Online Companion for

“Rendezvous Search on the Labeled Line”

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Appendix

1 An IP formulation

In this section, we will give an integer programming formulation of the rendezvous search problem on the labeled discrete line. All search decisions (i.e., whether one should move left or right in a particular period) can be represented as binary variables. Let $\alpha_{it} \in \{-1, 1\},\ i \in E, 1 \leq t \leq n$ denote the decision of Searcher 1 at time t if she started from the initial location i, where we adopt the convention that -1 denotes a left move and 1 denotes a right move. Let $\beta_{it} \in \{-1, 1\},\ i \in E, 1 \leq t \leq n$ denote the corresponding variable for Searcher 2.

Then,

$$\begin{aligned} x_{it} &= i + \sum_{s=1}^{t} \alpha_{it}, && (6) \\ y_{it} &= i + \sum_{s=1}^{t} \beta_{it} && (7) \end{aligned}$$

represent the locations of Searchers 1 and 2 at time t if they were initially placed at i. Then, the optimization problem is,

$$\begin{array}{lll} \min & \sum_{i,j \in E} m_{ij} p_i q_j & \\ & x_{it} = i + \sum_{s=1}^{t} \alpha_{is}, & \forall i \in E, 1 \leq t \leq n, \\ & y_{it} = i + \sum_{s=1}^{t} \beta_{is}, & \forall i \in E, 1 \leq t \leq n, \\ & \alpha_{it} \in \{-1, 1\} & \forall i \in E, 1 \leq t \leq n, \\ & \beta_{it} \in \{-1, 1\} & \forall i \in E, 1 \leq t \leq n, \end{array} \qquad (8)$$

where $m_{ij} = \min\{t : x_i(t) = y_j(t)\}$. We now introduce the following auxiliary variables to enforce this definition: Let z_{ij}^t be 0-1 variables for $i, j \in E, t = 1, \ldots, n$. We impose the following constraints:

$$\begin{aligned} 2n\left(1 - z_{ij}^t\right) &\geq x_i(t) - y_j(t),\ \forall i, j \in E, t = 1, \ldots, n && (9) \\ 2n\left(1 - z_{ij}^t\right) &\geq y_j(t) - x_i(t),\ \forall i, j \in E, t = 1, \ldots, n && (10) \\ \sum_{t=1}^{n} z_{ij}^t &= 1,\ \forall i, j \in E, && (11) \\ z_{ij}^t &\in \{0, 1\},\ \forall i, j \in E, t = 1, \ldots, n. && (12) \end{aligned}$$

Finally, we let

$$m_{ij} = \sum_{t=1}^{n} t z_{ij}^t,\ \forall i, j \in E. \qquad (13)$$

Inequalities (9) and (10) force z_{ij}^t to zero when $x_i(t) \neq y_j(t)$. Equations (11) and (13) make sure that m_{ij} is the time index for a period when x_i and y_j coincide and the minimization objective ensures that m_{ij} is assigned the minimum such value, which is the intended definition of this variable.

Thus, the problem (8) with the additional constraints (9)-(13) provides an integer programming formulation for the rendezvous search problem. This formulation can also be used when the initial placement of the two searchers have a joint rather than independent distributions (replace $p_i q_j$ terms with p_{ij} type terms).

2 Proof of Theorem 3.1

Our proof is adapted from the proof of a similar result in Howard (1999). Let us denote the deterministic strategy pair given in the theorem with $(\bar{x}, \bar{y})$ and assume that there exists a strategy pair $(\hat{x}, \hat{y})$ with a smaller expected meeting time. As in the discrete case, we use the notation $\hat{x}_i(t)$ to denote the location of Searcher 1 at time t if she started at point i and is using strategy $\hat{x}$.

