Appendix for "Managing Patient Service in a Diagnostic Medical Facility" by Linda V. Green, Sergei Savin, and Ben Wang

Proof of Proposition 1

Define the dynamic programming operator

$$T_i^a [v (n,s)] = -sw_s - nw_n + p_e p_n [(1 - p_s a_{i+1}) v (n+1, s) + p_s a_{i+1} v (n+1, s+1)]$$
$$+p_e (1 - p_n) [(1 - p_s a_{i+1}) v (n, s) + p_s a_{i+1} v (n, s+1)]$$
$$+p_n (1 - p_e) \left[(1 - p_s a_{i+1}) H_{i+1}^a (n+1, s) + p_s a_{i+1} H_{i+1}^a (n+1, s+1)\right]$$
$$+ (1 - p_e)(1 - p_n) \left[(1 - p_s a_{i+1}) H_{i+1}^a (n, s) + p_s a_{i+1} H_{i+1}^a (n, s+1)\right]. \quad (18)$$

For $i = 1, ..., N+1$, define G_i as the class of functions defined on S_i such that for every $g \in G_i$

$$g (n, s+1) - g (n+1, s) \leq g (n+1, s+1) - g (n+2, s), \quad (19)$$
$$g (n+1, s) - g (n, s+1) \leq g (n+1, s+1) - g (n, s+2), \quad (20)$$
$$g (n, s) - g (n, s+1) \leq g (n+1, s) - g (n+1, s+1), \quad (21)$$
$$g (n, s) - g (n+1, s) \leq g (n+1, s) - g (n+2, s), \quad (22)$$
$$g (n, s) - g (n, s+1) \leq g (n, s+1) - g (n, s+2), \quad (23)$$

where we have assumed that all the states for which g is evaluated belong to S_i. First, we will show that the class G_{i+1} is mapped onto the class G_i by the action of T_i^a:

Lemma A1

For any function $g \in G_{i+1}$ we have $T_i^a g \in G_i$, $i = 1, ..., N$

Proof of Lemma A1

First we will show that $T_i^a g (n+1, s) - T_i^a g (n, s+1) - T_i^a g (n+2, s) + T_i^a g (n+1, s+1) \geq 0$. We look at 4 separate cases: 1) $(n \geq 1, s \geq 1)$, 2) $(n \geq 1, s = 0)$, 3) $(n = 0, s \geq 1)$, 4)

$(n=0, s=0)$. Below we provide the proof for the case of $a_{i+1}=0$, since the proof for the case $a_{i+1}=1$ can be derived in the exactly same fashion.

First, for the case of $n \geq 1$ and $s \geq 1$, we get

$$\begin{aligned}
& T_i^a g(n+1, s+1) - T_i^a g(n+2, s) - T_i^a g(n, s+1) + T_i^a g(n+1, s) \\
= \; & p_e p_n \left[g(n+2, s+1) - g(n+3, s) - g(n+1, s+1) + g(n+2, s) \right] \\
& + p_e (1-p_n) \left[g(n+1, s+1) - g(n+2, s) - g(n, s+1) + g(n+1, s) \right] \\
& + p_n (1-p_e) \left[H_{i+1}^a(n+2, s+1) - H_{i+1}^a(n+3, s) - H_{i+1}^a(n+1, s+1) + H_{i+1}^a(n+2, s) \right] \\
& + (1 - p_e - p_n + p_e p_n) \\
& \times \left[H_{i+1}^a(n+1, s+1) - H_{i+1}^a(n+2, s) - H_{i+1}^a(n, s+1) + H_{i+1}^a(n+1, s) \right]. \qquad (24)
\end{aligned}$$

As $p_n(1-p_e)$ and $(1-p_e-p_n+p_e p_n)$ terms share a similar structure, below we look at the $p_n(1-p_e)$ term:

$$\begin{aligned}
& H_{i+1}^a(n+2, s+1) - H_{i+1}^a(n+3, s) - H_{i+1}^a(n+1, s+1) + H_{i+1}^a(n+2, s) \\
= \; & \max\left[g(n+1, s) + r_n, g(n+2, s-1) + r_s \right] \\
& + \max\left[g(n+1, s+1) + r_n, g(n+2, s) + r_s \right] \\
& - \max\left[g(n, s+1) + r_n, g(n+1, s) + r_s \right] \\
& - \max\left[g(n+2, s) + r_n, g(n+3, s-1) + r_s \right]. \qquad (25)
\end{aligned}$$

1) If $g(n, s+1) + r_n \geq g(n+1, s) + r_s$ and $g(n+2, s) + r_n \geq g(n+3, s-1) + r_s$,

$$\begin{aligned}
& H_{i+1}^a(n+2, s+1) - H_{i+1}^a(n+3, s) - H_{i+1}^a(n+1, s+1) + H_{i+1}^a(n+2, s) \\
= \; & \max\left[g(n+1, s) + r_n, g(n+2, s-1) + r_s \right] \\
& + \max\left[g(n+1, s+1) + r_n, g(n+2, s) + r_s \right] \\
& - g(n, s+1) - r_n - g(n+2, s) - r_n \\
\geq \; & g(n+1, s) + g(n+1, s+1) - g(n, s+1) - g(n+2, s) \geq 0. \qquad (26)
\end{aligned}$$

2) If $g(n,s+1)+r_n \le g(n+1,s)+r_s$ and $g(n+2,s)+r_n \le g(n+3,s-1)+r_s$,

$$
\begin{aligned}
& H_{i+1}^{a}(n+2,s+1) - H_{i+1}^{a}(n+3,s) - H_{i+1}^{a}(n+1,s+1) + H_{i+1}^{a}(n+2,s) \\
= \ & \max[g(n+1,s)+r_n, g(n+2,s-1)+r_s] \\
& +\max[g(n+1,s+1)+r_n, g(n+2,s)+r_s] \\
& -g(n+1,s)-r_s-g(n+3,s-1)-r_s \\
\ge \ & g(n+2,s-1)+g(n+2,s)-g(n+1,s)-g(n+3,s-1) \ge 0. \qquad (27)
\end{aligned}
$$

3) If $g(n,s+1)+r_n \ge g(n+1,s)+r_s$ and $g(n+2,s)+r_n \le g(n+3,s-1)+r_s$,

$$
\begin{aligned}
& H_{i+1}^{a}(n+2,s+1) - H_{i+1}^{a}(n+3,s) - H_{i+1}^{a}(n+1,s+1) + H_{i+1}^{a}(n+2,s) \\
= \ & \max[g(n+1,s)+r_n, g(n+2,s-1)+r_s] \\
& +\max[g(n+1,s+1)+r_n, g(n+2,s)+r_s] \\
& -g(n,s+1)-r_n-g(n+3,s-1)-r_s \\
\ge \ & g(n+2,s-1)+g(n+1,s+1)-g(n,s+1)-g(n+3,s-1) \\
\ge \ & g(n+1,s)+g(n+1,s+1)-g(n,s+1)-g(n+2,s) \ge 0. \qquad (28)
\end{aligned}
$$

4) If $g(n,s+1)+r_n \le g(n+1,s)+r_s$ and $g(n+2,s)+r_n \ge g(n+3,s-1)+r_s$,

$$
\begin{aligned}
& H_{i+1}^{a}(n+2,s+1) - H_{i+1}^{a}(n+3,s) - H_{i+1}^{a}(n+1,s+1) + H_{i+1}^{a}(n+2,s) \\
= \ & \max[g(n+1,s)+r_n, g(n+2,s-1)+r_s] \\
& +\max[g(n+1,s+1)+r_n, g(n+2,s)+r_s] \\
& -g(n+1,s)-r_s-g(n+2,s)-r_n \\
\ge \ & g(n+1,s)+g(n+2,s)-g(n+1,s)-g(n+2,s) = 0. \qquad (29)
\end{aligned}
$$

Second, for the case of $n \ge 1$ and $s = 0$,

$$
\begin{aligned}
& [T_i^a g(n+1,0) - T_i^a g(n,1)] - [T_i^a g(n+2,0) - T_i^a g(n+1,1)] \\
= \ & p_e p_n [g(n+2,0) - g(n+1,1) - g(n+3,0) + g(n+2,1)]
\end{aligned}
$$

$$+p_e\left(1-p_n\right)\left[g\left(n+1,0\right)-g\left(n,1\right)-g\left(n+2,0\right)+g\left(n+1,1\right)\right]$$

$$+p_n\left(1-p_e\right)\left[H_{i+1}^a\left(n+2,0\right)-H_{i+1}^a\left(n+1,1\right)-H_{i+1}^a\left(n+3,0\right)+H_{i+1}^a\left(n+2,1\right)\right]$$

$$+\left(1-p_e-p_n+p_e p_n\right)$$

$$\times\left[H_{i+1}^a\left(n+1,0\right)-H_{i+1}^a\left(n,1\right)-H_{i+1}^a\left(n+2,0\right)+H_{i+1}^a\left(n+1,1\right)\right]. \tag{30}$$

For the $p_n\left(1-p_e\right)$ term we get

$$\begin{aligned} & H_{i+1}^a\left(n+2,0\right)-H_{i+1}^a\left(n+1,1\right)-H_{i+1}^a\left(n+3,0\right)+H_{i+1}^a\left(n+2,1\right) \\ = \;& g\left(n+1,0\right)+r_n-H_{i+1}^a\left(n+1,1\right)-g\left(n+2,0\right)-r_n+H_{i+1}^a\left(n+2,1\right) \\ = \;& g\left(n+1,0\right)-\max[g\left(n,1\right)+r_n,g\left(n+1,0\right)+r_s]-g\left(n+2,0\right) \\ & +\max[g\left(n+1,1\right)+r_n,g\left(n+2,0\right)+r_s]. \end{aligned} \tag{31}$$

1) If $g\left(n,1\right)+r_n\geq g\left(n+1,0\right)+r_s$,

$$\begin{aligned} & H_{i+1}^a\left(n+2,0\right)-H_{i+1}^a\left(n+1,1\right)-H_{i+1}^a\left(n+3,0\right)+H_{i+1}^a\left(n+2,1\right) \\ \geq \;& g\left(n+1,0\right)-g\left(n,1\right)-g\left(n+2,0\right)+g\left(n+1,1\right)\geq 0. \end{aligned} \tag{32}$$

2) If $g\left(n,1\right)+r_n\leq g\left(n+1,0\right)+r_s$,

$$\begin{aligned} & H_{i+1}^a\left(n+2,0\right)-H_{i+1}^a\left(n+1,1\right)-H_{i+1}^a\left(n+3,0\right)+H_{i+1}^a\left(n+2,1\right) \\ \geq \;& g\left(n+1,0\right)-g\left(n+1,0\right)-g\left(n+2,0\right)+g\left(n+2,0\right)=0. \end{aligned} \tag{33}$$

Third, for the case of $n=0$ and $s\geq 1$,

$$\begin{aligned} & T_i^a g\left(n+1,s+1\right)-T_i^a g\left(n+2,s\right)-T_i^a g\left(n,s+1\right)+T_i^a g\left(n+1,s\right) \\ = \;& p_e p_n\left[g\left(2,s+1\right)-g\left(3,s\right)-g\left(1,s+1\right)+g\left(2,s\right)\right] \\ & +p_e\left(1-p_n\right)\left[g\left(1,s+1\right)-g\left(2,s\right)-g\left(0,s+1\right)+g\left(1,s\right)\right] \\ & +p_n\left(1-p_e\right)\left[H_{i+1}^a\left(2,s+1\right)-H_{i+1}^a\left(3,s\right)-H_{i+1}^a\left(1,s+1\right)+H_{i+1}^a\left(2,s\right)\right] \\ & +\left(1-p_e-p_n+p_e p_n\right) \\ & \times\left[H_{i+1}^a\left(1,s+1\right)-H_{i+1}^a\left(2,s\right)-H_{i+1}^a\left(0,s+1\right)+H_{i+1}^a\left(1,s\right)\right]. \end{aligned} \tag{34}$$

For the $(1-p_e-p_n+p_ep_n)$ term we have

$$\begin{aligned}
& H^a_{i+1}(1,s+1) - H^a_{i+1}(2,s) - H^a_{i+1}(0,s+1) + H^a_{i+1}(1,s) \\
= \; & \max[g(0,s+1)+r_n, g(1,s)+r_s] - \max[g(1,s)+r_n, g(2,s-1)+r_s] \\
& -g(0,s) - r_s + \max[g(0,s)+r_n, g(1,s-1)+r_s]. \qquad (35)
\end{aligned}$$

1) If $g(1,s)+r_n \geq g(2,s-1)+r_s$,

$$\begin{aligned}
& H^a_{i+1}(g(1,s+1)) - H^a_{i+1}(g(2,s)) - H^a_{i+1}(g(0,s+1)) + H^a_{i+1}(g(1,s)) \\
= \; & \max[g(0,s+1)+r_n, g(1,s)+r_s] - g(1,s) - r_n \\
& -g(0,s) - r_s + \max[g(0,s)+r_n, g(1,s-1)+r_s] \\
\geq \; & g(1,s)+r_s - g(1,s) - r_n - g(0,s) - r_s + g(0,s) + r_n = 0. \qquad (36)
\end{aligned}$$

2) If $g(1,s)+r_n \leq g(2,s-1)+r_s$,

$$\begin{aligned}
& H^a_{i+1}(1,s+1) - H^a_{i+1}(2,s) - H^a_{i+1}(0,s+1) + H^a_{i+1}(1,s) \\
= \; & \max[g(0,s+1)+r_n, g(1,s)+r_s] - g(2,s-1) - r_s \\
& -g(0,s) - r_s + \max[g(0,s)+r_n, g(1,s-1)+r_s] \\
\geq \; & g(1,s)+r_s - g(2,s-1) - r_s - g(0,s) - r_s + g(1,s-1) + r_s \\
\geq \; & 0. \qquad (37)
\end{aligned}$$

The non-negativity of the $p_n(1-p_e)$ term is proved in the exactly same fashion as in the case of $n \geq 1$ and $s \geq 1$.

Fourth, for the case of $n=0$ and $s=0$,

$$\begin{aligned}
= \; & T^a_i g(1,1) - T^a_i g(2,0) - T^a_i g(0,1) + T^a_i g(1,0) \\
= \; & p_ep_n[g(2,1) - g(3,0) - g(1,1) + g(2,0)] \\
& +p_e(1-p_n)[g(1,1) - g(2,0) - g(0,1) + g(1,0)] \\
& +p_n(1-p_e)\left[H^a_{i+1}(2,1) - H^a_{i+1}(3,0) - H^a_{i+1}(1,1) + H^a_{i+1}(2,0)\right]
\end{aligned}$$

$$+\left(1-p_e-p_n+p_e p_n\right)$$
$$\times\left[H_{i+1}^a(1,1)-H_{i+1}^a(2,0)-H_{i+1}^a(0,1)+H_{i+1}^a(1,0)\right]. \quad (38)$$

For the $p_n(1-p_e)$ term we get

$$\begin{aligned}
& H_{i+1}^a(2,1)-H_{i+1}^a(3,0)-H_{i+1}^a(1,1)+H_{i+1}^a(2,0)\\
= & \max[g(1,1)+r_n, g(2,0)+r_s]-g(2,0)-r_n\\
& -\max[g(0,1)+r_n, g(1,0)+r_s]+g(1,0)+r_n. \quad (39)
\end{aligned}$$

1) If $g(0,1)+r_n \geq g(1,0)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(2,1)-H_{i+1}^a(3,0)-H_{i+1}^a(1,1)+H_{i+1}^a(2,0)\\
\geq & g(1,1)+r_n-g(2,0)-g(0,1)-r_n+g(1,0) \geq 0. \quad (40)
\end{aligned}$$

2) If $g(0,1)+r_n \leq g(1,0)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(2,1)-H_{i+1}^a(3,0)-H_{i+1}^a(1,1)+H_{i+1}^a(2,0)\\
\geq & g(2,0)+r_s-g(2,0)-g(1,0)-r_s+g(1,0)=0. \quad (41)
\end{aligned}$$

Finally, for the $(1-p_e-p_n+p_e p_n)$ term we have

$$\begin{aligned}
& H_{i+1}^a(1,1)-H_{i+1}^a(2,0)-H_{i+1}^a(0,1)+H_{i+1}^a(1,0)\\
= & \max[g(0,1)+r_n, g(1,0)+r_s]-g(1,0)-r_n-g(0,0)-r_s+g(0,0)+r_n\\
\geq & g(1,0)+r_s-g(1,0)-g(0,0)-r_s+g(0,0)=0. \quad (42)
\end{aligned}$$

The property (20) in Proposition 1 is derived exactly in the same way as the proof of (19). We now show that $T_i^a g(n+1,s+1)-T_i^a g(n,s+2)-T_i^a g(n+1,s)+T_i^a g(n,s+1) \geq 0$. We provide the proof for the case $a_{i+1}=0$ and look at 4 separate cases: 1) $(n \geq 1, s \geq 1)$, 2) $(n \geq 1, s=0)$, 3) $(n=0, s \geq 1)$, 4) $(n=0, s=0)$. First, for the case of $n \geq 1$ and $s \geq 1$,

$$T_i^a g(n+1,s+1)+T_i^a g(n,s+1)-T_i^a g(n,s+2)-T_i^a g(n+1,s)$$

$$
\begin{aligned}
= \quad & p_e p_n \left[g(n+2,s+1)+g(n+1,s+1)-g(n+1,s+2)-g(n+2,s)\right] \\
& +p_e(1-p_n)\left[g(n+1,s+1)+g(n,s+1)-g(n,s+2)-g(n+1,s)\right] \\
& +p_n(1-p_e)\left[H_{i+1}^a(n+2,s+1)+H_{i+1}^a(n+1,s+1)-H_{i+1}^a(n+1,s+2)-H_{i+1}^a(n+2,s)\right] \\
& +(1-p_e-p_n+p_e p_n) \\
& \times\left[H_{i+1}^a(n+1,s+1)+H_{i+1}^a(n,s+1)-H_{i+1}^a(n,s+2)-H_{i+1}^a(n+1,s)\right]. \qquad (43)
\end{aligned}
$$

