Appendix to "On the (In)Efficiency of Sustainability-Linked Loans"

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Organization. Appendices A–D extend the main analysis to uncertain target achievement, a profit-maximizing bank, risk-averse borrowers, and partially observable sustainability effort; Appendix E contains the proofs of the results in the main text. A separate Supplementary Material, available at `https://ssrn.com/abstract=7573598`, contains further variants of government-subsidized SLLs and cash rebates (Supplement SA), the implementation of refined SLLs as bundled loans (Supplement SB), and the proofs of the results in Appendices A–D and Supplements SA–SB (Supplement SC).

Appendix A. Uncertainty in Target Achievement

For the ordinary loan, the introduction of $\rho(e|T)$ will not affect the model; therefore the optimal decision and profit of each player is the same as the original model.

Under the SLL, given the contract term (r, r_d, T), when considering the uncertainty of meeting the target, the firm's profit can be formulated as follows:

$$\begin{aligned}\pi(e|r,r_d,T) &= (1-\beta(R))\left(P(e)-(\mathbb{1}_{\{e\geq T\}}Y+\mathbb{1}_{\{e<T\}}(1+r))(C(e)-x)\right)-x\\ &= \begin{cases}(1-\beta(R))\left(P(e)-Y\cdot(C(e)-x)\right)-x, & \text{if } e\geq T;\\ (1-\beta(R))\left(P(e)-(1+r)(C(e)-x)\right)-x, & \text{if } e<T,\end{cases}\end{aligned}$$

where $Y=\rho(e|T)(1+r_d)+(1-\rho(e|T))(1+r)$ is the expected interest rate faced by the firm when $e\geq T$. The bank's constraint can be written as

$$(1+s(R))\cdot L=(1-\beta(R))[\mathbb{1}_{\{e\geq T\}}Y+\mathbb{1}_{\{e<T\}}(1+r)]\cdot L.$$

The optimal solution of the SLL model with uncertainty of meeting the target is already demonstrated in Proposition 6.

When the firm is able to access to an ordinary loan in the presence of the SLL, then the firm's profit under the SLL needs to be no less than that under an ordinary loan. In this scenario, we find that the firm's optimal effort under the SLL will be the same as that of an ordinary loan; this is equivalent to the original model. We present this result in the following proposition.

Proposition A1 *When there is uncertainty to meet the target, if the firm can access to an ordinary loan, then the bank's optimal SLL is as follows:*

(i) the discounted rate is set to be the same as an ordinary loan, but the regular interest rate is higher than an ordinary loan;

(ii) the target is set to be zero and the firm's optimal effort is the same as an ordinary loan;

(iii) both the bank and the firm earn the same profits as they would under an ordinary loan.

Different from the original model, when there is uncertainty to meet the target, the bank will set a zero target in the optimal SLL contract, which stems from the fact if the target is positive, then the firm may

be charged with the regular interest rate (by not meeting the target) that weakens his profit. Therefore, to ensure that the firm earns the same profit under the SLL as an ordinary loan, the bank should set a zero target such that there is no uncertainty to achieve the target.

We next consider the government subsidy. If subsidy A is provided to the firm, his profit is

$$\pi_s(e|r, r_d, T, A) = (1-\beta(R))\left(P(e) - (\mathbb{1}_{\{e\geq T\}}Y + \mathbb{1}_{\{e<T\}}(1+r))(C(e)-x)\right) - x + A,$$

where the bank's constraint is the unchanged.

On the other hand, if the subsidy is provided to the bank, then the firm's profit function is unchanged, while the bank's constraint becomes

$$(1+s(R))\cdot L = (1-\beta(R))[\mathbb{1}_{\{e\geq T\}}Y + \mathbb{1}_{\{e<T\}}(1+r)]\cdot L + A.$$

Proposition A2 *When there is uncertainty to achieve the target, if the government subsidy A is provided to the firm, under the optimal SLL contract,*

(i) the bank sets the target $T_s^u = T_s^*$, offers a regular interest rate $r_s^u \geq r_s'$, and provides a discounted interest rate $r_{ds}^u = \frac{1+s(R)-(1-\beta(R))[\rho(T_s^*|T_s^*)+(1-\rho(T_s^*|T_s^*))(1+r_s^u)]}{(1-\beta(R))\rho(T_s^*|T_s^*)}$. The bank's profit is $\pi_s^{bu} = s(R)(C(T_s^*)-x) > \pi_o^{b*}$.

(ii) the firm's optimal effort, borrowing amount and the utility are $e_s^u = T_s^*$, $L_s^u = C(T_s^*) - x$ and $\pi_s^u = \pi_o^*$. Moreover, e_s^u is increasing in A.

Proposition A3 *When there is uncertainty to achieve the target, if the government subsidy A is provided to the bank, under the optimal SLL contract,*

(i) the bank sets the target $\hat{T}_s^u = T_s^*$, offers a regular interest rate $\hat{r}_s^u \geq \hat{r}_s'$, and provides a discounted interest rate $\hat{r}_{ds}^u = \frac{1+s(R)-(1-\beta(R))[\rho(T_s^*|T_s^*)+(1-\rho(T_s^*|T_s^*))(1+r_s^u)]}{(1-\beta(R))\rho(T_s^*|T_s^*)} - \frac{A}{(1-\beta(R))\rho(T_s^*|T_s^*)(C(T_s^*)-x)}$. The bank's profit is $\hat{\pi}_s^{bu} = s(R)(C(T_s^*)-x) > \pi_o^{b*}$.

(ii) the firm's optimal effort, borrowing amount and the utility are $\hat{e}_s^u = T_s^*$, $\hat{L}_s^u = C(T_s^*) - x$ and $\hat{\pi}_s^u = \pi_o^*$. Moreover, $\hat{e}_s^u$ is increasing in A.

Recall that T_s^* is the firm's optimal effort with government subsidy in the original model; therefore, Propositions A2 and A3 manifest that, when there is uncertainty to achieve the target, the SLL with government subsidy can also achieve the same sustainability level as that of the original model. Furthermore, the firm's optimal efforts under both scenarios are equivalent no matter who receives the subsidy. Finally, since the optimal effort is increasing in subsidy A, we can also conclude that over- and under-investment of government subsidy will happen with respect to the most effective effort that guarantees each player's reservation profit.

For the refined SLL, the bank offers a lump-sum payment w to the firm, under which the firm's profit can be written as

$$\pi_t(e|r, r_d, T, w) = (1-\beta(R))\left(P(e) - (\mathbb{1}_{\{e\geq T\}}Y + \mathbb{1}_{\{e<T\}}(1+r))(C(e)-x)\right) - x + w.$$

The bank's constraint is the same as before, which is

$$(1+s(R))\cdot L=(1-\beta(R))[\mathbb{1}_{\{e\geq T\}}Y+\mathbb{1}_{\{e<T\}}(1+r)]\cdot L,$$

and the bank's profit becomes $s(R)(C(e)-x)-w$. In addition, we also need each player to obtain the same profit as that under an ordinary loan. We have the following result corresponding to the refined SLL.

Proposition A4 *When there is uncertainty to meet the target, under the optimal refined SLL,*

(i) the bank sets the target $T_t^u=T_t^*$, offers a regular interest rate $r_t^u\geq r_t'$, provides a discounted interest rate $r_d^u=\frac{1+s(R)-(1-\beta(R))[\rho(T_t^*|T_t^*)+(1-\rho(T_t^*|T_t^*))(1+r^u)]}{(1-\beta(R))\rho(T_t^*|T_t^*)}$ and the lump-sum payment as $w^u=s(R)(C(T_t^*)-C(e_o^*))$. The bank's profit is $\pi_t^{bu}=\pi_o^{b*}$.

(ii) the firm's optimal effort, loan size and utility are $e_t^u=T_t^*$, $L_t^u=C(T_t^*)-x$ and $\pi_t^u=\pi_o^*$.

