h} \le -\pi e^{-r(T-t)}\mathbb{P}\left(D_T^{d,\hat{p}} > I_0\right).$$

For any admissible price process, the controlled demand drift satisfies $\alpha(1-p_s) + kb \ge 0$, and therefore

$$D_T^{d,\hat{p}} = d + \int_t^T \left(\alpha(1-\hat{p}_s) + kb\right) ds + \sigma(B_T - B_t) \ge d + \sigma(B_T - B_t).$$

It follows that

$$\frac{U^b(t,d+h) - U^b(t,d)}{h} \le -\pi e^{-r(T-t)}\mathbb{P}\left(d + \sigma(B_T - B_t) > I_0\right).$$

Letting $h \downarrow 0$ and using the differentiability of U^b in d inherited from the classical-solution hypothesis above gives

$$-U_d^b(t,d) \ge \pi e^{-r(T-t)}\bar{\Phi}\left(\frac{I_0 - d}{\sigma\sqrt{T-t}}\right).$$

The displayed primitive condition therefore implies the price-cap condition (ii). *Q.E.D.*

OA2. Competition between Direct Sales Channel and Sales Channel through an Agent

In this section, we apply our analytical framework to examine the competition effect between two channels of sales – direct sales and sales through an agent. The firm uses these two channels to sell the product, as explored in Section 3. The cumulative demand quantity, D_t, for the product up to any given time $t \in [0, T]$, evolves according to

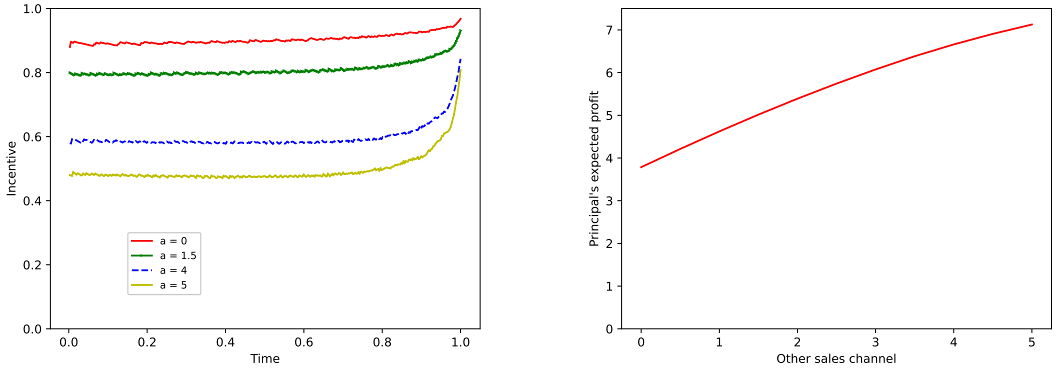
$$dD_t = (a + A_t)dt + \sigma dB_t, \tag{OA42}$$

where a corresponds to the firm's demand rate through its own direct sales channel, $A_t \geq 0$ is the agent's effort level, and σ is a constant diffusion term.

The agent's optimal effort level, given the incentive variable Z_t, is $\hat{A}_t(Z_t) = \eta Z_t$. We explore how the direct sales channel influences the agent's effort level. Up to time T, the total sales through the direct channel are aT. Therefore, the remaining inventory for the agent is $I_0 - aT$. As the direct sales channel a increases, the agent has less to sell. Equivalently, we may consider transferring the problem of increasing a to that of decreasing I_0.

From the principal's perspective, the incentive variable provided by the principal decreases with increasing direct sales a. Consequently, the expected optimal effort of the agent decreases, and thus, the expected sales, $E(\int_0^T \hat{A}_t(Z_t)dt)$ (i.e., $\int_0^T \eta Z_t dt$), also decrease with a. We numerically show how the agent's effort decreases with I_0. Similarly, we could conduct comparative statics numerically for a.

As the value of the other channel a increases, we observe from the numerical study presented in Figure OA1 a decrease in the average incentive offered to the sales agents. At the same time, the firm's expected profit shows an upward trend as a increases. With a higher presence of alternative sales channels, the firm reduces its dependence on the sales agent's effort. This reduction in dependence, in turn, leads to a reduction in the cost of incentives provided to motivate these agents. Consequently, as the influence of other channels a escalates, the firm's expected profit experiences a corresponding increase.



(a) **Incentive versus time**

(b) **The firm's expected profit versus other sales channel**

Figure OA1 Optimal incentives and the firm's expected profit under different alternative sales channels a. Parameter values : $r = 0.025$, $\eta = 8$, $I_0 = 10$, $\sigma = 2$, and $\pi = 2$.

