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**e-companion**

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# Electronic Companion—"A Framework Using Two-Factor Price Lattices for Generation Asset Valuation" by Chung-Li Tseng and Kyle Y. Lin, *Operations Research* 2007, 55(2) 234–251.

## Online Appendix

### Appendix A. Proofs

PROOF OF LEMMA 1. Replacing (37e) by (38) in $(Q)$, $(P_1)$ is a problem that optimizes the left-hand side of (38) subject to all other constraints (37a)–(37d), (37f) in $(Q)$. Obviously, $[X_{\min}, X_{\max}]$ provides the exact interval to bound the right-hand side of (38), $\rho/(c_1c_2)+\epsilon_1\epsilon_2$, such that $(Q)$ is feasible. □

PROOF OF LEMMA 2. (1) In this case, because $p_{\text{uu}} \leqslant \tilde{p}_{1\text{u}}$, $p_{\text{dd}} \leqslant \tilde{p}_{1\text{d}}$, $p_{\text{ud}} \geqslant 0$, $p_{\text{du}} \geqslant 0$, it is clear that $\tilde{p}_{1\text{u}}+\tilde{p}_{1\text{d}}$ is an upper bound for $X$. We can conclude that $X_{\max}=\tilde{p}_{1\text{u}}+\tilde{p}_{1\text{d}}$ by presenting the following feasible solution that achieves this upper bound:

$$\begin{bmatrix} p_{\text{du}} & p_{\text{mu}} & p_{\text{uu}} \\ p_{\text{dm}} & p_{\text{mm}} & p_{\text{um}} \\ p_{\text{dd}} & p_{\text{md}} & p_{\text{ud}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{p}_{2\text{u}}-\tilde{p}_{1\text{u}} & \tilde{p}_{1\text{u}} \\ 0 & \tilde{p}_{2\text{m}} & 0 \\ \tilde{p}_{1\text{d}} & \tilde{p}_{2\text{d}}-\tilde{p}_{1\text{d}} & 0 \end{bmatrix}. \tag{EC.A1}$$

(2) Similar to (i).
(3) First, note that

$$p_{\text{uu}} \leqslant \tilde{p}_{1\text{u}}, \qquad p_{\text{dd}} \leqslant \tilde{p}_{2\text{d}}, \qquad p_{\text{ud}} \geqslant 0.$$

On the other hand,

$$p_{\text{du}} = \tilde{p}_{2\text{u}} - p_{\text{uu}} - p_{\text{mu}} \geqslant \max\{0, \tilde{p}_{2\text{u}} - \tilde{p}_{1\text{u}} - \tilde{p}_{1\text{m}}\} = (\tilde{p}_{2\text{u}} + \tilde{p}_{1\text{d}} - 1)^+,$$

where $x^+ \equiv \max(0, x)$. Hence,

$$\begin{aligned} X &= p_{\text{uu}} + p_{\text{dd}} - p_{\text{ud}} - p_{\text{du}} \\ &\leqslant \tilde{p}_{1\text{u}} + \tilde{p}_{2\text{d}} - (\tilde{p}_{2\text{u}} + \tilde{p}_{1\text{d}} - 1)^+ \\ &= \min\{\tilde{p}_{1\text{u}} + \tilde{p}_{2\text{d}}, 1 - (\tilde{p}_{1\text{d}} - \tilde{p}_{2\text{d}}) - (\tilde{p}_{2\text{u}} - \tilde{p}_{1\text{u}})\}. \end{aligned}$$