Comparing to the original model of the refined SLL, with uncertainty, we find that the bank will set the same target and provide the same lump-sum payment, under which the firm exerts the same effort as that of the original model.

Appendix B. Alternative Bank Objective: Profit Maximization

Suppose now that the bank's profit enters her objective function. We use the superscript "m" to denote variables of interest in this extension.

Under the ordinary loan, the firm's model is unchanged, so his optimal response remains $e_o(r)$. Given the interest rate r and the firm's response $e_o(r)$, the bank's problem can be formulated as follows:

$$\max_{r\geq 0}\Big\{\Omega_o^b(r)=\pi_o^b(r)+\alpha F(e_o(r))\Big\}\quad\text{where}\quad\pi_o^b(r)=[(1-\beta(R))(1+r)-1]\big(C(e_o(r))-x\big)$$
$$s.t.\quad(1+s(R))L=(1-\beta(R))(1+r)L.$$

A more general form of the objective function is $\Omega_o^b(r)=\delta\pi_o^b(r)+(1-\delta)F(e)$, where $\delta\in(0,1)$. We simplify the model by letting $\alpha=\frac{1-\delta}{\delta}$. The constraint directly implies that the optimal interest rate under the ordinary loan equals $r_o^m=\frac{s(R)+\beta(R)}{1-\beta(R)}=r_o^*$ for any $\alpha\geq 0$, which then uniquely determines the firm's optimal effort $e_o^m=e_o(r_o^m)=e_o^*$. Therefore, under the ordinary loan, the optimal decisions and profits of the two players are the same as those of the original model, regardless of the bank's objective.

Under the SLL, the firm's model is unchanged. For the bank, given the contract (r,r_d,T), the firm's effort e, and his loan size L, we define the bank's objective function Ω^b as follows:

$$\Omega^b(r,r_d,T|e,L)=\pi^b(r,r_d,T|e,L)+\alpha F(e),$$

where $\pi^b(r,r_d,T|e,L)=[(1-\beta(R))(\mathbb{1}_{\{e\geq T\}}\cdot(1+r_d)+\mathbb{1}_{\{e<T\}}\cdot(1+r))-1]L$ is the bank's expected profit under the SLL. The bank's problem can be formulated as

$$\max_{r>r_d\geq 0,T\geq 0}\Omega^b(r,r_d,T|e,L)$$
$$s.t.\quad(1+s(R))L=(1-\beta(R))(\mathbb{1}_{\{e\geq T\}}\cdot(1+r_d)+\mathbb{1}_{\{e<T\}}\cdot(1+r))L.$$

Because the firm's model is unchanged, his best response is also the same as before. The following result characterizes the optimal SLL contract when the bank's profit enters her objective function.

Proposition B1 *Under the alternative bank objective, the optimal SLL contract* (r^m, r_d^m, T^m) *is equivalent to the optimal SLL contract* (r^*, r_d^*, T^*) *of the original model, i.e.,* $r^m = r^*$, $r_d^m = r_d^*$, *and* $T^m = T^*$. *Therefore, the bank's and the firm's profits are also the same as before:* $\pi^{bm} = \pi^{b*} = s(R)[C(T^*) - x]$ *and* $\pi^m = \pi^* = 0$.

Even though the bank incorporates her profit in the objective function, the optimal SLL contract remains unchanged compared with the original model. The intuition is that the bank's profit increases in the firm's effort: to exert a higher effort level, the firm needs to borrow more from the bank, which raises the bank's interest income. The bank's new objective is therefore aligned with the original objective of maximizing the firm's effort.

When the firm can access an ordinary loan in the presence of the SLL, we have the following result.

Proposition B2 *Under the alternative bank objective, if the firm can access an ordinary loan and an SLL simultaneously, the optimal SLL contract is the same as that of the original model. That is, the optimal SLL degenerates to an ordinary loan.*

With the competition of an ordinary loan, incorporating the bank's profit in her objective does not change the fact that standard SLLs fail to improve the firm's sustainability effort.

We next consider government subsidies. If the subsidy A is provided to the firm, the firm's profit is the same as that in the original model; if the subsidy is provided to the bank, the bank's objective becomes

$$\Omega_s^b(r, r_d, T|e, L) = \pi^b(r, r_d, T|e, L) + A + \alpha F(e),$$

and the competitive pricing condition becomes

$$(1 + s(R))L = (1 - \beta(R))\big(\mathbb{1}_{\{e \geq T\}} \cdot (1 + r_d) + \mathbb{1}_{\{e < T\}} \cdot (1 + r)\big)L + A.$$

Proposition B3 *Under the alternative bank objective, the equilibrium results of the government-subsidized SLL model are the same as those of the original model.*

Whether the subsidy is paid to the firm or to the bank, the optimal SLL contract specifies the same target, and therefore achieves the same effort as the original model. Note that substituting the pricing condition into Ω_s^b shows that the subsidy cancels out of the bank's profit, which remains $s(R)L$: the subsidy is passed through to loan terms and never becomes discretionary surplus for the bank to allocate, consistent with the discussion in Section 6.2.

For the refined SLL, with the lump-sum rebate w, the bank's objective function becomes

$$\Omega_t^b(r, r_d, T, w|e, L) = \pi^b(r, r_d, T|e, L) - w + \alpha F(e),$$

while the firm's profit function is unchanged. Let $\hat{T} = \arg\max_e \big\{\Pi(e) + \alpha F(e)\big\}$ and we have:

Proposition B4 *Under the alternative bank objective, under the optimal refined SLL,*

(i) the bank offers a discounted interest rate $r_{dt}^m = \frac{s(R)+\beta(R)}{1-\beta(R)}$ *and a regular rate* $r_t^m \geq r_{t2}$*, where* r_{t2} *satisfies* $\pi\left(g^{-1}(1+r_{t2}) \mid r_{t2}, r_{dt}^m, T\right) + w^m = \pi_o^*$ *with* $g^{-1}(1+r_{t2}) < T$*, sets the target to* $T_t^m = \min\{\hat{T}, T_t^*\}$*, and provides a lump-sum rebate* $w^m = (1+s(R))(C(T_t^m) - C(e_o^*)) - (1-\beta(R))(P(T_t^m) - P(e_o^*))$*, with her profit* $\pi_t^{bm} = s(R)(C(T_t^m) - x) - w^m \geq \pi_o^{b*}$*;*

(ii) the firm's optimal effort and profit are $e_t^m = T_t^m$ *and* $\pi_t^m = \pi_o^*$*, respectively;*

(iii) there exists a threshold $\bar{\alpha} > 0$ *such that when* $\alpha < \bar{\alpha}$*,* $w^m < w^*$ *and* $e_t^m < e_t^*$*; when* $\alpha \geq \bar{\alpha}$*,* $w^m = w^*$ *and* $e_t^m = e_t^*$*.*

Corollary B1 *When* $\alpha = 0$*, the optimal refined SLL sets* $T_t^m = e_{fb}^*$*, and the bank's profit satisfies* $\pi_t^{bm} = \Pi(e_{fb}^*) - \pi_o^* > \pi_o^{b*}$*. That is, even a purely profit-driven bank voluntarily offers the rebate, and her profit under the refined SLL strictly exceeds her profit under an ordinary loan.*

When the bank's profit enters her objective function, the refined SLL continues to improve the firm's sustainability effort. However, the improved effort may not be as high as in the original model, because the bank's lump-sum rebate is smaller: since the bank now values her own profit, she does not allocate all incremental surplus to improving sustainability effort. Nevertheless, if α is sufficiently large—that is, if the weight on sustainability in the bank's objective is sufficient—she commits the incremental surplus to sustainability, and the original refined SLL is recovered.

