Appendix to "A Branch-and-Cut Algorithm for the Multiple Depot Vehicle Scheduling Problem"

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Appendix A

A Facets of the MDVSP polytope

We assume that the submultigraph of G induced by the tasks is acyclic but not necessarily transitive. We also assume that all the arcs of the form (D_k, T_i) or (T_i, D_k) belong to G. Let Q denote the convex hull of the feasible solutions of (P), i.e., the feasible assignments. In this appendix we investigate the dimension of Q and prove that some inequalities arising from thorny submultigraphs represent facets of Q. In what follows we call an arc of the form (i, j, k), where i and j are at most n, an *intertask*. For the sake of clarity, it will be sometimes denoted by (T_i, T_j, k). We denote by IT the set of intertasks and by m the number of pairs (T_i, T_j) in the multigraph G. Thus the number of intertasks is equal to $m|K|$. The program (P) has $m|K| + 2n|K|$ variables and $n + (n+1)|K|$ equality constraints, but in each group of flow constraints, one constraint is redundant. It thus seems reasonable to conjecture that the dimension of Q is equal to $(m|K| + 2n|K|) - (n + n|K|) = m|K| + n(|K| - 1)$ (at least under certain conditions).

Let X be any feasible assignment. $v(X)$ will denote the number of routes (or vehicles) in X. Also v will denote $\sum_{k=1}^{|K|} v_k$ and $v_{min}(G)$ will denote $\min\{v(X) \,|\, X$ is a feasible assignment in $G\}$.

Theorem A.1 *The dimension of Q is $m|K| + n(|K| - 1)$ if $v_{min}(G) \leq v - 2$ and there exists an ℓ such that $v_\ell \geq 2$.*

Proof. We may assume, without loss of generality, that $v_{|K|}$ is at least 2. Consider a feasible assignment X^0 such that $v(X^0)$ is equal to $v_{min}(G)$. For $k = 1, 2, \ldots, |K|$, we denote by α_k the number of routes of colour k in X^0. Observe that the colour of a route may be changed, as long as the capacity constraints on the depots are satisfied. Without loss of generality, we may thus assume that $\alpha_{|K|}$ is at most $v_{|K|} - 2$ (since $v_{min}(G) \leq v - 2$). The r^{th} route of colour k, denoted Z_r^k, is $(D_k, T_1^{k,r}, T_2^{k,r}, \ldots, T_{t_r^k}^{k,r}, D_k)$, where $T_i^{k,r}$ denotes the i^{th} task in the route Z_r^k (and not the task T_i).

The gist of the proof is to construct $m|K| + n(|K| - 1)$ feasible assignments that are in one-to-one correspondence with the elements of an arc set denoted B (and such that $B \cap X^0$ is empty). The correspondence will be such that the s^{th} arc of B belongs to X^s (the s^{th} assignment) but not to $X^{s'}$ for any s' greater than s. It will follow that the X^s (for $s = 0, 1, \ldots, |B|$) are linearly, and hence affinely, independent. We now proceed to define B. We let

$$A_k^o = \{(D_k, T_i, k) \mid T_i \text{ is not of the form } T_1^{k,r} \text{ for some } r\},$$

$$L_{|K|} = \{(T_i, T_j, |K|) \mid (T_i, T_j, k) \text{ belongs to one of the } Z_r^k \text{ for some } k \neq |K|\},$$

$$A_{|K|}^d = \{(T_i^{k,r}, D_{|K|}, |K|) \mid k \neq |K| \text{ and } i < t_r^k\}$$

and

$$B = (IT \backslash (X^0 \cup L_{|K|})) \cup \left(\bigcup_{k=1}^{|K|} A_k^o \right) \cup A_{|K|}^d.$$

Note that in the above expression, X^0 is used to denote the set of arcs in X^0.

Since $|IT \cap X^0|$ is equal to $n - \sum_{k=1}^{|K|} \alpha_k$, $|L_{|K|}|$ to $|A_{|K|}^d|$ and $|A_k^o|$ to $n - \alpha_k$ for all k, the cardinality of B is exactly $m|K| + n(|K| - 1)$. Indeed, we have

$$\begin{aligned}
|B| &= \Big(|IT| - (|IT \cap X^0| + |L_{|K|}|)\Big) + \sum_{k=1}^{|K|} |A_k^o| + |A_{|K|}^d| \\
&= \left(m|K| - \left(n - \sum_{k=1}^{|K|} \alpha_k \right) - |A_{|K|}^d| \right) + \left(n|K| - \sum_{k=1}^{|K|} \alpha_k \right) + |A_{|K|}^d| \\
&= m|K| + n(|K| - 1).
\end{aligned}$$

We now describe, for each arc in B, a feasible assignment corresponding to this arc. This assignment is obtained from X^0 by modifying certain routes. Observe that once the assignment corresponding to an arc has been constructed, this arc will not appear in any assignment constructed thereafter.

Case 1 An arc in $IT \backslash (X^0 \cup L_{|K|})$ of the form $(T_i^{k,r}, T_j^{k,r}, \ell)$ (where $i < j - 1$)

1. If $k = \ell$, we replace the route Z_r^k in X^0 by the routes $(D_k, T_1^{k,r}, \ldots, T_i^{k,r}, T_j^{k,r}, \ldots, T_{t_r^k}^{k,r}, D_k)$ and $(D_{|K|}, T_{i+1}^{k,r}, \ldots, T_{j-1}^{k,r}, D_{|K|})$.