As $p_n(1-p_e)$ and $(1-p_e-p_n+p_e p_n)$ terms share a similar structure, we only look at the $(1-p_e-p_n+p_e p_n)$ term:

$$
\begin{aligned}
& H_{i+1}^a(n,s+1)-H_{i+1}^a(n+1,s)-H_{i+1}^a(n,s+2)+H_{i+1}^a(n+1,s+1) \\
= \quad & \max[g(n-1,s+1)+r_n, g(n,s)+r_s]-\max[g(n,s)+r_n, g(n+1,s-1)+r_s] \\
& -\max[g(n-1,s+2)+r_n, g(n,s+1)+r_s] \\
& +\max[g(n,s+1)+r_n, g(n+1,s)+r_s] \qquad (44)
\end{aligned}
$$

1) If $g(n,s)+r_n \geq g(n+1,s-1)+r_s$ and $g(n-1,s+2)+r_n \geq g(n,s+1)+r_s$,

$$
\begin{aligned}
& H_{i+1}^a(n,s+1)-H_{i+1}^a(n+1,s)-H_{i+1}^a(n,s+2)+H_{i+1}^a(n+1,s+1) \\
= \quad & \max[g(n-1,s+1)+r_n, g(n,s)+r_s]-g(n,s)-r_n \\
& -g(n-1,s+2)-r_n+\max[g(n,s+1)+r_n, g(n+1,s)+r_s] \\
\geq \quad & g(n-1,s+1)-g(n,s)-g(n-1,s+2)+g(n,s+1) \geq 0. \qquad (45)
\end{aligned}
$$

2) If $g(n,s)+r_n \leq g(n+1,s-1)+r_s$ and $g(n-1,s+2)+r_n \leq g(n,s+1)+r_s$,

$$
\begin{aligned}
& H_{i+1}^a(n,s+1)-H_{i+1}^a(n+1,s)-H_{i+1}^a(n,s+2)+H_{i+1}^a(n+1,s+1) \\
= \quad & \max[g(n-1,s+1)+r_n, g(n,s)+r_s]-g(n+1,s-1)-r_s \\
& -g(n,s+1)-r_s+\max[g(n,s+1)+r_n, g(n+1,s)+r_s] \\
\geq \quad & g(n,s)+r_s-g(n+1,s-1)-r_s-g(n,s+1)-r_s+g(n+1,s)+r_s \\
= \quad & g(n,s)-g(n+1,s-1)-g(n,s+1)+g(n+1,s) \geq 0. \qquad (46)
\end{aligned}
$$

3) If $g(n,s)+r_n \leq g(n+1,s-1)+r_s$ and $g(n-1,s+2)+r_n \geq g(n,s+1)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(n,s+1)-H_{i+1}^a(n+1,s)-H_{i+1}^a(n,s+2)+H_{i+1}^a(n+1,s+1) \\
= \; & \max[g(n-1,s+1)+r_n, g(n,s)+r_s]-g(n+1,s-1)-r_s \\
& -g(n-1,s+2)-r_n+\max[g(n,s+1)+r_n, g(n+1,s)+r_s] \\
\geq \; & g(n-1,s+1)-g(n+1,s-1)-g(n-1,s+2)+g(n+1,s) \\
\geq \; & g(n,s)-g(n+1,s-1)-g(n,s+1)+g(n+1,s) \geq 0. \qquad (47)
\end{aligned}$$

4) If $g(n,s)+r_n \geq g(n+1,s-1)+r_s$ and $g(n-1,s+2)+r_n \leq g(n,s+1)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(n,s+1)-H_{i+1}^a(n+1,s)-H_{i+1}^a(n,s+2)+H_{i+1}^a(n+1,s+1) \\
= \; & \max[g(n-1,s+1)+r_n, g(n,s)+r_s]-g(n,s)-r_n \\
& -g(n,s+1)-r_s+\max[g(n,s+1)+r_n, g(n+1,s)+r_s] \\
\geq \; & g(n,s)-g(n,s)-g(n,s+1)+g(n,s+1)=0. \qquad (48)
\end{aligned}$$

Second, for the case of $n \geq 1$ and $s=0$,

$$\begin{aligned}
& T_i^a g(n+1,1)+T_i^a g(n,1)-T_i^a g(n,2)-T_i^a g(n+1,0) \\
= \; & p_e p_n[g(n+1,1)+g(n,1)-g(n,2)-g(n+1,0)] \\
& +p_e(1-p_n)[g(n+1,1)+g(n,1)-g(n,2)-g(n+1,0)] \\
& +p_n(1-p_e)[H_{i+1}^a(n+2,1)+H_{i+1}^a(n+1,1)-H_{i+1}^a(n+1,2)-H_{i+1}^a(n+2,0)] \\
& +(1-p_e-p_n+p_e p_n) \\
& \times[H_{i+1}^a(n+1,1)+H_{i+1}^a(n,1)-H_{i+1}^a(n,2)-H_{i+1}^a(n+1,0)]. \qquad (49)
\end{aligned}$$

For the $p_n(1-p_e)$ term,

$$\begin{aligned}
& H_{i+1}^a(n+2,1)+H_{i+1}^a(n+1,1)-H_{i+1}^a(n+1,2)-H_{i+1}^a(n+2,0) \\
= \; & \max[g(n+1,1)+r_n, g(n+2,0)+r_s]+\max[g(n,1)+r_n, g(n+1,0)+r_s] \\
& -\max[g(n,2)+r_n, g(n+1,1)+r_s]-g(n+1,0)-r_n. \qquad (50)
\end{aligned}$$

1) If $g(n,2)+r_n \geq g(n+1,1)+r_s$,

$$\begin{aligned}
& H_{i+1}^{a}(n+2,1)+H_{i+1}^{a}(n+1,1)-H_{i+1}^{a}(n+1,2)-H_{i+1}^{a}(n+2,0) \\
= \; & \max[g(n+1,1)+r_n, g(n+2,0)+r_s]+\max[g(n,1)+r_n, g(n+1,0)+r_s] \\
& -g(n,2)-r_n-g(n+1,0)-r_n \\
\geq \; & g(n+1,1)+r_n+g(n,1)+r_n-g(n,2)-r_n-g(n+1,0)-r_n \geq 0. \qquad (51)
\end{aligned}$$

2) If $g(n,2)+r_n \leq g(n+1,1)+r_s$,

$$\begin{aligned}
& H_{i+1}^{a}(n+2,1)+H_{i+1}^{a}(n+1,1)-H_{i+1}^{a}(n+1,2)-H_{i+1}^{a}(n+2,0) \\
= \; & \max[g(n+1,1)+r_n, g(n+2,0)+r_s]+\max[g(n,1)+r_n, g(n+1,0)+r_s] \\
& -g(n+1,1)-r_s-g(n+1,0)-r_n \\
\geq \; & g(n+1,1)+g(n+1,0)-g(n+1,1)-g(n+1,0)=0. \qquad (52)
\end{aligned}$$

Third, for the case of $n=0$ and $s \geq 1$,

$$\begin{aligned}
& T_i^a g(1,s+1)+T_i^a g(0,s+1)-T_i^a g(0,s+2)-T_i^a g(1,s) \\
= \; & p_e p_n[g(1,s+1)+g(0,s+1)-g(0,s+2)-g(1,s)] \\
& +p_e(1-p_n)[g(1,s+1)+g(0,s+1)-g(0,s+2)-g(1,s)] \\
& +p_n(1-p_e)[H_{i+1}^{a}(2,s+1)+H_{i+1}^{a}(1,s+1)-H_{i+1}^{a}(1,s+2)-H_{i+1}^{a}(2,s)] \\
& +(1-p_e-p_n+p_e p_n) \\
& \times[H_{i+1}^{a}(1,s+1)+H_{i+1}^{a}(0,s+1)-H_{i+1}^{a}(0,s+2)-H_{i+1}^{a}(1,s)]. \qquad (53)
\end{aligned}$$

For the $(1-p_e-p_n+p_e p_n)$ term, we get

$$\begin{aligned}
& H_{i+1}^{a}(1,s+1)+H_{i+1}^{a}(0,s+1)-H_{i+1}^{a}(0,s+2)-H_{i+1}^{a}(1,s) \\
= \; & \max[g(0,s+1)+r_n, g(1,s)+r_s]+g(0,s)+r_s \\
& -g(0,s+1)-r_s-\max[g(0,s)+r_n, g(1,s-1)+r_s]. \qquad (54)
\end{aligned}$$

1) If $g(0,s)+r_n \geq g(1,s-1)+r_s$,

$$
\begin{aligned}
& H_{i+1}^{a}(1,s+1)+H_{i+1}^{a}(0,s+1)-H_{i+1}^{a}(0,s+2)-H_{i+1}^{a}(1,s) \\
= \; & \max[g(0,s+1)+r_n, g(1,s)+r_s]+g(0,s)-g(0,s+1)-g(0,s)-r_n \\
\geq \; & g(0,s+1)+r_n+g(0,s)-g(0,s+1)-g(0,s)-r_n=0. \qquad (55)
\end{aligned}
$$

2) If $g(0,s)+r_n \leq g(1,s-1)+r_s$,

$$
\begin{aligned}
& H_{i+1}^{a}(1,s+1)+H_{i+1}^{a}(0,s+1)-H_{i+1}^{a}(0,s+2)-H_{i+1}^{a}(1,s) \\
= \; & \max[g(0,s+1)+r_n, g(1,s)+r_s]+g(0,s)-g(0,s+1)-g(1,s-1)-r_s \\
\geq \; & g(1,s)+r_s+g(0,s)-g(0,s+1)-g(1,s-1)-r_s \geq 0. \qquad (56)
\end{aligned}
$$

Fourth, for the case of $n=0$ and $s=0$,

$$
\begin{aligned}
& T_i^a g(1,1)+T_i^a g(0,1)-T_i^a g(0,2)-T_i^a g(1,0) \\
= \; & p_e p_n [g(1,1)+g(0,1)-g(0,2)-g(1,0)] \\
& + p_e(1-p_n)[g(1,1)+g(0,1)-g(0,2)-g(1,0)] \\
& + p_n(1-p_e)\left[H_{i+1}^{a}(2,1)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,2)-H_{i+1}^{a}(2,0)\right] \\
& + (1-p_e-p_n+p_e p_n) \\
& \times \left[H_{i+1}^{a}(1,1)+H_{i+1}^{a}(0,1)-H_{i+1}^{a}(0,2)-H_{i+1}^{a}(1,0)\right]. \qquad (57)
\end{aligned}
$$

For the $p_n(1-p_e)$ term, we have

$$
\begin{aligned}
& H_{i+1}^{a}(2,1)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,2)-H_{i+1}^{a}(2,0) \\
= \; & \max[g(1,1)+r_n, g(2,0)+r_s]+\max[g(0,1)+r_n, g(1,0)+r_s] \\
& -\max[g(0,2)+r_n, g(1,1)+r_s]-g(1,0)-r_n \qquad (58)
\end{aligned}
$$

1) If $g(0,2)+r_n \geq g(1,1)+r_s$,

$$
H_{i+1}^{a}(2,1)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,2)-H_{i+1}^{a}(2,0)
$$

$$
\begin{aligned}
&= \max[g(1,1)+r_n, g(2,0)+r_s] + \max[g(0,1)+r_n, g(1,0)+r_s] \\
&\quad -g(0,2) - r_n - g(1,0) - r_n \\
&\geq g(1,1)+r_n+g(0,1)+r_n-g(0,2)-r_n-g(1,0)-r_n \geq 0. \qquad (59)
\end{aligned}
$$

2) If $g(0,2)+r_n \leq g(1,1)+r_s$,

$$
\begin{aligned}
&H_{i+1}^{a}(2,1)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,2)-H_{i+1}^{a}(2,0) \\
&= \max[g(1,1)+r_n, g(2,0)+r_s] + \max[g(0,1)+r_n, g(1,0)+r_s] \\
&\quad -g(1,1) - r_s - g(1,0) - r_n \\
&\geq g(1,1)+r_n+g(1,0)+r_s-g(1,1)-r_s-g(1,0)-r_n = 0. \qquad (60)
\end{aligned}
$$

Finally, for the $(1-p_e-p_n+p_ep_n)$ term, we get

$$
\begin{aligned}
&H_{i+1}^{a}(1,1)+H_{i+1}^{a}(0,1)-H_{i+1}^{a}(0,2)-H_{i+1}^{a}(1,0) \\
&= \max[g(0,1)+r_n, g(1,0)+r_s] + g(0,0)+r_s-g(0,1)-r_s-g(0,0)-r_n \\
&\geq g(0,1)+r_n-g(0,1)-r_n = 0. \qquad (61)
\end{aligned}
$$

For the proof of submodularity (21) we need an additional result:

Lemma A2

For every service period $i = 1, ..., N+1$ and every state of the system $(n,s) \in S_{i-1}$, from $r_n \geq g(n+1,s) - g(n,s)$, $r_s \geq g(n,s+1) - g(n,s)$ it follows that

$$
\begin{aligned}
r_n &\geq T_i^a g(n+1,s) - T_i^a g(n,s), \qquad (62) \\
r_s &\geq T_i^a g(n,s+1) - T_i^a g(n,s). \qquad (63)
\end{aligned}
$$

Proof of Lemma A2

For (62), we provide the proof for the case $a_{i+1} = 0$ and look at 4 separate cases: 1) $(n \geq 1, s \geq 1)$, 2) $(n \geq 1, s = 0)$, 3) $(n = 0, s \geq 1)$, 4) $(n = 0, s = 0)$. In general,

$$
r_n + T_i^a g(n,s) - T_i^a g(n+1,s)
$$

$$\begin{aligned}
&= r_n + w_n + p_e p_n \left[g(n+1,s) - g(n+2,s)\right] + p_e(1-p_n)\left[g(n,s) - g(n+1,s)\right] \\
&\quad + p_n(1-p_e)\left[H_{i+1}^a(n+1,s) - H_{i+1}^a(n+2,s)\right] \\
&\quad + (1-p_e-p_n+p_e p_n)\left[H_{i+1}^a(n,s) - H_{i+1}^a(n+1,s)\right] \\
&= p_e p_n \left[(r_n+w_n) + g(n+1,s) - g(n+2,s)\right] \\
&\quad + p_e(1-p_n)\left[(r_n+w_n) + g(n,s) - g(n+1,s)\right] \\
&\quad + p_n(1-p_e)\left[(r_n+w_n) + H_{i+1}^a(n+1,s) - H_{i+1}^a(n+2,s)\right] \\
&\quad + (1-p_e-p_n+p_e p_n)\left[(r_n+w_n) + H_{i+1}^a(n,s) - H_{i+1}^a(n+1,s)\right]. \qquad (64)
\end{aligned}$$

First, for the case of $n \geq 1$ and $s \geq 1$, the non-negativity of the $p_e p_n$, $p_e(1-p_n)$ terms can be directly derived from the assumption. Since the $p_n(1-p_e)$ and the $(1-p_e-p_n+p_e p_n)$ terms share a similar structure, we only look at the $p_n(1-p_e)$ term:

$$\begin{aligned}
&(r_n+w_n) + H_{i+1}^a(n+1,s) - H_{i+1}^a(n+2,s) \\
&= (r_n+w_n) + \max[g(n,s)+r_n, g(n+1,s-1)+r_s] \\
&\quad - \max[g(n+1,s)+r_n, g(n+2,s-1)+r_s] \qquad (65)
\end{aligned}$$

1) If $g(n+1,s)+r_n \geq g(n+2,s-1)+r_s$,

$$\begin{aligned}
&(r_n+w_n) + H_{i+1}^a(n+1,s) - H_{i+1}^a(n+2,s) \\
&= r_n + w_n + \max[g(n,s)+r_n, g(n+1,s-1)+r_s] - g(n+1,s) - r_n \\
&\geq r_n + w_n + g(n,s) + r_n - g(n+1,s) - r_n \geq r_n + w_n > 0. \qquad (66)
\end{aligned}$$

2) If $g(n+1,s)+r_n \leq g(n+2,s-1)+r_s$,

$$\begin{aligned}
&(r_n+w_n) + H_{i+1}^a(n+1,s) - H_{i+1}^a(n+2,s) \\
&= r_n + w_n + \max[g(n,s)+r_n, g(n+1,s-1)+r_s] - g(n+2,s-1) - r_s \\
&\geq r_n + w_n + g(n+1,s-1) + r_s - g(n+2,s-1) - r_s \\
&= r_n + w_n + g(n+1,s-1) - g(n+2,s-1) > 0. \qquad (67)
\end{aligned}$$

Second, for the case of $n \geq 1$ and $s = 0$,

$$\begin{aligned}
& r_n + g(n,0) - g(n+1,0) \\
= \; & p_e p_n \left[(r_n + w_n) + g(n+1,0) - g(n+2,0) \right] \\
& + p_e (1 - p_n) \left[(r_n + w_n) + g(n,0) - g(n+1,0) \right] \\
& + p_n (1 - p_e) \left[(r_n + w_n) + H^a_{i+1}(n+1,0) - H^a_{i+1}(n+2,0) \right] \\
& + (1 - p_e - p_n + p_e p_n) \left[(r_n + w_n) + H^a_{i+1}(n,0) - H^a_{i+1}(n+1,0) \right]
\end{aligned} \tag{68}$$

The non-negativity of the $p_e p_n$, $p_e(1-p_n)$ term can be directly derived from the assumption. As the $p_n(1-p_e)$ and the $(1 - p_e - p_n + p_e p_n)$ terms share a similar structure, we look at the $p_n(1-p_e)$ term:

$$(r_n + w_n) + H^a_{i+1}(n+1,0) - H^a_{i+1}(n+2,0) = r_n + w_n + g(n,0) - g(n+1,0) > 0. \tag{69}$$

Third, for the case of $n = 0$ and $s \geq 1$,

$$\begin{aligned}
& r_n + g(0,s) - g(1,s) \\
= \; & p_e p_n \left[(r_n + w_n) + g(1,s) - g(2,s) \right] + p_e (1 - p_n) \left[(r_n + w_n) + g(0,s) - g(1,s) \right] \\
& + p_n (1 - p_e) \left[(r_n + w_n) + H^a_{i+1}(1,s) - H^a_{i+1}(2,s) \right] \\
& + (1 - p_e - p_n + p_e p_n) \left[(r_n + w_n) + H^a_{i+1}(0,s) - H^a_{i+1}(1,s) \right].
\end{aligned} \tag{70}$$