Appendix C. Risk-Averse Borrowers

In this extension, we use the superscript "a" in the notations to represent the situation where the firm is risk averse. Under an ordinary loan, the bank's constraint is not affected by the firm's risk-aversion. Therefore, the optimal interest rate under the ordinary loan is $r_o^a = \frac{s(R)+\beta(R)}{1-\beta(R)}$. For the firm, with the interest rate r_o^a, his utility can be written as

$$\pi_o(e|r_o^a) = \mathbb{E}\left[K\left((\epsilon P(e) - (1+r_o^a)(C(e)-x))^+\right)\right] - x = (1-\beta(R))K\left(P(e) - (1+r_o^a)(C(e)-x)\right) - x.$$

Because $K(\cdot)$ is increasingly concave and $P(e) - (1+r_o^a)(C(e)-x)$ is concave in e, we can verify that $K\left(P(e) - (1+r_o^a)(C(e)-x)\right)$ is concave in e. In addition, since $K(\cdot)$ increases, to maximize $\pi_o(e|r_o^a)$, the firm suffices to maximize $P(e) - (1+r_o^a)(C(e)-x)$. The FOC yields $p(e) - (1+r_o^a)c(e) = 0$ and we can obtain that the firm's optimal effort under the ordinary loan is $e_o^a = g^{-1}(\frac{1+s(R)}{1-\beta(R)})$, which is the same as the original model. We summarize the result of the ordinary loan when the firm is risk averse as follows.

Lemma C1 *When the firm is risk averse, under the ordinary loan,*

(1) the bank's optimal interest rate and the expected profit are

$$r_o^a = \frac{s(R)+\beta(R)}{1-\beta(R)} \text{ and } \pi_o^{ba} = s(R)\left[C(g^{-1}(\frac{1+s(R)}{1-\beta(R)})) - x\right].$$

(2) the firm's optimal effort, the optimal loan amount and the utility are

$$e_o^a = g^{-1}(\frac{1+s(R)}{1-\beta(R)}), L_o^a = C(e_o^a) - x \text{ and } \pi_o^a = (1-\beta(R))K\left(P(e_o^a) - (1+r_o^a)(C(e_o^a)-x)\right) - x.$$

From Lemma C1, we find that the structure of the optimal solution under the ordinary loan is identical to that of the original model.

Under the SLL, given the contract term (r, r_d, T), the firm's utility is as follows:

$$\begin{aligned}\pi(e|r,r_d,T) &= \mathbb{E}\left[K\left((\epsilon P(e) - (\mathbb{1}_{\{e\geq T\}}(1+r_d) + \mathbb{1}_{\{e<T\}}(1+r))(C(e)-x)\right)^+\right)\right] - x \\ &= (1-\beta(R))K\left(P(e) - (\mathbb{1}_{\{e\geq T\}}(1+r_d) + \mathbb{1}_{\{e<T\}}(1+r))(C(e)-x)\right)\right) - x \\ &= \begin{cases} (1-\beta(R))K\left((P(e)-(1+r_d)(C(e)-x)\right)\right) - x, & \text{if } e \geq T; \\ (1-\beta(R))K\left((P(e)-(1+r)(C(e)-x)\right)\right) - x, & \text{if } e < T. \end{cases}\end{aligned}$$

The firm's utility function is similar to that of the original model, except that there is a function of $K(\cdot)$. Because $K(\cdot)$ is increasing, it will not affect the firm's best reponse to the SLL contract (r, r_d, T). Therefore, under (r, r_d, T), we can obtain the same result of Lemma 2 of the original model. For the bank, the interest rates are determined by the following equation:

$$(1+s(R))\cdot L = (1-\beta(R))[\mathbb{1}_{\{e\geq T\}}(1+r_d)L + \mathbb{1}_{\{e<T\}}(1+r)L],$$

which is also equivalent to the original model. We present the optimal SLL terms and the firm's corresponding optimal decision in the following proposition.

Proposition C1 *When the firm is risk averse, under the optimal SLL,*

(i) the bank offers a discounted rate $r_d^a = \frac{s(R)+\beta(R)}{1-\beta(R)}$ *and a regular interest rate* $r^a \geq r_n^a$*, and sets the target* $T^a = \bar{T}(r_n^a, r_d^a)$*, where* r_n^a *is the solution of* r *in* $\max_e\{\pi_o(e|r)\} = 0$*. The bank's profit is* $\pi^{ba} = s(R)(C(T^a)-x) > \pi_o^{ba}$*.*

(ii) the firm's optimal effort, loan size and utility are $e^a = T^a$*,* $L^a = C(T^a) - x$ *and* $\pi^a = 0$*.*

(iii) if the utility function satisfies $K(x) \leq x$*, then we have* $e^a \leq e^*$*.*

From Proposition C1, when the firm is risk averse, the optimal SLL contract forces the firm's utility to zero, while the discounted interest rate equals that of the ordinary loan, showing that the SLL uses the target and regular interest rate to jeopardize the firm's utility and maximize sustainability effort. This is equivalent to the result in the original model. Therefore, the firm's optimal effort exceeds that under an ordinary loan. Note that many utility functions satisfy $K(x) \leq x$—a condition met by common specifications including logarithmic utility $K(x) = \log(1+x)$, quadratic utility $K(x) = x - bx^2/2$, CARA $K(x) = (1-e^{-ax})/a$, and CRRA $K(x) = (x^{1-\gamma}-1)/(1-\gamma)$ for $\gamma \geq 0$.

When the firm is able to access to an ordinary loan in the presence of the SLL, it means that the firm's utility under the SLL is no less than the optimal utility under the ordinary loan, which indicates the following inequality must hold:

$$\max_{e\geq T}\{(1-\beta(R))K\left((P(e)-(1+r_d^a)(C(e)-x)\right)\right) - x\} \geq (1-\beta(R))K\left(P(e_o^a) - (1+r_o^a)(C(e_o^a)-x)\right) - x.$$

Because $r_d^a = \frac{s(R)+\beta(R)}{1-\beta(R)} = r_o^a$ and $e_o^a = \arg\max\{P(e) - (1+r_o^a)(C(e)-x)\}$, this suggests that with the competition of an ordinary loan, the firm's optimal effort must be equal to e_o^a, which further implies that

the optimal target under the SLL equals e_o^a. We thus conclude that with the competition of an ordinary loan, the optimal solution of each player under the SLL is the same as that under an ordinary loan.

We next consider government subsidy. If the subsidy is provided to the firm, his utility is

$$\begin{aligned}\pi_s(e|r,r_d,T,A) &= \mathbb{E}\left[K\left((\epsilon P(e)-(\mathbb{1}_{\{e\geq T\}}(1+r_d)+\mathbb{1}_{\{e<T\}}(1+r))(C(e)-x))^+ + A\right)\right]-x \\ &= (1-\beta(R))K\left(P(e)-(\mathbb{1}_{\{e\geq T\}}(1+r_d)+\mathbb{1}_{\{e<T\}}(1+r))(C(e)-x))+A\right)+\beta(R)K(A)-x,\end{aligned}$$

while the bank's constraint is the same as before.

On the other hand, if the subsidy is provided to the bank, the firm's utility is the same as before, while the bank's constraint becomes

$$(1+s(R))\cdot L = (1-\beta(R))[\mathbb{1}_{\{e\geq T\}}(1+r_d)L+\mathbb{1}_{\{e<T\}}(1+r)L]+A.$$

Denote e_s^a and $\hat{e}_s^a$ as the firm's optimal effort under the case when the subsidy is provided to the firm and the bank respectively. In the original model, we know that the firm's optimal efforts are equivalent despite who receives the subsidy; however, when the firm's risk averse, we find that this result does not hold.

Proposition C2 *When the firm is risk averse,*

(i) if the government subsidy A is provided to the firm, under the optimal SLL contract, the bank offers a discounted interest rate $r_{ds}^a = \frac{s(R)+\beta(R)}{1-\beta(R)}$ and a regular interest rate $r_s^a \geq r_{s1}$, where r_{s1} satisfies $\pi_s(g^{-1}(1+r_{s1})<T|r_{s1},r_{ds}^a,T,A)=\pi_o^a$, and sets the target to T_s^a which is the largest solution of T in $\pi_s(T|r_{s1},r_{ds}^a,T,A)=\pi_o^a$. Correspondingly, the bank's profit is $\pi_s^{ba}=s(R)(C(T_s^a)-x)>\pi_o^{ba}$; the firm's optimal effort and profit are $e_s^a=T_s^a$ and $\pi_s^a=\pi_o^a$. Moreover, e_s^a increases in A.