We can conclude that $X_{\max} = \min\{\tilde{p}_{1\text{u}} + \tilde{p}_{2\text{d}}, 1 - (\tilde{p}_{1\text{d}} - \tilde{p}_{2\text{d}}) - (\tilde{p}_{2\text{u}} - \tilde{p}_{1\text{u}})\}$ by presenting the following feasible solution that achieves this upper bound:

$$\begin{bmatrix} p_{\text{du}} & p_{\text{mu}} & p_{\text{uu}} \\ p_{\text{dm}} & p_{\text{mm}} & p_{\text{um}} \\ p_{\text{dd}} & p_{\text{md}} & p_{\text{ud}} \end{bmatrix} = \begin{bmatrix} (\tilde{p}_{1\text{d}} - \tilde{p}_{2\text{d}} - \tilde{p}_{2\text{m}})^+ & \tilde{p}_{2\text{u}} - \tilde{p}_{1\text{u}} - (\tilde{p}_{1\text{d}} - \tilde{p}_{2\text{d}} - \tilde{p}_{2\text{m}})^+ & \tilde{p}_{1\text{u}} \\ \min\{\tilde{p}_{2\text{m}}, \tilde{p}_{1\text{d}} - \tilde{p}_{2\text{d}}\} & (\tilde{p}_{2\text{m}} - (\tilde{p}_{1\text{d}} - \tilde{p}_{2\text{d}}))^+ & 0 \\ \tilde{p}_{2\text{d}} & 0 & 0 \end{bmatrix}. \tag{EC.A2}$$

(4) Similar to (iii). □

PROOF OF THEOREM 3. From Lemma 1, a lattice is feasible if the following inequality holds for all $|\epsilon_1|, |\epsilon_2| \leqslant 1/2$:

$$c_1c_2(X_{\min} - \epsilon_1\epsilon_2) \leqslant \rho \leqslant c_1c_2(X_{\max} - \epsilon_1\epsilon_2). \tag{EC.A3}$$

One can guarantee feasibility of the entire lattice if $\rho^{\min}(c_1, c_2) \leqslant \rho \leqslant \rho^{\max}(c_1, c_2)$, where

$$\rho^{\max}(c_1, c_2) \equiv \min_{-1/2 \leqslant \epsilon_1, \epsilon_2 \leqslant 1/2} c_1 c_2 (X_{\max} - \epsilon_1 \epsilon_2),$$

$$\rho^{\min}(c_1, c_2) \equiv \max_{-1/2 \leqslant \epsilon_1, \epsilon_2 \leqslant 1/2} c_1 c_2 (X_{\min} - \epsilon_1 \epsilon_2).$$

To determine $\rho^{\min}(c_1, c_2)$ and $\rho^{\max}(c_1, c_2)$, consider the four cases discussed in Lemma 2.

(1) If $\tilde{p}_{1u} \leqslant \tilde{p}_{2u}$ and $\tilde{p}_{1d} \leqslant \tilde{p}_{2d}$, from Lemma 2(i) and (27) for $|\epsilon_1|, |\epsilon_2| \leqslant 1/2$,

$$X_{\max} - \epsilon_1 \epsilon_2 = \tilde{p}_{1u} + \tilde{p}_{1d} - \epsilon_1 \epsilon_2 = \frac{1}{c_1^2} + \epsilon_1^2 - \epsilon_1 \epsilon_2 \geqslant \frac{1}{c_1^2} - \frac{1}{16},$$

where we use the fact that the global minimum for $\epsilon_1^2 - \epsilon_1 \epsilon_2$ subject to $|\epsilon_1|, |\epsilon_2| \leqslant 1/2$ is $-1/16$, achieved at $(\epsilon_1, \epsilon_2) = (-1/4, -1/2)$.