The non-negativity of $p_e(1-p_n)$ term can be directly derived from assumption. The proof of the non-negativity of the $p_n(1-p_e)$ term goes exactly as in the case of $n \geq 1$ and $s \geq 1$. For the $(1 - p_e - p_n + p_e p_n)$ term,

$$\begin{aligned}
& (r_n + w_n) + H^a_{i+1}(0,s) - H^a_{i+1}(1,s) \\
= \; & r_n + w_n + g(0,s-1) + r_s - \max[g(0,s) + r_n, g(1,s-1) + r_s] \\
= \; & \begin{cases} w_n + r_s + g(0,s-1) - g(0,s) > 0, & \text{if } g(0,s) + r_n \geq g(1,s-1) + r_s, \\ w_n + r_n + g(0,s-1) - g(1,s-1) > 0, & \text{if } g(0,s) + r_n \leq g(1,s-1) + r_s. \end{cases}
\end{aligned} \tag{71}$$

Fourth, for the case of $n = 0$ and $s = 0$,

$$\begin{aligned}
& r_n + g(0,0) - g(1,0) \\
= \; & p_e p_n \left[(r_n + w_n) + g(1,0) - g(2,0)\right] + p_e(1-p_n)\left[(r_n + w_n) + g(0,0) - g(1,0)\right] \\
& + p_n(1-p_e)\left[(r_n + w_n) + H_{i+1}^a(1,0) - H_{i+1}^a(2,0)\right] \\
& + (1 - p_e - p_n + p_e p_n)\left[(r_n + w_n) + H_{i+1}^a(0,0) - H_{i+1}^a(1,0)\right] \qquad (72)
\end{aligned}$$

The non-negativity of $p_e p_n, p_e(1-p_n)$ term follows directly from the assumption. The proof for the $p_n(1-p_e)$ term goes exactly as in the case of $n \geq 1$ and $s = 0$. Finally, for the $(1 - p_e - p_n + p_e p_n)$ term we get

$$(r_n + w_n) + H_{i+1}^a(0,0) - H_{i+1}^a(1,0) = r_n + w_n + g(0,0) - g(0,0) - r_n = w_n > 0. \qquad (73)$$

In a similar way we get $r_s + T_i^a g(n,s) - T_i^a g(n,s+1) \geq 0.\square$

The proof of submodularity is derived the same way as above. We consider 4 scenarios: 1) $(n \geq 1, s \geq 1)$, 2) $(n \geq 1, s = 0)$, 3) $(n = 0, s \geq 1)$, 4) $(n = 0, s = 0)$. Below we provide the proof for the case of $a_{i+1} = 0$ since the case of $a_i = 1$ is proved in a similar way. Now,

$$\begin{aligned}
& T_i^a g(n+1,s) + T_i^a g(n,s+1) - T_i^a g(n,s) - T_i^a g(n+1,s+1) \\
= \; & p_e p_n \left[g(n+2,s) + g(n+1,s+1) - g(n+1,s) - g(n+2,s+1)\right] \\
& + p_e(1-p_n)\left[g(n+1,s) + g(n,s+1) - g(n,s) - g(n+1,s+1)\right] \\
& + p_n(1-p_e)\left[H_{i+1}^a(n+2,s) + H_{i+1}^a(n+1,s+1) - H_{i+1}^a(n+1,s) - H_{i+1}^a(n+2,s+1)\right] \\
& + (1 - p_e - p_n + p_e p_n) \\
& \times \left[H_{i+1}^a(n+1,s) + H_{i+1}^a(n,s+1) - H_{i+1}^a(n,s) - H_{i+1}^a(n+1,s+1)\right] \qquad (74)
\end{aligned}$$

First, for the case of $n \geq 1$ and $s \geq 1$, the non-negativity of the p_e term can be directly derived from the assumption. As the $p_n(1-p_e)$ and the $(1 - p_e - p_n + p_e p_n)$ terms share a similar structure, we look only at the $p_n(1-p_e)$ term:

$$H_{i+1}^a(n+2,s) + H_{i+1}^a(n+1,s+1) - H_{i+1}^a(n+1,s) - H_{i+1}^a(n+2,s+1)$$

$$
\begin{aligned}
&= \max\left[g\left(n+1,s\right)+r_n, g\left(n+2,s-1\right)+r_s\right] \\
&\quad +\max\left[g\left(n,s+1\right)+r_n, g\left(n+1,s\right)+r_s\right] \\
&\quad -\max\left[g\left(n,s\right)+r_n, g\left(n+1,s-1\right)+r_s\right] \\
&\quad -\max\left[g\left(n+1,s+1\right)+r_n, g\left(n+2,s\right)+r_s\right]. \qquad (75)
\end{aligned}
$$

1) If $g\left(n,s\right)+r_n \geq g\left(n+1,s-1\right)+r_s$ and $g\left(n+1,s+1\right)+r_n \geq g\left(n+2,s\right)+r_s$,

$$
\begin{aligned}
&H_{i+1}^{a}\left(n+2,s\right)+H_{i+1}^{a}\left(n+1,s+1\right)-H_{i+1}^{a}\left(n+1,s\right)-H_{i+1}^{a}\left(n+2,s+1\right) \\
&= \max\left[g\left(n+1,s\right)+r_n, g\left(n+2,s-1\right)+r_s\right] \\
&\quad +\max\left[g\left(n,s+1\right)+r_n, g\left(n+1,s\right)+r_s\right] \\
&\quad -g\left(n,s\right)-g\left(n+1,s+1\right)-r_n-r_n \\
&\geq g\left(n+1,s\right)+g\left(n,s+1\right)-g\left(n,s\right)-g\left(n+1,s+1\right) \geq 0. \qquad (76)
\end{aligned}
$$

2) If $g\left(n,s\right)+r_n \leq g\left(n+1,s-1\right)+r_s$ and $g\left(n+1,s+1\right)+r_n \leq g\left(n+2,s\right)+r_s$,

$$
\begin{aligned}
&H_{i+1}^{a}\left(n+2,s\right)+H_{i+1}^{a}\left(n+1,s+1\right)-H_{i+1}^{a}\left(n+1,s\right)-H_{i+1}^{a}\left(n+2,s+1\right) \\
&= \max\left[g\left(n+1,s\right)+r_n, g\left(n+2,s-1\right)+r_s\right] \\
&\quad +\max\left[g\left(n,s+1\right)+r_n, g\left(n+1,s\right)+r_s\right] \\
&\quad -g\left(n+1,s-1\right)-r_s-g\left(n+2,s\right)-r_s \\
&\geq g\left(n+2,s-1\right)+g\left(n+1,s\right)-g\left(n+1,s-1\right)-g\left(n+2,s\right) \geq 0. \qquad (77)
\end{aligned}
$$

3) If $g\left(n,s\right)+r_n \geq g\left(n+1,s-1\right)+r_s$ and $g\left(n+1,s+1\right)+r_n \leq g\left(n+2,s\right)+r_s$,

$$
\begin{aligned}
&H_{i+1}^{a}\left(n+2,s\right)+H_{i+1}^{a}\left(n+1,s+1\right)-H_{i+1}^{a}\left(n+1,s\right)-H_{i+1}^{a}\left(n+2,s+1\right) \\
&= \max\left[g\left(n+1,s\right)+r_n, g\left(n+2,s-1\right)+r_s\right] \\
&\quad +\max\left[g\left(n,s+1\right)+r_n, g\left(n+1,s\right)+r_s\right] \\
&\quad -g\left(n,s\right)-r_n-g\left(n+2,s\right)-r_s \\
&\geq g\left(n+1,s\right)+g\left(n+1,s\right)-g\left(n,s\right)-g\left(n+2,s\right) \\
&= 2g\left(n+1,s\right)-g\left(n,s\right)-g\left(n+2,s\right) \geq 0, \qquad (78)
\end{aligned}
$$

4) If $g(n,s)+r_n \leq g(n+1,s-1)+r_s$ and $g(n+1,s+1)+r_n \geq g(n+2,s)+r_s$,

$$
\begin{aligned}
& H_{i+1}^a(n+2,s)+H_{i+1}^a(n+1,s+1)-H_{i+1}^a(n+1,s)-H_{i+1}^a(n+2,s+1) \\
= \; & \max\left[g(n+1,s)+r_n, g(n+2,s-1)+r_s\right] \\
& +\max\left[g(n,s+1)+r_n, g(n+1,s)+r_s\right] \\
& -g(n+1,s-1)-r_s-g(n+1,s+1)-r_n \\
\geq \; & g(n+1,s)+r_n+g(n+1,s)+r_s-g(n+1,s-1)-r_s-g(n+1,s+1)-r_n \\
= \; & 2g(n+1,s)-g(n+1,s-1)-g(n+1,s+1) \geq 0,
\end{aligned}
\tag{79}
$$

Second, for the case that $n \geq 1$ and $s=0$,

$$
\begin{aligned}
& T_i^a g(n+1,0)+T_i^a g(n,1)-T_i^a g(n,0)-T_i^a g(n+1,1) \\
= \; & p_e p_n\left[g(n+2,0)+g(n+1,1)-g(n+1,0)-g(n+2,1)\right] \\
& +p_e(1-p_n)\left[g(n+1,0)+g(n,1)-g(n,0)-g(n+1,1)\right] \\
& +p_n(1-p_e)\left[H_{i+1}^a(n+2,0)+H_{i+1}^a(n+1,1)-H_{i+1}^a(n+1,0)-H_{i+1}^a(n+2,1)\right] \\
& +(1-p_e-p_n+p_e p_n) \\
& \times\left[H_{i+1}^a(n+1,0)+H_{i+1}^a(n,1)-H_{i+1}^a(n,0)-H_{i+1}^a(n+1,1)\right].
\end{aligned}
\tag{80}
$$

The non-negativity of p_e terms can be directly derived from our assumption. As the $p_n(1-p_e)$ and the $(1-p_e-p_n+p_e p_n)$ terms share a similar structure, it is sufficient to look at the $p_n(1-p_e)$ term:

$$
\begin{aligned}
& H_{i+1}^a(n+2,0)+H_{i+1}^a(n+1,1)-H_{i+1}^a(n+1,0)-H_{i+1}^a(n+2,1) \\
= \; & g(n+1,0)+r_n+\max[g(n,1)+r_n, g(n+1,0)+r_s] \\
& -g(n,0)-r_n-\max[g(n+1,1)+r_n, g(n+2,0)+r_s].
\end{aligned}
\tag{81}
$$

1) If $g(n+1,1)+r_n \geq g(n+2,0)+r_s$,

$$
H_{i+1}^a(n+2,0)+H_{i+1}^a(n+1,1)-H_{i+1}^a(n+1,0)-H_{i+1}^a(n+2,1)
$$

$$\begin{aligned}
&\geq g(n+1,0)+r_n+g(n,1)+r_n-g(n,0)-r_n-g(n+1,1)-r_n \\
&= g(n+1,0)+g(n,1)-g(n,0)-g(n+1,1)\geq 0. \qquad (82)
\end{aligned}$$

2) If $g(n+1,1)+r_n\leq g(n+2,0)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(n+2,0)+H_{i+1}^a(n+1,1)-H_{i+1}^a(n+1,0)-H_{i+1}^a(n+2,1) \\
\geq\ & g(n+1,0)+r_n+g(n+1,0)+r_s-g(n,0)-r_n-g(n+2,0)-r_s \\
=\ & 2g(n+1,0)-g(n,0)-g(n+2,0)\geq 0, \qquad (83)
\end{aligned}$$

Third, for the case that $n=0$ and $s\geq 1$,

$$\begin{aligned}
& T_i^a g(1,s)+T_i^a g(0,s+1)-T_i^a g(0,s)-T_i^a g(1,s+1) \\
=\ & p_e p_n\left[g(2,s)+g(1,s+1)-g(1,s)-g(2,s+1)\right] \\
& +p_e(1-p_n)\left[g(1,s)+g(0,s+1)-g(0,s)-g(1,s+1)\right] \\
& +p_n(1-p_e)\left[H_{i+1}^a(2,s)+H_{i+1}^a(1,s+1)-H_{i+1}^a(1,s)-H_{i+1}^a(2,s+1)\right] \\
& +(1-p_e-p_n+p_e p_n) \\
& \times\left[H_{i+1}^a(1,s)+H_{i+1}^a(0,s+1)-H_{i+1}^a(0,s)-H_{i+1}^a(1,s+1)\right]. \qquad (84)
\end{aligned}$$

The non-negativity of p_e term can be directly derived from the assumption. The proof for the $p_n(1-p_e)$ term is exactly same as the one for the case of $n\geq 1$ and $s\geq 1$. For the $(1-p_e-p_n+p_e p_n)$ term, we get

$$\begin{aligned}
& H_{i+1}^a(1,s)+H_{i+1}^a(0,s+1)-H_{i+1}^a(0,s)-H_{i+1}^a(1,s+1) \\
=\ & \max[g(0,s)+r_n, g(1,s-1)+r_s]+g(0,s)+r_s-g(0,s-1)-r_s \\
& -\max[g(0,s+1)+r_n, g(1,s)+r_s]. \qquad (85)
\end{aligned}$$

1) If $g(0,s+1)+r_n\geq g(1,s)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(1,s)+H_{i+1}^a(0,s+1)-H_{i+1}^a(0,s)-H_{i+1}^a(1,s+1) \\
\geq\ & g(0,s)+r_n+g(0,s)-g(0,s-1)-g(0,s+1)-r_n \\
=\ & 2g(0,s)-g(0,s-1)-g(0,s+1)\geq 0, \qquad (86)
\end{aligned}$$

2) If $g(0,s+1)+r_n \leq g(1,s)+r_s$,

$$\begin{aligned} & H_{i+1}^{a}(1,s)+H_{i+1}^{a}(0,s+1)-H_{i+1}^{a}(0,s)-H_{i+1}^{a}(1,s+1) \\ \geq\ & g(1,s-1)+r_s+g(0,s)-g(0,s-1)-g(1,s)-r_s \\ =\ & g(1,s-1)+g(0,s)-g(0,s-1)-g(1,s) \geq 0. \end{aligned} \tag{87}$$

Fourth, for the case of $n=0$ and $s=0$,

$$\begin{aligned} & T_i^a g(1,0)+T_i^a g(0,1)-T_i^a g(0,0)-T_i^a g(1,1) \\ =\ & p_e p_n \left[g(2,0)+g(1,1)-g(1,0)-g(2,1)\right] \\ & +p_e(1-p_n)\left[g(1,0)+g(0,1)-g(0,0)-g(1,1)\right] \\ & +p_n(1-p_e)\left[H_{i+1}^{a}(2,0)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,0)-H_{i+1}^{a}(2,1)\right] \\ & +(1-p_e-p_n+p_e p_n) \\ & \times\left[H_{i+1}^{a}(1,0)+H_{i+1}^{a}(0,1)-H_{i+1}^{a}(0,0)-H_{i+1}^{a}(1,1)\right]. \end{aligned} \tag{88}$$

The non-negativity of the p_e term can be directly derived from our assumption. For the $p_n(1-p_e)$ term, we have

$$\begin{aligned} & H_{i+1}^{a}(2,0)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,0)-H_{i+1}^{a}(2,1) \\ =\ & g(1,0)+r_n+\max[g(0,1)+r_n, g(1,0)+r_s] \\ & -g(0,0)-r_n-\max[g(1,1)+r_n, g(2,0)+r_s]. \end{aligned} \tag{89}$$

1) If $g(1,1)+r_n \geq g(2,0)+r_s$,

$$\begin{aligned} & H_{i+1}^{a}(2,0)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,0)-H_{i+1}^{a}(2,1) \\ =\ & g(1,0)+r_n+\max[g(0,1)+r_n, g(1,0)+r_s]-g(0,0)-r_n-g(1,1)-r_n \\ \geq\ & g(1,0)+g(0,1)-g(0,0)-g(1,1) \geq 0. \end{aligned} \tag{90}$$

2) If $g(1,1)+r_n \leq g(2,0)+r_s$,

$$H_{i+1}^{a}(2,0)+H_{i+1}^{a}(1,1)-H_{i+1}^{a}(1,0)-H_{i+1}^{a}(2,1)$$

$$
\begin{aligned}
&= g(1,0) + r_n + \max[g(0,1) + r_n, g(1,0) + r_s] - g(0,0) - r_n - g(2,0) - r_s \\
&\geq 2g(1,0) - g(0,0) - g(2,0) \geq 0,
\end{aligned} \tag{91}
$$

For the $(1 - p_e - p_n)$ term,

$$
\begin{aligned}
& H_{i+1}^{a}(1,0) + H_{i+1}^{a}(0,1) - H_{i+1}^{a}(0,0) - H_{i+1}^{a}(1,1) \\
&= g(0,0) + r_n + g(0,0) + r_s - g(0,0) - \max[g(0,1) + r_n, g(1,0) + r_s].
\end{aligned} \tag{92}
$$

1) If $g(0,1) + r_n \geq g(1,0) + r_s$, then using Lemma A2, we get

$$
\begin{aligned}
& H_{i+1}^{a}(1,0) + H_{i+1}^{a}(0,1) - H_{i+1}^{a}(0,0) - H_{i+1}^{a}(1,1) \\
&= g(0,0) + r_n + g(0,0) + r_s - g(0,0) - g(0,1) - r_n \\
&= g(0,0) + r_s - g(0,1) \geq 0.
\end{aligned} \tag{93}
$$

2) If $g(0,1) + r_n \leq g(1,0) + r_s$, then using Lemma A2, we get

$$
\begin{aligned}
& H_{i+1}^{a}(1,0) + H_{i+1}^{a}(0,1) - H_{i+1}^{a}(0,0) - H_{i+1}^{a}(1,1) \\
&= g(0,0) + r_n + g(0,0) + r_s - g(0,0) - g(1,0) - r_s \\
&= g(0,0) + r_n - g(1,0) \geq 0.
\end{aligned} \tag{94}
$$

For the concavity of g, if we have $g(n,s) + g(n+2,s) \leq 2g(n+1,s)$ and $g(n,s) + g(n,s+2) \leq 2g(n,s+1)$, by adding up (19) and (21) we get

$$
T_i^a g(n,s) + T_i^a g(n+2,s) \leq 2T_i^a g(n+1,s), \tag{95}
$$

and by adding up (20) and (21) we get

$$
T_i^a g(n,s) + T_i^a g(n,s+2) \leq 2T_i^a g(n,s+1). \tag{96}
$$

This concludes the proof of Lemma A1. Now we turn to the proof of the statement of Proposition.

a) For a given service period i, we observe that the structure of the capacity allocation operator $H_i^a(n,s)$ implies that, when in the state (n,s) (such that $n,s \geq 1$) an outpatient is chosen for service if and only if $V_i^a(n,s-1) - V_i^a(n-1,s) \geq r_n - r_s$. Since, according to (19), $V_i^a(n-1,s) - V_i^a(n,s-1)$ is an decreasing function of n, we can define

$$n_i^a(s) = \begin{cases} i+1, & \Delta V_i^a \geq r_n - r_s, \\ \min\left(n | V_i(n,s-1) - V_i(n-1,s) < r_n - r_s\right), & \Delta V_i^a < r_n - r_s, \end{cases} \tag{97}$$

where $\Delta V_i^a = V_i^a(i,s-1) - V_i^a(i-1,s)$, so that outpatients are serviced if and only if $n < n_i^a(s)$.

b) Considering $n_i^a(s)$ and $n_i^a(s+1)$, we observe that, as it follows from (20) and the definition of $n_i^a(s)$,

$$\begin{aligned} V_i^a(n_i^a(s+1)+1,s-1) - V_i^a(n_i^a(s+1),s) &\geq V_i^a(n_i^a(s+1)+1,s) - V_i^a(n_i^a(s+1),s+1) \\ &\geq r_n - r_s \end{aligned} \tag{98}$$

Thus, $n_i^a(s) \leq n_i^a(s+1)$.