OA3. Comparison of the Dynamic Contract with Other Contracts

In this section, we contrast our dynamic contract with other forms of contracts. We are particularly interested in exploring the differences in the principal's value, the incentive variables, and the agent's optimal effort. First, we consider a contract of the form $\xi_T^Q = e^{rT}B + \int_0^T e^{-r(t-T)}RdD_t - M(D_T - I_0)^+$, where B is the base pay, R denotes the commission rate, and M is the marginal overbook penalty with $M > R$. This contract only depends on the terminal demand D_T and also takes limited inventory into account. We can numerically search for the optimal B, R, and M for this form of contract and compare the results with our dynamic contract. Second, we examine contracts in which the firm does not compensate the agent for demand that exceeds the capacity. At time $\hat{\mathbb{T}} = \{t \geq 0 : D_t \geq I_0\}$, we set the incentive variable $Z_t = 0$ for $t \in \hat{\mathbb{T}}$. This scenario can be seen as a subset of our contract, similar to adding an additional constraint where $Z_t = 0$ when demand exceeds capacity.

Linear Penalty Contract. We start with considering the linear penalty contract that makes the agent share the overbooking penalty with the principal. We consider the contract $\xi_T^Q = e^{rT}B + \int_0^T e^{-r(t-T)}RdD_t - M(D_T - I_0)^+$, where B is the base pay, R denotes the commission rate, and M is the marginal overbook penalty with $M > R$. This contract is only dependent on the terminal demand D_T and also takes the limited inventory into account. We could numerically search for the optimal B, R, and M in this form of contract and compare it with our dynamic contract.

Corresponding to the demand process (1) and given the contract ξ_T^Q, the agent chooses optimal effort $A = \{A_t, 0 \leq t \leq T\}$ by maximizing his expected utility:

$$\begin{aligned} V^{\mathrm{a}} = & \sup_{\{A_t\}_{\{0\leq t\leq T\}}} E\left[e^{-rT}\xi_T^Q - \int_0^T e^{-rt}c(A_t)dt\right], \\ = & \sup_{\{A_t\}_{\{0\leq t\leq T\}}} E\left[B + \int_0^T e^{-rt}\left(RA_t - c(A_t)\right)dt - Me^{-rT}(D_T - I_0)^+\right], \end{aligned} \tag{OA43}$$

and without loss of generality, let $B = 0$ and $a = 0$ as in the main text.

Then, we write the agent's value function as

$$V_t^{\mathrm{a}} = \sup_{\{A_s\}_{\{t\leq s\leq T\}}} E\left[\int_t^T e^{-r(s-t)}\left(RA_s - A_s^2/(2\eta)\right)ds - Me^{-r(T-t)}(D_T - I_0)^+\right], \tag{OA44}$$

where the base pay B could be normalized to zero.

Then, we have the following HJB equation for the agent's problem:

$$\sup_{A_t}\left\{RA_t - A_t^2/(2\eta) - rV_t^{\mathrm{a}} + \frac{\partial V_t^{\mathrm{a}}}{\partial t} + \frac{\partial V_t^{\mathrm{a}}}{\partial D}A_t + \frac{1}{2}\frac{\partial^2 V_t^{\mathrm{a}}}{\partial D^2}\sigma^2\right\} = 0, \tag{OA45}$$

which gives the following theorem.

Theorem EC.1. *The agent's optimal effort* A_t *satisfies*

$$\sup_{A_t}\left\{RA_t - c(A_t) + \frac{\partial V_t^{\mathrm{a}}}{\partial D}A_t\right\} \tag{OA46}$$

and is given by $\hat{A}_t = \eta\left(R + \frac{\partial V_t^{\mathrm{a}}}{\partial D}\right)$.

Then, the firm 's value function is given by

$$\begin{aligned} V^{\mathrm{p}} = & \sup_{\xi_T^Q} E\left[\int_0^T e^{-rt}dD_t - e^{-rT}\xi_T^Q - \pi e^{-rT}\left(D_T - I_0\right)^+\right] \\ = & \sup_{\{p_t\}_{t\in[0,T]},(R,M)} E\left[\int_0^T e^{-rt}(1-R)\left[a + \hat{A}_t\right]dt - (\pi - M)e^{-rT}\left(D_T - I_0\right)^+\right], \end{aligned} \tag{OA47}$$

where $R \leq M \leq \pi$. The firm's expected profit function at $t \in [0,T]$ is given by