When the searchers use the strategy $(\bar{x}, \bar{y})$, the meeting occurs at time $\max\{|i|, |j|\}$ if searcher 1 is initially placed at i and searcher 2 is initially placed at j. The expected search time for this pair of strategies is

$$\mathbf{E}_\Pi(\bar{x}, \bar{y}) := \mathbf{E}_{\Pi,\Pi}(\bar{x}, \bar{y}) \quad = \quad \int_{-1}^{1}\int_{-1}^{1} \max\{|z|, |w|\} f(z) f(w) dz dw.$$

Then, we must have an $\varepsilon > 0$ such that

$$\mathbf{E}_\Pi(\hat{x}, \hat{y}) \quad = \quad -\varepsilon + \int_{-1}^{1}\int_{-1}^{1} \max\{|z|, |w|\} f(z) f(w) dz dw.$$

We now consider a discretization of the interval $[-1, 1]$ using $2n+1$ points labeled from $-n$ and n and let $E = \{-n, -n+2, -n+4, \ldots, n-2, n\}$ as before, the initial location distribution for this discretization can be obtained as follows:

$$p_k \quad = \quad \begin{cases} \int_{\frac{k-1}{n}}^{\frac{k+1}{n}} f(z) dz & k \in E \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

From (4) it follows that $p_k = p_{-k}$ for all $k \in E$ as well as $p_k \leq p_l$ for all $k, l \in E$ such that $|k| < |l|$. Therefore, p is symmetric and monotone.

We will consider the locations of the searchers at time points

$$t_i = \frac{i}{n}, \text{ for } i = 1, \ldots, n.$$

Until now, we only considered pure search strategies, i.e., deterministic search strategies that prescribe a search path for all possible starting points. A *mixed* search strategy is a probability distribution on the set of all possible pure search strategies. We define (mixed) search strategies $(\tilde{x}, \tilde{y})$ for the discrete problem as follows: $\tilde{x}_i(j)$ denotes the location of Searcher 1 at time j if she started at initial location $i \in E$ and has the following distribution:

$$\mathbf{P}(\tilde{x}_i(u) = k) \quad = \quad \begin{cases} \int_{\frac{i-1}{n}}^{\frac{i+1}{n}} \mathbf{P}\left(\hat{x}_s(t_u) \in [\frac{k-1}{n}, \frac{k+1}{n}]\right) f(s) ds & i \in E, n+k+u \text{ even} \\ 0 & \text{otherwise,} \end{cases}$$

for $k \in \{-n, -n+1, \ldots, n-1, n\}$. The strategy $\tilde{y}$ for the second searcher is defined identically using $\hat{y}$. We make the following observations: When the searchers follow $(\tilde{x}, \tilde{y})$, the expected time until they meet can not exceed the expected time until they meet in the continuous case (using $(\hat{x}, \hat{y})$) by one more than one period, namely $\frac{1}{n}$ time units. Thus, we must have that

$$\frac{1}{n}\mathbf{E}_{\Pi'}(\tilde{x}, \tilde{y}) \quad \leq \quad \frac{1}{n} - \varepsilon + \int_{-1}^{1}\int_{-1}^{1} \max\{|z|, |w|\} f(z) f(w) dz dw.$$

Above, we introduced the factor $\frac{1}{n}$ to $\mathbf{E}_{\Pi'}$ since each time period in the discretized problem has length $\frac{1}{n}$. We also have that a central strategy (x, y) is optimal for the discretized problem by Theorem 2.1. Since $m_{ij}^{(x,y)}$ are as in (3), combining, we obtain the following inequality:

$$\begin{aligned}
\frac{1}{n}\mathbf{E}_{\Pi'}(x, y) = \frac{1}{n}\sum_{i,j \in E} \max\{|i|, |j|\} p_i p_j - \frac{1}{n} & \sum_{\substack{i,j \in E \\ j > |i|, \text{ or } i > |j|}} p_i p_j \; - \frac{1}{n}\sum_{i \in E} |i| p_i^2 \\
\leq \quad & \frac{1}{n}\mathbf{E}_{\Pi'}(\tilde{x}, \tilde{y}) \\
\leq \quad & \frac{1}{n} - \varepsilon + \int_{-1}^{1}\int_{-1}^{1} \max\{|z|, |w|\} f(z) f(w) \; dz \; dw \qquad (15)
\end{aligned}$$