Proof of Proposition 2

We start by proving the monotonicity of the critical indices with respect to π_n. Consider two different values of the penalty costs π_n and $\bar{\pi}_n$ such that $\pi_n < \bar{\pi}_n$. Let $V_i^a(n,s,\pi_n)$ and $V_i^a(n,s,\bar{\pi}_n)$ be the optimal value functions for these respective values of the penalty costs. The monotonicity of the critical index $n_i^a(s)$ would follow if we show that for every i $(i = 1,...,N)$ and every $(n,s) \in S_i$,

$$V_i^a(n,s+1,\pi_n) - V_i^a(n+1,s,\pi_n) \leq V_i^a(n,s+1,\bar{\pi}_n) - V_i^a(n+1,s,\bar{\pi}_n). \tag{99}$$

The proof of this statement is derived by induction over the service period index i. For $i = N+1$, $V_{N+1}^a(n,s+1,\pi_n) - V_{N+1}^a(n+1,s,\pi_n) - V_{N+1}^a(n,s+1,\bar{\pi}_n) - V_{N+1}^a(n+1,s,\bar{\pi}_n) = \pi_n - \bar{\pi}_n < 0$. Assuming that $V_{i+1}^a(n,s+1,\pi_n) - V_{i+1}^a(n+1,s,\pi_n) \leq V_{i+1}^a(n,s+1,\bar{\pi}_n) - V_{i+1}^a(n+1,s,\bar{\pi}_n)$ for some service period index, we want to prove that $V_i^a(n,s+1,\pi_n) -$

$V_i^a(n+1,s,\pi_n) \leq V_i^a(n,s+1,\bar{\pi}_n) - V_i^a(n+1,s,\bar{\pi}_n)$. We need to look at 4 separate cases: 1) $(n \geq 1, s \geq 1)$, 2) $(n \geq 1, s = 0)$, 3) $(n = 0, s \geq 1)$, 4) $(n = 0, s = 0)$. The proofs for the cases of $a_i = 0$ and $a_i = 1$ are similar, so we only look at the case of $a_i = 0$ below:

$$
\begin{aligned}
& [V_i^a(n,s+1,\pi_n) - V_i^a(n+1,s,\pi_n)] - [V_i^a(n,s+1,\bar{\pi}_n) - V_i^a(n+1,s,\bar{\pi}_n)] \\
= \; & p_e p_n \left(V_{i+1}^a(n+1,s+1,\pi_n) - V_{i+1}^a(n+2,s,\pi_n) - V_{i+1}^a(n+1,s+1,\bar{\pi}_n) + V_{i+1}^a(n+2,s,\bar{\pi}_n)\right) \\
& + p_e(1-p_n)\left(V_{i+1}^a(n,s+1,\pi_n) - V_{i+1}^a(n+1,s,\pi_n) - V_{i+1}^a(n,s+1,\bar{\pi}_n) + V_{i+1}^a(n+1,s,\bar{\pi}_n)\right) \\
& + p_n(1-p_e)\left(H_{i+1}^a(n+1,s+1,\pi_n) - H_{i+1}^a(n+2,s,\pi_n) - H_{i+1}^a(n+1,s+1,\bar{\pi}_n)\right. \\
& \left. + H_{i+1}^a(n+2,s,\bar{\pi}_n)\right) + (1-p_e-p_n+p_e p_n)\left(H_{i+1}^a(n,s+1,\pi_n) - H_{i+1}^a(n+1,s,\pi_n)\right. \\
& \left. - H_{i+1}^a(n,s+1,\bar{\pi}_n) + H_{i+1}^a(n+1,s,\bar{\pi}_n)\right) \qquad (100)
\end{aligned}
$$

The non-positivity of the p_e term can be directly derived from the induction assumption. As the $p_n(1-p_e)$ and the $(1-p_e-p_n+p_e p_n)$ terms share a similar structure, we take the $(1-p_e-p_n+p_e p_n)$ term for example.

First, for the case of $n \geq 1$ and $s \geq 1$, we get

$$
\begin{aligned}
& H_{i+1}^a(n,s+1,\pi_n) - H_{i+1}^a(n+1,s,\pi_n) - H_{i+1}^a(n,s+1,\bar{\pi}_n) + H_{i+1}^a(n+1,s,\bar{\pi}_n) \\
= \; & \max\left[V_{i+1}^a(n-1,s+1,\pi_n) + r_n, V_{i+1}^a(n,s,\pi_n) + r_s\right] \\
& - \max\left[V_{i+1}^a(n,s,\pi_n) + r_n, V_{i+1}^a(n+1,s-1,\pi_n) + r_s\right] \\
& - \max\left[V_{i+1}^a(n-1,s+1,\bar{\pi}_n) + r_n, V_{i+1}^a(n,s,\bar{\pi}_n) + r_s\right] \\
& + \max\left[V_{i+1}^a(n,s,\bar{\pi}_n) + r_n, V_{i+1}^a(n+1,s-1,\bar{\pi}_n) + r_s\right] \qquad (101)
\end{aligned}
$$

1) If $V_{i+1}^a(n-1,s+1,\pi_n) + r_n \geq V_{i+1}^a(n,s,\pi_n) + r_s$ and $V_{i+1}^a(n,s,\bar{\pi}_n) + r_n \geq V_{i+1}^a(n+1,s-1,\bar{\pi}_n) + r_s$

$$
\begin{aligned}
& H_{i+1}^a(n,s+1,\pi_n) - H_{i+1}^a(n+1,s,\pi_n) - H_{i+1}^a(n,s+1,\bar{\pi}_n) + H_{i+1}^a(n+1,s,\bar{\pi}_n) \\
= \; & V_{i+1}^a(n-1,s+1,\pi_n) + r_n + V_{i+1}^a(n,s,\bar{\pi}_n) + r_n \\
& - \max\left[V_{i+1}^a(n,s,\pi_n) + r_n, V_{i+1}^a(n+1,s-1,\pi_n) + r_s\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\max\left[V_{i+1}^{a}\left(n-1,s+1,\bar{\pi}_n\right)+r_n, V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)+r_s\right]\\
\leq\ & V_{i+1}^{a}\left(n-1,s+1,\pi_n\right)+V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)-V_{i+1}^{a}\left(n,s,\pi_n\right)-V_{i+1}^{a}\left(n-1,s+1,\bar{\pi}_n\right)\\
\leq\ & 0. \qquad (102)
\end{aligned}
$$

2) If $V_{i+1}^{a}\left(n-1,s+1,\pi_n\right)+r_n \leq V_{i+1}^{a}\left(n,s,\pi_n\right)+r_s$ and $V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)+r_n \leq V_{i+1}^{a}\left(n+1,s-1,\bar{\pi}_n\right)+r_s$

$$
\begin{aligned}
& H_{i+1}^{a}\left(n,s+1,\pi_n\right)-H_{i+1}^{a}\left(n+1,s,\pi_n\right)-H_{i+1}^{a}\left(n,s+1,\bar{\pi}_n\right)+H_{i+1}^{a}\left(n+1,s,\bar{\pi}_n\right)\\
=\ & V_{i+1}^{a}\left(n,s,\pi_n\right)+r_s+V_{i+1}^{a}\left(n+1,s-1,\bar{\pi}_n\right)+r_s\\
&-\max\left[V_{i+1}^{a}\left(n,s,\pi_n\right)+r_n, V_{i+1}^{a}\left(n+1,s-1,\pi_n\right)+r_s\right]\\
&-\max\left[V_{i+1}^{a}\left(n-1,s+1,\bar{\pi}_n\right)+r_n, V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)+r_s\right]\\
=\ & V_{i+1}^{a}\left(n,s,\pi_n\right)+V_{i+1}^{a}\left(n+1,s-1,\bar{\pi}_n\right)-V_{i+1}^{a}\left(n+1,s-1,\pi_n\right)-V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)\\
\leq\ & 0. \qquad (103)
\end{aligned}
$$

3) If $V_{i+1}^{a}\left(n-1,s+1,\pi_n\right)+r_n \geq V_{i+1}^{a}\left(n,s,\pi_n\right)+r_s$ and $V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)+r_n \leq V_{i+1}^{a}\left(n+1,s-1,\bar{\pi}_n\right)+r_s$, then, using the induction assumption we have

$$
V_{i+1}^{a}\left(n-1,s+1,\pi_n\right)-V_{i+1}^{a}\left(n,s,\pi_n\right)-V_{i+1}^{a}\left(n-1,s+1,\bar{\pi}_n\right)+V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)\leq 0, \quad (104)
$$

and

$$
V_{i+1}^{a}\left(n,s,\pi_n\right)-V_{i+1}^{a}\left(n+1,s-1,\pi_n\right)-V_{i+1}^{a}\left(n,s,\bar{\pi}_n\right)+V_{i+1}^{a}\left(n+1,s-1,\bar{\pi}_n\right)\leq 0. \quad (105)
$$

Adding up (104) and (105), we get

$$
\begin{aligned}
& V_{i+1}^{a}\left(n-1,s+1,\pi_n\right)-V_{i+1}^{a}\left(n+1,s-1,\pi_n\right)-V_{i+1}^{a}\left(n-1,s+1,\bar{\pi}_n\right)+V_{i+1}^{a}\left(n+1,s-1,\bar{\pi}_n\right)\\
\leq\ & 0. \qquad (106)
\end{aligned}
$$

Therefore,

$$
H_{i+1}^{a}\left(n,s+1,\pi_n\right)-H_{i+1}^{a}\left(n+1,s,\pi_n\right)-H_{i+1}^{a}\left(n,s+1,\bar{\pi}_n\right)+H_{i+1}^{a}\left(n+1,s,\bar{\pi}_n\right)
$$

$$
\begin{aligned}
&= V_{i+1}^{a}(n-1,s+1,\pi_n)+r_n-\max\left[V_{i+1}^{a}(n,s,\pi_n)+r_n,V_{i+1}^{a}(n+1,s-1,\pi_n)+r_s\right] \\
&\quad +V_{i+1}^{a}(n+1,s-1,\bar{\pi}_n)+r_s-\max\left[V_{i+1}^{a}(n-1,s+1,\bar{\pi}_n)+r_n,V_{i+1}^{a}(n,s,\bar{\pi}_n)+r_s\right] \\
&\leq V_{i+1}^{a}(n-1,s+1,\pi_n)-V_{i+1}^{a}(n+1,s-1,\pi_n)-V_{i+1}^{a}(n-1,s+1,\bar{\pi}_n) \\
&\quad +V_{i+1}^{a}(n+1,s-1,\bar{\pi}_n)\leq 0.
\end{aligned}
\tag{107}
$$

4) If $V_{i+1}^{a}(n-1,s+1,\pi_n)+r_n \leq V_{i+1}^{a}(n,s,\pi_n)+r_s$ and $V_{i+1}^{a}(n,s,\bar{\pi}_n)+r_n \geq V_{i+1}^{a}(n+1,s-1,\bar{\pi}_n)+r_s$,

$$
\begin{aligned}
& H_{i+1}^{a}(n,s+1,\pi_n)-H_{i+1}^{a}(n+1,s,\pi_n)-H_{i+1}^{a}(n,s+1,\bar{\pi}_n)+H_{i+1}^{a}(n+1,s,\bar{\pi}_n) \\
&= V_{i+1}^{a}(n,s,\pi_n)+r_s+V_{i+1}^{a}(n,s,\bar{\pi}_n)+r_n \\
&\quad -\max\left[V_{i+1}^{a}(n,s,\pi_n)+r_n,V_{i+1}^{a}(n+1,s-1,\pi_n)+r_s\right] \\
&\quad -\max\left[V_{i+1}^{a}(n-1,s+1,\bar{\pi}_n)+r_n,V_{i+1}^{a}(n,s,\bar{\pi}_n)+r_s\right] \\
&\leq V_{i+1}^{a}(n,s,\pi_n)+V_{i+1}^{a}(n,s,\bar{\pi}_n)-V_{i+1}^{a}(n,s,\pi_n)-V_{i+1}^{a}(n,s,\bar{\pi}_n)=0.
\end{aligned}
\tag{108}
$$

Second, for the case of $n \geq 1$ and $s = 0$,

$$
\begin{aligned}
& H_{i+1}^{a}(n,1,\pi_n)-H_{i+1}^{a}(n+1,0,\pi_n)-H_{i+1}^{a}(n,1,\bar{\pi}_n)+H_{i+1}^{a}(n+1,0,\bar{\pi}_n) \\
&= \max\left[V_{i+1}^{a}(n-1,1,\pi_n)+r_n,V_{i+1}^{a}(n,0,\pi_n)+r_s\right]-V_{i+1}^{a}(n,0,\pi_n)-r_n \\
&\quad -\max\left[V_{i+1}^{a}(n-1,1,\bar{\pi}_n)+r_n,V_{i+1}^{a}(n,0,\bar{\pi}_n)+r_s\right]+V_{i+1}^{a}(n,0,\bar{\pi}_n)+r_n
\end{aligned}
\tag{109}
$$

1) If $V_{i+1}^{a}(n-1,1,\pi_n)+r_n \leq V_{i+1}^{a}(n,0,\pi_n)+r_s$,

$$
\begin{aligned}
& H_{i+1}^{a}(n,1,\pi_n)-H_{i+1}^{a}(n+1,0,\pi_n)-H_{i+1}^{a}(n,1,\bar{\pi}_n)+H_{i+1}^{a}(n+1,0,\bar{\pi}_n) \\
&\leq V_{i+1}^{a}(n,0,\pi_n)-V_{i+1}^{a}(n,0,\pi_n)-V_{i+1}^{a}(n,0,\bar{\pi}_n)+V_{i+1}^{a}(n,0,\bar{\pi}_n)=0.
\end{aligned}
\tag{110}
$$

2) If $V_{i+1}^{a}(n-1,1,\pi_n)+r_n \geq V_{i+1}^{a}(n,0,\pi_n)+r_s$, and, using the induction assumption

$$
\begin{aligned}
& H_{i+1}^{a}(n,1,\pi_n)-H_{i+1}^{a}(n+1,0,\pi_n)-H_{i+1}^{a}(n,1,\bar{\pi}_n)+H_{i+1}^{a}(n+1,0,\bar{\pi}_n) \\
&\leq V_{i+1}^{a}(n-1,1,\pi_n)-V_{i+1}^{a}(n,0,\pi_n)-V_{i+1}^{a}(n-1,1,\bar{\pi}_n)+V_{i+1}^{a}(n,0,\bar{\pi}_n) \\
&\leq 0.
\end{aligned}
\tag{111}
$$

Third, for the case of $n = 0$ and $s \geq 1$,

$$
\begin{aligned}
& H_{i+1}^{a}\left(0, s+1, \pi_{n}\right) - H_{i+1}^{a}\left(1, s, \pi_{n}\right) - H_{i+1}^{a}\left(0, s+1, \bar{\pi}_{n}\right) + H_{i+1}^{a}\left(1, s, \bar{\pi}_{n}\right) \\
= \; & V_{i+1}^{a}\left(0, s, \pi_{n}\right) + r_{s} - \max\left[V_{i+1}^{a}\left(0, s, \pi_{n}\right) + r_{n}, V_{i+1}^{a}\left(1, s-1, \pi_{n}\right) + r_{s}\right] \\
& -V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) - r_{s} + \max\left[V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) + r_{n}, V_{i+1}^{a}\left(1, s-1, \bar{\pi}_{n}\right) + r_{s}\right] \qquad (112)
\end{aligned}
$$

1) If $V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) + r_{n} \geq V_{i+1}^{a}\left(1, s-1, \bar{\pi}_{n}\right) + r_{s}$,

$$
\begin{aligned}
& H_{i+1}^{a}\left(0, s+1, \pi_{n}\right) - H_{i+1}^{a}\left(1, s, \pi_{n}\right) - H_{i+1}^{a}\left(0, s+1, \bar{\pi}_{n}\right) + H_{i+1}^{a}\left(1, s, \bar{\pi}_{n}\right) \\
= \; & V_{i+1}^{a}\left(0, s, \pi_{n}\right) + r_{s} - \max\left[V_{i+1}^{a}\left(0, s, \pi_{n}\right) + r_{n}, V_{i+1}^{a}\left(1, s-1, \pi_{n}\right) + r_{s}\right] \\
& -V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) - r_{s} + V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) + r_{n} \\
\leq \; & V_{i+1}^{a}\left(0, s, \pi_{n}\right) - V_{i+1}^{a}\left(0, s, \pi_{n}\right) - V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) + V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) = 0. \qquad (113)
\end{aligned}
$$

