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APPENDIX

In the following, "," corresponds to the logical operator "and".

A Proof of Lemma 3.1

Let λ_i, $i = 1, 2$, and μ denote the Lagrangian multipliers corresponding to constraints (9) and (10) respectively in Problem P_2'. If a feasible solution (p_1, p_2) is a maximizer to Problem P_2', then the first-order KKT necessary conditions require that there exists a vector $(\lambda_1,\ \lambda_2,\ \mu) \geq 0$ such that

$$\epsilon_1 - 2\alpha_1 p_1 + \alpha_1\lambda_1 + \alpha_1\mu = 0 \Rightarrow p_1 = \frac{\epsilon_1 + \alpha_1\lambda_1 + \alpha_1\mu}{2\alpha_1} \tag{19}$$

$$\epsilon_2 - 2\alpha_2 p_2 + \alpha_2\lambda_2 + \alpha_2\mu = 0 \Rightarrow p_2 = \frac{\epsilon_2 + \alpha_2\lambda_2 + \alpha_2\mu}{2\alpha_2} \tag{20}$$

$$\lambda_1\ (\alpha_1 p_1 - \epsilon_1 + K_1 + K_f) = 0 \tag{21}$$

$$\lambda_2\ (\alpha_2 p_2 - \epsilon_2 + K_2 + K_f) = 0 \tag{22}$$

$$\mu\ (\alpha_1 p_1 + \alpha_2 p_2 - \epsilon_1 - \epsilon_2 + K_1 + K_2 + K_f) = 0. \tag{23}$$

We define sets $E = \{\epsilon_1 \leq 2K_1 + 2K_f,\quad \epsilon_2 \leq 2K_2 + 2K_f,\quad \epsilon_1 + \epsilon_2 \leq 2K_1 + 2K_2 + 2K_f\}$ and $F = \{2\alpha_1 K_2 - 2\alpha_2(K_1 + K_f) \leq \alpha_1\epsilon_2 - \alpha_2\epsilon_1 \leq -2\alpha_2 K_1 + 2\alpha_1(K_2 + K_f)\}$. Observe that the optimal solution to the unconstrained problem, $p_i^U = \frac{\epsilon_i}{2\alpha_i}$, $i = 1, 2$, is optimal for Problem P_2' (and $(\mu, \lambda_1, \lambda_2) = 0$) if it satisfies all constraints of Problem P_2'; that is, if $\{\epsilon_1 \leq 2K_1 + 2K_f,\ \epsilon_2 \leq 2K_2 + 2K_f,\ \epsilon_1 + \epsilon_2 \leq 2K_1 + 2K_2 + 2K_f\}$. **Otherwise** (if E^c), the optimal solution will be on a boundary line of the feasible region (and its corresponding Lagrangian multiplier will be positive); see KKT conditions (21)–(23). In what follows, we consider event E^c and analyze the solution on one of the boundary lines. The other cases can be analyzed similarly.

Consider event E^c and the solutions on the boundary line $\alpha_1 p_1 + \alpha_2 p_2 = \epsilon_1 + \epsilon_2 - K_1 - K_2 - K_f$ ($\mu \geq 0$ and $\lambda_i = 0$, $i = 1, 2$). Setting $\lambda_i = 0$, $i = 1, 2$, in (19) and (20), we find the maximizing solution on this boundary line as:

$$p_i = \frac{\epsilon_i}{2\alpha_i} + \frac{\mu}{2},\ i = 1, 2, \quad \text{where} \tag{24}$$

$$\mu = \frac{\epsilon_1 + \epsilon_2 - 2(K_1 + K_2 + K_f)}{\alpha_1 + \alpha_2}. \tag{25}$$

Substituting (25) in (24), we obtain:

$$p_i = \frac{\epsilon_i}{2\alpha_i} + \frac{\epsilon_1 + \epsilon_2 - 2(K_1 + K_2 + K_f)}{2(\alpha_1 + \alpha_2)}, \quad \text{for } i = 1, 2. \tag{26}$$

If this solution also satisfies constraints (9), then it will be optimal for Problem P_2'. That is, if

$$\frac{\epsilon_1 - K_1 - K_f}{\alpha_1} \leq \frac{\epsilon_1}{2\alpha_1} + \frac{\epsilon_1 + \epsilon_2 - 2(K_1 + K_2 + K_f)}{2(\alpha_1 + \alpha_2)} \quad \text{and}$$

$$\frac{\epsilon_2 - K_2 - K_f}{\alpha_2} \le \frac{\epsilon_2}{2\alpha_2} + \frac{\epsilon_1 + \epsilon_2 - 2(K_1 + K_2 + K_f)}{2(\alpha_1 + \alpha_2)}$$
$$\Leftrightarrow 2\alpha_1 K_2 - 2\alpha_2(K_1 + K_f) \le \alpha_1\epsilon_2 - \alpha_2\epsilon_1 \le -2\alpha_2 K_1 + 2\alpha_1(K_2 + K_f), \tag{27}$$

or equivalently, if $E^c F$, then it is the optimal solution.