where, once again, $\frac{1}{n}$ factors are used because of the scaling of the length of each period. We will bound several of the terms in the above inequality to contradict the assumption that $\varepsilon > 0$. The following inequalities hold:

$$\sum_{\substack{i,j \in E \\ j > |i|, \text{ or } i > |j|}} p_i p_j \quad \leq \quad \sum_{i,j \in E} p_i p_j = 1, \qquad (16)$$

$$\begin{aligned}
\sum_{i \in E} |i| p_i^2 \quad & \leq \quad \max_j \{p_j\} \sum_{i \in E} |i| p_i \leq \max_j \{p_j\} \sum_{i \in E} \left(\sum_{k \in E, -|i| \leq k < |i|} p_k \right) \\
& \leq \quad \max_j \{p_j\} \sum_{i \in E} \left(\mathbf{1} - p_{|i|} \right) = n \max_j \{p_j\}. \qquad (17)
\end{aligned}$$

Further, we have

$$\begin{aligned}
\int_{\frac{i-1}{n}}^{\frac{i+1}{n}} \int_{\frac{j-1}{n}}^{\frac{j+1}{n}} \max\{|z|, |w|\} f(z) f(w) dz dw \quad & \leq \quad \int_{\frac{i-1}{n}}^{\frac{i+1}{n}} \int_{\frac{j-1}{n}}^{\frac{j+1}{n}} \frac{1 + \max\{|i|, |j|\}}{n} f(z) f(w) \; dz \; dw \\
& = \quad \frac{1 + \max\{|i|, |j|\}}{n} p_i p_j, \qquad (18) \\
\int_{-1}^{1}\int_{-1}^{1} \max\{|z|, |w|\} f(z) f(w) dz dw \quad & = \quad \sum_{i,j \in E} \left(\int_{\frac{i-1}{n}}^{\frac{i+1}{n}} \int_{\frac{j-1}{n}}^{\frac{j+1}{n}} \max\{|z|, |w|\} f(z) f(w) dz dw \right) \\
& \leq \quad \frac{1}{n} + \frac{1}{n} \sum_{i,j \in E} \max\{|i|, |j|\} p_i p_j \qquad (19)
\end{aligned}$$

where (19) follows from (18) and $\sum_{i,j \in E} p_i p_j = 1$. Combining the inequalities (15)–(19) and simplifying we get

$$\varepsilon \quad \leq \quad \frac{3}{n} + \max_j \{p_j\}. \qquad (20)$$

Note, however, that both terms on the right-hand-side can be made arbitrarily close to zero by choosing n large enough. This contradicts the assumption that $\varepsilon > 0$ and proves that the strategy pair $(\bar{x}, \bar{y})$ is optimal for the rendezvous search problem on $[-1, 1]$. $\square$

3 Proof of Theorem 3.2

Let (x, y) denote the strategy pair given in the theorem and let (u, v) be any alternative pair of strategies for this problem. For any interval $[-k, k]$, where $k \in (0, \infty)$, we can associate a truncated problem (on the interval) and for any pair of strategies (u, v) for the problem on the infinite line, we can associate a truncated strategy (u^k, v^k) corresponding to this truncated problem: Let Π be the initial placement distribution of the problem given in the statement of the theorem. Let $k \in (0, \infty)$ be given. Consider the rendezvous search problem on $[-k, k]$ whose initial placement distribution, Π^k, is given by

$$f^k(x) \quad = \quad \begin{cases} \frac{f(x)}{\int_{-k}^{k} f(t)dt} & x \in [-k, k] \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

For each strategy pair (u, v) for the problem on the infinite line, the associated truncated strategy (u^k, v^k) is formed simply by using the strategies deemed by (u, v) for the initial locations in $[-k, k]$.