(ii) if the government subsidy A is provided to the bank, under the optimal SLL contract, the bank offers a discounted interest rate $\hat{r}_{ds}^a = \frac{s(R)+\beta(R)}{1-\beta(R)} - \frac{A}{(1-\beta(R))(C(\hat{T}_s^a)-x)}$ and a regular interest rate $\hat{r}_s^a \geq r_{s2}$, where r_{s2} satisfies $\pi(g^{-1}(1+r_{s2})<T|r_{s2},\hat{r}_{ds}^a,T)=\pi_o^a$, and sets the target to $\hat{T}_s^a$ which is given by $\hat{T}_s^a = \max\left\{e|(1-\beta(R))K\left(P(e)-(1+r_o^a)(C(e)-x)\right)+\frac{A}{1-\beta(R)}\right)-x=\pi_o^a\right\}$. Correspondingly, the bank's profit is $\hat{\pi}_s^{ba}=s(R)(C(\hat{T}_s^a)-x)>\pi_o^{ba}$; the firm's optimal effort and profit are $\hat{e}_s^a=\hat{T}_s^a$ and $\hat{\pi}_s^a=\pi_o^a$. Moreover, $\hat{e}_s^a$ increases in A.

(iii) it holds that $r_{ds}^a > \hat{r}_{ds}^a$ and $e_s^a \geq \hat{e}_s^a > e_o^a$.

Proposition C2 demonstrates the firm's optimal efforts under the government subsidy, and we also find that the firm can achieve greater effort when the subsidy is offered to himself than to the bank. This is because when the subsidy A is provided to the firm, the firm can still earn an amount of A even if his revenue is wiped out with probability $\beta(R)$. Therefore, the firm's risk is diversified. However, when the subsidy is provide to the bank, the bank will certainly reduce the interest rate and pass through to the firm; however, the firm will still be bankrupt with probability $\beta(R)$. Because the firm is risk averse, he certainly prefers the case when the government provides subsidy to him and therefore, the SLL can force the firm to exert more effort in this case.

According to the definition of e_s^a and $\hat{e}_s^a$, one can easily verify that the firm's optimal effort under government subsidy is increasing in A; therefore, over- and under-investment of government subsidy will also happen when there exists an effort level that is maximized while guarantees each player's profit as in the ordinary loan.

Under the refined SLL, with the cash rebate w, the firm's utility can be written as follows:

$$\begin{aligned}\pi_t(e|r,r_d,T,w) &= \mathbb{E}\left[K\left((\epsilon P(e)-(\mathbb{1}_{\{e\geq T\}}(1+r_d)+\mathbb{1}_{\{e<T\}}(1+r))(C(e)-x))^+ + w\right)\right]-x \\ &= (1-\beta(R))K\left(P(e)-(\mathbb{1}_{\{e\geq T\}}(1+r_d)+\mathbb{1}_{\{e<T\}}(1+r))(C(e)-x))+w\right)+\beta(R)K(w)-x.\end{aligned}$$

The bank's constraint is the same as before, which is

$$(1+s(R))\cdot L = (1-\beta(R))[\mathbb{1}_{\{e\geq T\}}(1+r_d)L+\mathbb{1}_{\{e<T\}}(1+r)L],$$

while the bank's profit becomes $s(R)(C(e)-x)-w$. We have the following result.

Proposition C3 *When the firm is risk averse, under the optimal refined SLL,*

(1) the bank offers a discounted rate $r_{dt}^a = \frac{s(R)+\beta(R)}{1-\beta(R)}$, sets the target T_t^a that satisfies

$$\max_{e\geq T_t^a}\{(1-\beta(R))K\left(P(e)-\frac{1+s(R)\beta(R)}{1-\beta(R)}(C(e)-x))-\pi_o^{ba}\right)+\beta(R)K\left(s(R)(C(e)-x))-\pi_o^{ba}\right)-x\}=\pi_o^a,$$

provides a lump-sum rebate $w^a = s(R)(C(T_t^a)-x)-\pi_o^{ba}$; the bank's profit is $\pi_t^{ba}=\pi_o^{ba}$.

(2) the firm's optimal effort and profit are $e_t^a = T_t^a > e_o^a$ and $\pi_t^a = \pi_o^a$, respectively.

Proposition C3 shows that when the firm is risk averse, under the refined SLL, the optimal contract structure is similar to that of the original model. It's also noteworthy that under the optimal refined SLL contract, the firm's utility function is very similar to that when the government subsidy is provided to the firm, except the the rebate w^a is replaced by A. Therefore, if the amount of subsidy is exactly equal to w^a, we can also conclude that the optimal refined SLL achieves the same sustainability level as the one when government provides subsidy to the firm.

Appendix D. Partially Observable Sustainability Effort

Suppose that, with probability λ, the market observes the firm's sustainability effort and pays the price $P(e)$; with probability $1-\lambda$, the effort is not observed and consumers' willingness to pay is a constant $\bar{P}$, independent of e. Under this setting, the no-financing, ordinary-loan, SLL, subsidy, and refined-SLL models all remain unchanged except that the firm's expected selling price changes from $P(e)$ to $\lambda P(e)+(1-\lambda)\bar{P}$ in his profit function. In this extension, all functions of interest (e.g., Π, π_o, $\bar{T}$) denote their counterparts under the modified selling price, and we use the superscript "k" for optimal decisions and profits. To rule out trivial cases, we assume that $\bar{P}$ is large enough that production remains profitable for the firm.

For the no-financing model, the firm's profit function becomes

$$\Pi(e) = (1-\beta(R))[\lambda P(e)+(1-\lambda)\bar{P}]-C(e),$$

which remains concave in e; the optimal effort therefore equals $e^k_{fb} = g^{-1}\big(\frac{1}{\lambda(1-\beta(R))}\big)$. We find that $e^k_{fb} < e^*_{fb}$: since only a fraction of consumers pay a price that reflects the firm's sustainability effort, the marginal revenue of effort is scaled down by λ, which weakens the firm's incentive to invest in sustainability.

For the ordinary loan, the bank's pricing condition

$$(1+s(R))L = (1-\beta(R))(1+r)L$$

does not involve λ, so the optimal interest rate equals $r^k_o = \frac{s(R)+\beta(R)}{1-\beta(R)} = r^*_o$, the same as in the original model. The following lemma summarizes the ordinary-loan outcome.

Lemma D1 *When the firm's effort is not fully observed by the market, under the ordinary loan,*

(1) the bank's optimal interest rate is r^k_o, and her expected profit is

$$\pi^{bk}_o = [(1-\beta(R))(1+r^k_o) - 1](C(e^k_o) - x);$$

(2) the firm's optimal effort, loan size, and profit are

$$e^k_o = g^{-1}\Big(\frac{1+s(R)}{\lambda(1-\beta(R))}\Big), \quad L^k_o = C(e^k_o) - x, \text{ and } \pi^k_o = (1-\beta(R))[\lambda P(e^k_o) + (1-\lambda)\bar{P} - (1+r^k_o)(C(e^k_o)-x)] - x.$$

Under the SLL, the modified selling price does not alter the structure of the firm's profit function; the firm's best response therefore takes the same form as in Lemma 3, which allows us to characterize the optimal decisions of the two players under the SLL.