If not (if $E^c F^c$), then the optimal solution must be on one of the boundary lines $p_i = \frac{\epsilon_i - K_i - K_f}{\alpha_i}$, $i = 1, 2$, of the feasible region. These cases can be analyzed similarly; see Bish and Wang (2002) for details. This completes the proof. ∎

B Proof of Theorem 4.1

In what follows, we first prove that $V(\vec{K})$ is strictly jointly concave in the investment vector $\vec{K} = (K_1,\ K_2,\ K_f)$ for any continuous distribution of $\xi_i, i = 1, 2$ having positive support. Let g denote the joint pdf of the random variables ξ_1 and ξ_2. We define the following elements:

$$\begin{aligned}
a &\equiv \frac{2}{(\alpha_1 + \alpha_2)} \int\int_{\Omega_2} g(\epsilon_1, \epsilon_2) d\epsilon_1\ d\epsilon_2 \\
b &\equiv \frac{2}{\alpha_1} \int\int_{\Omega_3} g(\epsilon_1, \epsilon_2) d\epsilon_1\ d\epsilon_2 + \frac{2}{\alpha_1} \int\int_{\Omega_6} g(\epsilon_1, \epsilon_2) d\epsilon_1\ d\epsilon_2 \\
c &\equiv \frac{2}{\alpha_1} \int\int_{\Omega_4} g(\epsilon_1, \epsilon_2) d\epsilon_1\ d\epsilon_2; \qquad d \equiv \frac{2}{\alpha_2} \int\int_{\Omega_3} g(\epsilon_1, \epsilon_2) d\epsilon_1\ d\epsilon_2 \\
e &\equiv \frac{2}{\alpha_2} \int\int_{\Omega_4} g(\epsilon_1, \epsilon_2) d\epsilon_1\ d\epsilon_2 + \frac{2}{\alpha_2} \int\int_{\Omega_5} g(\epsilon_1, \epsilon_2) d\epsilon_1\ d\epsilon_2.
\end{aligned}$$

Then, we can write $-\mathbf{H}$, negative of the Hessian matrix of $V(\vec{K})$ corresponding to $\vec{K}$, as follows:

$$-\mathbf{H} = \begin{pmatrix} a+b+c & a & a+b \\ a & a+d+e & a+e \\ a+b & a+e & a+b+e \end{pmatrix}$$

Next we apply the super diagonalization algorithm to check the positive definiteness of $-\mathbf{H}$ [see, for instance, Bazaraa, Sherali, and Shetty (1993)]. Observing that all elements on the diagonal are positive, $-\mathbf{H}$ reduces to the following form by elementary row operations:

$$\begin{pmatrix} a+b+c & a & a+b \\ 0 & a+d+e-\frac{a^2}{a+b+c} & a+e-\frac{a(a+b)}{a+b+c} \\ 0 & a+e-\frac{a(a+b)}{a+b+c} & a+b+e-\frac{(a+b)^2}{a+b+c} \end{pmatrix}$$

Let

$$\mathbf{G_{new}} \equiv \begin{pmatrix} a+d+e-\frac{a^2}{a+b+c} & a+e-\frac{a(a+b)}{a+b+c} \\ a+e-\frac{a(a+b)}{a+b+c} & a+b+e-\frac{(a+b)^2}{a+b+c} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where $g_{12} = g_{21}$.

Then, since $a+b+c > 0$, $-\mathbf{H}$ is positive definite if and only if $\mathbf{G_{new}}$ is positive definite [Theorem

3.3.12, Bazaraa, Sherali, and Shetty (1993), p.96]. Observing that

$$g_{11} = \frac{a(b+c+d+e)+(b+c)(d+e)}{a+b+c} > 0; \quad g_{22} = \frac{(a+b)(c+e)+ce}{a+b+c} > 0 \tag{28}$$

$$g_{11}g_{22} - g_{12}^2 = \frac{abc+abe+acd+ade+bcd+bce+bde+cde}{a+b+c} > 0, \tag{29}$$

we conclude that $\mathbf{G_{new}}$ is positive definite, and therefore, $-\mathbf{H}$ is positive definite for any continuous distribution of $\xi_i, i = 1, 2$, having positive support. Thus, $\mathbf{H}$ is negative definite, and therefore, $V(\vec{K}) = E[\Pi(\vec{K},\ \vec{\xi})] - \sum_{i=1,2,f} c_i K_i$ is strictly jointly concave in $\vec{K}$. Thus, the optimal investment vector is unique and the first-order KKT conditions, given in the theorem, are necessary and sufficient for optimality. This completes the proof. ∎

In the following, let $\Omega_j^D, \Omega_j^F, \Omega_j^{1F}, \Omega_j^{2F}$, and Ω_j^A denote set Ω_j, $j = 1, \cdots, 6$, given in Eq. (17), at boundary solutions $\vec{K}^D, \vec{K}^F, \vec{K}^{1F}, \vec{K}^{2F}$, and $\vec{K}^A$, respectively.

The following result will be used in the subsequent proofs.

Lemma B.1 *Consider the boundary solutions $\vec{K}^F$, $\vec{K}^{1F}$, $\vec{K}^{2F}$, and $\vec{K}^A$. We have the following properties:*

1. $\vec{K}^F = (K_1^F = 0,\ K_2^F = 0,\ K_f^F > 0)$ *is not a possible solution if* $\{(\Omega_3^F = \emptyset)$ *or* $(\Omega_4^F = \emptyset)\}$;
2. $\vec{K}^{1F} = (K_1^{1F} > 0,\ K_2^{1F} = 0,\ K_f^{1F} > 0)$ *is not a possible solution if* $\{(\Omega_3^{1F} = \emptyset)$ *or* $(\Omega_4^{1F} = \emptyset,\ \Omega_5^{1F} = \emptyset)\}$;
3. $\vec{K}^{2F} = (K_1^{2F} = 0,\ K_2^{2F} > 0,\ K_f^{2F} > 0)$ *is not a possible solution if* $\{(\Omega_4^{2F} = \emptyset)$ *or* $(\Omega_3^{2F} = \emptyset,\ \Omega_6^{2F} = \emptyset)\}$;
4. $\vec{K}^A = (K_1^A > 0,\ K_2^A > 0,\ K_f^A > 0)$ *is not a possible solution if* $\{(\Omega_4^A = \emptyset, \Omega_5^A = \emptyset)$ *or* $(\Omega_3^A = \emptyset,\ \Omega_6^A = \emptyset)\}$.

Proof: In what follows, we prove the result for boundary solution $\vec{K}^{2F}$. The other cases can be analyzed similarly; see Bish and Wang (2002) for details.