2. If $k \neq \ell$, we can "rearrange" X^0 as follows. If the capacity constraint for D_ℓ is not satisfied at equality, we replace Z_r^k by $Z_r^k \backslash \{D_k\} \cup \{D_\ell\}$. If the capacity constraint is satisfied at equality, we replace the routes Z_r^k and $Z_{r'}^\ell$ (for some r') by $Z_r^k \backslash \{D_k\} \cup \{D_\ell\}$ and $Z_{r'}^\ell \backslash \{D_\ell\} \cup \{D_k\}$. In either case, we can then replace the route of colour ℓ by two routes in the same fashion as in Case 1.1.

Case 2 An arc in $IT \backslash (X^0 \cup L_{|K|})$ of the form $(T_i^{k,r}, T_j^{k',r'}, \ell)$ (where $r \neq r'$ if $k = k'$)

If $k = \ell$, we replace the routes Z_r^k and $Z_{r'}^{k'}$ by the routes $(D_k, T_1^{k,r}, \ldots, T_i^{k,r}, T_j^{k',r'}, \ldots, T_{t_{r'}^{k'}}^{k',r'}, D_k)$, $(D_{k'}, T_1^{k',r'}, \ldots, T_{j-1}^{k',r'}, D_{k'})$ and $(D_{|K|}, T_{i+1}^{k,r}, \ldots, T_{t_r^k}^{k,r}, D_{|K|})$. If $k \neq \ell$, one can rearrange X^0 so that k becomes equal to ℓ (as in Case 1.2).

Case 3 An arc in $\bigcup_{k=1}^{|K|-1} A_k^o$ of the form $(D_k, T_i^{k,r}, k)$

Note that $i \geq 2$ by definition of the A_k^o. We replace the route Z_r^k in X^0 by the routes $(D_k, T_i^{k,r}, \ldots, T_{t_r^k}^{k,r}, D_k)$ and $(D_{|K|}, T_1^{k,r}, \ldots, T_{i-1}^{k,r}, D_{|K|})$.

Case 4 An arc in $\bigcup_{k=1}^{|K|-1} A_k^o$ of the form $(D_k, T_i^{k',r'}, k)$ (where $k \neq k'$ and $i \geq 2$)

If there is at least one route of colour k in X^0, we replace the routes Z_1^k and $Z_{r'}^{k'}$ by the routes $(D_k, T_i^{k',r'}, \ldots, T_{t_{r'}^{k'}}^{k',r'}, D_k)$, $(D_{k'}, T_1^{k',r'}, \ldots, T_{i-1}^{k',r'}, D_{k'})$ and $(D_{|K|}, T_1^{k,1}, \ldots, T_{t_1^k}^{k,1}, D_{|K|})$. If there is no route of colour k, it suffices to replace $Z_{r'}^{k'}$ by the first two routes given above.

Case 5 Arcs in $IT \backslash (X^0 \cup L_{|K|})$ of the form $(T_i^{k,r}, T_{i+1}^{k,r}, \ell)$ (where $i < t_r^k$)

For each $Z^{k,r}$ we consider the indices $t_r^k - 1, t_r^k - 2, \ldots, 2$ and 1 in that order. For each i in $\{t_r^k - 1, t_r^k - 2, \ldots, 1\}$, we replace the routes Z_r^k and Z_1^ℓ by the routes $(D_\ell, T_1^{k,r}, \ldots, T_{i+1}^{k,r}, D_\ell)$, $(D_{|K|}, T_{i+2}^{k,r}, \ldots, T_{t_r^k}^{k,r}, D_{|K|})$ and $(D_{|K|}, T_1^{\ell,1}, \ldots, T_{t_1^\ell}^{\ell,1}, D_{|K|})$. Note that if $i = t_r^k - 1$, the second route is vacuous, and that if Z_1^ℓ does not exist, it suffices to replace Z_r^k by the first two routes given above.

Case 6 An arc in $\bigcup_{k=1}^{|K|-1} A_k^o$ of the form $(D_k, T_1^{k',r'}, k)$ (where $k \neq k'$)

We replace the routes $Z_{r'}^{k'}$ and Z_1^k by the routes $(D_k, T_1^{k',r'}, D_k)$, $(D_{|K|}, T_2^{k',r'}, \ldots, T_{t_{r'}^{k'}}^{k',r'}, D_{|K|})$ and $(D_{|K|}, T_1^{k,1}, \ldots, T_{t_1^k}^{k,1}, D_{|K|})$. If Z_1^k does not exist, it suffices to replace $Z_{r'}^{k'}$ by the first two routes given above.

Case 7 An arc in $A^d_{|K|}$

This arc is of the form $(T^{k,r}_i, D_{|K|}, |K|)$. We replace Z^k_r by the routes $(D_{|K|}, T^{k,r}_1, \ldots, T^{k,r}_i, D_{|K|})$ and $(D_{|K|}, T^{k,r}_{i+1}, \ldots, T^{k,r}_{t^k_r}, D_{|K|})$.