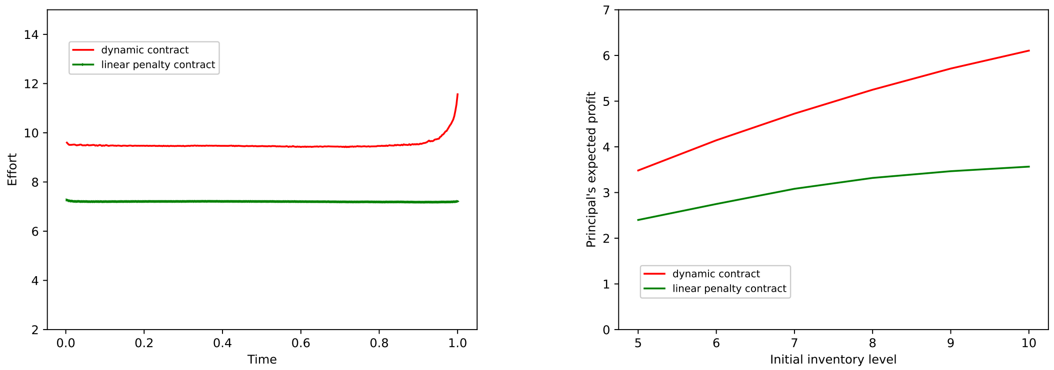
$$\begin{aligned} V_t^{\mathrm{p}} &= \sup_{\xi_T^Q} E\left[\int_t^T e^{-r(s-t)} dD_t - e^{-r(T-t)} \xi_T^Q - \pi e^{-r(T-t)} \left(D_T - I_0\right)^+\right] \\ &= \sup_{(R,M)} E\left[\int_t^T e^{-r(s-t)} (1-R) \left[a + \hat{A}_s\right] ds - (\pi - M) e^{-r(T-t)} \left(D_T - I_0\right)^+\right], \end{aligned} \tag{OA48}$$

where $R \leq M \leq \pi$ and $\hat{A}_s$ is given by Theorem EC.1.

THEOREM EC.2. *The firm's expected profit function in V_t^{p} under the dynamic contract dominates the firm's expected profit under the linear penalty contract ξ_T^Q.*

THEOREM EC.3. *The agent's value under the linear penalty contract ξ_T^Q weakly dominates the agent's value under our dynamic contract ξ_T.*

We now provide a numerical illustration of the agent's actions under the two contracts, detailing both the path of effort and the path of incentives under the benchmark parameters. As shown in Figure OA2, the sales agent's effort under the linear penalty contract is lower. Given that the sales agent shares the penalty cost, the linear penalty contract is not as effective at inducing effort. Moreover, the principal's value under a linear penalty contract is lower than it is under our dynamic contract. As inventory levels increase, this disparity grows more pronounced, primarily because the linear penalty contract proves to be more costly per unit of demand generated. On average, the value difference between the two contracts is approximately 40%.

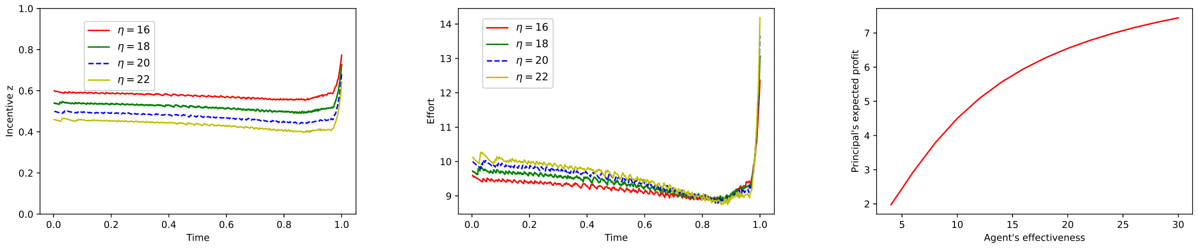


(a) **Agent's effort versus time**

(b) **The firm's expected profit versus initial inventory level**

Figure OA2 **Agent's effort, and the firm's expected profit for different initial inventory I_0. Parameter values :** $r = 0.025$, $\eta = 15$, $\sigma = 2$, $I_0 = 10$ **and** $\pi = 2$. **The left side represents the comparison between linear penalty contract and dynamic contract for** $I_0 = 10$. **In this case, the optimal linear penalty contract is given by** $(R, M) = (0.5, 0.6)$.

The Case of a Hard Inventory Constraint. Next, we analyze the case with a hard inventory constraint. We use numerical experiments to compare the gap between our model and that under a hard inventory constraint. As shown in Figure OA4, the firm's expected profit under the dynamic contract is higher than that under the hard inventory constraint contract. The incentive effectiveness is lower because the firm cannot do overbooking and the demand that exceeds the inventory becomes lost sales. Therefore, at the beginning, faced with ample inventory, the principal provides lower-powered incentives to smooth the increment of demand, and then uses higher-powered incentives to clear the remaining inventory at the end.