Proposition D1 *When the firm's effort is not fully observed by the market, under the optimal SLL,*

(i) the bank offers a discounted rate $r^k_d = \frac{s(R)+\beta(R)}{1-\beta(R)}$ and a regular interest rate $r^k \geq r^k_n$, and sets the target $T^k = \bar{T}(r^k_n, r^k_d)$, where r^k_n is the solution of r in $\max_e\{\pi_o(e|r)\} = 0$. The bank's profit is $\pi^{bk} = s(R)(C(T^k) - x) > \pi^{bk}_o$;

(ii) the firm's optimal effort, loan size, and profit are $e^k = T^k$, $L^k = C(T^k) - x$, and $\pi^k = 0$.

When the firm can access an ordinary loan and an SLL simultaneously, the firm's expected profit under the SLL must be no less than that under the optimal ordinary loan. Since the structure of the firm's profit function is unchanged, the degeneracy result of the original model carries over, as shown in the following proposition.

Proposition D2 *When the firm's effort is not fully observed by the market, if the firm can access an ordinary loan and an SLL simultaneously, then the bank writes the SLL contract so that*

(i) the discounted interest rate is the same as under an ordinary loan, i.e., $\bar{r}^k_d = r^*_o$, but the regular interest rate is higher, i.e., $\bar{r}^k > r^*_o$;

(ii) the target induces the same effort as an ordinary loan, i.e., $\bar{T}^k = \bar{e}^k = e^k_o$;

(iii) both the bank and the firm earn the same profits as they would under an ordinary loan.

For government subsidies, we again examine the two scenarios in which the subsidy is provided to the firm or to the bank.

Proposition D3 *When the firm's effort is not fully observed by the market,*

(i) if the government subsidy A is provided to the firm, under the optimal SLL contract, the bank offers a discounted interest rate $r_{ds}^k = \frac{s(R)+\beta(R)}{1-\beta(R)}$ and a regular interest rate $r_s^k \geq r_{s1}^k$, where r_{s1}^k satisfies $\pi_s(g^{-1}(1+r_{s1}^k) \mid r_{s1}^k, r_{ds}^k, T, A) = \pi_o^k$ with $g^{-1}(1+r_{s1}^k) < T$, and sets the target to T_s^k, the largest solution of T in $\pi_s(T \mid r_s^k, r_{ds}^k, T, A) = \pi_o^k$. Correspondingly, the bank's profit is $\pi_s^{bk} = s(R)(C(T_s^k) - x) > \pi_o^{bk}$; the firm's optimal effort and profit are $e_s^k = T_s^k$ and $\pi_s^k = \pi_o^k$. Moreover, e_s^k increases in A;

(ii) if the government subsidy A is provided to the bank, under the optimal SLL contract, the bank offers a discounted interest rate $\hat{r}_{ds}^k = \frac{s(R)+\beta(R)}{1-\beta(R)} - \frac{A}{(1-\beta(R))(C(\hat{T}_s^k)-x)}$ and a regular interest rate $\hat{r}_s^k \geq r_{s2}^k$, where r_{s2}^k satisfies $\pi(g^{-1}(1+r_{s2}^k) \mid r_{s2}^k, \hat{r}_{ds}^k, T) = \pi_o^k$ with $g^{-1}(1+r_{s2}^k) < T$, and sets the target $\hat{T}_s^k = \max\{T : \pi(T \mid \hat{r}_s^k, \hat{r}_{ds}^k, T) = \pi_o^k\}$. Correspondingly, the bank's profit is $\hat{\pi}_s^{bk} = s(R)(C(\hat{T}_s^k) - x) > \pi_o^{bk}$; the firm's optimal effort and profit are $\hat{e}_s^k = \hat{T}_s^k$ and $\hat{\pi}_s^k = \pi_o^k$. Moreover, $\hat{e}_s^k$ increases in A.

With government subsidies, our results therefore remain robust when the firm's effort cannot be fully observed by the market. Finally, we turn to the refined SLL.

Proposition D4 *When the firm's effort is not fully observed by the market, under the optimal refined SLL, (i) the bank offers a discounted interest rate $r_{dt}^k = \frac{s(R)+\beta(R)}{1-\beta(R)}$ and a regular rate $r_t^k \geq r_{t1}$, where r_{t1} satisfies $\pi_t(g^{-1}(1+r_{t1}) \mid r_{t1}, r_{dt}^k, T, w^k) = \pi_o^k$ with $g^{-1}(1+r_{t1}) < T$, sets the target to T_t^k, the largest solution of e in $\Pi(e) = \Pi(e_o^k)$, and provides a lump-sum rebate $w^k = s(R)(C(T_t^k) - C(e_o^k))$, with her profit $\pi_t^{bk} = \pi_o^{bk}$; (ii) the firm's optimal effort and profit are $e_t^k = T_t^k > e_o^k$ and $\pi_t^k = \pi_o^k$, respectively.*

Appendix E. Proofs

This appendix proves the results in the main text. The proofs of the results in Appendices A–D are in Supplement SC of the Supplementary Material.

Proof of Lemma 3. The firm's net profit function is

$$\pi(e|r, r_d, T) = \begin{cases} (1-\beta(R))\left[P(e) - (1+r_d)(C(e)-x)\right] - x & \text{if } e \geq T; \\ (1-\beta(R))\left[P(e) - (1+r)(C(e)-x)\right] - x & \text{if } e < T. \end{cases}$$

To derive the firm's optimal effort e^*, we consider two cases: (1) $e \geq T$ and (2) $e < T$. For case (1), if $e \geq T$, then $\pi(e|r, r_d, T) = (1-\beta(R))[P(e) - (1+r_d)(C(e)-x)] - x$. The first-order condition (FOC) is $(1-\beta(R))[p(e) - (1+r_d)c(e)] = 0$, implying that the optimal effort level (which is denoted by e_1^* under this case) equals $\max\{e_d, T\}$. The firm's corresponding profit is (which is denoted by π_1 under this case)

$$\pi_1 = (1-\beta(R))[P(e_1^*) - (1+r_d)(C(e_1^*)-x)] - x.$$

For case (2) with $e < T$, we have

$$\pi(e|r, r_d, T) = (1-\beta(R))[P(e) - (1+r)(C(e)-x)] - x,$$

and therefore the FOC is $(1-\beta(R))[y(e)-(1+r)f(e)]=0$, implying that the optimal effort level (which is denoted by e_2^* under this case) equals $\min\{e_r, T\}$. The corresponding profit of the firm is (which is denoted by π_2 under this case)

$$\pi_2 = (1-\beta(R))[P(e_2^*)-(1+r)(C(e_2^*)-x)]-x.$$

We next need to compare π_1 and π_2 to derive the firm's optimal effort level. It's obvious that $e_r < e_d$ since $r > r_d$. Several cases are discussed as follows.

Case (i): $T \le e_r$. In this case, $e_1^* = e_d$ and $e_2^* = T$. Thus,

$$\begin{aligned}\pi_1 - \pi_2 &= (1-\beta(R))[P(e_1^*)-(1+r_d)(C(e_1^*)-x)-(P(T)-(1+r)(C(T)-x))]\\ &> (1-\beta(R))[(r-r_d)(C(T)-x)]\\ &> 0,\end{aligned}$$

where the first inequality is because $P(e_1^*)-(1+r_d)(C(e_1^*)-x) > P(T)-(1+r_d)(C(T)-x)$ (since $P(e)-(1+r_d)(C(e)-x)$ is maximized at $e=e_d$ and $T \le e_r \le e_d$) and the second inequality is because $C(T) > C(0) > x$. Therefore, we can conclude that the optimal effort level is $e^* = e_d$.

Case (ii): $e_r < T \le e_d$. In this case, $e_1^* = e_d$ and $e_2^* = e_r$. Thus,

$$\begin{aligned}\pi_1 - \pi_2 &= (1-\beta(R))[P(e_1^*)-(1+r_d)(C(e_1^*)-x)-(P(e_2^*)-(1+r)(C(e_2^*)-x))]\\ &> (1-\beta(R))[(r-r_d)(C(e_2^*)-x)]\\ &> 0,\end{aligned}$$

where the first inequality is because $P(e_1^*)-(1+r_d)(C(e_1^*)-x) > P(e_2^*)-(1+r_d)(C(e_2^*)-x)$. Therefore, in this case, we can conclude that the optimal ESG level is $e^* = e_d$.