Case 8 An arc in $A^o_{|K|}$

This arc is of the form $(D_{|K|}, T^{k,r}_i, |K|)$. We replace Z^k_r by the routes $(D_k, T^{k,r}_1, \ldots, T^{k,r}_{i-1}, D_k)$ and $(D_{|K|}, T^{k,r}_i, \ldots, T^{k,r}_{t^k_r}, D_{|K|})$. □

Observe that if $v_{min}(G) = v$, the dimension of Q is less than $m|K| + n(|K| - 1)$ (because the capacity constraints are satisfied at equality for any feasible assignment). If $v_{min}(G) = v - 1$, the dimension of Q may or may not be equal to $m|K| + n(|K| - 1)$. For instance, if the MDVSP instance consists of two incompatible tasks and two depots of capacity 1 and 2 (respectively), the dimension of Q is still equal to $m|K| + n(|K| - 1)$. If the MDVSP instance consists of two depots of capacity 1 and two compatible tasks, the dimension of Q is smaller than $m|K| + n(|K| - 1)$. Finally, the above proposition may hold even if there does not exist an ℓ such that $v_\ell \geq 2$. If the MDVSP instance consists of three depots of capacity 1 and two compatible tasks, the dimension of Q is equal to $m|K| + n(|K| - 1)$.

We now show that some valid inequalities actually represent facets of Q.

Theorem A.2 *Let us assume that there exists an ℓ such that $v_\ell \geq 2$. For some intertask (i, j, k), let H denote the submultigraph obtained from G by removing the arcs of the form (i, j, k') (for $k' = 1, 2, \ldots, |K|$). If $v \geq v_{min}(H) + 2$, the inequality $X^k_{ij} \geq 0$ represents a facet of Q.*

Proof. First note that $v_{min}(G) \leq v_{min}(H) \leq v - 2$ and let X^0 denote a feasible assignment contained in H and such that $v(X^0) = v_{min}(H)$. Since X^0 is also a feasible assignment in G, we may apply the construction of Theorem A.1 to obtain a collection of feasible assignments (denoted C). Because every arc of the form (i, j, ℓ) (for $\ell = 1, 2, \ldots, |K|$) belongs to $IT \backslash (X^0 \cup L_{|K|})$, the arc (i, j, k) belongs to exactly one assignment in C (see Cases 1 and 2 of Theorem A.1). Therefore C includes a collection of $m|K| + n(|K| - 1)$ affinely independent assignments in G, none of which contains the arc (i, j, k). This proves that $X^k_{ij} \geq 0$ represents a facet of Q. □

If (i, j, k) is not an intertask, the conclusion of Theorem A.2 does not hold in general. Consider for instance the arc (D_k, T_i, k) and assume that G contains two arcs of tail T_i, say, the arcs (T_i, T_j, k) and $(T_i, T_{j'}, k)$. Then the face $\{X \mid X^k_{n+k,i} = 0\}$ is included in the intersection of the faces $X^k_{ij} = 0$ and $X^k_{ij'} = 0$. Hence the inequality $X^k_{n+k,i} \geq 0$ cannot represent a facet in this case. A facet will be called *trivial* if it can be represented by an inequality of the form $X^k_{ij} \geq 0$. Note that Löbel (1997) also gives a result on trivial facets (see Lemma 9.7), but only for the uncapacitated version of the MDVSP.

We now consider the valid inequalities introduced in Section 4. In matching theory, an inequality of this type represents a facet only if the underlying subgraph is 2-connected. For the MDVSP also, we need "connectivity conditions" (see Definition A.4 and Theorem A.10 below). In the sequel G_S will denote the submultigraph of G induced by S (for any directed multigraph G and any subset S of vertices of G). $G \backslash F$ will denote the submultigraph obtained from G by removing the edges in F, and $Conf(i, j, k)$ will denote the set of arcs that are in conflict with the arc (i, j, k) (see Section 4).

Definition A.3 *Let G' be a submultigraph of G and S_1 a subset of vertices of G'. An* ear*, denoted $O_{G',S_1}(i, j, k)$, is a path of colour k from i to j in G' such that $O_{G',S_1}(i, j, k) \cap S_1 = \{i, j\}$.*

Definition A.4 *Let G be a directed multigraph and $H = (S_1 \cup S_2 \cup S_3, F_1 \cup F_2)$ a thorny submultigraph of G. Let G' denote the submultigraph $G_{S_1 \cup S_2}$. We say that H is* normal *if*

- *$col(\delta^-_{F_2}(T_i)) \neq \emptyset$ for every $T_i \in S_2$,*
- *$col(\delta^-_{F_1}(T_i)) = col(\delta^+_{F_1}(T_i))$ for every $T_i \in S_2$, and*
- *G' contains the arc (T_i, T_j, k) whenever $O_{G',S_1}(i, j, k)$ is an ear of G'.*

Because G is an acyclic multigraph, the second condition of the above definition is equivalent to the following: "each arc of F_1 belongs to an ear of H of the form $O_{H,S_1}(i, j, k)$". The third condition is satisfied whenever G is transitive. Clearly, any conflictual submultigraph is normal. In Theorem A.10 we consider thorny submultigraphs that are maximal. $H = (S, F)$ is said to be *maximal* if no submultigraph H'

of the form (S, F'), where F' contains F properly, is a thorny submultigraph. Observe that in Definition A.4, neither of the multigraphs H and G' need be contained in the other, but that if H is maximal, the third condition of Definition A.4 is also satisfied by $(S_1 \cup S_2, F_1)$, which is a submultigraph of both H and G'.

Finally, we note that in Definition A.4, we do not require that $col(\delta^-_{F_1}(T_i))$ be nonempty for every $T_i \in S_2$. Indeed, if H is normal and maximal and $col(\delta^-_{F_1}(T_i))$ is empty, the valid inequality corresponding to H is equivalent to the valid inequality corresponding to H' (where H' is obtained from H by removing T_i). This is not surprising since the MDVSP polytope is not full-dimensional. In order to state Theorem A.10, we recall a few definitions from matching theory.