At solution $\vec{K}^{2F} = (K_1^{2F} = 0,\ K_2^{2F} > 0,\ K_f^{2F} > 0)$, which implies that $v_1 \geq 0$, $v_2 = 0$, $v_f = 0$, the demand space reduces to the following:

$$\begin{aligned} &\Omega_1^{2F} = \{\xi_1 < 2K_f^{2F},\ \xi_1 + \xi_2 < 2K_2^{2F} + 2K_f^{2F}\} \\ &\Omega_2^{2F} = \{\xi_1 + \xi_2 > 2K_2^{2F} + 2K_f^{2F},\ 2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F} < \alpha_1\xi_2 - \alpha_2\xi_1 < 2\alpha_1(K_2^{2F} + K_f^{2F})\} \\ &\Omega_3^{2F} = \{\xi_2 > 2K_2^{2F},\ \alpha_1\xi_2 - \alpha_2\xi_1 < 2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F}\};\ \Omega_4^{2F} = \{\alpha_1\xi_2 - \alpha_2\xi_1 > 2\alpha_1(K_2^{2F} + K_f^{2F})\} \\ &\Omega_5^{2F} = \emptyset; \quad \Omega_6^{2F} = \{\xi_1 > 2K_f^{2F},\ \xi_2 < 2K_2^{2F}\}. \end{aligned} \tag{30}$$

Thus, the first-order KKT necessary & sufficient conditions reduce to the following:

$$\mathbf{KKT-1}: E[\frac{\xi_1 + \xi_2 - 2K_2^{2F} - 2K_f^{2F}}{\alpha_1 + \alpha_2}|\Omega_2^{2F}]\Pr(\Omega_2^{2F}) + E[\frac{\xi_1 - 2K_f^{2F}}{\alpha_1}|\Omega_3^{2F}]\Pr(\Omega_3^{2F})$$

$$+E[\frac{\xi_1}{\alpha_1}|\Omega_4^{2F}]\Pr(\Omega_4^{2F}) + E[\frac{\xi_1 - 2K_f^{2F}}{\alpha_1}|\Omega_6^{2F}]\Pr(\Omega_6^{2F}) = c_1 - v_1 \tag{31}$$

$$\mathbf{KKT-2}: E[\frac{\xi_1+\xi_2-2K_2^{2F}-2K_f^{2F}}{\alpha_1+\alpha_2}|\Omega_2^{2F}]\Pr(\Omega_2^{2F}) + E[\frac{\xi_2 - 2K_2^{2F}}{\alpha_2}|\Omega_3^{2F}]\Pr(\Omega_3^{2F})$$

$$+E[\frac{\xi_2 - 2K_2^{2F} - 2K_f^{2F}}{\alpha_2}|\Omega_4^{2F}]\Pr(\Omega_4^{2F}) = c_2 \tag{32}$$

$$\mathbf{KKT-3}: E[\frac{\xi_1+\xi_2-2K_2^{2F}-2K_f^{2F}}{\alpha_1+\alpha_2}|\Omega_2^{2F}]\Pr(\Omega_2^{2F}) + E[\frac{\xi_1 - 2K_f^{2F}}{\alpha_1}|\Omega_3^{2F}]\Pr(\Omega_3^{2F})$$

$$+E[\frac{\xi_2 - 2K_2^{2F} - 2K_f^{2F}}{\alpha_2}|\Omega_4^{2F}]\Pr(\Omega_4^{2F}) + E[\frac{\xi_1 - 2K_f^{2F}}{\alpha_1}|\Omega_6^{2F}]\Pr(\Omega_6^{2F}) = c_f. \tag{33}$$

Observe that when $\Omega_4^{2F} = \emptyset$, KKT conditions (31) and (33) imply that $c_f = c_1 - v_1 \le c_1$, which is a contradiction, since $c_f > c_1$ by definition. Similarly, when $(\Omega_3^{2F} = \emptyset, \Omega_6^{2F} = \emptyset)$, by KKT conditions (32) and (33), $c_f = c_2$, which is a contradiction, since $c_f > c_2$ by definition. Thus, if $\{\Omega_4^{2F} = \emptyset \text{ or } (\Omega_3^{2F} = \emptyset, \Omega_6^{2F} = \emptyset)\}$, then $\vec{K}^{2F} = (K_1^{2F} = 0,\ K_2^{2F} > 0,\ K_f^{2F} > 0)$ is **not** a possible solution. This completes the proof. ∎

C Proof of Theorem 4.2

Consider that $\Pr(\frac{\xi_1}{\alpha_1} < \frac{\xi_2}{\alpha_2}) = 1$. Since $\alpha_1\xi_2 - \alpha_2\xi_1 > 0$ with probability 1, $\Omega_3^F = \{\alpha_1\xi_2 - \alpha_2\xi_1 < -2\alpha_2 K_f^F\} = \emptyset$ for any $K_f^F > 0$, where Ω_3^F is obtained by setting $K_f^F > 0$, $K_1^F = 0$, $K_2^F = 0$ in Ω_3, given in Eq. (17). Thus, it follows, by Lemma B.1, that $\vec{K}^F$ is not a possible solution in this case. Similarly, $\Omega_3^{1F} = \{\alpha_1\xi_2 - \alpha_2\xi_1 < -2\alpha_2(K_1^{1F} + K_f^{1F})\} = \emptyset$ for any $K_1^{1F} > 0$ and $K_f^{1F} > 0$, and hence $\vec{K}^{1F}$ is not a possible solution by Lemma B.1. Thus, when $c_f < \underline{c}_f$, Lemma 4.1 implies that $K_f > 0$ in the optimal solution, which must correspond to either boundary solution $\vec{K}^{2F}$ or $\vec{K}^A$.