2) If $V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) + r_{n} \leq V_{i+1}^{a}\left(1, s-1, \bar{\pi}_{n}\right) + r_{s}$, and, using the induction assumption,

$$
\begin{aligned}
& H_{i+1}^{a}\left(0, s+1, \pi_{n}\right) - H_{i+1}^{a}\left(1, s, \pi_{n}\right) - H_{i+1}^{a}\left(0, s+1, \bar{\pi}_{n}\right) + H_{i+1}^{a}\left(1, s, \bar{\pi}_{n}\right) \\
= \; & V_{i+1}^{a}\left(0, s, \pi_{n}\right) + r_{s} - \max\left[V_{i+1}^{a}\left(0, s, \pi_{n}\right) + r_{n}, V_{i+1}^{a}\left(1, s-1, \pi_{n}\right) + r_{s}\right] \\
& -V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) - r_{s} + V_{i+1}^{a}\left(1, s-1, \bar{\pi}_{n}\right) + r_{s} \\
\leq \; & V_{i+1}^{a}\left(0, s, \pi_{n}\right) - V_{i+1}^{a}\left(1, s-1, \pi_{n}\right) - V_{i+1}^{a}\left(0, s, \bar{\pi}_{n}\right) + V_{i+1}^{a}\left(1, s-1, \bar{\pi}_{n}\right) \\
\leq \; & 0. \qquad (114)
\end{aligned}
$$

Fourth, for the case of $n = 0$ and $s = 0$,

$$
\begin{aligned}
& H_{i+1}^{a}\left(0, 1, \pi_{n}\right) - H_{i+1}^{a}\left(1, 0, \pi_{n}\right) - H_{i+1}^{a}\left(0, 1, \bar{\pi}_{n}\right) + H_{i+1}^{a}\left(1, 0, \bar{\pi}_{n}\right) \\
= \; & V_{i+1}^{a}\left(0, 0, \pi_{n}\right) + r_{s} - V_{i+1}^{a}\left(0, 0, \pi_{n}\right) - r_{n} - V_{i+1}^{a}\left(0, 0, \bar{\pi}_{n}\right) - r_{s} + V_{i+1}^{a}\left(0, 0, \bar{\pi}_{n}\right) + r_{n} \\
= \; & 0. \qquad (115)
\end{aligned}
$$

The monotonicity of the critical indices with respect to π_s is established in the similar fashion. As far as the monotonicity with respect to w_n is concerned, we use the same set-up for $w_n < \bar{w}_n$ to observe that at the initial step of induction (for $i = N + 1$),

$V_{N+1}^{a}(n,s+1,w_n) - V_{N+1}^{a}(n+1,s,w_n) - V_{N+1}^{a}(n,s+1,\bar{w}_n) + V_{N+1}^{a}(n+1,s,\bar{w}_n) = 0.$ Considering the main induction step, we have

$$\begin{aligned}
& [V_i^a(n,s+1,w_n) - V_i^a(n+1,s,w_n)] - [V_i^a(n,s+1,\bar{w}_n) - V_i^a(n+1,s,\bar{w}_n)] \\
= \; & (w_n - \bar{w}_n) \\
& + p_e p_n (V_{i+1}^a(n+1,s+1,w_n) - V_{i+1}^a(n+2,s,w_n) - V_{i+1}^a(n+1,s+1,\bar{w}_n) \\
& + V_{i+1}^a(n+2,s,\bar{w}_n)) \\
& + p_e(1-p_n)(V_{i+1}^a(n,s+1,w_n) - V_{i+1}^a(n+1,s,w_n) - V_{i+1}^a(n,s+1,\bar{w}_n) \\
& + V_{i+1}^a(n+1,s,\bar{w}_n)) \\
& + p_n(1-p_e)(H_{i+1}^a(n+1,s+1,w_n) - H_{i+1}^a(n+2,s,w_n) - H_{i+1}^a(n+1,s+1,\bar{w}_n) \\
& + H_{i+1}^a(n+2,s,\bar{w}_n)) \\
& + (1-p_e-p_n+p_e p_n)(H_{i+1}^a(n,s+1,w_n) - H_{i+1}^a(n+1,s,w_n) - H_{i+1}^a(n,s+1,\bar{w}_n) \\
& + H_{i+1}^a(n+1,s,\bar{w}_n))
\end{aligned} \tag{116}$$

It's clear that the first term $(w_n - \bar{w}_n) < 0$. The proof of non-positivity of other terms follows from the induction assumption in exactly the same way as in the proof of monotonicity with respect to π_n. Similarly, the monotonicity with respect to w_s is established.

Further, let us look at the monotonicity with respect to p_n. As above, we consider two different values for this probability $p_n < \overline{p}_n$ and define $V_i^a(n,s,p_n)$ and $V_i^a(n,s,\overline{p}_n)$ to be the optimal value functions for these respective values of the probability of a non-scheduled arrival. The proof of this statement is derived by induction over the service period index i. For $i = N+1$, $V_{N+1}^a(n,s+1,p_n) - V_{N+1}^a(n+1,s,p_n) - V_{N+1}^a(n,s+1,\overline{p}_n) + V_{N+1}^a(n+1,s,\overline{p}_n) = 0$. Assuming that for some service period index i, $V_{i+1}^a(n,s+1,p_n) - V_{i+1}^a(n+1,s,p_n) \leq V_{i+1}^a(n,s+1,\bar{p}_n) - V_{i+1}^a(n+1,s,\bar{p}_n)$, we will show that $V_i^a(n,s+1,p_n) - V_i^a(n+1,s,p_n) \leq V_i^a(n,s+1,\bar{p}_n) - V_i^a(n+1,s,\bar{p}_n)$. As before, we only provide the proof for the case of $a_i = 0$ since the case of $a_i = 1$ is analyzed in a similar manner. Now,

$$
\begin{aligned}
& \left[V_i^a(n, s+1, p_n) - V_i^a(n+1, s, p_n)\right] - \left[V_i^a(n, s+1, \bar{p}_n) - V_i^a(n+1, s, \bar{p}_n)\right] \\
= \; & p_e p_n \left(V_{i+1}^a(n+1, s+1, p_n) - V_{i+1}^a(n+2, s, p_n) - V_{i+1}^a(n+1, s+1, \bar{p}_n) + V_{i+1}^a(n+2, s, \bar{p}_n)\right) \\
& + p_e(\bar{p}_n - p_n)\left(-V_{i+1}^a(n+1, s+1, \bar{p}_n) + V_{i+1}^a(n+2, s, \bar{p}_n)\right) \\
& + p_e(1 - p_n)\left(V_{i+1}^a(n, s+1, p_n) - V_{i+1}^a(n+1, s, p_n) - V_{i+1}^a(n, s+1, \bar{p}_n) + V_{i+1}^a(n+1, s, \bar{p}_n)\right) \\
& - p_e(\bar{p}_n - p_n)\left(-V_{i+1}^a(n, s+1, \bar{p}_n) + V_{i+1}^a(n+1, s, \bar{p}_n)\right) \\
& + p_n(1 - p_e)\left[H_{i+1}^a(n+1, s+1, p_n) - H_{i+1}^a(n+2, s, p_n) - H_{i+1}^a(n+1, s+1, \bar{p}_n)\right. \\
& \left. + H_{i+1}^a(n+2, s, \bar{p}_n)\right] \\
& - (\bar{p}_n - p_n)(1 - p_e)\left[H_{i+1}^a(n+1, s+1, \bar{p}_n) - H_{i+1}^a(n+2, s, \bar{p}_n)\right] \\
& + (1 - p_e - p_n + p_e p_n)\left[H_{i+1}^a(n, s+1, p_n) - H_{i+1}^a(n+1, s, p_n) - H_{i+1}^a(n, s+1, \bar{p}_n)\right. \\
& \left. + H_{i+1}^a(n+1, s, \bar{p}_n)\right] \\
& - (\bar{p}_n - p_n)(1 - p_e)\left[H_{i+1}^a(n, s+1, \bar{p}_n) - H_{i+1}^a(n+1, s, \bar{p}_n)\right] \qquad (117)
\end{aligned}
$$

The non-positivity of the $p_e p_n$, $p_e(1 - p_n)$ terms above directly follows from the induction assumption. For the $p_n(1 - p_e)$ and the $(1 - p_e - p_n + p_e p_n)$ terms, the non-positivity is established in exactly the same way as in the proof of monotonicity with respect to w_n. Finally, the non-positivity of the $(\bar{p}_n - p_n)$ terms were established in the proof of Proposition 1.

The monotonicity with respect to p_s is established as follows. As before, we consider two different values for this probability $p_s < \bar{p}_s$ and define $V_i^a(n, s, p_s)$ and $V_i^a(n, s, \bar{p}_s)$ as the respective optimal value functions. For $i = N+1$, $V_{N+1}^a(n, s+1, p_s) - V_{N+1}^a(n+1, s, p_s) - V_{N+1}^a(n, s+1, \bar{p}_s) + V_{N+1}^a(n+1, s, \bar{p}_s) = 0$. Assuming that for some service period index i, $V_{i+1}^a(n, s+1, p_s) - V_{i+1}^a(n+1, s, p_s) \geq V_{i+1}^a(n, s+1, \bar{p}_s) - V_{i+1}^a(n+1, s, \bar{p}_s)$, we will show that $V_i^a(n, s+1, p_s) - V_i^a(n+1, s, p_s) \geq V_i^a(n, s+1, \bar{p}_s) - V_i^a(n+1, s, \bar{p}_s)$. Since we only need to consider the case of $a_i = 1$, we get

$$
\begin{aligned}
& \left[V_i^a(n, s+1, p_s) - V_i^a(n+1, s, p_s)\right] - \left[V_i^a(n, s+1, \bar{p}_s) - V_i^a(n+1, s, \bar{p}_s)\right] \\
= \; & p_e p_n (1-p_s) \begin{pmatrix} V_{i+1}^a(n+1, s+1, p_s) - V_{i+1}^a(n+2, s, p_s) \\ -V_{i+1}^a(n+1, s+1, \bar{p}_s) + V_{i+1}^a(n+2, s, \bar{p}_s) \end{pmatrix} \\
& + p_e p_n p_s \begin{pmatrix} V_{i+1}^a(n+1, s+2, p_s) - V_{i+1}^a(n+2, s+1, p_s) \\ -V_{i+1}^a(n+1, s+2, \bar{p}_s) + V_{i+1}^a(n+2, s+1, \bar{p}_s) \end{pmatrix} \\
& + p_e (1-p_n)(1-p_s) \begin{pmatrix} V_{i+1}^a(n, s+1, p_s) - V_{i+1}^a(n+1, s, p_s) \\ -V_{i+1}^a(n, s+1, \bar{p}_s) + V_{i+1}^a(n+1, s, \bar{p}_s) \end{pmatrix} \\
& + p_e (1-p_n) p_s \begin{pmatrix} V_{i+1}^a(n, s+2, p_s) - V_{i+1}^a(n+1, s+1, p_s) \\ -V_{i+1}^a(n, s+2, \bar{p}_s) + V_{i+1}^a(n+1, s+1, \bar{p}_s) \end{pmatrix} \\
& + p_n (1-p_e)(1-p_s) \begin{pmatrix} H_{i+1}^a(n+1, s+1, p_s) - H_{i+1}^a(n+2, s, p_s) \\ -H_{i+1}^a(n+1, s+1, \overline{p}_s) + H_{i+1}^a(n+2, s, \overline{p}_s) \end{pmatrix} \\
& + p_n (1-p_e) p_s \begin{pmatrix} H_{i+1}^a(n+1, s+2, p_s) - H_{i+1}^a(n+2, s+1, p_s) \\ -H_{i+1}^a(n+1, s+2, \overline{p}_s) + H_{i+1}^a(n+2, s+1, \overline{p}_s) \end{pmatrix} \\
& + (1-p_e-p_n+p_e p_n)(1-p_s) \begin{pmatrix} H_{i+1}^a(n, s+1, p_s) - H_{i+1}^a(n+1, s, p_s) \\ -H_{i+1}^a(n, s+1, \overline{p}_s) + H_{i+1}^a(n+1, s, \overline{p}_s) \end{pmatrix} \\
& + (1-p_e-p_n+p_e p_n) p_s \begin{pmatrix} H_{i+1}^a(n, s+2, p_s) - H_{i+1}^a(n+1, s+1, p_s) \\ -H_{i+1}^a(n, s+2, \overline{p}_s) + H_{i+1}^a(n+1, s+1, \overline{p}_s) \end{pmatrix} \\
& + (\overline{p}_s - p_s) p_e p_n \begin{pmatrix} V_{i+1}^a(n+1, s+1, \bar{p}_s) - V_{i+1}^a(n+2, s, \bar{p}_s) \\ -V_{i+1}^a(n+1, s+2, \bar{p}_s) + V_{i+1}^a(n+2, s+1, \bar{p}_s) \end{pmatrix} \\
& + (\overline{p}_s - p_s) p_e (1-p_n) \begin{pmatrix} V_{i+1}^a(n, s+1, \bar{p}_s) - V_{i+1}^a(n+1, s, \bar{p}_s) \\ -V_{i+1}^a(n, s+2, \bar{p}_s) + V_{i+1}^a(n+1, s+1, \bar{p}_s) \end{pmatrix} \\
& + (\overline{p}_s - p_s) p_n (1-p_e) \begin{pmatrix} H_{i+1}^a(n+1, s+1, \overline{p}_s) - H_{i+1}^a(n+2, s, \overline{p}_s) \\ -H_{i+1}^a(n+1, s+2, \overline{p}_s) + H_{i+1}^a(n+2, s+1, \overline{p}_s) \end{pmatrix}
\end{aligned}
$$

$$+ \left(\overline{p}_s - p_s\right)\left(1 - p_e - p_n + p_e p_n\right) \times \begin{pmatrix} H_{i+1}^a\left(n, s+1, \overline{p}_s\right) - H_{i+1}^a\left(n+1, s, \overline{p}_s\right) \\ -H_{i+1}^a\left(n, s+2, \overline{p}_s\right) + H_{i+1}^a\left(n+1, s+1, \overline{p}_s\right) \end{pmatrix}. \tag{118}$$

On the one hand, the non-negativity of the first 8 terms above follows from the induction assumption and is shown in exactly the same way as in the proof of the monotonicity with respect to w_n. On the other hand, the non-negativity of the last 4 terms with $(\overline{p}_s - p_s)$ follows from Proposition 1. Thus, $V_i^a(n, s+1, p_s) - V_i^a(n+1, s, p_s) \geq V_i^a(n, s+1, \bar{p}_s) - V_i^a(n+1, s, \bar{p}_s)$, and the monotonicity with respect to p_s is established.

Now, let us look at the monotonicity with respect to r_n. We consider two different values $r_n < \overline{r}_n$ and define $V_i^a(n, s, r_n)$ and $V_i^a(n, s, \overline{r}_n)$ to be the respective value functions. In order to prove the monotonicity, we need to show that $V_{i+1}^a(n, s+1, r_n) - V_{i+1}^a(n+1, s, r_n) + r_n \leq V_{i+1}^a(n, s+1, \overline{r}_n) - V_{i+1}^a(n+1, s, \overline{r}_n) + \overline{r}_n$ implies that $V_i^a(n, s+1, r_n) - V_i^a(n+1, s, r_n) + r_n \leq V_i^a(n, s+1, \overline{r}_n) - V_i^a(n+1, s, \overline{r}_n) + \overline{r}_n$. Then, for the main induction step, we get $(a_i = 0)$

$$\begin{aligned}
& \left[V_i^a(n, s+1, r_n) - V_i^a(n+1, s, r_n) + r_n\right] - \left[V_i^a(n, s+1, \overline{r}_n) - V_i^a(n+1, s, \overline{r}_n) + \overline{r}_n\right] \\
= \;& p_e p_n [V_{i+1}^a(n+1, s+1, r_n) - V_{i+1}^a(n+2, s, r_n) + r_n - V_{i+1}^a(n+1, s+1, \overline{r}_n) \\
& + V_{i+1}^a(n+2, s, \overline{r}_n) - \overline{r}_n] \\
& + p_e(1 - p_n)[V_{i+1}^a(n, s+1, r_n) - V_{i+1}^a(n+1, s, r_n) + r_n - V_{i+1}^a(n, s+1, \overline{r}_n) \\
& + V_{i+1}^a(n+1, s, \overline{r}_n) - \overline{r}_n] \\
& + p_n(1 - p_e)[H_{i+1}^a(n+1, s+1, r_n) - H_{i+1}^a(n+2, s, r_n) + r_n - H_{i+1}^a(n+1, s+1, \overline{r}_n) \\
& + H_{i+1}^a(n+2, s, \overline{r}_n) - \overline{r}_n] \\
& + (1 - p_e - p_n + p_e p_n)[H_{i+1}^a(n, s+1, r_n) - H_{i+1}^a(n+1, s, r_n) + r_n - H_{i+1}^a(n, s+1, \overline{r}_n) \\
& + H_{i+1}^a(n+1, s, \overline{r}_n) - \overline{r}_n].
\end{aligned} \tag{119}$$

The non-positivity of the p_e term follows from the induction assumption. Since the $p_n(1-p_e)$ and the $(1 - p_e - p_n + p_e p_n)$ terms share similar structure, we look at the

$(1-p_e-p_n+p_ep_n)$ term:

$$\begin{aligned}
& H_{i+1}^a(n+1,s+1,r_n)-H_{i+1}^a(n+2,s,r_n)+r_n \\
& -H_{i+1}^a(n+1,s+1,\overline{r}_n)+H_{i+1}^a(n+2,s,\overline{r}_n)-\overline{r}_n \\
= \; & r_n+\max\left[V_{i+1}^a(n,s+1,r_n)+r_n, V_{i+1}^a(n+1,s,r_n)+r_s\right] \\
& -\max\left[V_{i+1}^a(n+1,s,r_n)+r_n, V_{i+1}^a(n+2,s-1,r_n)+r_s\right] \\
& -\overline{r}_n-\max\left[V_{i+1}^a(n,s+1,\overline{r}_n)+\overline{r}_n, V_{i+1}^a(n+1,s,\overline{r}_n)+r_s\right] \\
& +\max\left[V_{i+1}^a(n+1,s,\overline{r}_n)+\overline{r}_n, V_{i+1}^a(n+2,s-1,\overline{r}_n)+r_s\right]. \qquad (120)
\end{aligned}$$