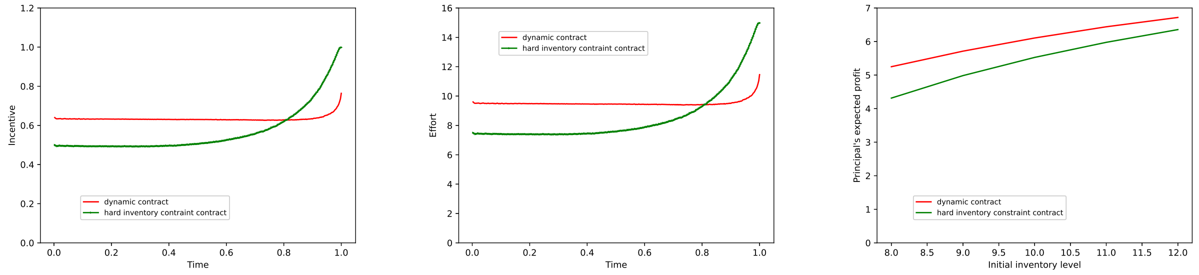


(a) **Incentive versus time**

(b) **Agent's effort versus time**

(c) **The firm's expected profit versus agent's effectiveness**

Figure OA3 Incentive, agent effort, and the firm's expected profit for different levels of salesforce effectiveness η.
Parameter values : $r = 0.025$, $I_0 = 10$, $\sigma = 2$, and $\pi = 2$.



(a) **Incentive versus time**

(b) **Agent's effort versus time**

(c) **The firm's expected profit versus initial inventory level**

Figure OA4 Incentive, agent effort, and the firm's expected profit for different initial inventory I_0.
Parameter values : $r = 0.025$, $\eta = 15$, $\sigma = 2$, $I_0 = 10$, and $\pi = 2$.

OA4. Digital Marketing versus Traditional Marketing

In this extension, we apply our general framework to compare and understand how digital marketing and traditional marketing vary in influencing the decision-making processes of the principal and the agent.

Digital marketing, characterized by unique features such as customer tracking over visits, comparison shopping, and knowledge of customer choice sets, provides the agent with customer visit data. Leveraging these data sources for targeted strategies, the agent can attract customers more efficiently. Consequently, the agent's effectiveness parameter, η, tends to be higher with digital marketing than with traditional marketing.

We conducted a numerical analysis to assess the impact of the effectiveness parameter on the decisions of the principal and the agent. As shown in Figure OA3, the average incentive decreases in the agent's effectiveness parameter η. A higher effectiveness parameter η means the principal needs less incentive to induce the same level of effort from the agent to meet the inventory level, because the agent's optimal effort is expressed as $\hat{A}(Z_t) = \eta Z_t$. Additionally, the agent's average effort first increases then decreases in η over time. Interestingly, the principal's expected profit increases in the agent's effectiveness parameter η.

Figure OA3 illustrates that the average incentive provided to the sales agent in digital marketing is smaller than that in traditional marketing under limited inventory conditions. This difference can be attributed to the lower costs and increased effectiveness of attracting customers through digital marketing. At the onset of the time horizon, when the firm has ample inventory, it prefers high sales effort to deplete the inventory. Thus, it provides a high-powered compensation plan to the more effective sales agent in the digital marketing channel, resulting in higher cumulative demand

and lower remaining inventory. However, as the end of the time horizon approaches, and inventory depletes, the firm becomes wary of overbooking. In response, it offers a lower-power compensation plan to the digital marketing agent. Also note the firm's expected profit is higher in the digital marketing scenario.

OA5. Single-Period Benchmark

This appendix motivates the continuous-time formulation by introducing a single-period benchmark for the contracting problem under limited inventory. We solve the benchmark and then compare its implications with those of our dynamic model, highlighting what is gained by allowing demand (and the firm's controls) to evolve over time.

Consider a single period model with random demand $D = \alpha + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$ and σ is the demand volatility. The agent controls effort α, which is not observed by the firm. The firm observes the realized demand and designs a contract given to the sales agent. After the demand is realized, the agent receives a payoff ξ_T.