Observe that by Lemma B.1, $\vec{K}^A$ is a possible solution only when $\{(\Omega_4^A \neq \emptyset \textit{ or } \Omega_5^A \neq \emptyset) \textit{ and } (\Omega_3^A \neq \emptyset \textit{ or } \Omega_6^A \neq \emptyset)\}$. We can write the second condition as:

$$\begin{aligned}&(\Omega_3^A \neq \emptyset \textit{ or } \Omega_6^A \neq \emptyset) = \{(\Omega_3^A \textit{ or } \Omega_6^A) \neq \emptyset\}\\ &= \{\xi_1 > 2K_1^A + 2K_f^A,\ \alpha_1\xi_2 - \alpha_2\xi_1 < 2\alpha_1 K_2^A - 2\alpha_2(K_1^A + K_f^A)\} \neq \emptyset, \quad \text{(see Eq. (17))}\end{aligned}$$

or equivalently, $\Pr\{\xi_1 > 2K_1^A + 2K_f^A,\ \alpha_1\xi_2 - \alpha_2\xi_1 < 2\alpha_1 K_2^A - 2\alpha_2(K_1^A + K_f^A)\} \neq 0$. Since $\alpha_1\xi_2 - \alpha_2\xi_1 > 0$ with probability 1, we must have $2\alpha_1 K_2^A - 2\alpha_2(K_1^A + K_f^A) > 0$. Thus, $K_2^A > \frac{\alpha_2}{\alpha_1}(K_1^A + K_f^A)$, if $\vec{K}^A$ is the optimal solution. By a similar argument, we can show that if $\vec{K}^{2F}$ is the optimal solution, then it must satisfy $K_2^{2F} > \frac{\alpha_2}{\alpha_1} K_f^{2F}$. This completes the proof. ∎

D Proof of Theorem 5.1

Consider that $Pr(\xi_1 = a\xi_2) = 1$, for some constant $a > 0$; that is, $\rho = +1$. For the sake of simplicity in notation, let $\xi_2 = \xi$, and thus $\xi_1 = a\xi$. In this case, the first two optimality conditions at boundary solution $\vec{K}^D = (K_1^D > 0,\ K_2^D > 0,\ K_f^D = 0)$, given in Theorem 4.1, reduce to the following:

$$E[\xi - \frac{2K_1^D}{a} | \xi > \frac{2K_1^D}{a}] \Pr(\xi > \frac{2K_1^D}{a}) = \frac{\alpha_1 c_1}{a} \tag{34}$$

$$E[\xi - 2K_2^D | \xi > 2K_2^D] \Pr(\xi > 2K_2^D) = \alpha_2 c_2. \tag{35}$$

In what follows, we first prove part (1) of the theorem.

Proof of Theorem 5.1, Part (1)

Recall that by Lemma 4.1, $K_f > 0$ in the optimal solution only if $c_f < \underline{c}_f$. Thus, in the following, we analyze whether or not this is possible for the $\frac{\alpha_1}{a\alpha_2} > \frac{c_2}{c_1}$ case. The other cases can be analyzed similarly; see Bish and Wang (2002) for details.

Since $\frac{\alpha_1}{a\alpha_2} > \frac{c_2}{c_1}$, it follows by Eq.s (34) and (35) that $\frac{K_1^D}{a} < K_2^D$.

D.0.1 Subcase 1: $\alpha_1 > a\alpha_2$

In this case, at boundary solution $\vec{K}^D$ the demand space reduces to the following:

$$\begin{aligned}
\Omega_1^D &= \{\xi < \frac{2K_1^D}{a},\ \xi < 2K_2^D\} = \{\xi < \frac{2K_1^D}{a}\}; \quad \Omega_2^D = \emptyset \\
\Omega_3^D &= \{\xi > 2K_2^D,\ \xi(\alpha_1 - a\alpha_2) < 2\alpha_1 K_2^D - 2\alpha_2 K_1^D\} = \{2K_2^D < \xi < \frac{2\alpha_1 K_2^D - 2\alpha_2 K_1^D}{\alpha_1 - a\alpha_2}\} \\
\Omega_4^D &= \{\xi > \frac{2K_1^D}{a},\ \xi(\alpha_1 - a\alpha_2) > 2\alpha_1 K_2^D - 2\alpha_2 K_1^D\} = \{\xi > \frac{-2\alpha_2 K_1^D + 2\alpha_1 K_2^D}{\alpha_1 - a\alpha_2}\} \\
\Omega_5^D &= \{\xi < \frac{2K_1^D}{a},\ \xi > 2K_2^D\} = \emptyset \\
\Omega_6^D &= \{\xi > \frac{2K_1^D}{a},\ \xi < 2K_2^D\} = \{\frac{2K_1^D}{a} < \xi < 2K_2^D\}.
\end{aligned}$$

Thus, the first-order KKT necessary & sufficient conditions for boundary solution $\vec{K}^D$, given in Theorem 4.1, reduce to the following:

KKT − 1 :

$$E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_3^D] \Pr(\Omega_3^D) + E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_4^D] \Pr(\Omega_4^D) + E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_6^D] \Pr(\Omega_6^D) = c_1 \tag{36}$$

KKT − 2 :

$$E[\frac{\xi - 2K_2^D}{\alpha_2} | \Omega_3^D] \Pr(\Omega_3^D) + E[\frac{\xi - 2K_2^D}{\alpha_2} | \Omega_4^D] \Pr(\Omega_4^D) = c_2 \tag{37}$$

KKT − 3 :

$$E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_3^D] \Pr(\Omega_3^D) + E[\frac{\xi - 2K_2^D}{\alpha_2} | \Omega_4^D] \Pr(\Omega_4^D) + E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_6^D] \Pr(\Omega_6^D)$$

$$= c_f - v_f \equiv \underline{c}_f \tag{38}$$

$$\Rightarrow \underline{c}_f = c_1 + c_2 - \{ \ \frac{a}{\alpha_1} E[\xi - \frac{2K_1^D}{a} | \Omega_4^D] \Pr(\Omega_4^D) + \frac{1}{\alpha_2} E[\xi - 2K_2^D | \Omega_3^D] \Pr(\Omega_3^D) \ \}.$$