Definition A.5 *Let H be an undirected multigraph. A* matching *M in H is a set of edges of H with the property that no two edges in M have a vertex in common.*

Definition A.6 *Let H be an undirected multigraph and u a vertex of H. A* near perfect matching *of H deficient at u is a matching M in H such that every vertex in $S \backslash \{u\}$ belongs to an edge of M.*

Definition A.7 *An undirected multigraph H is said to be* critical *if for every vertex u of H, H contains a near perfect matching deficient at u.*

Before embarking upon the proof of Theorem A.10, we state two lemmas. The proof of the first one can be found in Pulleyblank and Edmonds (1974).

Lemma A.8 *Any 2-connected critical multigraph with ℓ edges contains ℓ linearly independent near perfect matchings.*

Lemma A.9 *Let G be a directed multigraph and $H = (S_1 \cup S_2 \cup S_3, F_1 \cup F_2)$ a normal submultigraph of G that is maximal. Let us assume that the arc (T_i, T_j, k) does not belong to $F_1 \cup F_2$ but $T_j \in S_2$. Then*

1. *$k \in col(\delta^-_{F_1}(T_j))$,*
2. *$k \in col(\delta^-_{F_1}(T_i))$ if $T_i \in S_1$ and*
3. *$k \in col(\delta^-_{F_2}(T_i))$ if $T_i \in S_2$.*

Let us assume that (T_i, T_j, k) does not belong to $F_1 \cup F_2$ but $T_j \in S_1$ and $T_i \in S_2$. Then $k \in col(\delta^+_{F_1}(T_j))$ if $k \in col(\delta^-_{F_1}(T_i))$.

Proof. Observe that any multigraph obtained from H by including an arc of $G_{S_1 \cup S_2}$ into H satisfies the first and third conditions of Definition A.4.

1. If k did not belong to $col(\delta^-_{F_1}(T_j))$, one could consider (T_i, T_j, k) as a thorn and include it into H. Then $H \cup \{(T_i, T_j, k)\}$ would be a normal submultigraph, contradicting the maximality of H.
2. Suppose that T_i belongs to S_1 and k does not belong to $col(\delta^-_{F_1}(T_i))$. Then $H \cup \{(T_i, T_j, k)\}$ is a thorny submultigraph since k belongs to $col(\delta^-_{F_1}(T_j))$ and (T_i, T_j, k) is in conflict with all the other arcs incident to T_i. This contradicts the assumption that H is maximal and we conclude that k belongs to $col(\delta^-_{F_1}(T_i))$.
3. Suppose that T_i belongs to S_2 and k does not belong to $col(\delta^-_{F_2}(T_i))$. Then k must belong to $col(\delta^-_{F_1}(T_i))$ since otherwise, $H \cup \{(D_k, T_i, k)\}$ would be a normal submultigraph, contradicting the assumption that H is maximal. But if k belongs to $col(\delta^-_{F_1}(T_i))$, the multigraph $H \cup \{(T_i, T_j, k)\}$ itself is normal, contradicting again the assumption that H is maximal. Thus k belongs to $col(\delta^-_{F_2}(T_i))$.
4. Suppose that k belongs to $col(\delta^-_{F_1}(T_i))$ but not to $col(\delta^+_{F_1}(T_j))$. Then k belongs to $col(\delta^+_{F_1}(T_i))$ by Definition A.4. Since k does not belong to $col(\delta^+_{F_1}(T_j))$, the multigraph $H \cup \{(T_i, T_j, k)\}$ is normal, contradicting the assumption that H is maximal. We conclude that k belongs to $col(\delta^+_{F_1}(T_j))$ if it belongs to $col(\delta^-_{F_1}(T_i))$.

□

Theorem A.10 *Let G be a directed multigraph and $H = (S, F)$ a normal submultigraph of G that is maximal (where $S = S_1 \cup S_2 \cup S_3$ and $F = F_1 \cup F_2$). Assume that $v_{|K|} \geq n - \lfloor |S_1|/2 \rfloor$ and $v_k \geq \lfloor |S_1|/2 \rfloor + |S_2| + 1$ for every $k < |K|$. Then the inequality $\sum_{(i,j,k) \in F} X^k_{ij} \leq \lfloor |S_1|/2 \rfloor + |S_2|$ represents a non trivial facet of Q if*

1. *H_{S_1} (considered as an undirected multigraph) is critical and 2-connected, and*
2. *for any arc (T_i, T_j, k) in $G_{S_1} \backslash F_1$, $H_{S_1} \backslash Conf(i,j,k)$ contains a matching of cardinality $\lfloor |S_1|/2 \rfloor$.*

Proof. The assumptions imply that $v_{min}(G) \leq v - 2$ and there exists an ℓ such that

$v_\ell \geq 2$. Therefore Theorem A.1 holds. As was the case in Theorem A.1, the gist of the proof is to construct $m|K|+n(|K|-1)$ linearly independent assignments that satisfy the equation $\sum_{(i,j,k)\in F} X_{ij}^k = \lfloor |S_1|/2 \rfloor + |S_2|$ and are in one-to-one correspondence with the elements of an arc set denoted B. The correspondence will be such that with the exception of the arcs in H_{S_1}, the s^{th} arc of B belongs to X^s (the s^{th} assignment) but not to $X^{s'}$ for any s' greater than s. For each T_i in S_2 we choose a color k belonging to $col(\delta^-_{F_2}(T_i))$ and denote it by $col(T_i)$. B contains the following arcs:

- the arcs in IT,
- the arcs of the form (D_k, T_i, k) such that $k \neq |K|$ and $T_i \notin S_1 \cup S_2$,
- the arcs of the form (D_k, T_i, k) such that $k \neq col(T_i)$ and $T_i \in S_2$, and
- an arc of the form (D_k, T_i, k) or (T_i, D_k, k) for each $k \neq |K|$ and each i in S_1.