1) If $V_{i+1}^a(n,s+1,r_n)+r_n \geq V_{i+1}^a(n+1,s,r_n)+r_s$ and $V_{i+1}^a(n+1,s,\overline{r}_n)+\overline{r}_n \geq V_{i+1}^a(n+2,s-1,\overline{r}_n)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(n+1,s+1,r_n)-H_{i+1}^a(n+2,s,r_n)+r_n \\
& -H_{i+1}^a(n+1,s+1,\overline{r}_n)+H_{i+1}^a(n+2,s,\overline{r}_n)-\overline{r}_n \\
= \; & r_n+V_{i+1}^a(n,s+1,r_n)+r_n-\max\left[V_{i+1}^a(n+1,s,r_n)+r_n, V_{i+1}^a(n+2,s-1,r_n)+r_s\right] \\
& -\overline{r}_n-\max\left[V_{i+1}^a(n,s+1,\overline{r}_n)+\overline{r}_n, V_{i+1}^a(n+1,s,\overline{r}_n)+r_s\right]+V_{i+1}^a(n+1,s,\overline{r}_n)+\overline{r}_n \\
\leq \; & r_n+V_{i+1}^a(n,s+1,r_n)-V_{i+1}^a(n+1,s,r_n)-\overline{r}_n-V_{i+1}^a(n,s+1,\overline{r}_n)+V_{i+1}^a(n+1,s,\overline{r}_n) \\
\leq \; & 0. \qquad (121)
\end{aligned}$$

2) If $V_{i+1}^a(n,s+1,r_n)+r_n \leq V_{i+1}^a(n+1,s,r_n)+r_s$ and $V_{i+1}^a(n+1,s,\overline{r}_n)+\overline{r}_n \leq V_{i+1}^a(n+2,s-1,\overline{r}_n)+r_s$,

$$\begin{aligned}
& H_{i+1}^a(n+1,s+1,r_n)-H_{i+1}^a(n+2,s,r_n)+r_n \\
& -H_{i+1}^a(n+1,s+1,\overline{r}_n)+H_{i+1}^a(n+2,s,\overline{r}_n)-\overline{r}_n \\
= \; & r_n+V_{i+1}^a(n+1,s,r_n)+r_s-\max\left[V_{i+1}^a(n+1,s,r_n)+r_n, V_{i+1}^a(n+2,s-1,r_n)+r_s\right] \\
& -\overline{r}_n-\max\left[V_{i+1}^a(n,s+1,\overline{r}_n)+\overline{r}_n, V_{i+1}^a(n+1,s,\overline{r}_n)+r_s\right]+V_{i+1}^a(n+2,s-1,\overline{r}_n)+r_s \\
\leq \; & r_n+V_{i+1}^a(n+1,s,r_n)-V_{i+1}^a(n+2,s-1,r_n)-\overline{r}_n-V_{i+1}^a(n+1,s,\overline{r}_n) \\
& +V_{i+1}^a(n+2,s-1,\overline{r}_n) \leq 0. \qquad (122)
\end{aligned}$$

3) If $V_{i+1}^{a}(n,s+1,r_n)+r_n \geq V_{i+1}^{a}(n+1,s,r_n)+r_s$ and $V_{i+1}^{a}(n+1,s,\overline{r}_n)+\overline{r}_n \leq V_{i+1}^{a}(n+2,s-1,\overline{r}_n)+ r_s$,

$$\begin{aligned}
& H_{i+1}^{a}(n+1,s+1,r_n) - H_{i+1}^{a}(n+2,s,r_n) + r_n \\
& -H_{i+1}^{a}(n+1,s+1,\overline{r}_n) + H_{i+1}^{a}(n+2,s,\overline{r}_n) - \overline{r}_n \\
= \;& r_n + V_{i+1}^{a}(n,s+1,r_n) + r_n - \max\left[V_{i+1}^{a}(n+1,s,r_n) + r_n, V_{i+1}^{a}(n+2,s-1,r_n) + r_s\right] \\
& -\overline{r}_n - \max\left[V_{i+1}^{a}(n,s+1,\overline{r}_n) + \overline{r}_n, V_{i+1}^{a}(n+1,s,\overline{r}_n) + r_s\right] + V_{i+1}^{a}(n+2,s-1,\overline{r}_n) + r_s \\
\leq \;& r_n + V_{i+1}^{a}(n,s+1,r_n) - V_{i+1}^{a}(n+1,s,r_n) - \overline{r}_n - V_{i+1}^{a}(n+1,s,\overline{r}_n) + V_{i+1}^{a}(n+2,s-1,\overline{r}_n) \\
\leq \;& r_n + V_{i+1}^{a}(n+1,s,r_n) - V_{i+1}^{a}(n+2,s-1,r_n) - \overline{r}_n - V_{i+1}^{a}(n+1,s,\overline{r}_n) \\
& +V_{i+1}^{a}(n+2,s-1,\overline{r}_n) \leq 0, \qquad (123)
\end{aligned}$$

where we have used the result of Proposition 1.

4) If $V_{i+1}^{a}(n,s+1,r_n)+r_n \leq V_{i+1}^{a}(n+1,s,r_n)+r_s$ and $V_{i+1}^{a}(n+1,s,\overline{r}_n)+\overline{r}_n \geq V_{i+1}^{a}(n+2,s-1,\overline{r}_n)+ r_s$,

$$\begin{aligned}
& H_{i+1}^{a}(n+1,s+1,r_n) - H_{i+1}^{a}(n+2,s,r_n) + r_n \\
& -H_{i+1}^{a}(n+1,s+1,\overline{r}_n) + H_{i+1}^{a}(n+2,s,\overline{r}_n) - \overline{r}_n \\
= \;& r_n + V_{i+1}^{a}(n+1,s,r_n) + r_s - \max\left[V_{i+1}^{a}(n+1,s,r_n) + r_n, V_{i+1}^{a}(n+2,s-1,r_n) + r_s\right] \\
& -\overline{r}_n - \max\left[V_{i+1}^{a}(n,s+1,\overline{r}_n) + \overline{r}_n, V_{i+1}^{a}(n+1,s,\overline{r}_n) + r_s\right] + V_{i+1}^{a}(n+1,s,\overline{r}_n) + \overline{r}_n \\
\leq \;& r_n + V_{i+1}^{a}(n+1,s,r_n) + r_s - V_{i+1}^{a}(n+1,s,r_n) - r_n - \overline{r}_n - V_{i+1}^{a}(n+1,s,\overline{r}_n) - r_s \\
& +V_{i+1}^{a}(n+1,s,\overline{r}_n) + \overline{r}_n = 0. \qquad (124)
\end{aligned}$$

Finally, the monotonicity with respect to r_s is established in exactly the same way as the monotonicity with respect to r_n.

Proof of Proposition 3

We first need the following definitions. Let $G_{j,i},\ j=1,2,3$ be the classes of functions defined on S_i such that for every $g(n,s) \in G_{1,i}$,

$$g(n-1,s) - g(n,s-1) \leq r_s - r_n, \tag{125}$$

and for every $g(n,s) \in G_{2,i}$,

$$g(n-1,s) - g(n,s-1) \geq r_s - r_n, \tag{126}$$

while for every $g(n,s) \in G_{3,i}$,

$$g(n-1,s) - g(n,s-1) = g(n-1,s+1) - g(n,s), \tag{127}$$

for every $(n,s) \in S_i$, $n \geq 1, s \geq 1, i = 1, ..., N+1$. We observe that if the optimal value function of (1), $V_i^a(n,s)$, belongs to class $G_{1,i}$, then it implies that it is always optimal to serve outpatients at the i-th service slot irrespective of the state the system. Similarly, if $V_i^a(n,s) \in G_{2,i}$, then inpatients will have service priority at the i-th service slot. Finally, $V_i^a(n,s) \in G_{3,i}$ implies that the decision on which patient type to serve at the i-th service depends only on the number of inpatients waiting for service and not on the number of waiting outpatients.

The proof of part a) is derived using induction over the service period index i. Using (3), for $a_N = 0$ (similar result is obtained for $a_N = 1$), we get

$$\begin{aligned}
& V_N^a(n,s+1) - V_N^a(n+1,s) - V_{N-1}^a(n,s+1) + V_{N-1}^a(n+1,s) \\
= \; & [-(s+1)(w_s+\pi_s) - n(w_n+\pi_n) - p_n\pi_n] \\
& - [-s(w_s+\pi_s) - (n+1)(w_n+\pi_n) - p_n\pi_n] \\
& - \begin{bmatrix} -(s+1)w_s - nw_n + p_e p_n V_N^a(n+1,s+1) + p_e(1-p_n)V_N^a(n,s+1) \\ +p_n(1-p_e)H_N^a(n+1,s+1) + (1-p_e-p_n+p_e p_n)H_N^a(n,s+1) \end{bmatrix} \\
& + \begin{bmatrix} -sw_s - (n+1)w_n + p_e p_n V_N^a(n+2,s) + p_e(1-p_n)V_N^a(n+1,s) \\ +p_n(1-p_e)H_N^a(n+2,s) + (1-p_e-p_n+p_e p_n)H_N^a(n+1,s) \end{bmatrix} \\
= \; & \pi_n - \pi_s - p_e p_n [V_N^a(n+1,s+1) - V_N^a(n+2,s)] - p_e(1-p_n)[V_N^a(n,s+1) - V_N^a(n+1,s)]
\end{aligned}$$

$$
\begin{aligned}
& -p_n (1-p_e) \left[H_N^a (n+1, s+1) - H_N^a (n+2, s)\right] \\
& -(1-p_e-p_n+p_e p_n) \left[H_N^a (n, s+1) - H_N^a (n+1, s)\right] \\
= \; & \pi_n - \pi_s - p_e p_n \left[V_N^a (n+1, s+1) - V_N^a (n+2, s)\right] - p_e (1-p_n) \left[V_N^a (n, s+1) - V_N^a (n+1, s)\right] \\
& -p_n (1-p_e) \left[V_N^a (n, s+1) - V_N^a (n+1, s)\right] \\
& -(1-p_e-p_n+p_e p_n) \left[V_N^a (n-1, s+1) - V_N^a (n, s)\right] \\
= \; & (\pi_n - \pi_s) - p_e p_n (w_n + \pi_n - w_s - \pi_s) - p_e (1-p_n) (w_n + \pi_n - w_s - \pi_s) \\
& -p_n (1-p_e) (w_n + \pi_n - w_s - \pi_s) - (1-p_e-p_n+p_e p_n) (w_n + \pi_n - w_s - \pi_s) \\
= \; & w_s - w_n \geq 0. \qquad (128)
\end{aligned}
$$

For the case $n = 0$, $s = 0$,

$$
\begin{aligned}
& V_N^a (0,1) - V_N^a (1,0) - V_{N-1}^a (0,1) + V_{N-1}^a (1,0) \\
= \; & \pi_n - \pi_s - p_e p_n \left[V_N^a (0,1) - V_N^a (1,0)\right] - p_e (1-p_n) \left[V_N^a (0,1) - V_N^a (1,0)\right] \\
& -p_n (1-p_e) \left[H_N^a (1,1) - H_N^a (2,0)\right] - (1-p_e-p_n+p_e p_n) \left[H_N^a (0,1) - H_N^a (1,0)\right] \\
= \; & \pi_n - \pi_s - p_e p_n (\pi_n + w_n - \pi_s - w_s) - p_e (1-p_n) (\pi_n + w_n - \pi_s - w_s) \\
& -p_n (1-p_e) (\pi_n + w_n - \pi_s - w_s) - (1-p_e-p_n+p_e p_n) (r_s - r_n) \\
= \; & (1-p_e-p_n+p_e p_n) (\pi_n + r_n - \pi_s - r_s) + (p_e + p_n - p_e p_n) (w_s - w_n) \\
\geq \; & 0. \qquad (129)
\end{aligned}
$$

For the case $n \geq 1$, $s = 0$,

$$
\begin{aligned}
& V_N^a (n,1) - V_N^a (n+1,0) - V_{N-1}^a (n,1) + V_{N-1}^a (n+1,0) \\
= \; & \pi_n - \pi_s - p_e p_n \left[V_N^a (n+1,1) - V_N^a (n+2,0)\right] - p_e (1-p_n) \left[V_N^a (n,1) - V_N^a (n+1,0)\right] \\
& -p_n (1-p_e) \left[H_N^a (n+1,1) - H_N^a (n+2,0)\right] - (1-p_e-p_n+p_e p_n) \left[H_N^a (n,1) - H_N^a (n+1,0)\right] \\
= \; & w_s - w_n \geq 0. \qquad (130)
\end{aligned}
$$

For the case $n = 0$, $s \geq 1$,

$$
V_N^a (0, s+1) - V_N^a (1, s) - V_{N-1}^a (0, s+1) + V_{N-1}^a (1, s)
$$

$$
\begin{aligned}
&= \pi_n - \pi_s - p_e p_n \left[V_N^a(1, s+1) - V_N^a(2, s)\right] - p_e(1-p_n)\left[V_N^a(0, s+1) - V_N^a(1, s)\right] \\
&\quad - p_n(1-p_e)\left[H_N^a(1, s+1) - H_N^a(2, s)\right] - (1-p_e-p_n+p_e p_n)\left[H_N^a(0, s+1) - H_N^a(1, s)\right] \\
&= (1-p_e-p_n+p_e p_n)(\pi_n + r_n - \pi_s - r_s) + (p_e+p_n-p_e p_n)(w_s - w_n) \geq 0. \qquad (131)
\end{aligned}
$$

Suppose that, for some i and any $(n, s) \in S$, $V_{i+2}^a(n, s+1) - V_{i+2}^a(n+1, s) - V_{i+1}^a(n, s+1) + V_{i+1}^a(n+1, s) \geq 0$. Below we will show that $V_{i+1}^a(n, s+1) - V_{i+1}^a(n+1, s) - V_i^a(n, s+1) + V_i^a(n+1, s) \geq 0$.

We look at 4 cases: 1) $(n \geq 1, s \geq 1)$, 2) $(n \geq 1, s = 0)$, 3) $(n = 0, s \geq 1)$, 4) $(n = 0, s = 0)$. For each of those cases, we consider 2 scenarios: $a_{i+1} = a_{i+2}$ and $a_{i+1} = 1$, $a_{i+2} = 0$. In addition, the proofs for the scenarios $a_{i+1} = a_{i+2} = 0$ and $a_{i+1} = a_{i+2} = 1$ are similar, so below we only provide the proof for $a_{i+1} = a_{i+2} = 0$. For this scenario, for the case of $n \geq 1$ and $s \geq 1$,

$$
\begin{aligned}
&V_{i+1}^a(n, s+1) - V_{i+1}^a(n+1, s) - V_i^a(n, s+1) + V_i^a(n+1, s) \\
= \; & p_e p_n \left[V_{i+2}^a(n+1, s+1) - V_{i+2}^a(n+2, s) - V_{i+1}^a(n+1, s+1) + V_{i+1}^a(n+2, s)\right] \\
&+ p_e(1-p_n)\left[V_{i+2}^a(n, s+1) - V_{i+2}^a(n+1, s) - V_{i+1}^a(n, s+1) + V_{i+1}^a(n+1, s)\right] \\
&+ p_n(1-p_e)\left[H_{i+2}^a(n+1, s+1) - H_{i+2}^a(n+2, s) - H_{i+1}^a(n+1, s+1) + H_{i+1}^a(n+2, s)\right] \\
&+ (1-p_e-p_n+p_e p_n) \\
&\times \left[H_{i+2}^a(n, s+1) - H_{i+2}^a(n+1, s) - H_{i+1}^a(n, s+1) + H_{i+1}^a(n+1, s)\right]. \qquad (132)
\end{aligned}
$$

The non-negativity of the p_e terms can be directly derived from the induction assumption. As the $p_n(1-p_e)$ and the $(1-p_e-p_n+p_e p_n)$ terms share a similar structure, we only look at the $p_n(1-p_e)$ term:

$$
\begin{aligned}
&H_{i+2}^a(n+1, s+1) - H_{i+2}^a(n+2, s) - H_{i+1}^a(n+1, s+1) + H_{i+1}^a(n+2, s) \\
= \; & \max\left[V_{i+2}^a(n, s+1) + r_n, V_{i+2}^a(n+1, s) + r_s\right] \\
&- \max\left[V_{i+2}^a(n+1, s) + r_n, V_{i+2}^a(n+2, s-1) + r_s\right]
\end{aligned}
$$

$$-\max\left[V_{i+1}^{a}\left(n,s+1\right)+r_{n},V_{i+1}^{a}\left(n+1,s\right)+r_{s}\right]$$

$$+\max\left[V_{i+1}^{a}\left(n+1,s\right)+r_{n},V_{i+1}^{a}\left(n+2,s-1\right)+r_{s}\right]. \quad (133)$$

1) If $V_{i+2}^{a}\left(n+1,s\right)+r_{n}\geq V_{i+2}^{a}\left(n+2,s-1\right)+r_{s}$ and $V_{i+1}^{a}\left(n,s+1\right)+r_{n}\geq V_{i+1}^{a}\left(n+1,s\right)+r_{s}$,

$$\begin{aligned}
& H_{i+2}^{a}\left(n+1,s+1\right)-H_{i+2}^{a}\left(n+2,s\right)-H_{i+1}^{a}\left(n+1,s+1\right)+H_{i+1}^{a}\left(n+2,s\right)\\
= & \max\left[V_{i+2}^{a}\left(n,s+1\right)+r_{n},V_{i+2}^{a}\left(n+1,s\right)+r_{s}\right]\\
& -V_{i+2}^{a}\left(n+1,s\right)-r_{n}-V_{i+1}^{a}\left(n,s+1\right)-r_{n}\\
& +\max\left[V_{i+1}^{a}\left(n+1,s\right)+r_{n},V_{i+1}^{a}\left(n+2,s-1\right)+r_{s}\right]\\
\geq & V_{i+2}^{a}\left(n,s+1\right)-V_{i+2}^{a}\left(n+1,s\right)-V_{i+1}^{a}\left(n,s+1\right)+V_{i+1}^{a}\left(n+1,s\right)\\
\geq & 0.
\end{aligned} \quad (134)$$