Recall that $K_f > 0$ in the optimal solution if $c_f < \underline{c}_f$, as stated in Lemma 4.1. Also, by definition, $c_1, \ c_2 < c_f < c_1 + c_2$. Thus, in order to show that $K_f > 0$ is possible in the optimal solution, we need to show that:

$$\frac{a}{\alpha_1} E[\xi - \frac{2K_1^D}{a} | \Omega_4^D] \Pr(\Omega_4^D) + \frac{1}{\alpha_2} E[\xi - 2K_2^D | \Omega_3^D] \Pr(\Omega_3^D) < \min(c_1, \ c_2).$$

(i) We first show that $\frac{a}{\alpha_1} E[\xi - \frac{2K_1^D}{a} | \Omega_4^D] \Pr(\Omega_4^D) + \frac{1}{\alpha_2} E[\xi - 2K_2^D | \Omega_3^D] \Pr(\Omega_3^D) < c_1$. Observe that by Eq. (36), this is equivalent to showing that

$$\underbrace{E[\xi(\alpha_1 - a\alpha_2) - 2\alpha_1 K_2^D + 2\alpha_2 K_1^D | \Omega_3^D]}_{<0(\text{by definition of } \Omega_3^D)} \ Pr(\Omega_3^D) < a\alpha_2 \ \underbrace{E[\xi - \frac{2K_1^D}{a} | \Omega_6^D]}_{>0(\text{by definition of } \Omega_6^D)} \ Pr(\Omega_6^D),$$

which always holds by definitions of Ω_3^D and Ω_6^D, thus proving the first part.

(ii) Similarly, we next show that $\frac{a}{\alpha_1} E[\xi - \frac{2K_1^D}{a} | \Omega_4^D] \Pr(\Omega_4^D) + \frac{1}{\alpha_2} E[\xi - 2K_2^D | \Omega_3^D] \Pr(\Omega_3^D) < c_2$. Note that, by Eq. (37), this is equivalent to showing that

$$E[\xi(\alpha_1 - a\alpha_2) - 2\alpha_1 K_2^D + 2\alpha_2 K_1^D | \Omega_4^D] \ Pr(\Omega_4^D) > 0,$$

which always holds by definition of Ω_4^D, thus proving the second part.

Hence, $K_f > 0$ is possible in the optimal solution in this case.

D.0.2 Subcase 2: $\alpha_1 = a\alpha_2$

In this case, the demand space reduces to the following:

$$\begin{aligned}
&\Omega_1^D = \{\xi < \frac{2K_1^D}{a}\}; \quad \Omega_2^D = \emptyset \\
&\Omega_3^D = \{\xi > 2K_2^D\}; \quad \Omega_4^D = \emptyset \\
&\Omega_5^D = \emptyset; \quad \Omega_6^D = \{\frac{2K_1^D}{a} < \xi < 2K_2^D\}.
\end{aligned}$$

Thus, we have the following first-order KKT necessary & sufficient conditions:

$$\mathbf{KKT-1}: \ E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_3^D] \Pr(\Omega_3^D) + E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_6^D] \Pr(\Omega_6^D) = c_1 \tag{39}$$

$$\mathbf{KKT-2}: \ E[\frac{\xi - 2K_2^D}{\alpha_2} | \Omega_3^D] \Pr(\Omega_3^D) = c_2 \tag{40}$$

$$\mathbf{KKT-3}: \ E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_3^D] \Pr(\Omega_3^D) + E[\frac{a\xi - 2K_1^D}{\alpha_1} | \Omega_6^D] \Pr(\Omega_6^D) = c_f - v_f = \underline{c}_f. \tag{41}$$

KKT conditions (39) and (41) imply that $\underline{c}_f = c_1$. By Lemma 4.1, $K_f > 0$ in the optimal solution only if $c_f < \underline{c}_f = c_1$. However, by definition, $c_f > c_1 = \underline{c}_f$. Thus, $K_f = 0$ in the optimal solution.

D.0.3 Subcase 3: $\alpha_1 < a\alpha_2$

In this case, the demand space reduces to the following:

$$\begin{aligned}
\Omega_1^D &= \{\xi < \frac{2K_1^D}{a}\}; \quad \Omega_2^D = \emptyset \\
\Omega_3^D &= \{\xi > 2K_2^D,\ \xi(\alpha_1 - a\alpha_2) < -2\alpha_2 K_1^D + 2\alpha_1 K_2^D\} \\
\Omega_4^D &= \{\xi > \frac{2K_1^D}{a},\ \xi(\alpha_1 - a\alpha_2) > -2\alpha_2 K_1^D + 2\alpha_1 K_2^D\} \\
\Omega_5^D &= \emptyset; \quad \Omega_6^D = \{\frac{2K_1^D}{a} < \xi < 2K_2^D\}.
\end{aligned}$$

Let $\Omega^{\cup}$ represent the universal set. There are two cases to consider:

1. $2\alpha_1 K_2^D - 2\alpha_2 K_1^D \geq 0 \Rightarrow \{\xi(\alpha_1 - a\alpha_2) \leq -2\alpha_2 K_1^D + 2\alpha_1 K_2^D\} = \Omega^{\cup}$, since $\alpha_1 < a\alpha_2$ and $\xi > 0$. Thus, $\Omega_3^D = \{\xi > 2K_2^D\}$ and $\Omega_4^D = \emptyset$.

2. $2\alpha_1 K_2^D - 2\alpha_2 K_1^D < 0 \Rightarrow \frac{2\alpha_2 K_1^D - 2\alpha_1 K_2^D}{a\alpha_2 - \alpha_1} < \frac{2a\alpha_2 \frac{K_1^D}{a} - 2\alpha_1 \frac{K_1^D}{a}}{a\alpha_2 - \alpha_1} = \frac{2K_1^D}{a}$. Thus, $\Omega_3^D = \{\xi > 2K_2^D\}$ and $\Omega_4^D = \emptyset$.