Let (u, v) be a strategy that yields a finite expected meeting time:

$$\mathbf{E}_{\Pi}(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m_{(t,s)}^{(u,v)} f(t) f(s) dt ds < \infty \tag{22}$$

where $m_{(t,s)}^{(u,v)}$ is the meeting time of a pair of searchers, one starting at t, using strategy u, the other at s, using strategy v.

Fix $\varepsilon > 0$. There exists $M_0 > 0$ such that $\forall k \geq M_0$,

$$\mathbf{E}_{\Pi}(u, v) - \int_{-k}^{k} \int_{-k}^{k} m_{(t,s)}^{(u,v)} f(t) f(s) dt ds < \varepsilon/4. \tag{23}$$

In the truncated case,

$$\begin{aligned} \mathbf{E}_{\Pi^k}(u^k, v^k) \quad &= \quad \int_{-k}^{k} \int_{-k}^{k} m_{(t,s)}^{(u^k,v^k)} f^k(t) f^k(s) dt ds & (24) \\ &= \quad \frac{1}{(\int_{-k}^{k} f(t)dt)^2} \int_{-k}^{k} \int_{-k}^{k} m_{(t,s)}^{(u,v)} f(t) f(s) dt ds. & (25) \end{aligned}$$

Since $f(x)$ describes a probability distribution, we know that

$$0 \leq \int_{-k}^{k} f(x)dx \leq 1 \tag{26}$$

Also,

$$\lim_{k \to \infty} \int_{-k}^{k} f(x)dx = 1. \tag{27}$$

It follows that there exists some M_1 such that $\forall k \geq M_1$,

$$1 - \left(\int_{-k}^{k} f(x)dx \right)^2 < \min\{\frac{\varepsilon}{4\mathbf{E}_\Pi(u,v)}, \frac{1}{2}\} \tag{28}$$

Henceforth, let

$$A_k = \int_{-k}^{k} f(x)dx \tag{29}$$

Let $M = \max\{M_0, M_1\}$. For all $k > M$, we have

$$\begin{aligned} A_k^2 |\mathbf{E}_\Pi(u,v) - \mathbf{E}_{\Pi^k}(u^k,v^k)| &= \left|(A_k^2 - 1)\mathbf{E}_\Pi(u,v) + \mathbf{E}_\Pi(u,v) - A_k^2 \mathbf{E}_{\Pi^k}(u^k,v^k)\right| \\ &\leq (1 - A_k^2)\mathbf{E}_\Pi(u,v) + \left| \mathbf{E}_\Pi(u,v) - \int_{-k}^{k}\int_{-k}^{k} m_{(t,s)}^{(u,v)} f(t)f(s)dtds \right| \\ &< \frac{\varepsilon}{4\mathbf{E}_\Pi(u,v)} \mathbf{E}_\Pi(u,v) + \varepsilon/4 \\ &= \varepsilon/4 + \varepsilon/4 \\ &= \varepsilon/2 \end{aligned}$$

It follows that

$$\begin{aligned} |\mathbf{E}_\Pi(u,v) - \mathbf{E}_{\Pi^k}(u^k,v^k)| &< \frac{\varepsilon}{2(\int_{-k}^{k} f(x)dx)^2} \\ &< \varepsilon. \end{aligned}$$

Thus

$$\lim_{k \to \infty} \mathbf{E}_{\Pi^k}(u^k, v^k) = \mathbf{E}_\Pi(u,v) \tag{30}$$

for any strategy, (u,v) that yields a finite expected meeting time. For any truncated strategy (u^k, v^k), using Theorem 3.1, we have that

$$\mathbf{E}_{\Pi^k}(x^k, y^k) \leq \mathbf{E}_{\Pi^k}(u^k, v^k). \tag{31}$$

where (x^k, y^k) is the truncation of the strategy (x,y). Therefore, by (30),

$$\mathbf{E}_\Pi(x,y) \leq \mathbf{E}_\Pi(u,v) \quad \forall (u,v). \tag{32}$$

□