Case (iii): $T > e_d$. In this case, $e_1^* = T$ and $e_2^* = e_r$. Thus,

$$\pi_1 - \pi_2 = (1-\beta(R))[P(T)-(1+r_d)(C(T)-x)-(P(e_2^*)-(1+r)(C(e_2^*)-x))].$$

Clearly, $P(T)-(1+r_d)(C(T)-x)$ is decreasing in T for $T > e_d$. Therefore, $\pi_1 - \pi_2$ is decreasing in T for $T > e_d$. In addition, when $T = e_d$, we know $\pi_1 - \pi_2 > 0$; when T goes to $+\infty$, $\pi_1 - \pi_2$ goes to $-\infty$. Therefore, there must exist a value $\bar{T}(r, r_d) > e_d$ such that when $T = \bar{T}(r, r_d)$, $\pi_1 - \pi_2 = 0$. That is, when $T \in (e_d, \bar{T}(r, r_d)]$, $\pi_1 - \pi_2 \ge 0$, therefore the optimal effort level is $e^* = T$. When $T > \bar{T}(r, r_d)$, $\pi_1 - \pi_2 < 0$, and the optimal effort level would be $e^* = e_r$. Note that in this case $\pi_1 = \pi(T|r, r_d, T)$ and $\pi_2 = (1-\beta(R))[P(e_r)-(1+r)(C(e_r)-x)]-x$, implying that $\bar{T}(r, r_d)$ satisfies

$$\bar{T}(r, r_d) = \max\{T : \pi(T|r, r_d, T) = \pi_o(e_r|r)\}.$$

Since $\pi(T|r, r_d, T)$ is independent of r and is decreasing in T for $T > e_d$, while $\pi_o(e_r|r)$ is decreasing in r, we thus conclude that $\bar{T}(r, r_d)$ is increasing in r. The proof is complete.

Proof of Proposition 1. The bank's objective is to maximize the firm's effort. From Lemma 3 and Figure 2, it's easily verified that the firm's effort will be maximized only when he meets the target

(specifically, when $e = \bar{T}(r, r_d)$). That is, the bank should design the SLL contract parameters such that the firm is willing to meet the target. Therefore, when meeting the target, the firm's net profit is

$$\pi(e|r, r_d, T) = (1-\beta(R))[(P(e) - (1+r_d)(C(e)-x))] - x.$$

And the bank's problem can be rewritten as

$$\max_{r>r_d>0, T>0} \quad e \qquad \text{s.t.} \quad (1+s(R))L = (1-\beta(R))(1+r_d)L.$$

This immediately indicates that the optimal discounted interest rate is

$$r_d^* = \frac{s(R)+\beta(R)}{1-\beta(R)}.$$

Next, we derive the bank's the optimal target to induce a maximum effort level. With r_d^*, the firm's maximum profit, which is denoted as π^*, is

$$\pi^* = \max_{e\geq 0}\{\pi(e|r, r_d, T)\} = \max_{e\geq T}\Big\{(1-\beta(R))[P(e) - \frac{1+s(R)}{1-\beta(R)}(C(e)-x)]\} - x\Big\}.$$

The firm's reservation profit is zero, therefore the bank needs to ensure $\pi^* \geq 0$. It can be easily verified that π^* is decreasing in T while the firm's optimal effort is increasing in T (due to the constraint of $e \geq T$). This implies that the bank's optimal target T^* must satisfies

$$\max_{e\geq T^*}\Big\{(1-\beta(R))[P(e) - \frac{1+s(R)}{1-\beta(R)}(C(e)-x)]\} - x\Big\} = 0. \tag{E-2}$$

On the other hand, the bank needs to ensure that the firm's profit under the regular interest rate (which is $\pi_o(e_r|r)$) is no larger than zero, otherwise the firm will choose to miss the target. Since $\pi_o(e_r|r)$ is decreasing in r, the definition of r_n indicates that the bank's optimal regular interest rate r^* must satisfy $r^* \geq r_n$. In addition, from (E-2) and the definition of $\bar{T}(r, r_d)$, we immediately have $T^* = \bar{T}(r_n, r_d^*)$. Finally, the firm's optimal effort and borrowing amount are $e^* = T^*$ and $L^* = C(e^*) - x$, respectively.

Proof of Corollary 3. According to the formulation of $\Pi(e)$, we have $e_{fb}^* = g^{-1}(\frac{1}{1-\beta(R)})$, which is increasing in R since $\beta(R)$ is decreasing in R and $g(\cdot)$ is a decreasing function. The firm's optimal effort under the SLL e^* satisfies $(1-\beta(R))[P(e^*) - \frac{1+s(R)}{1-\beta(R)}(C(e^*)-x)]\} - x = 0$, and the proof of Corollary 2 indicates that e^* is increasing in R. The firm's optimal effort under the ordinary loan is $e_o^* = g^{-1}(\frac{1+s(R)}{1-\beta(R)})$, which is smaller than e_{fb}^*. In addition, the firm's profit under the ordinary loan is

$$\pi_o^* = (1-\beta(R))[P(e_o^*) - \frac{1+s(R)}{1-\beta(R)}(C(e_o^*)-x)]\} - x,$$

which is increasing in R. Clearly, when $R \to 0$ (such that $\beta(R) \to 1$ and $s(R) \to \infty$), $\pi_o^* < 0$. Therefore, there must exists a $R' > 0$ to make $\pi_o^* = 0$. In this case, according to the proof of Proposition 1, with $R = R'$, the firm's optimal effort e^* must equal e_o^*, which is smaller than e_{fb}^*. In addition, when $R \to \infty$ (such that $\beta(R) \to 0$ and $s(R) \to 0$), we can see that e^* satisfies

$$(1-\beta(R))[P(e^*) - \frac{1+s(R)}{1-\beta(R)}(C(e^*)-x)]\} - x = P(e^*) - C(e^*) = 0,$$

and $e^* = \max\{e : P(e) - C(e) = 0\}$. Since $e^*_{fb} = \arg\max\{P(e) - C(e)\}$ for $R \to \infty$, therefore it must have $e^* > e^*_{fb}$ for $R \to \infty$. Combining that both e^* and e^*_{fb} are increasing in R and $e^* < e^*_{fb}$ for $R = R'$, there must exist a $\widehat{R} > R' > 0$ such that $e^* = e^*_{fb}$, which means that $e^* \le e^*_{fb}$ for $R \le \widehat{R}$, otherwise $e^* > e^*_{fb}$.

Proof of Corollary 4. The result of $\pi^* = 0 < \pi^*_o$ is directly from the proof of Proposition 1. Under the optimal SLL contract, the bank's profit is $\pi^{b*} = s(R)(C(e^*) - x)$. Since $e^* > e^*_o$, therefore $\pi^{b*} = s(R)(C(e^*) - x) > s(R)(C(e^*_o) - x) = \pi^{b*}_o$.

Proof of Proposition 2. Similar to the proof of Proposition 1, under the SLL, the firm's effort level will be maximized only when the firm reaches the target (specifically, when $e = \bar{T}(r, r_d)$). Therefore, the firm's net profit is

$$\pi(e|r, r_d, T) = (1 - \beta(R))(P(e) - (1 + r_d)(C(e) - x))) - x.$$

For the bank, her constraint becomes $(1 + s(R))L = (1 - \beta(R))(1 + r_d)L$, which implies that the optimal discounted interest rate is $\bar{r}^*_d = \frac{s(R)+\beta(R)}{1-\beta(R)}$. Thus the firm's problem can be rewritten as

$$\max_{e \ge 0}\{\pi(e|r, r_d, T)\} = \max_{e \ge T}\{(1 - \beta(R))[P(e) - \frac{1 + s(R)}{1 - \beta(R)}(C(e) - x)] - x\}. \tag{E-3}$$

The bank needs to ensure that the firm's profit is no less than $\pi \ge \pi^*_o$. Note that

$$(1 - \beta(R))[P(e) - \frac{1 + s(R)}{1 - \beta(R)}(C(e) - x)] - x$$

is maximized at $e = e^*_o$ and $(1 - \beta(R))[P(e^*_o) - \frac{1+s(R)}{1-\beta(R)}(C(e^*_o) - x)] - x = \pi^*_o$; this means that the bank's target should be $\bar{T}^* = e^*_o$, under which the firm's optimal effort level is $\bar{e}^* = e^*_o$. In addition, the optimal regular interest rate should satisfy $\bar{r}^* > \bar{r}^*_d = r^*_o$. Clearly, both the firm and the bank earn the same profits as they would under an ordinary loan.