Thus, the first-order KKT necessary and sufficient conditions are the same as in subcase 2, which imply that $\underline{c}_f = c_1$. Therefore, $K_f = 0$ in the optimal solution.

The cases where $\alpha_1 = a\alpha_2$ and $\alpha_1 < a\alpha_2$ are analyzed similarly; see Bish and Wang (2002) for details.

Proof of Theorem 5.1, Part (2)

When $c_f < \underline{c}_f$, it follows, by Lemma 4.1, that $K_f > 0$ in the optimal solution. Thus, in this case, the optimal strategy must correspond to one of the boundary solutions $\vec{K}^F$, $\vec{K}^{1F}$, $\vec{K}^{2F}$, or $\vec{K}^A$. In what follows, we analyze solution $\vec{K}^F$ for the case where $\frac{\alpha_1}{a\alpha_2} > \max\{\frac{c_2}{c_1}, 1\}$. The other cases can be analyzed similarly; see Bish and Wang (2002) for details.

When $\xi_1 = a\xi_2$, Ω_3^F, at boundary solution $\vec{K}^F$, reduces to

$$\Omega_3^F = \{\xi \underbrace{(\alpha_1 - a\alpha_2)}_{>0} < \underbrace{-2\alpha_2 K_f^F}_{<0}\} = \emptyset.$$

Thus, by Lemma B.1, $\vec{K}^F$ is **not** a possible solution in this case.

Similarly, we can show that solution $\vec{K}^{1F}$ is not possible, while solutions $\vec{K}^{2F}$ and $\vec{K}^A$ are possible. Consequently, when $\{\frac{\alpha_1}{a\alpha_2} > \max\{\frac{c_2}{c_1}, 1\},\ c_f < \underline{c}_f\}$, the optimal solution must be either $\vec{K}^{2F}$ or $\vec{K}^A$.

D.1 Proof of 2(a)

In the following, we first prove that $\frac{K_f^{2F}}{a} < K_2^{2F}$, and then, using this result, we prove the other inequalities.

D.1.1 Proof of $\frac{K_f^{2F}}{a} < K_2^{2F}$

Suppose that $\vec{K}^{2F}$ is the optimal solution. Suppose, to the contrary, that $\frac{K_f^{2F}}{a} \geq K_2^{2F}$. In addition, recall that $\alpha_1 > a\alpha_2$. Then, we can show that the following inequalities hold:

$$\frac{2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F}}{\alpha_1 - a\alpha_2} \leq 2K_2^{2F} \leq \frac{2K_2^{2F} + 2K_f^{2F}}{1+a} \leq \frac{2K_f^{2F}}{a}; \textit{ and } \frac{2\alpha_1(K_2^{2F} + K_f^{2F})}{\alpha_1 - a\alpha_2} \geq \frac{2K_2^{2F} + 2K_f^{2F}}{1+a}.$$

Thus, if $\frac{K_f^{2F}}{a} \geq K_2^{2F}$, then $\Omega_3^{2F} = \{\xi > 2K_2^{2F},\ \xi < \frac{2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F}}{\alpha_1 - a\alpha_2}\} = \emptyset$ and $\Omega_6^{2F} = \{\xi > \frac{2K_f^{2F}}{a},\ \xi < 2K_2^{2F}\} = \emptyset$, which follow by Eq. (30). Hence, by Lemma B.1, $\vec{K}^{2F}$ is not a possible solution, which is a contradiction. Thus, we must have $\frac{K_f^{2F}}{a} < K_2^{2F}$. This implies the following inequalities:

$$\frac{2K_f^{2F}}{a} < \frac{2K_2^{2F} + 2K_f^{2F}}{1+a} < 2K_2^{2F} < \frac{2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F}}{\alpha_1 - a\alpha_2}.$$

Thus, our demand space reduces to the one depicted in Figure 7.

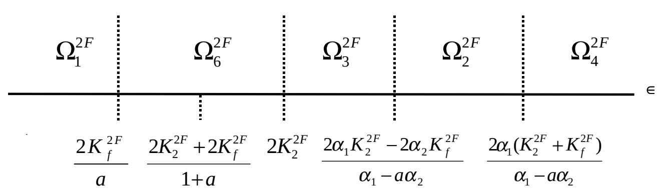

Figure 7: The demand space at solution $\vec{K}^{2F}$ for perfectly positively correlated demand patterns

D.1.2 Proof of $K_2^{2F} < K_2^I$

Observe that we can write:

$$E[\frac{\xi(1+a) - 2K_2^{2F} - 2K_f^{2F}}{\alpha_1+\alpha_2}|\Omega_2^{2F}]$$
$$= E[\frac{\xi - 2K_2^{2F}}{\alpha_2}|\Omega_2^{2F}] + E[\frac{2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F} - \xi(\alpha_1 - a\alpha_2)}{\alpha_2(\alpha_1+\alpha_2)}|\Omega_2^{2F}]. \tag{42}$$

Using Eq. (42), KKT condition, given in Eq. (32), can be written as:

$$E[\xi - 2K_2^{2F}|\Omega_2^{2F}]\Pr(\Omega_2^{2F}) + E[\frac{2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F} - \xi(\alpha_1 - a\alpha_2)}{\alpha_1+\alpha_2}|\Omega_2^{2F}]\Pr(\Omega_2^{2F})$$
$$+E[\xi - 2K_2^{2F}|\Omega_3^{2F}]\Pr(\Omega_3^{2F}) + E[\xi - 2K_2^{2F}|\Omega_4^{2F}]\Pr(\Omega_4^{2F}) - 2K_f^{2F}\Pr(\Omega_4^{2F}) = \alpha_2 c_2. \tag{43}$$