It is easy to verify that the cardinality of B is exactly $m|K| + n(|K|-1)$. We now consider several cases and give the corresponding assignments. Observe that the list of cases is exhaustive (because the maximality of H implies that $col(\delta^-_{F_1}(T_i)) \cup col(\delta^-_{F_2}(T_i)) = K$ for any $T_i \in S_2$), and that any intertask, with the exception of the arcs of F_1, belongs to a single X^s. In what follows the arc set of H_{S_1} is denoted A_1.

Case 1 The arcs of the submultigraph H_{S_1}

Since H_{S_1} is critical and 2-connected, Lemma A.8 implies that there exist $|A_1|$ linearly independent matchings of cardinality $\lfloor |S_1|/2 \rfloor$ in H_{S_1}. Let us denote these matchings $M_1, M_2, \ldots, M_{|A_1|}$. Since M_s (for $s = 1, 2, \ldots, |A_1|$) is a near perfect matching of H_{S_1}, there is a single vertex, denoted $u(M_s)$, that does not belong to an edge in M_s. Without loss of generality, for any T_i in S_1, there exists a matching M_s such that $T_i = u(M_S)$. To each M_s corresponds the assignment X^s defined by the routes

$$(D_k, T_i, T_j, D_k) \text{ for any } (T_i, T_j, k) \in M_s,$$

$$(D_{|K|}, u(M_s), D_{|K|}),$$

$$(D_{col(T_i)}, T_i, D_{col(T_i)}) \text{ for any } T_i \in S_2$$

and

$$(D_{|K|}, T_i, D_{|K|}) \text{ for any } T_i \in V \backslash (S_1 \cup S_2).$$

In the sequel we will denote $\hat{X}$ the collection of assignments $X^1, X^2, \ldots, X^{|A_1|}$. Also we note that if $v_k(X)$ denotes the number of vehicles leaving from the k^{th} depot in X, then for any s, $v_{|K|}(X^s) \leq v_{|K|}$ and $v_k(X^s)$ is at most $v_k - 1$ for $k = 1, 2, \ldots, |K| - 1$.

Case 2 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i \notin S_1 \cup S_2$ and $T_j \notin S_1 \cup S_2$

To this arc corresponds an assignment obtained from any assignment in $\hat{X}$ by replacing the routes $(D_{|K|}, T_i, D_{|K|})$ and $(D_{|K|}, T_j, D_{|K|})$ by the route (D_k, T_i, T_j, D_k).

Case 3 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i \notin S_1 \cup S_2$ and $T_j \in S_1$, **or** $T_i \in S_1$ and $T_j \notin S_1 \cup S_2$, **or** $T_i \in S_2$, $T_j \in S_1$ and $k \in col(\delta^-_{F_2}(T_i))$

Let us assume that T_j (resp. T_i, T_j) belongs to S_1 and let M_s be a matching such that $T_j = u(M_s)$ (resp. $T_i = u(M_s)$, $T_j = u(M_s)$). If $T_i \notin S_2$, we construct an assignment by replacing the routes $(D_{|K|}, T_i, D_{|K|})$ and $(D_{|K|}, T_j, D_{|K|})$ in X^s by the route (D_k, T_i, T_j, D_k). If $T_i \in S_2$, we replace the routes $(D_{col(T_i)}, T_i, D_{col(T_i)})$ and $(D_{|K|}, T_j, D_{|K|})$ by the route (D_k, T_i, T_j, D_k).

Case 4 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i \notin S_1 \cup S_2$ and $T_j \in S_2$

By Lemma A.9, k is a colour in $col(\delta^-_{F_1}(T_j))$, and thus T_j belongs to an ear of colour k and of the form $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$. Let q be the index such that $T_{i_q} = T_j$ and M_s a matching such that $T_{i_r} = u(M_s)$. The assignment corresponding to (T_i, T_j, k) is obtained from X^s by replacing the routes $(D_{|K|}, u(M_s), D_{|K|})$, $(D_{|K|}, T_i, D_{|K|})$ and $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_j to $T_{i_{r-1}}$) by the single route $(D_k, T_i, T_j, T_{i_{q+1}}, \ldots, T_{i_r}, D_k)$.

Case 5 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i \in S_2$ and $T_j \notin S_1 \cup S_2$

Recall that $col(\delta^-_{F_1}(T_i) \cup \delta^-_{F_2}(T_i)) = K$ since H is maximal. If k belongs to $col(\delta^-_{F_2}(T_i))$, we choose the assignment obtained from any assignment in $\hat{X}$ by replacing the routes $(D_{|K|}, T_j, D_{|K|})$ and $(D_{col(T_i)}, T_i, D_{col(T_i)})$ by the route (D_k, T_i, T_j, D_k).

If k belongs to $col(\delta^-_{F_1}(T_i))$, T_i belongs to an ear of colour k and of the form $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$. Let q be the index such that $T_{i_q} = T_i$ and M_s a matching such that $T_{i_1} = u(M_s)$. We choose the assignment obtained from X^s by replacing the routes

$(D_{|K|}, u(M_s), D_{|K|})$, $(D_{|K|}, T_j, D_{|K|})$ and $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_{i_2} to T_{i_q}) by the single route $(D_k, T_{i_1}, \ldots, T_{i_{q-1}}, T_i, T_j, D_k)$.