Proof of Proposition 3. Since $\pi_t(e|r, r_d, T, w) = \pi(e|r, r_d, T) + w$, the firm's optimal response to the refined SLL is not affected by the lump-sum rebate w, and is the same as that stated in Lemma 3. Therefore, to maximize the firm's effort level, the bank will design the contract so that the firm is induced to reach the target. Thus the firm's profit becomes

$$\pi_t(e|r, r_d, T, w) = (1 - \beta(R))[P(e) - (1 + r_d)(C(e) - x)] - x + w;$$

and the bank's constraint becomes

$$(1 + s(R))L = (1 - \beta(R))(1 + r_d)L,$$

implying that the optimal discounted interest rate is $r^*_{dt} = \frac{s(R)+\beta(R)}{1-\beta(R)}$. Therefore, the firm's problem can be rewritten as

$$\pi_t = \max_{e \ge T}\Big\{(1 - \beta(R))P(e) - (1 + s(R))(C(e) - x) + w - x\Big\}.$$

The bank's constraint $\pi^b_t \ge \pi^{b*}_o$ implies that $w \le s(R)(C(e) - x) - \pi^{b*}_o$. On the one hand, the bank needs to ensure $\pi_t \ge \pi^*_o$. Since π_t is decreasing in T, therefore a maximized target T^*_t should satisfy $\pi_t = \pi^*_o$, meaning that

$$\pi_t = \max_{e \geq T_t^*} \Big\{ (1-\beta(R))P(e) - (1+s(R))(C(e)-x) + w - x \Big\} = \pi_o^*,$$

On the other hand, the function $(1-\beta(R))P(e)-(1+s(R))(C(e)-x)+w-x$ is increasing in w; to increase T_t^*, the bank should offer a greatest lump-sum rebate w to the firm. Since $w \leq s(R)(C(e)-x)-\pi_o^{b*}$, this means that the optimal rebate should be $s(R)(C(e)-x)-\pi_o^{b*}$. Plugging into π_t, then the optimal target satisfies

$$\pi_t = \max_{e \geq T_t^*} \Big\{ (1-\beta(R))P(e) - C(e) - \pi_o^{b*} \Big\} = \pi_o^*.$$

which indicates that

$$\begin{aligned} \max_{e \geq T_t^*} \Big\{ \Pi(e) \Big\} &= \pi_o^* + \pi_o^{b*} \\ &= (1-\beta(R))P(e_o^*) - (1+s(R))(C(e_o^*)-x) - x + s(R)(C(e_o^*)-x) \\ &= (1-\beta(R))P(e_o^*) - C(e_o^*) = \Pi(e_o^*), \end{aligned}$$

indicating that T_t^* is the largest solution of e in $\Pi(e)=\Pi(e_o^*)$. Accordingly, the optimal effort is $e_t^* = T_t^*$. And the optimal lump-sum rebate is

$$w^* = s(R)(C(T_t^*)-x) - \pi_o^{b*} = s(R)(C(T_t^*)-x) - s(R)(C(e_o^*)-x) = s(R)(C(T_t^*)-C(e_o^*)),$$

and the firm's and bank's profit are $\pi_t = \pi_o^*$ and $\pi_t^b = \pi_o^{b*}$ respectively. Finally, the bank sets the regular interest rate such that the firm will never choose not to reach the target. According to the definition of r_t in (20), the optimal interest rate r_t^* must be no smaller than r_t.

Proof of Proposition 4. We already know from Proposition 3 that $\Pi(e_t^*) = \Pi(e_o^*)$, therefore the concavity of $\Pi(e)$ implies that $e_t^* > e_{fb}^* > e_o^*$. In addition, according to the proof of Corollary 4, when R is smaller enough, e^* equals e_o^* and therefore $e^* < e_t^*$. When $R \to \infty$, we have $e_o^* = e_{fb}^* = e_t^*$ and therefore $e^* > e_t^*$. We already know e^* increases in R; however, since we do not know the monotonicity of e_t^* on R, hence there exists at least one intersection between e^* and e_t^*. We next show that there is only one intersection point. Denote

$$f(e) = (1-\beta(R))P(e) - (1+s(R))(C(e)-x) - x.$$

Note that $e_o^* = \arg\max_e \{f(e)\} = g^{-1}(\frac{1+s(R)}{1-\beta(R)})$ and $e^* = \max\{e : f(e) = 0\}$. Since $\beta(R)$ and $s(R)$ are decreasing in R, one can verify that when R increases, e_o^* increase. In addition, since $\frac{\partial f(e)}{\partial e}\Big|_{e=e_o^*} = 0$ and $\frac{\partial f(e)}{\partial e}\Big|_{e=e^*} < 0$, we can conclude that $\frac{\partial e^*}{\partial R} > \frac{\partial e_o^*}{\partial R}$. On the other hand, $\Pi(e_t^*) = \Pi(e_o^*)$ implies that

$$(1-\beta(R))P(e_t^*) - C(e_t^*) = (1-\beta(R))P(e_o^*) - C(e_o^*). \tag{E-4}$$

Since $e_o^* < e_{fb}^* = \arg\max_e \{(1-\beta(R))P(e) - C(e)\}$, therefore the RHS of (E-4) is increasing in R. In addition, it's easily verified that $\frac{\partial \Pi(e)}{\partial e}\Big|_{e=e_o^*} > 0$ while $\frac{\partial \Pi(e)}{\partial e}\Big|_{e=e_t^*} < 0$. Suppose now R increases to $R+\Delta$, where $\Delta > 0$. Correspondingly, e_o^* increases to $e_o^* + H(\Delta)$ where $H(\Delta) > 0$, and e_t^* changes to $e_t^* + \bar{H}(\Delta)$. Then (E-4) becomes

$$(1-\beta(R+\Delta))P(e_t^* + \bar{H}(\Delta)) - C(e_t^* + \bar{H}(\Delta)) = (1-\beta(R+\Delta))P(e_o^* + H(\Delta)) - C(e_o^* + H(\Delta)).$$