Given a general contract $\xi_{\mathrm{a}}(D)$ that depends on the demand D, the sales agent's problem is given by

$$V_{\mathrm{a}}(\xi_{\mathrm{a}}) = \sup_{\alpha} \mathbb{E}\left[\xi_{\mathrm{a}}(D) - \frac{\alpha^2}{2\eta}\right] = \sup_{\alpha} \int_{-\infty}^{+\infty} \left(\xi_{\mathrm{a}}(\alpha + y) - \frac{\alpha^2}{2\eta}\right) f(y) dy, \tag{OA49}$$

where $\alpha^2/(2\eta)$ is the cost of effort α and $f(y)$ is normal density function for ϵ. By the first order condition, the optimal effort α^* satisfies:

$$\int_{-\infty}^{+\infty} f(y) \left[\xi_{\mathrm{a}}'(\alpha^* + y) - \alpha^*/\eta\right] dy = 0. \tag{OA50}$$

A sufficient condition for (OA50) is given by $\xi_{\mathrm{a}}'(\alpha^* + y) - \alpha^*/\eta = 0$ for all y, that is, for all outcomes of ϵ. Then for all D we have $\xi_{\mathrm{a}}'(D) = \alpha^*/\eta$, and therefore $\xi_{\mathrm{a}} = K + D\alpha^*/\eta$, where K is a constant determined by the sales agent's participation constraint.

Under the contract $\xi_{\mathrm{a}} = K + D\alpha^*/\eta$, α^* satisfies the first-order condition in (OA50) and thus solves the sales agent's problem in (OA49). Thus, this linear contract is incentive compatible for the agent's problem and implements optimal effort. The firm chooses the incentive pay (the slope $z = \alpha^*/\eta$) and induces the corresponding response $\alpha^* = \eta z$. Given this, let us assume that the compensation contract is linear in demand, i.e., $\xi_{\mathrm{a}} = Y_0 + zD$, where Y_0 is the base salary and z is the incentive pay. However, note that the linear contract form is not necessarily optimal for the firm designing the contract for the sales agent. Now we can solve (OA49) as follows:

$$V_{\mathrm{a}}(\xi_{\mathrm{a}}) = \sup_{\alpha} \mathbb{E}\left[\xi_{\mathrm{a}} - \frac{\alpha^2}{2\eta}\right] = \sup_{\alpha} \mathbb{E}\left[Y_0 + zD - \frac{\alpha^2}{2\eta}\right] = \sup_{\alpha} \mathbb{E}\left[Y_0 + z\alpha - \frac{\alpha^2}{2\eta}\right], \tag{OA51}$$

which gives the sales agent's optimal effort $a^* = \arg\sup_{\alpha} \left\{z\alpha - \frac{\alpha^2}{2\eta}\right\} = \eta z$. Note that this solution is the same as in our continuous time model (see (8)).

The firm's problem is to choose contract (Y_0, z) for the manager:

$$V_{\mathrm{p}}(\xi_{\mathrm{a}}) = \sup_{z, Y_0} \mathbb{E}\left[D - \xi_{\mathrm{a}} - \pi(D - I_0)^+\right], \tag{OA52}$$

where ξ_{a} satisfies the sales agent's participation constraint $Y_0 \geq \rho_{\mathrm{a}}$, and ρ_{a} is the participation level.

If we have restricted the contract to be linear, it is easy to identify the optimal incentive z and Y_0 for the agent. However, as mentioned earlier, the linear contract is not necessarily optimal. That is, in this single-period model, it is difficult to solve for the optimal contract form. This further motivates our continuous time model for the dynamic contracting problem under limited inventory.

OA6. Pricing–Effort Interaction

This section extends the baseline model by allowing pricing and salesforce effort to interact in generating demand. The goal is to capture environments in which the salesperson is more effective when the firm offers a better deal (e.g., discounts facilitate conversion), so the marginal impact of effort depends on the price path.

The cumulative demand D_t up to time $t \in [0,T]$ evolves as

$$dD_t = \Big(\alpha(1-p_t) + \beta A_t + m(1-p_t)A_t\Big)\,dt + \sigma\,dB_t, \tag{OA53}$$

where $m \geq 0$ measures the strength of the interaction between pricing and effort. When $m = 0$, (OA53) reduces to the baseline demand process; when $m > 0$, effort becomes more productive when the firm sets a lower price (since $1 - p_t$ increases).