2) If $V_{i+2}^{a}\left(n+1,s\right)+r_{n}\geq V_{i+2}^{a}\left(n+2,s-1\right)+r_{s}$ and $V_{i+1}^{a}\left(n,s+1\right)+r_{n}\leq V_{i+1}^{a}\left(n+1,s\right)+r_{s}$,

$$\begin{aligned}
& H_{i+2}^{a}\left(n+1,s+1\right)-H_{i+2}^{a}\left(n+2,s\right)-H_{i+1}^{a}\left(n+1,s+1\right)+H_{i+1}^{a}\left(n+2,s\right)\\
= & \max\left[V_{i+2}^{a}\left(n,s+1\right)+r_{n},V_{i+2}^{a}\left(n+1,s\right)+r_{s}\right]\\
& -V_{i+2}^{a}\left(n+1,s\right)-r_{n}-V_{i+1}^{a}\left(n+1,s\right)-r_{s}\\
& +\max\left[V_{i+1}^{a}\left(n+1,s\right)+r_{n},V_{i+1}^{a}\left(n+2,s-1\right)+r_{s}\right]\\
\geq & V_{i+2}^{a}\left(n+1,s\right)-V_{i+1}^{a}\left(n+1,s\right)-V_{i+2}^{a}\left(n+1,s\right)-V_{i+1}^{a}\left(n+1,s\right)=0.
\end{aligned} \quad (135)$$

3) If $V_{i+2}^{a}\left(n+1,s\right)+r_{n}\leq V_{i+2}^{a}\left(n+2,s-1\right)+r_{s}$ and $V_{i+1}^{a}\left(n,s+1\right)+r_{n}\geq V_{i+1}^{a}\left(n+1,s\right)+r_{s}$,

$$\begin{aligned}
& H_{i+2}^{a}\left(n+1,s+1\right)-H_{i+2}^{a}\left(n+2,s\right)-H_{i+1}^{a}\left(n+1,s+1\right)+H_{i+1}^{a}\left(n+2,s\right)\\
= & \max\left[V_{i+2}^{a}\left(n,s+1\right)+r_{n},V_{i+2}^{a}\left(n+1,s\right)+r_{s}\right]\\
& -V_{i+2}^{a}\left(n+2,s-1\right)-r_{s}-V_{i+1}^{a}\left(n,s+1\right)-r_{n}
\end{aligned}$$

$$
\begin{aligned}
& +\max\left[V_{i+1}^{a}\left(n+1,s\right)+r_{n},V_{i+1}^{a}\left(n+2,s-1\right)+r_{s}\right] \\
\geq\ & V_{i+2}^{a}\left(n,s+1\right)-V_{i+2}^{a}\left(n+2,s-1\right)-V_{i+1}^{a}\left(n,s+1\right)+V_{i+1}^{a}\left(n+2,s-1\right) \\
\geq\ & 0. \qquad (136)
\end{aligned}
$$

4) If $V_{i+2}^{a}\left(n+1,s\right)+r_{n}\leq V_{i+2}^{a}\left(n+2,s-1\right)+r_{s}$ and $V_{i+1}^{a}\left(n,s+1\right)+r_{n}\leq V_{i+1}^{a}\left(n+1,s\right)+r_{s}$,

$$
\begin{aligned}
& H_{i+2}^{a}\left(n+1,s+1\right)-H_{i+2}^{a}\left(n+2,s\right)-H_{i+1}^{a}\left(n+1,s+1\right)+H_{i+1}^{a}\left(n+2,s\right) \\
=\ & \max\left[V_{i+2}^{a}\left(n,s+1\right)+r_{n},V_{i+2}^{a}\left(n+1,s\right)+r_{s}\right] \\
& -V_{i+2}^{a}\left(n+2,s-1\right)-r_{s}-V_{i+1}^{a}\left(n+1,s\right)-r_{s} \\
& +\max\left[V_{i+1}^{a}\left(n+1,s\right)+r_{n},V_{i+1}^{a}\left(n+2,s-1\right)+r_{s}\right] \\
\geq\ & V_{i+2}^{a}\left(n+1,s\right)-V_{i+2}^{a}\left(n+2,s-1\right)-V_{i+1}^{a}\left(n+1,s\right)+V_{i+1}^{a}\left(n+2,s-1\right) \\
\geq\ & 0. \qquad (137)
\end{aligned}
$$

Second, for the case of $n\geq 1$ and $s=0$,

$$
\begin{aligned}
& V_{i+1}^{a}\left(n,1\right)-V_{i+1}^{a}\left(n+1,0\right)-V_{i}^{a}\left(n,1\right)+V_{i}^{a}\left(n+1,0\right) \\
=\ & p_{e}p_{n}\left[V_{i+2}^{a}\left(n+1,1\right)-V_{i+2}^{a}\left(n+2,0\right)-V_{i+1}^{a}\left(n+1,1\right)+V_{i+1}^{a}\left(n+2,0\right)\right] \\
& +p_{e}\left(1-p_{n}\right)\left[V_{i+2}^{a}\left(n,1\right)-V_{i+2}^{a}\left(n+1,0\right)-V_{i+1}^{a}\left(n,1\right)+V_{i+1}^{a}\left(n+1,0\right)\right] \\
& +p_{n}\left(1-p_{e}\right)\left[H_{i+2}^{a}\left(n+1,1\right)-H_{i+2}^{a}\left(n+2,0\right)-H_{i+1}^{a}\left(n+1,1\right)+H_{i+1}^{a}\left(n+2,0\right)\right] \\
& +\left(1-p_{e}-p_{n}+p_{e}p_{n}\right) \\
& \times\left[H_{i+2}^{a}\left(n,1\right)-H_{i+2}^{a}\left(n+1,0\right)-H_{i+1}^{a}\left(n,1\right)+H_{i+1}^{a}\left(n+1,0\right)\right] \qquad (138)
\end{aligned}
$$

The non-negativity of the p_e terms follows directly from the induction assumption. Again, because of the similar structure of the $p_{n}\left(1-p_{e}\right)$ and the $\left(1-p_{e}-p_{n}+p_{e}p_{n}\right)$ terms, we only consider the $\left(1-p_{e}-p_{n}+p_{e}p_{n}\right)$ term:

$$
H_{i+2}^{a}\left(n,1\right)-H_{i+2}^{a}\left(n+1,0\right)-H_{i+1}^{a}\left(n,1\right)+H_{i+1}^{a}\left(n+1,0\right)
$$

$$= \max[V^a_{i+2}(n-1,1)+r_n, V^a_{i+2}(n,0)+r_s] - V^a_{i+2}(n,0) - r_n$$

$$- \max[V^a_{i+1}(n-1,1)+r_n, V^a_{i+1}(n,0)+r_s] + V^a_{i+1}(n,0) + r_n \quad (139)$$

1) If $V^a_{i+1}(n-1,1)+r_n \geq V^a_{i+1}(n,0)+r_s$,

$$H^a_{i+2}(n,1) - H^a_{i+2}(n+1,0) - H^a_{i+1}(n,1) + H^a_{i+1}(n+1,0)$$

$$= \max[V^a_{i+2}(n-1,1)+r_n, V^a_{i+2}(n,0)+r_s] - V^a_{i+2}(n,0) - V^a_{i+1}(n-1,1) - r_n + V^a_{i+1}(n,0)$$

$$\geq V^a_{i+2}(n-1,1) - V^a_{i+2}(n,0) - V^a_{i+1}(n-1,1) + V^a_{i+1}(n,0) \geq 0. \quad (140)$$

2) If $V^a_{i+1}(n-1,1)+r_n \leq V^a_{i+1}(n,0)+r_s$,

$$H^a_{i+2}(n,1) - H^a_{i+2}(n+1,0) - H^a_{i+1}(n,1) + H^a_{i+1}(n+1,0)$$

$$= \max[V^a_{i+2}(n-1,1)+r_n, V^a_{i+2}(n,0)+r_s] - V^a_{i+2}(n,0) - V^a_{i+1}(n,0) - r_s + V^a_{i+1}(n,0)$$

$$\geq V^a_{i+2}(n,0) - V^a_{i+2}(n,0) - V^a_{i+1}(n,0) + V^a_{i+1}(n,0) = 0. \quad (141)$$

Third, for the case of $n = 0$ and $s \geq 1$,

$$V^a_{i+1}(0,s+1) - V^a_{i+1}(1,s) - V^a_i(0,s+1) + V^a_i(1,s)$$

$$= p_e p_n \left[V^a_{i+2}(1,s+1) - V^a_{i+2}(2,s) - V^a_{i+1}(1,s+1) + V^a_{i+1}(2,s)\right]$$

$$+ p_e(1-p_n)\left[V^a_{i+2}(0,s+1) - V^a_{i+2}(1,s) - V^a_{i+1}(0,s+1) + V^a_{i+1}(1,s)\right]$$

$$+ p_n(1-p_e)\left[H^a_{i+2}(1,s+1) - H^a_{i+2}(2,s) - H^a_{i+1}(1,s+1) + H^a_{i+1}(2,s)\right]$$

$$+ (1-p_e-p_n+p_e p_n)$$

$$\times \left[H^a_{i+2}(0,s+1) - H^a_{i+2}(1,s) - H^a_{i+1}(0,s+1) + H^a_{i+1}(1,s)\right]. \quad (142)$$

The non-negativity of the p_e terms can be directly derived from the induction assumption. The non-negativity of the $p_n(1-p_e)$ term can be shown in exactly the same way as in the case of $n \geq 1$, $s \geq 1$. For the $(1-p_e-p_n+p_e p_n)$ term we get:

$$H^a_{i+2}(0,s+1) - H^a_{i+2}(1,s) - H^a_{i+1}(0,s+1) + H^a_{i+1}(1,s)$$

$$= V^a_{i+2}(0,s) + r_s - \max[V^a_{i+2}(0,s)+r_n, V^a_{i+2}(1,s-1)+r_s]$$

$$- V^a_{i+1}(0,s) - r_s + \max[V^a_{i+1}(0,s)+r_n, V^a_{i+1}(1,s-1)+r_s]. \quad (143)$$

1) If $V_{i+2}^{a}(0,s)+r_n \geq V_{i+2}^{a}(1,s-1)+r_s$,

$$\begin{aligned} & H_{i+2}^{a}(0,s+1)-H_{i+2}^{a}(1,s)-H_{i+1}^{a}(0,s+1)+H_{i+1}^{a}(1,s) \\ = \ & V_{i+2}^{a}(0,s)-V_{i+2}^{a}(0,s)-r_n-V_{i+1}^{a}(0,s)+\max[V_{i+1}^{a}(0,s)+r_n, V_{i+1}^{a}(1,s-1)+r_s] \\ \geq \ & V_{i+2}^{a}(0,s)-V_{i+2}^{a}(0,s)-r_n-V_{i+1}^{a}(0,s)+V_{i+1}^{a}(0,s)+r_n=0. \qquad (144) \end{aligned}$$

2) If $V_{i+2}^{a}(0,s)+r_n \leq V_{i+2}^{a}(1,s-1)+r_s$,

$$\begin{aligned} & H_{i+2}^{a}(0,s+1)-H_{i+2}^{a}(1,s)-H_{i+1}^{a}(0,s+1)+H_{i+1}^{a}(1,s) \\ = \ & V_{i+2}^{a}(0,s)-V_{i+2}^{a}(1,s-1)-r_s-V_{i+1}^{a}(0,s)+\max[V_{i+1}^{a}(0,s)+r_n, V_{i+1}^{a}(1,s-1)+r_s] \\ \geq \ & V_{i+2}^{a}(0,s)-V_{i+2}^{a}(1,s-1)-r_s-V_{i+1}^{a}(0,s)+V_{i+1}^{a}(1,s-1)+r_s \geq 0. \qquad (145) \end{aligned}$$

Fourth, for the case of $n=0$ and $s=0$,

$$\begin{aligned} & V_{i+1}^{a}(0,1)-V_{i+1}^{a}(1,0)-V_{i}^{a}(0,1)+V_{i}^{a}(1,0) \\ = \ & p_e p_n \left[V_{i+2}^{a}(1,1)-V_{i+2}^{a}(2,0)-V_{i+1}^{a}(1,1)+V_{i+1}^{a}(2,0)\right] \\ & +p_e(1-p_n)\left[V_{i+2}^{a}(0,1)-V_{i+2}^{a}(1,0)-V_{i+1}^{a}(0,1)+V_{i+1}^{a}(1,0)\right] \\ & +p_n(1-p_e)\left[H_{i+2}^{a}(1,1)-H_{i+2}^{a}(2,0)-H_{i+1}^{a}(1,1)+H_{i+1}^{a}(2,0)\right] \\ & +(1-p_e-p_n+p_e p_n) \\ & \times\left[H_{i+2}^{a}(0,1)-H_{i+2}^{a}(1,0)-H_{i+1}^{a}(0,1)+H_{i+1}^{a}(1,0)\right]. \qquad (146) \end{aligned}$$

The non-negativity of p_e terms is obvious. As the $p_n(1-p_e)$ and the $(1-p_e-p_n+p_e p_n)$ terms share a similar structure, we look at the $p_n(1-p_e)$ term:

$$\begin{aligned} & H_{i+2}^{a}(1,1)-H_{i+2}^{a}(2,0)-H_{i+1}^{a}(1,1)+H_{i+1}^{a}(2,0) \\ = \ & \max[V_{i+2}^{a}(0,1)+r_n, V_{i+2}^{a}(1,0)+r_s]-V_{i+2}^{a}(1,0)-r_n \\ & -\max[V_{i+1}^{a}(0,1)+r_n, V_{i+1}^{a}(1,0)+r_s]+V_{i+1}^{a}(1,0)+r_n. \qquad (147) \end{aligned}$$

1) If $V_{i+1}^{a}(0,1)+r_n \geq V_{i+1}^{a}(1,0)+r_s$,

$$H_{i+2}^{a}(1,1)-H_{i+2}^{a}(2,0)-H_{i+1}^{a}(1,1)+H_{i+1}^{a}(2,0)$$

$$
\begin{aligned}
&= \max[V_{i+2}^{a}(0,1)+r_n, V_{i+2}^{a}(1,0)+r_s] - V_{i+2}^{a}(1,0) - V_{i+1}^{a}(0,1) - r_n + V_{i+1}^{a}(1,0) \\
&\geq V_{i+2}^{a}(0,1) - V_{i+2}^{a}(1,0) - V_{i+1}^{a}(0,1) + V_{i+1}^{a}(1,0) \geq 0. \qquad (148)
\end{aligned}
$$

2) If $V_{i+1}^{a}(0,1)+r_n \leq V_{i+1}^{a}(1,0)+r_s$,

$$
\begin{aligned}
& H_{i+2}^{a}(1,1) - H_{i+2}^{a}(2,0) - H_{i+1}^{a}(1,1) + H_{i+1}^{a}(2,0) \\
&= \max[V_{i+2}^{a}(0,1)+r_n, V_{i+2}^{a}(1,0)+r_s] - V_{i+2}^{a}(1,0) - V_{i+1}^{a}(1,0) - r_s + V_{i+1}^{a}(1,0) \\
&\geq V_{i+2}^{a}(1,0) - V_{i+2}^{a}(1,0) - V_{i+1}^{a}(1,0) + V_{i+1}^{a}(1,0) = 0. \qquad (149)
\end{aligned}
$$

Now we turn to the scenario for which $a_{i+1}=1$, $a_{i+2}=0$. For $n \geq 1$, $s \geq 1$, the $p_e(1-p_n)$ term in $V_{i+1}^{a}(n,s+1) - V_{i+1}^{a}(n+1,s) - V_{i}^{a}(n,s+1) + V_{i}^{a}(n+1,s)$ looks like

$$
\begin{aligned}
& \left[(1-p_s a_{i+2}) V_{i+2}^{a}(n,s+1) + p_s a_{i+2} V_{i+2}^{a}(n,s+2)\right] \\
& - \left[(1-p_s a_{i+2}) V_{i+2}^{a}(n+1,s) + p_s a_{i+2} V_{i+2}^{a}(n+1,s+1)\right] \\
& - \left[(1-p_s a_{i+1}) V_{i+1}^{a}(n,s+1) + p_s a_{i+1} V_{i+1}^{a}(n,s+2)\right] \\
& + \left[(1-p_s a_{i+1}) V_{i+1}^{a}(n+1,s) + p_s a_{i+1} V_{i+1}^{a}(n+1,s+1)\right] \\
&= V_{i+2}^{a}(n,s+1) - V_{i+2}^{a}(n+1,s) - \left[(1-p_s) V_{i+1}^{a}(n,s+1) + p_s V_{i+1}^{a}(n,s+2)\right] \\
& + \left[(1-p_s) V_{i+1}^{a}(n+1,s) + p_s V_{i+1}^{a}(n+1,s+1)\right] \\
&= V_{i+2}^{a}(n,s+1) - V_{i+2}^{a}(n+1,s) - V_{i+1}^{a}(n,s+1) + V_{i+1}^{a}(n+1,s) \\
& + p_s \left[V_{i+1}^{a}(n,s+1) - V_{i+1}^{a}(n,s+2) - V_{i+1}^{a}(n+1,s) + V_{i+1}^{a}(n+1,s+1)\right] \\
&\geq 0. \qquad (150)
\end{aligned}
$$

We can get same result for $p_e p_n$ term. For the $(1-p_e-p_n+p_e p_n)$ term we get

$$
\begin{aligned}
& \left[(1-p_s a_{i+2}) H_{i+2}^{a}(n,s+1) + p_s a_{i+2} H_{i+2}^{a}(n,s+2)\right] \\
& - \left[(1-p_s a_{i+2}) H_{i+2}^{a}(n+1,s) + p_s a_{i+2} H_{i+2}^{a}(n+1,s+1)\right] \\
& - \left[(1-p_s a_{i+1}) H_{i+1}^{a}(n,s+1) + p_s a_{i+1} H_{i+1}^{a}(n,s+2)\right] \\
& + \left[(1-p_s a_{i+1}) H_{i+1}^{a}(n+1,s) + p_s a_{i+1} H_{i+1}^{a}(n+1,s+1)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= H_{i+2}^{a}(n,s+1) - H_{i+2}^{a}(n+1,s) - H_{i+1}^{a}(n,s+1) + H_{i+1}^{a}(n+1,s) \\
&\quad + p_s\left[H_{i+1}^{a}(n,s+1) - H_{i+1}^{a}(n,s+2) - H_{i+1}^{a}(n+1,s) + H_{i+1}^{a}(n+1,s+1)\right] \\
&\geq 0. \qquad (151)
\end{aligned}
$$

For $n = 0,\ s \geq 1$, the p_e terms are non-negative, just like in the case of $n \geq 1,\ s \geq 1$. For the $(1 - p_e - p_n + p_e p_n)$ term we have

$$
\begin{aligned}
&H_{i+2}^{a}(0,s+1) - H_{i+2}^{a}(1,s) - H_{i+1}^{a}(0,s+1) + H_{i+1}^{a}(1,s) \\
&+ p_s\left[H_{i+1}^{a}(0,s+1) - H_{i+1}^{a}(0,s+2) - H_{i+1}^{a}(1,s) + H_{i+1}^{a}(1,s+1)\right] \geq 0. \qquad (152)
\end{aligned}
$$

Similarly, for $n \geq 1,\ s = 0$ and for $n = 0,\ s = 0$, the p_e terms have the same structure as above. For the $(1 - p_e - p_n + p_e p_n)$ term, we get

$$
\begin{aligned}
&H_{i+2}^{a}(n,1) - H_{i+2}^{a}(n+1,0) - H_{i+1}^{a}(n,1) + H_{i+1}^{a}(n+1,0) \\
&+ p_s\left[H_{i+1}^{a}(n,1) - H_{i+1}^{a}(n,2) - H_{i+1}^{a}(n+1,0) + H_{i+1}^{a}(n+1,1)\right] \geq 0 \qquad (153)
\end{aligned}
$$

for $n \geq 1,\ s = 0$, and

$$
\begin{aligned}
&H_{i+2}^{a}(0,1) - H_{i+2}^{a}(1,0) - H_{i+1}^{a}(0,1) + H_{i+1}^{a}(1,0) \\
&+ p_s\left[H_{i+1}^{a}(0,1) - H_{i+1}^{a}(0,2) - H_{i+1}^{a}(1,0) + H_{i+1}^{a}(1,1)\right] \geq 0 \qquad (154)
\end{aligned}
$$

for $n = 0,\ s = 0$.