We next show that to make the above equation hold, it must have $\bar{H}(\Delta) < H(\Delta)$. We prove the result by contradiction. Suppose $\bar{H}(\Delta) \geq H(\Delta)$. First consider the case of $\bar{H}(\Delta) = H(\Delta)$. Since $e_t^* > e_o^*$ and $P(\cdot)$ is increasing concave and $C(\cdot)$ is increasing convex, then

$$P(e_t^* + H(\Delta)) - P(e_t^*) < P(e_o^* + H(\Delta)) - P(e_o^*) \text{ and } C(e_t^* + H(\Delta)) - C(e_t^*) > C(e_o^* + H(\Delta)) - C(e_o^*),$$

suggesting that

$$(1-\beta(R+\Delta))P(e_t^* + \bar{H}(\Delta)) - C(e_t^* + \bar{H}(\Delta)) < (1-\beta(R+\Delta))P(e_o^* + H(\Delta)) - C(e_o^* + H(\Delta))$$

when R increases to $R+\Delta$. Therefore, it's a contradiction. Next, consider $\bar{H}(\Delta) > H(\Delta)$. Since $\frac{\partial \Pi(e)}{\partial e}\Big|_{e=e_t^*} < 0$, then we have

$$\begin{aligned}(1-\beta(R+\Delta))P(e_t^* + \bar{H}(\Delta)) - C(e_t^* + \bar{H}(\Delta)) &< (1-\beta(R+\Delta))P(e_t^* + H(\Delta)) - C(e_t^* + H(\Delta)) \\ &< (1-\beta(R+\Delta))P(e_o^* + H(\Delta)) - C(e_o^* + H(\Delta)),\end{aligned}$$

which is also a contradiction. Therefore, we conclude that when R increases to $R+\Delta$, the change of e_t^* must be smaller than $H(\Delta)$, which means that $\bar{H}(\Delta) < H(\Delta)$. This implies that $\frac{\partial e_t^*}{\partial R} < \frac{\partial e_o^*}{\partial R}$. Combining $\frac{\partial e^*}{\partial R} > \frac{\partial e_o^*}{\partial R}$, we have $\frac{\partial e_t^*}{\partial R} < \frac{\partial e^*}{\partial R}$, showing that when R increases, e^* increases faster than e_t^* (no matter e_t^* increases or not). Therefore, there must exit only one intersection between e^* and e_t^*, which is denoted as $\bar{R} > 0$. Hence, when $R < \bar{R}$, $e_t^* > e^*$; otherwise $e_t^* \leq e^*$.

Proof of Proposition 5. Note that $w^* = s(R)(C(e_t^*) - C(e_o^*))$, where $e_t^* = T_t^*$. Since $e_s^* = T_s^*$, when $A = \bar{A} = w^*$, we have

$$\max_{e \geq e_s^*}\{(1-\beta(R))P(e) - (1+s(R))(C(e)-x)] - x + \bar{A}\} = \pi_o^*,$$

which can be rewritten as

$$\begin{aligned}&\max_{e \geq e_s^*}\{(1-\beta(R))P(e) - (1+s(R))(C(e)-x)] - x\} \\ &= \bar{A} + \pi_o^* \\ &= s(R)(C(e_t^*) - C(e_o^*)) + (1-\beta(R))P(e_o^*) - (1+s(R))(C(e_o^*)-x)] - x \\ &= s(R)C(e_t^*) + (1-\beta(R))P(e_o^*) - C(e_o^*) + (1+s(R))x - x \\ &= s(R)C(e_t^*) + (1-\beta(R))P(e_t^*) - C(e_t^*) + (1+s(R))x - x \\ &= (1-\beta(R))P(e_t^*) - (1+s(R))(C(e_t^*)-x)] - x,\end{aligned}$$

where the fourth equation is from $\Pi(e_t^*) = \Pi(e_o^*)$. Therefore, we have $e_t^* = e_s^*$. Since e_s^* is increasing in A and e_t^* is independent of A, we can conclude that when $A \leq \bar{A}$, $e_t^* \geq e_s^*$; otherwise $e_t^* < e_s^*$.

Proof of Proposition 6. Similar to the original model, it can be verified that the firm's effort will be maximized only when $e \geq T$. Therefore, the firm's profit becomes

$$\begin{aligned}\pi(e) &= (1-\beta(R))[P(e) - (\rho(e|T)(1+r_d) + (1-\rho(e|T))(1+r))(C(e)-x)] - x \\ &= (1-\beta(R))[P(e) - (1+r-(r-r_d)\rho(e|T))(C(e)-x)] - x.\end{aligned}$$

The bank's constraint becomes

$$(1+s(R))\cdot L=(1-\beta(R))[\mathbb{1}_{\{e\geq T\}}Y+\mathbb{1}_{\{e<T\}}(1+r)]\cdot L=(1-\beta(R))[1+r-(r-r_d)\rho(e|T)]\cdot L$$

We next prove that T^* is the maximum effort the SLL can achieve. Plugging the bank's constraint into the firm's profit function $\pi(e)$, we obtain

$$\pi(e)=(1-\beta(R))[P(e)-\frac{1+s(R)}{1-\beta(R)}(C(e)-x)]-x.$$

According to the definition of T^*, we know $\pi(e=T^*)=0$; because $\pi(e)$ is decreasing in e for $e\geq T$ and the firm's reservation profit is zero, therefore the maximum effort cannot exceed T^*.

We next show that under the SLL contract (r^u,r_d^u,T^u), the firm will exert an effort level T^*. When facing the contract (r^u,r_d^u,T^u), the firm has two choices: if the firm chooses $e<T^u$, then his profit will be no larger than zero since the regular interest rate satisfies $r^u\geq r_n$. If the firm chooses $e\geq T^u$, then his problem can be written as

$$\max_{e\geq T^u}\pi(e)=\max_{e\geq T^u}\Big\{(1-\beta(R))[P(e)-(1+r^u-(r^u-r_d^u)\rho(e|T^u))(C(e)-x)]-x\Big\}.$$

Let $\rho^u=\rho(T^u|T^u)$. Note that

$$r^u-r_d^u=1-\frac{2+s(R)-\rho^u}{\rho^u(1-\beta(R))}+(1-\frac{1-\rho^u}{\rho^u(1-\beta(R))})r^u=A+B\cdot r^u,$$

where $B<1$. Plugging into the firm's profit function, we have

$$\pi(e)=(1-\beta(R))[P(e)-(1+(1+B\rho(e|T^u)r^u-A\rho(e|T^u))(C(e)-x)]-x.$$

It can be verified that

$$\frac{\partial^2\pi(e)}{\partial e\partial r^u}=-(1-B\rho'(e|T^u))(C(e)-x)-(1-B\rho(e|T^u))c(e)<0,$$

which is because $B<1$, $\rho(e|T^u)\in[0,1]$ and $\rho'(e|T^u)\in[0,1]$. This indicates that $\pi(e)$ is submodular in (e,r^u). Denote $e^u(r^u)=\arg\max_{e\geq 0}\{\pi(e)\}$, and it holds that $e^u(r^u)$ is decreasing in r^u. In addition, when $r^u\to\infty$, it must have $e^u(r^u)\to 0$. Let $\bar{r}_n^u$ satisfies the condition $e^u(\bar{r}_n^u)\leq T^u$. Then when $r^u\geq\bar{r}_n^u$, the firm's optimal effort under the SLL contract (r^u,r_d^u,T^u) will be equal to T^u, under which we have

$$\begin{aligned}\pi(e=T^u)&=(1-\beta(R))[(P(T^u)-(1+r^u-(r^u-r_d^u)\rho(T^u|T^u))(C(T^u)-x))]-x\\&=(1-\beta(R))[(P(T^u)-\frac{1+s(R)}{1-\beta(R)}(C(T^u)-x))]-x=0.\end{aligned}$$

Denote $r_n^u=\max\{\bar{r}_n^u,r_n\}$; the condition $r^u\geq r_n^u$ ensures that the firm will exert effort T^u under the SLL contract (r^u,r_d^u,T^u). The proof is complete.

Proof of Corollary 5. Notice that $r_d^*=\frac{s(R)+\beta(R)}{1-\beta(R)}$, then

$$\begin{aligned}r_d^*-r_d^u&=\frac{s(R)+\beta(R)}{1-\beta(R)}-\frac{1+s(R)-(1-\beta(R))[\rho(T^*|T^*)+(1-\rho(T^*|T^*))(1+r^u)]}{(1-\beta(R))\rho(T^*|T^*)}\\&=\frac{1-\rho(T^*|T^*)}{\rho(T^*|T^*)}\cdot\frac{(1-\beta(R))(1+r^u)-s(R)-1}{1-\beta(R)}>0,\end{aligned}$$

where the inequality is due to the fact that $r_d^u<r^u$ and therefore $1+r^u>\frac{1+s(R)}{1-\beta(R)}$. In addition, it can be verified that $r_d^*-r_d^u$ is decreasing in $\rho(T^*|T^*)$.