Given a contract characterized by $(Y_0, \{Z_t\}_{t\in[0,T]})$ and a price path $\{p_t\}_{t\in[0,T]}$, the agent chooses effort pointwise over time. The associated Hamiltonian is

$$H_{\mathrm{a}}(z;p_t) = \sup_{A\geq 0}\Big\{z\big[\alpha(1-p_t) + (\beta + m(1-p_t))A\big] - c(A)\Big\}, \tag{OA54}$$

where $c(A) = A^2/(2\eta)$. The first-order condition yields the agent's optimal effort

$$\hat{A}_t(z) = \eta\big(\beta + m(1-p_t)\big)z, \qquad H_{\mathrm{a}}(z;p_t) = z\alpha(1-p_t) + \frac{\eta}{2}\big(\beta + m(1-p_t)\big)^2 z^2. \tag{OA55}$$

Substituting (OA55) (and the induced dynamics of $Y_T^{Y_0,Z}$) into the principal's problem, the firm's value can be written in a form parallel to (16) and (17):

$$V^{\mathrm{p}} = \sup_{\{p_t\}_{t\in[0,T]},\{Z_t\}_{t\in[0,T]}} \mathbb{E}\Bigg[\int_0^T e^{-rt}\Big(p_t\,\eta\big(\beta + m(1-p_t)\big)^2 Z_t + \alpha p_t - \alpha p_t^2\Big)\,dt - \int_0^T e^{-rt}\,\frac{\eta}{2}\big(\beta + m(1-p_t)\big)^2 Z_t^2\,dt - \pi e^{-rT}(D_T - I_0)^+\Bigg]. \tag{OA56}$$

Relative to the baseline case, the interaction term rescales both the marginal benefit of inducing demand (the term proportional to $p_t Z_t$) and the convex incentive cost (the term proportional to Z_t^2) by the same state-dependent factor $\big(\beta + m(1-p_t)\big)^2$. Consequently, pricing and contracting become more tightly linked when m is large: the firm can increase the productivity of incentives by lowering price, but doing so also changes margins and inventory exposure. As in Section 5, we compare four strategies: (i) fully dynamic (dynamic pricing and dynamic contracting), (ii) dynamic-pricing-only (dynamic pricing with static contracting), (iii) dynamic-contracting-only (dynamic contracting with static pricing), and (iv) fully static (static pricing and static contracting).

We denote by $V_{\mathrm{p}}^{\mathrm{fd}}$, $V_{\mathrm{p}}^{\mathrm{dp}}$, $V_{\mathrm{p}}^{\mathrm{dc}}$, and $V_{\mathrm{p}}^{\mathrm{fs}}$ the firm's expected profit under the fully-dynamic, dynamic-pricing-only, dynamic-contracting-only, and fully-static strategies. To streamline notation, define

$$\kappa(p) \equiv \big(\beta + m(1-p)\big)^2, \qquad \mathcal{L}(p,u;\chi) \equiv p\,\eta\,\kappa(p)\,u + \alpha p - \alpha p^2 - \chi\,\eta\,\kappa(p)\,u^2, \tag{OA57}$$

where u denotes the incentive intensity (either Z_t or a constant b_0) and $\chi \in \{\frac{1}{2}, 1\}$ indexes whether the quadratic term is $\frac{1}{2}u^2$ (dynamic contracting) or u^2 (static contracting), matching (OA56). The value functions can then be written compactly as

$$V_{\mathrm{p}}^{\mathrm{fd}} = \sup_{\hat{p},\,Z} \mathbb{E}\left[\int_0^T e^{-rt}\,\mathcal{L}\big(p_t, Z_t; \tfrac{1}{2}\big)\,dt - \pi e^{-rT}\big(D_T^{\hat{p},Z} - I_0\big)^+\right], \tag{OA58}$$

$$V_{\mathrm{p}}^{\mathrm{dp}} = \sup_{\hat{p},\,b_0} \mathbb{E}\left[\int_0^T e^{-rt}\,\mathcal{L}(p_t, b_0; 1)\,dt - \pi e^{-rT}\big(D_T^{\hat{p},b_0} - I_0\big)^+\right], \tag{OA59}$$

$$V_{\mathrm{p}}^{\mathrm{dc}} = \sup_{p,\,Z} \mathbb{E}\left[\int_0^T e^{-rt}\,\mathcal{L}\big(p, Z_t; \tfrac{1}{2}\big)\,dt - \pi e^{-rT}\big(D_T^{p,Z} - I_0\big)^+\right], \tag{OA60}$$

$$V_{\mathrm{p}}^{\mathrm{fs}} = \sup_{p,\,b_0} \mathbb{E}\left[\int_0^T e^{-rt}\,\mathcal{L}(p, b_0; 1)\,dt - \pi e^{-rT}\big(D_T^{p,b_0} - I_0\big)^+\right]. \tag{OA61}$$

Here $\hat{p} = \{p_t\}_{t\in[0,T]}$ and $Z = \{Z_t\}_{t\in[0,T]}$ are adapted controls, while p and b_0 are constants. The terminal demand variables $D_T^{\hat{p},Z}$, $D_T^{\hat{p},b_0}$, $D_T^{p,Z}$, and D_T^{p,b_0} follow (OA53) under the corresponding strategy.