(2) If $\tilde{p}_{1u} > \tilde{p}_{2u}$ and $\tilde{p}_{1d} > \tilde{p}_{2d}$, similar to (i) one can obtain

$$X_{\max} - \epsilon_1 \epsilon_2 = \tilde{p}_{2u} + \tilde{p}_{2d} - \epsilon_1 \epsilon_2 = \frac{1}{c_2^2} + \epsilon_2^2 - \epsilon_1 \epsilon_2 \geqslant \frac{1}{c_2^2} - \frac{1}{16}.$$

(3) If $\tilde{p}_{1u} \leqslant \tilde{p}_{2u}$ and $\tilde{p}_{1d} > \tilde{p}_{2d}$, from Lemma 2(iii) and (27), we have

$$X_{\max} - \epsilon_1 \epsilon_2 = \min \begin{cases} \dfrac{1}{2}\left(\dfrac{1}{c_1^2} + \epsilon_1^2 + \epsilon_1\right) + \dfrac{1}{2}\left(\dfrac{1}{c_2^2} + \epsilon_2^2 - \epsilon_2\right) - \epsilon_1 \epsilon_2 \geqslant \dfrac{1}{2}\left(\dfrac{1}{c_1^2} + \dfrac{1}{c_2^2}\right) - \dfrac{1}{8}, \\ 1 + \epsilon_1 - \epsilon_2 - \epsilon_1 \epsilon_2 \geqslant \dfrac{1}{4}, \end{cases}$$

where we use the fact that $-1/8$ and $-3/4$ are the global minima for $0.5(\epsilon_1^2 + \epsilon_1 + \epsilon_2^2 - \epsilon_2) - \epsilon_1 \epsilon_2$ and $\epsilon_1 - \epsilon_2 - \epsilon_1 \epsilon_2$, respectively, subject to $|\epsilon_1|, |\epsilon_2| \leqslant 1/2$.

(4) Same as case (iii), which obtains the same lower bound for $X_{\max}$.

Summarizing all four possible cases above, we conclude that

$$\begin{aligned} \rho^{\max} &= c_1 c_2 \cdot \min\left\{\frac{1}{c_1^2} - \frac{1}{16}, \frac{1}{c_2^2} - \frac{1}{16}, \frac{1}{2}\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right) - \frac{1}{8}, \frac{1}{4}\right\} \\ &= \min\left\{\frac{c_2}{c_1} - \frac{c_1 c_2}{16}, \frac{c_1}{c_2} - \frac{c_1 c_2}{16}, \frac{1}{2}\left(\frac{c_2}{c_1} + \frac{c_1}{c_2}\right) - \frac{c_1 c_2}{8}, \frac{c_1 c_2}{4}\right\}. \end{aligned} \tag{EC.A4}$$

Similarly, we can show that

$$\rho^{\min} = -\rho^{\max}.$$

The proof is completed. □

## Appendix B. Convergence Analysis

In this section, we shall show that the proposed two-factor lattice converges to the diffusion processes that it approximates as the number of periods $N \to \infty$.

Let $\{(y_1(t), y_2(t)) \mid 0 \leqslant t \leqslant T\}$ denote the solution to (19) and (20) with initial values $(y_1(t), y_2(t)) = (y_1(0), y_2(0))$ at $t = 0$. Consider a Markov chain $\{(\bar{Y}_{1,n}, \bar{Y}_{2,n}) \mid n = 0, 1, \ldots, N\}$ that approximates $\{(y_1(t), y_2(t)) \mid 0 \leqslant t \leqslant T\}$, and let $\Delta t = T/N$. Deng and Oren (2003) in Proposition 3.1 provides a sufficient condition such that $\{(\bar{Y}_{1,n}, \bar{Y}_{2,n})\}$ converges to $\{(y_1(t), y_2(t))\}$ if the following four conditions hold:

Condition 1. The local expectations of the discrete lattice match those of the diffusion process to the dominant term:

$$\mathrm{E}[\bar{Y}_{i,n+1} \mid \bar{Y}_{i,n} = \hat{y}] = \mu_i(\hat{y})\Delta t + o(\Delta t) \quad \forall n,\ i = 1, 2. \tag{EC.B1}$$