The proof of parts b),c) and d) is also derived using induction over the service period index i. For part b), we need to show that there exists an index n_i^* such that

$$
V_i^a(n,s) \in \begin{cases} G_{2,i}, G_{3,i} & n \geq n_i^*, \\ G_{1,i}, G_{3,i} & n = n_i^* - 1, \\ G_{1,i}, & n < n_i^* - 1. \end{cases} \qquad (155)
$$

We consider the case of $a_{i+1} = 1$ only since for the case $a_{i+1} = 0$ the structure of the proof is the same (results for this case can also be obtained by letting $p_s = 0$).

For $i = N$, we have $V_N^a(n-1,s) - V_N^a(n,s-1) = \pi_n + w_n - \pi_s - w_s \geq r_s - r_n$, $\forall (n,s) \in S_N$. Therefore, for $n \geq 0$, $V_N^a \in G_{2,N}$, $V_N^a \in G_{3,N}$. Equivalently, we can set $n_N^* = 0$.

Suppose at the $(i+1)$-th slot we have n_{i+1}^* such that: for $n \geq n_{i+1}^*$, $V_{i+1}^a \in G_{2,i+1}, G_{3,i+1}$, while for $n < n_{i+1}^*$, $V_{i+1}^a \in G_{1,i+1}$. We want to prove that at i-th slot, we have $n_i^* \geq n_{i+1}^*$ such that for any $n \geq n_i^*$, $V_i^a \in G_{2,i}$, $G_{3,i}$, and $\forall n < n_i^*$, $V_i^a \in G_{1,i}$. We observe that for $n < n_{i+1}^*$, it directly follows from the result of part a) that $V_i^a \in G_{1,i}$.

Define $D_{i+1}(n,s) = V_{i+1}^a(n-1,s) - V_{i+1}^a(n,s-1)$ and consider, for $n > n_{i+1}^*$,

$$\begin{aligned}
& V_i^a(n-1,s) - V_i^a(n,s-1) \\
= \; & -w_s + w_n + p_e p_n \left[(1-p_s a_{i+1}) V_{i+1}^a(n,s) + p_s a_{i+1} V_{i+1}^a(n,s+1)\right] \\
& -p_e p_n \left[(1-p_s a_{i+1}) V_{i+1}^a(n+1,s-1) + p_s a_{i+1} V_{i+1}^a(n+1,s)\right] \\
& +p_e(1-p_n) \left[(1-p_s a_{i+1}) V_{i+1}^a(n-1,s) + p_s a_{i+1} V_{i+1}^a(n-1,s+1)\right] \\
& -p_e(1-p_n) \left[(1-p_s a_{i+1}) V_{i+1}^a(n,s-1) + p_s a_{i+1} V_{i+1}^a(n,s)\right] \\
& +p_n(1-p_e) \left[(1-p_s a_{i+1}) H_{i+1}^a(n,s) + p_s a_{i+1} H_{i+1}^a(n,s+1)\right] \\
& -p_n(1-p_e) \left[(1-p_s a_{i+1}) H_{i+1}^a(n+1,s-1) + p_s a_{i+1} H_{i+1}^a(n+1,s)\right] \\
& +(1-p_e-p_n+p_e p_n) \left[(1-p_s a_{i+1}) H_{i+1}^a(n-1,s) + p_s a_{i+1} H_{i+1}^a(n-1,s+1)\right] \\
& -(1-p_e-p_n+p_e p_n) \left[(1-p_s a_{i+1}) H_{i+1}^a(n,s-1) + p_s a_{i+1} H_{i+1}^a(n,s)\right] \\
= \; & -w_s + w_n + p_e p_n \begin{bmatrix} (1-p_s)\left(V_{i+1}^a(n,s) - V_{i+1}^a(n+1,s-1)\right) \\ +p_s\left(V_{i+1}^a(n,s+1) - V_{i+1}^a(n+1,s)\right) \end{bmatrix} \\
& +p_e(1-p_n) \begin{bmatrix} (1-p_s)\left(V_{i+1}^a(n-1,s) - V_{i+1}^a(n,s-1)\right) \\ +p_s\left(V_{i+1}^a(n-1,s+1) - V_{i+1}^a(n,s)\right) \end{bmatrix} \\
& +p_n(1-p_e) \begin{bmatrix} (1-p_s)\left(H_{i+1}^a(n,s) - H_{i+1}^a(n+1,s-1)\right) \\ +p_s\left(H_{i+1}^a(n,s+1) - H_{i+1}^a(n+1,s)\right) \end{bmatrix}
\end{aligned}$$

$$+\left(1-p_e-p_n+p_ep_n\right)\begin{bmatrix}\left(1-p_s\right)\left(H^a_{i+1}\left(n-1,s\right)-H^a_{i+1}\left(n,s-1\right)\right)\\+p_s\left(H^a_{i+1}\left(n-1,s+1\right)-H^a_{i+1}\left(n,s\right)\right)\end{bmatrix}\tag{156}$$

In the analysis of the last expression, let us look at 2 separate cases. For $n \geq 2$, we have

$$\begin{aligned}
& V^a_i\left(n-1,s\right)-V^a_i\left(n,s-1\right)-r_s+r_n\\
= & -w_s+w_n+p_ep_n\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(n,s\right)-V^a_{i+1}\left(n+1,s-1\right)\right)\\+p_s\left(V^a_{i+1}\left(n,s+1\right)-V^a_{i+1}\left(n+1,s\right)\right)\end{bmatrix}\\
& +p_e\left(1-p_n\right)\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(n-1,s\right)-V^a_{i+1}\left(n,s-1\right)\right)\\+p_s\left(V^a_{i+1}\left(n-1,s+1\right)-V^a_{i+1}\left(n,s\right)\right)\end{bmatrix}\\
& +p_n\left(1-p_e\right)\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(n-1,s\right)-V^a_{i+1}\left(n,s-1\right)\right)\\+p_s\left(V^a_{i+1}\left(n-1,s+1\right)-V^a_{i+1}\left(n,s\right)\right)\end{bmatrix}\\
& +\left(1-p_e-p_n+p_ep_n\right)\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(n-2,s\right)-V^a_{i+1}\left(n-1,s-1\right)\right)\\+p_s\left(V^a_{i+1}\left(n-2,s+1\right)-V^a_{i+1}\left(n-1,s\right)\right)\end{bmatrix}-r_s+r_n\\
= & -w_s+w_n+p_ep_nD_{i+1}\left(n+1,s\right)+p_e\left(1-p_n\right)D_{i+1}\left(n,s\right)+p_n\left(1-p_e\right)D_{i+1}\left(n,s\right)\\
& +\left(1-p_e-p_n+p_ep_n\right)D_{i+1}\left(n-1,s\right)-r_s+r_n
\end{aligned}\tag{157}$$

On the other hand, if $n = 1$,

$$\begin{aligned}
& V^a_i\left(0,s\right)-V^a_i\left(1,s-1\right)-r_s+r_n\\
= & -w_s+w_n+p_ep_n\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(1,s\right)-V^a_{i+1}\left(2,s-1\right)\right)\\+p_s\left(V^a_{i+1}\left(1,s+1\right)-V^a_{i+1}\left(2,s\right)\right)\end{bmatrix}\\
& +p_e\left(1-p_n\right)\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(0,s\right)-V^a_{i+1}\left(1,s-1\right)\right)\\+p_s\left(V^a_{i+1}\left(0,s+1\right)-V^a_{i+1}\left(1,s\right)\right)\end{bmatrix}\\
& +p_n\left(1-p_e\right)\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(0,s\right)-V^a_{i+1}\left(1,s-1\right)\right)\\+p_s\left(V^a_{i+1}\left(0,s+1\right)-V^a_{i+1}\left(1,s\right)\right)\end{bmatrix}\\
& +\left(1-p_e-p_n+p_ep_n\right)\begin{bmatrix}\left(1-p_s\right)\left(V^a_{i+1}\left(0,s-1\right)+r_s-V^a_{i+1}\left(0,s-1\right)-r_n\right)\\+p_s\left(V^a_{i+1}\left(0,s\right)+r_s-V^a_{i+1}\left(0,s\right)-r_n\right)\end{bmatrix}-r_s+r_n
\end{aligned}$$

$$= -w_s + w_n + p_e p_n D_{i+1}(2,s) + p_e(1-p_n) D_{i+1}(1,s) + p_n(1-p_e) D_{i+1}(1,s)$$
$$+(-p_e - p_n + p_e p_n)(r_s - r_n) \quad (158)$$

As $D_{i+1}(n+1,s)$, $D_{i+1}(n,s)$ and $D_{i+1}(n-1,s)$ do not depend on the value of s as long as $n > n^*_{i+1}$, for both $n = 1$ and $n \geq 2$ cases, we have $V^a_i \in G_{3,i}$ for $n > n^*_{i+1}$. Using (19) we observe that values of $D_{i+1}(n+1,s)$, $D_{i+1}(n,s)$ and $D_{i+1}(n-1,s)$ increase with n, so that for $n > n^*_{i+1}$, there would be an index n^*_i such that since values $V^a_i \in G_{2,i}$ for $n \geq n^*_i$ and $V^a_i \in G_{1,i}$ for $n < n^*_i$. The above results also imply that if $w_s \leq w_n$, we will always have $V^a_i(n-1,s) - V^a_i(n,s-1) - r_s + r_n \geq 0$, i.e., $V^a_i \in G_{2,i}$ for all system states. Therefore, in this case, inpatients are always served whenever there is one waiting. We observe that the expression $V^a_i(n-1,s) - V^a_i(n,s-1) - r_s + r_n$ does not contain p_s. Thus, the switching curves are independent of the value of p_s, and, therefore, of the selected threshold appointment policy.

Proof of Proposition 4

In order to show that the dynamic programming operator mapping $\widehat{V}^a_{i+1}(n,s)$ into $\widehat{V}^a_i(n,s)$ preserves the linearity of the value function, we consider $\widehat{V}^a_{i+1}(n,s) = \alpha_{i+1}n + \beta_{i+1}s + \gamma_{i+1}$ and obtain

$$\widehat{H}^a_{i+1}(n,s) = \max[\widehat{V}^a_{i+1}(n-1,s) + r_n, \widehat{V}^a_{i+1}(n,s-1) + r_s] = \widehat{V}^a_{i+1}(n,s) + \delta_{i+1}, \quad (159)$$

where $\delta_{i+1} = \max[r_n - \alpha_{i+1}, r_s - \beta_{i+1}]$, so that

$$\begin{aligned}\widehat{V}^a_i(n,s) &= -w_n n - w_s s + \widehat{V}^a_{i+1}(n,s) + p_e p_n [\alpha_{i+1} + p_s a_{i+1}\beta_{i+1}] \\ &\quad + p_e(1-p_n)[p_s a_{i+1}\beta_{i+1}] + p_n(1-p_e)[\alpha_{i+1} + p_s a_{i+1}\beta_{i+1} + \delta_{i+1}] \\ &\quad + (1-p_e)(1-p_n)[p_s a_{i+1}\beta_{i+1} + \delta_{i+1}] \\ &= -w_n n - w_s s + \widehat{V}^a_{i+1}(n,s) + p_n\alpha_{i+1} + p_s a_{i+1}\beta_{i+1} + (1-p_e)\delta_{i+1}. \quad (160)\end{aligned}$$

Thus, $\alpha_i = \alpha_{i+1} - w_n$, $\beta_i = \beta_{i+1} - w_s$, $\gamma_i = \gamma_{i+1} + p_n\alpha_{i+1} + p_s a_{i+1}\beta_{i+1} + (1-p_e)\delta_{i+1}$. Noting that $\alpha_{N+1} = -\pi_n$, $\beta_{N+1} = -\pi_s$, we obtain by induction $\alpha_i = -\pi_n - (N+1-i)w_n$,

$\beta_i = -\pi_s - (N+1-i)w_s,\ \delta_i = \max\left[r_n + \pi_n + (N+1-i)w_n, r_s + \pi_s + (N+1-i)w_s\right]$, $i = 1, ..., N$.

Note on derivation of the "newsvendor" heuristic

For the case of $w_n = w_s = 0$, Proposition 3 indicates that the optimal tactical capacity allocation policy is either "inpatients first" (if $r_n + \pi_n \geq r_s + \pi_s$) or "outpatients first" (if $r_n + \pi_n < r_s + \pi_s$). For the case of $r_n + \pi_n < r_s + \pi_s$, we use the following approximation to the expected daily profits under threshold a^*:

$$\begin{aligned} V_h^*(r_n + \pi_n \quad < \quad r_s + \pi_s) &= r_s p_s a^* + r_n E\left[\min\left(\left(1-p_e\right)N - p_s a^*, D_n\right)\right] \\ & -\pi_n E\left[D_n - \min\left(\left(1-p_e\right)N - p_s a^*, D_n\right)\right], \end{aligned} \tag{161}$$

where D_n is the total (random) number of inpatients arriving during the day. According to our assumptions, D_n is distributed as a binomial random variable with parameters N and p_n: $D_n \sim B\left(N, p_n\right)$. (161) assumes that inpatients are served using the "residual" capacity $\left(1-p_e\right)N - p_s a^*$ left after serving the emergency patients and outpatients. In order to further simplify the analysis and to obtain closed-form expressions for the heuristic threshold level a^*_{N}, we approximate D_n as a normal random variable with expectation Np_n and variance $Np_n\left(1-p_n\right)$. Then, differentiating (161) with respect to a^* and introducing $\Phi\left(x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_x^{+\infty} e^{-\frac{t^2}{2}} dt$, we obtain, after some algebra

$$a^*_{\mathrm{N}}(r_n + \pi_n \quad < \quad r_s + \pi_s) = \arg \max_{0 \leq a^* \leq (1-p_e)N} V_h^*$$

$$= \begin{cases} 0, & \text{for } \frac{r_s}{r_n+\pi_n} \leq \Phi\left(\frac{(1-p_e-p_n)\sqrt{N}}{\sqrt{p_n(1-p_n)}}\right), \\ \frac{(1-p_e-p_n)N - \sqrt{Np_n(1-p_n)}\Phi^{-1}\left(\frac{r_s}{r_n+\pi_n}\right)}{p_s}, & \begin{array}{c}\text{for } \Phi\left(\frac{(1-p_e-p_n)\sqrt{N}}{\sqrt{p_n(1-p_n)}}\right) \leq \frac{r_s}{r_n+\pi_n} \\ \text{and } \frac{r_s}{r_n+\pi_n} \leq \Phi\left(\frac{-(p_e+p_n+p_s-1)\sqrt{N}}{\sqrt{p_n(1-p_n)}}\right),\end{array} \\ N, & \text{for } \frac{r_s}{r_n+\pi_n} \geq \Phi\left(\frac{-(p_e+p_n+p_s-1)\sqrt{N}}{\sqrt{p_n(1-p_n)}}\right). \end{cases} \tag{162}$$

When $r_n + \pi_n \geq r_s + \pi_s$, we approximate the expected daily profits under threshold a^*

as follows:

$$V_h^*\left(r_n+\pi_n \geq r_s+\pi_s\right) = r_n p_n N + r_s E\left[\min\left(\left(1-p_e-p_n\right)N, D_s\right)\right] \\ -\pi_s E\left[D_s - \min\left(\left(1-p_e-p_n\right)N, D_s\right)\right], \tag{163}$$

where D_s is the total (random) number of outpatients arriving during the day, $D_s \sim B\left(a^*, p_s\right)$. (163) implies that the outpatients are served using the "residual" capacity $\left(1-p_e-p_n\right)N$ left after serving the emergency patients and the inpatients. Approximating D_s as a normal random variable with expectation $p_s a^*$ and variance $p_s\left(1-p_s\right)a^*$, we get a closed-form expression for the expected daily profits:

$$\begin{aligned} & V_h^*\left(r_n+\pi_n \geq r_s+\pi_s\right) \\ = \; & r_n p_n N - \left(r_s+\pi_s\right)\frac{p_s\left(1-p_s\right)a^*}{\sqrt{2\pi}} e^{-\frac{\left(\left(1-p_e-p_n\right)N-p_s a^*\right)^2}{2p_s\left(1-p_s\right)a^*}} - \pi_s p_s a^* + \left(r_s+\pi_s\right)p_s a^* \\ & \times\left(\sqrt{p_s\left(1-p_s\right)a^*} + \left(\left(1-p_e-p_n\right)N - p_s a^*\right)\Phi\left(\left(1-p_e-p_n\right)N - p_s a^*\right)\right) \end{aligned} \tag{164}$$

While the transcendental form of (164) does not allow for a closed-form expression of the optimal threshold, a_{N}^* can be easily computed by comparing the value of V_h^* for $N+1$ potential thresholds $0, ..., N$.