Next, we compare the four strategies—as defined in (OA58) to (OA61)—under various overbooking penalties, initial inventory levels, demand volatility, and price-effect parameters. The results in Figures OA5 to OA8 support the main directional patterns from Section 5 under this alternative demand specification.

As shown in Figure OA5, under the parameters drawn from the airline industry, the dynamic-pricing-only strategy dominates the dynamic-contracting-only strategy over various overbooking penalty parameter when we consider an interactive effect of price and effort. The result is robust and indicates the dynamic-pricing-only strategy performs nearly as well as the fully dynamic strategy when we consider an interactive effect of pricing and effort for the airline industry. The reason is the effect of pricing on demand is greater than that of the sales incentives. As such, dynamic pricing is more effective than dynamic contracting. More specifically, the dynamic-pricing-only strategy is more effective in smoothing the uncertain demand and avoiding overbooking.

The effect of limited inventory on the performance of various strategies is also present under the interaction extension. Figure OA6 shows that as the inventory constraint becomes tighter (i.e., as I_0 decreases), the performance differences between various strategies widen.

Figure OA7 illustrates how demand volatility influences different strategies when we consider an interactive effect of pricing and effort. The results are also consistent with Section 5 As demand volatility rises, the performance differences between the fully dynamic strategy, the two partially dynamic strategies (esp. the dynamic-contracting-only strategy) and the fully static strategy widen. In addition, Figure OA7 shows that when demand is sufficiently volatile, the dynamic-pricing-only strategy dominates the dynamic-contracting-only strategy. The reason is that when the price-effect parameter α is relatively high (as in the case of the airline industry), dynamic pricing is more effective in smoothing the demand process than dynamic contracting.

Next, as Figure OA8 suggests, the relative magnitude of the price and effort effects plays an instrumental role in influencing the performance differences across strategies. The results are similar as Section 5. Figure OA8 shows that when the price effect α is low, dynamic contracting with static pricing dominates the strategy of static contracting with dynamic pricing. Note that when the price-effect parameter is low, pricing has little effect on demand. As a result, the firm prefers to influence demand over the planning horizon through dynamic incentive design.

Thus, the interaction extension supports the airline-calibrated conclusions and the main directional comparisons from Section 5; we do not claim a literal one-for-one robustness result for every comparative static.

OA7. First Best

As a benchmark, we consider the first-best problem in which the firm can directly choose (and effectively observe) the effort process $\{A_t\}_{t\in[0,T]}$. The firm maximizes expected discounted profit, defined as discounted revenue from demand increments net of effort cost, minus the terminal overbooking penalty:

$$V^{\mathrm{p,FB}} = \sup_{\{A_t\}_{t\in[0,T]}} \mathbb{E}\left[\int_0^T e^{-rt}\,dD_t - \int_0^T e^{-rt}\,\frac{A_t^2}{2\eta}\,dt - \pi e^{-rT}\big(D_T - I_0\big)^+\right]. \tag{OA62}$$

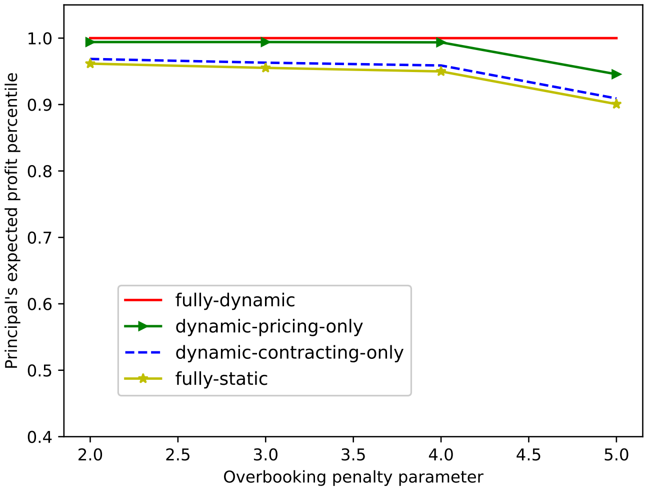


Figure OA5 **The firm's expected profit versus overbooking penalty parameter π. The parameter values, as calibrated from Perera and Tan (2019), are $r = 0.025$, $\eta = 10$, $I_0 = 150$, $\sigma = 13$, $\alpha = 354$, $\beta = 1$ and $m = 0.2$.**