Condition 2. The local variances and covariances of the discrete lattice match those of the diffusions processes to the dominant term. That is,

$$\mathrm{Var}(\bar{Y}_{i,n+1} \mid \bar{Y}_{i,n}) = \sigma_i^2 \Delta t + o(\Delta t) \quad \forall n,\ i = 1, 2 \tag{EC.B2}$$

and

$$\mathrm{Cov}(\bar{Y}_{1,n+1}, \bar{Y}_{2,n+1} \mid \bar{Y}_{1,n}, \bar{Y}_{2,n}) = \rho\sigma_1\sigma_2 \Delta t + o(\Delta t) \quad \forall n. \tag{EC.B3}$$

Condition 3. There exists a function $z(\Delta t)$ such that with probability 1,

$$|\bar{Y}_{i,n+1} - \bar{Y}_{i,n}| \leqslant z(\Delta t) \quad \forall n,\ i = 1, 2. \tag{EC.B4}$$

In addition, $\lim_{\Delta t \to 0} z(\Delta t) = 0$.

CONDITION 4. The initial condition is satisfied:

$$(\bar{Y}_{1,0}, \bar{Y}_{2,0}) = (y_1(0), y_2(0)). \tag{EC.B5}$$

Next, we state the following proposition of convergence by showing that the setup of our lattice satisfies the four conditions (EC.B1)–(EC.B4).

PROPOSITION 1. *Let* $\{(y_1(t), y_2(t)) \mid 0 \leqslant t \leqslant T\}$ *denote the solution to* (19) *and* (20) *with initial values* $(y_1(t), y_2(t)) = (y_1(0), y_2(0))$ *at* $t = 0$. *The Markov chain* $\{(Y_{1,n}, Y_{2,n}) \mid n = 0, 1, \ldots, N\}$ *constructed by solving* ($Q$), (37a)–(37f) *in* §3.3, *with initial values* $(Y_{1,0}, Y_{2,0}) = (y_1(0), y_2(0))$, *converges in distribution to* $\{(y_1(t), y_2(t)) \mid 0 \leqslant t \leqslant T\}$ *as* $\Delta t \to 0$, *if* $y_i(t)$, $i = 1, 2$, *satisfies one of the following conditions*:

(1) $\mu_i(y)$ *is bounded.*

(2) $y_i(t)$ *is an MR process.*

(3) $\mu_i(y)$ *is Lipschitz continuous and there exist two constants* $\bar{y}_i > \hat{y}_i$, *such that* $\mu_i(y) < 0$ *and decreases in* $y$ *for* $y > \bar{y}_i$, *and* $\mu_i(y) > 0$ *and decreases in* $y$ *for* $y < \hat{y}_i$.

PROOF. To prove this proposition, we need to verify Conditions (1)–(4). First, Conditions (1) and (2) are satisfied because ($Q$) ensures that the branch probabilities of $\{(Y_{1,n}, Y_{2,n})\}$ match the local mean, local variance, and local covariance of the Markov chain to the dominant terms of those in the diffusion process $\{(y_1(t), y_2(t))\}$ by (24a), (24b), and (37e), respectively. Condition (EC.B4) is satisfied because of the initial condition $(Y_{1,0}, Y_{2,0}) = (y_1(0), y_2(0))$.

To show that Condition (3) is satisfied, we need to show that the random variable $|Y_{i,n+1} - Y_{i,n}|$ is uniformly bounded by a function that tends to zero as $\Delta t \to 0$. Because Condition (3) concerns each process individually rather than by the joint process, we omit the subscript $i$ to simplify the notation in the following. Consider each case separately.