Case 6 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i, T_j \in S_2$

By Lemma A.9, k belongs to $col(\delta^-_{F_2}(T_i))$. From this one concludes that (D_k, T_i, k) belongs to F_2. As in Case 4, T_j belongs to an ear of colour k and of the form $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$. Let q be the index such that $T_{i_q} = T_j$ and M_s a matching such that $T_{i_r} = u(M_s)$. The assignment corresponding to (T_i, T_j, k) is obtained from X^s by replacing the routes $(D_{|K|}, u(M_s), D_{|K|})$, $(D_{col(T_i)}, T_i, D_{col(T_i)})$ and $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_j to $T_{i_{r-1}}$) by the single route $(D_k, T_i, T_j, T_{i_{q+1}}, \ldots, T_{i_r}, D_k)$.

Case 7 An arc (D_k, T_i, k) not in $F_1 \cup F_2$ such that $T_i \in S_2$ and $k \neq col(T_i)$

As in Case 4, k is a colour in $col(\delta^-_{F_1}(T_i))$ and T_i belongs to an ear of colour k and of the form $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$. Let q be the index such that $T_{i_q} = T_i$ and M_s a matching such that $T_{i_r} = u(M_s)$. The assignment corresponding to (D_k, T_i, k) is obtained from X^s by replacing the routes $(D_{|K|}, u(M_s), D_{|K|})$ and $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_i to $T_{i_{r-1}}$) by the single route $(D_k, T_i, T_{i_{q+1}}, \ldots, T_{i_r}, D_k)$.

Case 8 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i \in S_1$ and $T_j \in S_2$

By Lemma A.9, $k \in col(\delta^-_{F_1}(T_i))$ and $k \in col(\delta^-_{F_1}(T_j))$. Since $k \in col(\delta^-_{F_1}(T_j))$, T_j belongs to an ear of colour k, say, $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$. Let q be the index such that $T_{i_q} = T_j$ and $1 < q < r$. Because $T_i, T_j, T_{i_{q+1}}, \ldots, T_{i_r}$ is also an ear of $G_{S_1 \cup S_2}$, the arc (T_i, T_{i_r}, k) exists (since H is normal) and belongs to $G_{S_1} \backslash F_1$ (since $k \in col(\delta^-_{F_1}(T_i))$). By assumption $H_{S_1} \backslash Conf(i, i_r, k)$ contains a matching of cardinality $\lfloor |S_1|/2 \rfloor$, denoted by M. Thus $M \cup \{(T_i, T_{i_r}, k)\}$ contains no conflict and can be transformed into an assignment (called X') by the same method as in Case 1. Within X' the arc (T_i, T_{i_r}, k) is contained in a route of colour k. Finally, the assignment corresponding to the arc (T_i, T_j, k) is obtained from X' by replacing the arc (T_i, T_{i_r}, k) and the route $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_j to $T_{i_{r-1}}$) by the path $T_i, T_j, T_{i_{q+1}}, \ldots, T_{i_r}$.

Case 9 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i \in S_2$, $T_j \in S_1$ and $k \in col(\delta^-_{F_1}(T_i))$

This case is similar to the previous one.

Case 10 An arc (T_i, T_j, k) not in $F_1 \cup F_2$ such that $T_i, T_j \in S_1$

By assumption $H_{S_1} \backslash Conf(i, j, k)$ contains a matching of cardinality $\lfloor |S_1|/2 \rfloor$ (denoted by M). Thus $M \cup \{(T_i, T_j, k)\}$ contains no conflict and can be transformed into an assignment by the same method as in Case 1.

Case 11 The arcs in F_2 that belong to $H_{S_1 \cup S_2}$

Let F' be the set of all such arcs. Since the submultigraph defined by F' is acyclic, there exists a list of the arcs in F', denoted by $e_1, e_2, \ldots, e_{|F'|}$, such that i is greater than j whenever e_i occurs before e_j in a directed path. We consider the arcs in F' in the order given by such a list. Let e be an arc of colour k in F'. There exists a path of colour k, denoted $T_{j_1}, T_{j_2}, \ldots, T_{j_t}$, such that:

- e is the last arc of the path, i.e., $e = (T_{j_{t-1}}, T_{j_t}, k)$,
- each arc of the path belongs to F', and
- the path is maximal (i.e., either $T_{j_1} \in S_2$ **and** $k \in col(\delta^-_{F_1}(T_{j_1}))$, or $T_{j_1} \in S_1$).

Let us assume first that $T_{j_1} \in S_2$ and $k \in col(\delta^-_{F_1}(T_{j_1}))$. Then T_{j_1} belongs to an ear of colour k and of the form $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$. Let q be the index such that $T_{i_q} = T_{j_1}$ and M_s a matching such that $T_{i_1} = u(M_s)$. The assignment corresponding to $e = (T_{j_{t-1}}, T_{j_t}, k)$ is obtained from X^s by replacing the routes $(D_{|K|}, u(M_s), D_{|K|})$ and $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_{i_2} to $T_{i_{q-1}}$ and the path from T_{j_1} to T_{j_t}) by the single route $(D_k, T_{i_1}, \ldots, T_{i_{q-1}}, T_{j_1}, \ldots, T_{j_t}, D_k)$. Finally, if T_{j_1} belongs to S_1, we can use a similar argument; indeed, it suffices to observe that the ear reduces to T_{j_1} in this case.