Observe that $(\Omega_2^{2F} \cup \Omega_3^{2F} \cup \Omega_4^{2F}) = \{\xi > 2K_2^{2F}\}$ (see Figure 7). Thus, we can write Eq. (43) as:

$$E[\xi - 2K_2^{2F}|\xi > 2K_2^{2F}]\Pr(\xi > 2K_2^{2F})$$
$$= \alpha_2 c_2 + 2K_f^{2F}\Pr(\Omega_4^{2F}) - \frac{1}{\alpha_1+\alpha_2}\underbrace{E[2\alpha_1 K_2^{2F} - 2\alpha_2 K_f^{2F} - \xi(\alpha_1 - a\alpha_2)|\Omega_2^{2F}]}_{<0(\text{by definition of } \Omega_2^{2F})}\Pr(\Omega_2^{2F})$$
$$> \alpha_2 c_2.$$

We now compare this optimality condition at solution $\vec{K}^{2F}$ with the optimality condition at solution K_2^I, given as follows (see Eq. (16)):

$$E[\xi - 2K_2^I|\xi > 2K_2^I]\Pr(\xi > 2K_2^I) = \alpha_2 c_2.$$

Hence,

$$E[\xi - 2K_2^{2F}|\xi > 2K_2^{2F}]\Pr(\xi > 2K_2^{2F}) > E[\xi - 2K_2^I|\xi > 2K_2^I]\Pr(\xi > 2K_2^I) \ \Rightarrow K_2^{2F} < K_2^I.$$

D.1.3 Proof of $K_f^{2F} > K_1^I$

Observe that:

$$E[\frac{\xi(1+a) - 2K_2^{2F} - 2K_f^{2F}}{\alpha_1+\alpha_2}|\Omega_2^{2F}]\Pr(\Omega_2^{2F})$$
$$= E[\frac{a\xi - 2K_f^{2F}}{\alpha_1}|\Omega_2^{2F}]\Pr(\Omega_2^{2F}) + E[\frac{2\alpha_2 K_f^{2F} - 2\alpha_1 K_2^{2F} + \xi(\alpha_1 - a\alpha_2)}{\alpha_1(\alpha_1+\alpha_2)}|\Omega_2^{2F}]\Pr(\Omega_2^{2F}). \tag{44}$$

Using Eq. (44), we can write the KKT condition in Eq. (31) as:

$$\frac{a}{\alpha_1}E[\xi - \frac{2K_f^{2F}}{a}|\Omega_2^{2F}]\Pr(\Omega_2^{2F}) + E[\frac{2\alpha_2 K_f^{2F} - 2\alpha_1 K_2^{2F} + \xi(\alpha_1 - a\alpha_2)}{\alpha_1(\alpha_1+\alpha_2)}|\Omega_2^{2F}]\Pr(\Omega_2^{2F})$$

$$+\frac{a}{\alpha_1}E[\xi-\frac{2K_f^{2F}}{a}|\Omega_3^{2F}]\Pr(\Omega_3^{2F})+\frac{a}{\alpha_1}E[\xi-\frac{2K_f^{2F}}{a}|\Omega_4^{2F}]\Pr(\Omega_4^{2F})+\frac{2K_f^{2F}}{\alpha_1}\Pr(\Omega_4^{2F})$$
$$+\frac{a}{\alpha_1}E[\xi-\frac{2K_f^{2F}}{a}|\Omega_6^{2F}]\Pr(\Omega_6^{2F})=c_1-v_1. \tag{45}$$

Observe that $(\Omega_2^{2F}\cup\Omega_3^{2F}\cup\Omega_4^{2F}\cup\Omega_6^{2F})=\{\xi>\frac{2K_f^{2F}}{a}\}$ (see Figure 7). Thus, Eq. (45) can be written as:

$$E[\xi-\frac{2K_f^{2F}}{a}|\xi>\frac{2K_f^{2F}}{a}]\Pr(\xi>\frac{2K_f^{2F}}{a})$$
$$=\frac{\alpha_1}{a}(c_1-v_1)-\frac{2K_f^{2F}}{a}\Pr(\Omega_4^{2F})-\frac{1}{a(\alpha_1+\alpha_2)}|\Omega_2^{2F}]\underbrace{E[2\alpha_2K_f^{2F}-2\alpha_1K_2^{2F}+\xi(\alpha_1-a\alpha_2)|\Omega_2^{2F}]}_{>0(\text{by definition of }\Omega_2^{2F})}\Pr(\Omega_2^{2F})$$
$$<\frac{\alpha_1c_1}{a}.$$

Next, we compare this optimality condition at solution $\vec{K}^{2F}$ with the optimality condition at solution K_1^I, given as follows (see Eq. (16)):

$$E[\xi-\frac{2K_1^I}{a}|\xi>\frac{2K_1^I}{a}]\Pr(\xi>\frac{2K_1^I}{a})=\frac{\alpha_1c_1}{a}.$$

Thus, $E[\xi-\frac{2K_f^{2F}}{a}|\xi>\frac{2K_f^{2F}}{a}]\Pr(\xi>\frac{2K_f^{2F}}{a})<E[\xi-\frac{2K_1^I}{a}|\xi>\frac{2K_1^I}{a}]\Pr(\xi>\frac{2K_1^I}{a})$, which implies that $K_f^{2F}>K_1^I$.

Recall that we have already shown that $\frac{K_f^{2F}}{a}<K_2^{2F}$ and $K_2^{2F}<K_2^I$. Thus,

$$\frac{K_1^I}{a}<\frac{K_f^{2F}}{a}<K_2^{2F}<K_2^I,$$

which completes the proof of part 2(a) of Theorem 5.1.