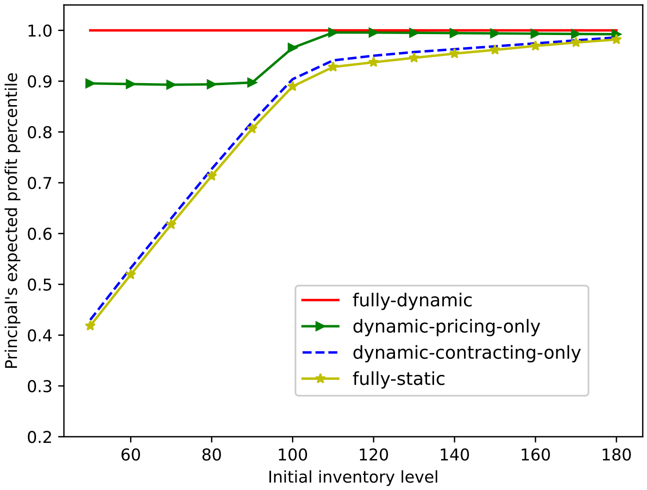


Figure OA6 **The firm's expected profit percentile versus the initial inventory level (I_0). All the parameter values are the same as in Figure 11 except that we vary I_0 between 50 and 180.**

Substitute the demand dynamics in (1). Then, at a given time $t \in [0,T]$, the firm's first-best expected profit can be expressed as:

$$V^{\mathrm{p}} = \sup_{\{A_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}(A_t - \frac{A_t^2}{2\eta})dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right], \tag{OA63}$$

where $dD_t = A_t dt + \sigma dB_t$. For the second best case, the firm's problem is defined as

$$V^{\mathrm{p}} = \sup_{\{Z_t\}_{t\in[0,T]}} E\left[\int_0^T e^{-rt}\eta Z_t dt - \int_0^T (e^{-rt}\eta Z_t^2/2)dt - \pi e^{-rT}\left(D_T - I_0\right)^+\right], \tag{OA64}$$

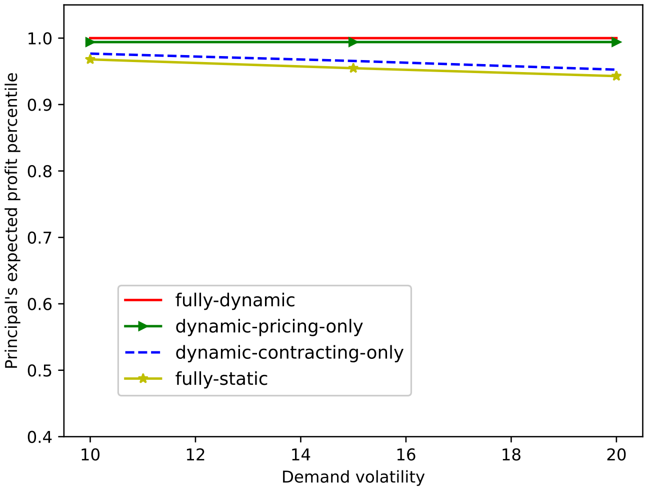


Figure OA7 **The firm's expected profit percentile versus demand volatility (σ). All parameter values are the same as in Figure 11 except that we vary σ between 10 and 20**

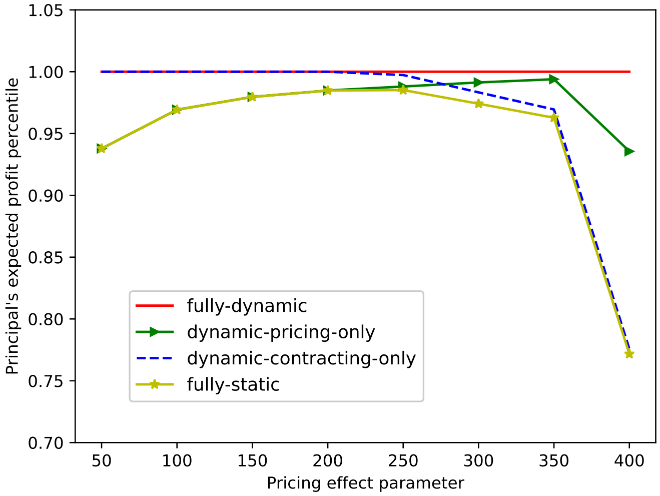


Figure OA8 **The firm's expected profit percentile versus price-effect parameter (α). All parameter values are the same as in Figure 11 except that we vary α between 50 and 400. In particular, the initial inventory level $I_0 = 150$.**

where $dD_t = \eta Z_t dt + \sigma dB_t$.