Case 12 An arc (T_i, T_j, k) in F_2 such that $T_i \in S_3$ and $T_j \in S_2$

To this arc corresponds the assignment obtained from any assignment in $\hat{X}$ by replacing the routes $(D_{|K|}, T_i, D_{|K|})$ and $(D_{col(T_j)}, T_j, D_{col(T_j)})$ by the route (D_k, T_i, T_j, D_k).

Case 13 An arc (D_k, T_i, k) not in $F_1 \cup F_2$ such that $k \neq |K|$ and $T_i \notin S_1 \cup S_2$

To this arc corresponds an assignment obtained from any assignment in $\hat{X}$ by replacing the route $(D_{|K|}, T_i, D_{|K|})$ by the route (D_k, T_i, D_k).

Case 14 An arc (D_k, T_i, k) in F_2 such that $T_i \in S_2$ and $k \neq col(T_i)$

To this arc corresponds the assignment obtained from any assignment in $\hat{X}$ by replacing the route $(D_{col(T_i)}, T_i, D_{col(T_i)})$ by the route (D_k, T_i, D_k).

Case 15 The arcs in F_1 that are not arcs of H_{S_1}

Let F' be the set of all such arcs. Since the submultigraph defined by F' is acyclic, there exists a list of the arcs in F', denoted by $e_1, e_2, \ldots, e_{|F'|}$, such that i is greater than j whenever e_i occurs before e_j in a directed path. We consider the arcs in F' in the order given by such a list. Let (T_i, T_j, k) be an arc in F'. Because H is normal, there exists an ear of $H_{S_1 \cup S_2}$ containing this arc. Let $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$ be such an ear and assume that $(T_i, T_j, k) = (T_{i_{t-1}}, T_{i_t}, k)$. Note that since H is maximal, the arc (T_{i_1}, T_{i_r}, k) belongs to F_1.

If $t \leq r-1$, we let M_s be a matching such that $T_{i_1} = u(M_s)$. We then construct an assignment corresponding to (T_i, T_j, k) by replacing, in X^s, the routes $(D_{|K|}, u(M_s), D_{|K|})$ and $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_{i_2} to T_{i_t}) by the single route $(D_k, T_{i_1}, \ldots, T_{i_t}, D_k)$.

If $t = r$, we let M_s be the matching containing (T_{i_1}, T_{i_r}, k) and construct the required assignment by replacing, in X^s, the routes $(D_k, T_{i_1}, T_{i_r}, D_k)$ and $(D_{col(T_{j'})}, T_{j'}, D_{col(T_{j'})})$ (for every vertex $T_{j'}$ on the subpath from T_{i_2} to $T_{i_{r-1}}$) by the single route $(D_k, T_{i_1}, \ldots, T_{i_r}, D_k)$.

Case 16 An arc of the form (D_k, T_i, k) or (T_i, D_k, k) for each $k \neq |K|$ and each T_i in S_1

Let M_s be a matching such that $T_i = u(M_s)$. We choose the assignment obtained from X_s by replacing the route $(D_{|K|}, T_i, D_{|K|})$ by the route (D_k, T_i, D_k). Observe that since $col(\delta^-_{F_1}(T_i)) \cap col(\delta^+_{F_1}(T_i)) = \emptyset$, if k does not belong to $col(\delta^-_{F_1}(T_i))$ (resp. $col(\delta^+_{F_1}(T_i))$), then the arc (D_k, T_i, k) (resp. (T_i, D_k, k)) does not belong to any assignment of the form X^s. □

We now proceed to show that two of the conditions in Theorem A.10 are necessary.

Theorem A.11 *Let G be a directed multigraph and $H = (S, F)$ a normal submultigraph of G that is maximal (where $S = S_1 \cup S_2 \cup S_3$ and $F = F_1 \cup F_2$). Assume that $v_{|K|} \geq n - \lfloor |S_1|/2 \rfloor$ and $v_k \geq \lfloor |S_1|/2 \rfloor + |S_2| + 1$ for every $k < |K|$. If the inequality $\sum_{(i,j,k) \in F} X^k_{ij} \leq \lfloor |S_1|/2 \rfloor + |S_2|$ represents a non trivial facet of Q, then*

1. *H_{S_1} (considered as an undirected multigraph) is critical, and*
2. *for any arc (T_i, T_j, k) in $G_{S_1} \backslash F_1$, $H_{S_1} \backslash Conf(i,j,k)$ contains a matching of cardinality $\lfloor |S_1|/2 \rfloor$.*

Proof. Let Q' denote the face of Q represented by the equation $\sum_{(i,j,k)\in F} X_{ij}^k \leq \lfloor |S_1|/2 \rfloor + |S_2|$. First we note that H_{S_1} does not contain an isolated vertex (otherwise Q' would be empty and thus not a facet). Assume that the subgraph induced by H_{S_1} is not critical, i.e., there exists a vertex T_i in S_1 such that $H_{S_1 \backslash \{T_i\}}$ does not include a matching of cardinality $\lfloor |S_1|/2 \rfloor$. We let ℓ denote a colour in the complement of $col(\delta_H^+(T_i))$ (if it exists). Since T_i is not an isolated vertex, any assignment X in Q' contains an intertask (T_i, T_j, k) (or (T_j, T_i, k)) belonging to $H_{S_1 \cup S_2}$. It follows that $X_{n+\ell,i}^\ell = 0$ for any assignment X in Q', that is, Q' is contained in a trivial face. If the complement of $col(\delta_H^+(T_i))$ is empty, we can choose a colour ℓ in the complement of $col(\delta_H^-(T_i))$ and conclude that Q' is contained in the face $X_{i,n+\ell}^\ell = 0$.