The proof of part 2(b) is similar, and hence omitted. Similarly, the proof of part (3) is also in the same spirit as the proof of part (2), and follows due to Lemma B.1, since $\Omega_4^F=\emptyset$ (corresponding to boundary solution $\vec{K}^F$) and $\Omega_4^{2F}=\emptyset$ (corresponding to boundary solution $\vec{K}^{2F}$) when $\alpha_1<a\alpha_2$; see Wang (2002) for details. ∎

E Proof of Theorem 5.2

Let $\xi_2=\xi$. Then, $\xi_1=a-\xi$. The demand space at boundary solution $\vec{K}^D$ reduces to:

$$\Omega_1^D=\{a-2K_1^D<\xi<2K_2^D\}$$
$$\Omega_2^D=\{a>2K_1^D+2K_2^D,\ \xi=\frac{\alpha_2(a-2K_1^D)+\alpha_1(2K_2^D)}{\alpha_1+\alpha_2}\}=\emptyset$$
(since ξ is a continuous random variable)

$$\Omega_3^D = \{\xi > 2K_2^D,\ \xi(\alpha_1+\alpha_2) < 2\alpha_1 K_2^D - 2\alpha_2 K_1^D + a\alpha_2\}$$
$$\Omega_4^D = \{\xi < a - 2K_1^D,\ \xi(\alpha_1+\alpha_2) > 2\alpha_1 K_2^D - 2\alpha_2 K_1^D + a\alpha_2\}$$
$$\Omega_5^D = \{\xi > a - 2K_1^D,\ \xi > 2K_2^D\}; \quad \Omega_6^D = \{\xi < a - 2K_1^D,\ \xi < 2K_2^D\}.$$

Thus, at boundary solution $\vec{K}^D$, the first-order KKT necessary & sufficient conditions, given in Theorem 4.1, reduce to the following:

$$\mathbf{KKT-1}: E[\frac{a-2K_1^D-\xi}{\alpha_1}|\Omega_3^D]\Pr(\Omega_3^D) + E[\frac{a-2K_1^D-\xi}{\alpha_1}|\Omega_4^D]\Pr(\Omega_4^D)$$
$$+E[\frac{a-2K_1^D-\xi}{\alpha_1}|\Omega_6^D]\Pr(\Omega_6^D) = c_1 \tag{46}$$
$$\mathbf{KKT-2}: E[\frac{\xi-2K_2^D}{\alpha_2}|\Omega_3^D]\Pr(\Omega_3^D) + E[\frac{\xi-2K_2^D}{\alpha_2}|\Omega_4^D]\Pr(\Omega_4^D) \tag{47}$$
$$+E[\frac{\xi-2K_2^D}{\alpha_2}|\Omega_5^D]\Pr(\Omega_5^D) = c_2 \tag{48}$$
$$\mathbf{KKT-3}: E[\frac{a-2K_1^D-\xi}{\alpha_1}|\Omega_3^D]\Pr(\Omega_3^D) + E[\frac{\xi-2K_2^D}{\alpha_2}|\Omega_4^D]\Pr(\Omega_4^D)$$
$$+E[\frac{\xi-2K_2^D}{\alpha_2}|\Omega_5^D]\Pr(\Omega_5^D) + E[\frac{a-2K_1^D-\xi}{\alpha_1}|\Omega_6^D]\Pr(\Omega_6^D) = c_f - v_f = \underline{c}_f. \tag{49}$$

Consider the optimal solution, $\vec{K}^I = (K_1^I, K_2^I)$, to the *dedicated system*. By definition, solution (K_1^I, K_2^I) satisfies conditions KKT-1 and KKT-2 (see Eq. (16)). If this solution also satisfies condition KKT-3, then by Theorem 4.1, the optimal solution must be boundary solution $\vec{K}^D = (K_1^I, K_2^I, 0)$. Otherwise, the optimal solution must have $K_f > 0$. Thus, in the following, letting $(K_1^D, K_2^D) = (K_1^I, K_2^I)$, we analyze whether or not this solution satisfies KKT-3. There are three possible cases:

1. If $2K_1^I + 2K_2^I = a$, then $\Omega_1^D = \Omega_2^D = \Omega_3^D = \Omega_4^D = \emptyset$. Then, KKT conditions in Eq.s (46)-(49) imply that $\underline{c}_f = c_1 + c_2$. Since $c_f < c_1 + c_2$ by definition, it follows by Lemma 4.1 that $K_f > 0$ in the optimal solution.

2. If $2K_1^I + 2K_2^I > a$, then $\Omega_2^D = \Omega_3^D = \Omega_4^D = \emptyset$. Then, KKT conditions in Eq.s (46)-(49) imply that $\underline{c}_f = c_1 + c_2$, and therefore, it follows by Lemma 4.1 that $K_f > 0$ in the optimal solution.

3. If $2K_1^I + 2K_2^I < a$, then $\Omega_1^D = \Omega_2^D = \emptyset$. Then, KKT conditions in Eq.s (46)-(49) imply that $\underline{c}_f = c_1 + c_2 - \{\frac{1}{\alpha_1}E[a - 2K_1^I - \xi|\Omega_4^D]\Pr(\Omega_4^D) + \frac{1}{\alpha_2}E[\xi - 2K_2^I|\Omega_3^D]\Pr(\Omega_3^D)\}$. By Lemma 4.1, $K_f > 0$ in the optimal solution only if $c_f < \underline{c}_f$. Also, by our assumption, $c_1, c_2 < c_f < c_1 + c_2$. Thus, in order to show that K_f can be positive in the optimal solution, we need to show that:

$$\frac{1}{\alpha_1}E[a - 2K_1^I - \xi|\Omega_4^D]\Pr(\Omega_4^D) + \frac{1}{\alpha_2}E[\xi - 2K_2^I|\Omega_3^D]\Pr(\Omega_3^D) < \min(c_1,\ c_2).$$

   Observe that this follows by Eq.s (46) and (48), and the definitions of Ω_3^D, Ω_4^D, Ω_5^D, and Ω_6^D. Thus, K_f may be positive in the optimal solution.

This completes the proof. ∎