Let us assume now that the arc (T_i, T_j, k) belongs to $G_{S_1} \backslash F_1$ and Q' is a non trivial facet of Q. Then Q' contains a feasible assignment X such that $X_{ij}^k = 1$. Let H' denote the subgraph of H consisting of the edges $(T_{i'}, T_{j'}, k')$ such that $X_{i'j'}^{k'} = 1$. It follows from the definition of Q' that the connected components of H' are either arcs belonging to F_2 or paths whose endpoints belong to S_1. Since $X_{ij}^k = 1$, H' does not contain any arc that is in conflict with (T_i, T_j, k). Because H is normal, it must contain the set of all the arcs $(T_{i'}, T_{j'}, k')$ such that i' and j' are the endpoints of some connected component of H' (recall the third point of Definition A.4). Therefore H_{S_1} contains a matching whose arcs are not in conflict with (T_i, T_j, k), and we conclude that $H_{S_1} \backslash Conf(i,j,k)$ contains a matching of cardinality $\lfloor |S_1|/2 \rfloor$. □

Corollary A.12 *Let G be a directed multigraph and $H = (S, F)$ a conflictual submultigraph of G. Assume that $v_{|K|} \geq n - \lfloor |S|/2 \rfloor$ and $v_k \geq \lfloor |S|/2 \rfloor + 1$ for every $k < |K|$. Then the inequality $\sum_{(i,j,k)\in F} X_{ij}^k \leq \lfloor |S|/2 \rfloor$ represents a non trivial facet of Q if and only if*

1. *H (considered as an undirected multigraph) is critical and 2-connected, and*
2. *for any arc (T_i, T_j, k) in $G_S \backslash F$, $H \backslash Conf(i,j,k)$ contains a matching of cardinality $\lfloor |S|/2 \rfloor$.*

Proof. H is normal because it is a conflictual submultigraph (i.e., $S_2 = S_3 = \emptyset$ and $F_2 = \emptyset$). The second condition implies that it is also maximal. Indeed, if $G_S \backslash F$ contained an arc (T_i, T_j, k) with the property that $H \cup \{(T_i, T_j, k)\}$ is a conflictual submultigraph, (T_i, T_j, k) would be in conflict with all the arcs incident to T_i or T_j in H and the nodes T_i and T_j would be isolated in $H \backslash Conf(i, j, k)$. This would contradict the second condition. Therefore H is maximal and Theorem A.10 implies that the conditions are sufficient. On the other hand, by Theorem A.11, if the inequality $\sum_{(i,j,k)\in F} X_{ij}^k \leq \lfloor |S|/2 \rfloor$ represents a non trivial facet, the second condition holds and H is a critical multigraph. If H contained a cut-vertex, H could be decomposed into two connected components (S', F') and (S'', F'') such that $|S' \cap S''| = 1$, $S' \cup S'' = S$, $F' \cup F'' = F$ and $|S'|$ and $|S''|$ are odd (since H is critical). Then the inequality $\sum_{(i,j,k)\in F} X_{ij}^k \leq \lfloor |S|/2 \rfloor$ would be the sum of the inequalities $\sum_{(i,j,k)\in F'} X_{ij}^k \leq \lfloor |S'|/2 \rfloor$ and $\sum_{(i,j,k)\in F''} X_{ij}^k \leq \lfloor |S''|/2 \rfloor$, contradicting the assumption that it represents a facet. We conclude that H is critical **and** 2-connected. □

Corollary A.13 *Let G be a directed multigraph and $H = (S, F)$ an elementary normal submultigraph of G that is maximal (where $S = S_1 \cup S_2 \cup S_3$ and $F = F_1 \cup F_2$). Assume that $v_{|K|} \geq n - \lfloor |S_1|/2 \rfloor$ and $v_k \geq \lfloor |S_1|/2 \rfloor + |S_2| + 1$ for every $k < |K|$. Then the inequality $\sum_{(i,j,k)\in F} X_{ij}^k \leq \lfloor |S_1|/2 \rfloor + |S_2|$ defines a non trivial facet of Q.*

Proof. Let H' denote the submultigraph $(S_1 \cup S_2, F_1)$. By the definition of an elementary submultigraph, H' contains an undirected hamiltonian cycle (denoted by C). Therefore H_{S_1} itself contains a hamiltonian cycle and satisfies the first condition of Theorem A.10. On the other hand, every source and every sink of H' belongs to S_1 (since H is normal). Because H is maximal, any arc in G_{S_1} joining a source to a sink of H' belongs to F_1. Furthermore, if T_i denotes the only vertex in S_1 that is neither a source nor a sink of H', $col(\delta^-_{F_1}(T_i)) \cup col(\delta^+_{F_1}(T_i))$ is the set of all colours.

Let H_B denote the submultigraph of H whose arcs either belong to C or are parallel to some arc of C. For any arc (T_i, T_j, k) in $G_{S_1} \backslash F_1$, either $H_B \backslash Conf(i, j, k)$ contains a hamiltonian path or $H_B \backslash Conf(i, j, k)$ has at most one component with an odd number of vertices (actually, this component is an isolated vertex). We conclude that H satisfies both conditions of Theorem A.10 and the corollary follows. □