(1) Because $\mu(y)$ is bounded, suppose that there exists a constant $M$ such that $|\mu(y)| < M$. From (23), we have that

$$\frac{\mu(y)\Delta t}{h} - \frac{1}{2} < \kappa \leqslant \frac{\mu(y)\Delta t}{h} + \frac{1}{2}.$$

Therefore,

$$|\kappa| \leqslant \left|\frac{\mu(y)\Delta t}{h}\right| + \frac{1}{2} < \frac{M\Delta t}{c\sigma\sqrt{\Delta t}} + \frac{1}{2} = \frac{M}{c\sigma}\sqrt{\Delta t} + \frac{1}{2}.$$

Because $\kappa$ is an integer, the preceding implies that if

$$\Delta t < \left(\frac{c\sigma}{2M}\right)^2, \tag{EC.B6}$$

then $\kappa = 0$. Hence, as $\Delta t \to 0$, the random variable $|Y_{n+1} - Y_n| \leqslant h = c\sigma\sqrt{\Delta t}$, so Condition (EC.B3) is satisfied.

(2) Suppose that $y$ is an MR process and by definition $\mu(y)$ is Lipschitz continuous. There exists a constant $L$ such that for any $\Delta y > 0$,

$$|\mu(y + \Delta y) - \mu(y)| \leqslant L|\Delta y|. \tag{EC.B7}$$

Suppose that at some node $Y_n = \bar{y}$, its corresponding $\kappa$ is equal to zero, or equivalently,

$$-\frac{1}{2} \leqslant \frac{\mu(\bar{y})\Delta t}{h} < \frac{1}{2}. \tag{EC.B8}$$

Then, in the next time period $Y_{n+1}$ can take on three possible values: $\bar{y} + h$, $\bar{y}$, and $\bar{y} - h$.

Consider the following three cases:

(a) $Y_{n+1} = \bar{y} + h$: Because $\mu(y)$ decreases in $y$ (by the definition of an MR process) and $\mu(y)$ is Lipschitz continuous, using (EC.B7) we have that

$$\mu(\bar{y}) - Lh \leqslant \mu(\bar{y} + h) \leqslant \mu(\bar{y}).$$

Multiplying the preceding by $\Delta t/h$ and applying (EC.B8), we then have

$$-L\Delta t - \frac{1}{2} \leqslant \frac{\mu(\bar{y} + h)\Delta t}{h} < \frac{1}{2}.$$

Therefore, if $-L\Delta t > -1$, or equivalently,

$$\Delta t < \frac{1}{L}, \tag{EC.B9}$$

the corresponding $\kappa$ for $Y_{n+1} = \bar{y} + h$ is either 0 or $-1$.

(b) $Y_{n+1} = \bar{y}$: In this case, the corresponding $\kappa$ for $Y_{n+1} = \bar{Y}$ is 0.

(c) $Y_{n+1} = \bar{y} - h$: Similar to (a). If $\Delta t$ satisfies (EC.B9), the corresponding $\kappa$ for $Y_{n+1} = \bar{y} - h$ is either 0 or 1.

Summarizing from these three cases, for $\Delta t$ satisfying (EC.B9), if $\kappa = 0$ in one time period, then in the next time period there will be only three possible values for $\kappa$, namely, $-1$, 0, and 1. By defining

$$r_u(\Delta t) \equiv \inf\left\{ y \,\middle|\, \frac{\mu(y)\Delta t}{h} \leqslant -\frac{1}{2} \right\}$$

and

$$r_l(\Delta t) \equiv \sup\left\{ y \,\middle|\, \frac{\mu(y)\Delta t}{h} \geqslant \frac{1}{2} \right\},$$

we can assert that if $Y_0 = y(0) \in [r_l(\Delta t) - h, r_u(\Delta t) + h]$, then the whole process $\{Y_n \mid n = 0, 1, \ldots, N\}$ will always lie in $[r_l(\Delta t) - h, r_u(\Delta t) + h]$. Consequently, in the whole lattice there are only three possible values for $\kappa$, namely, $-1$, 0, and 1. Hence, for $\Delta t$ satisfying (EC.B9), $|Y_{n+1} - Y_n|$ is uniformly bounded by $2h = 2c\sigma\sqrt{\Delta t}$, which tends to 0 as $\Delta t \to 0$.

To complete the proof, note that $r_u(\Delta t) \to \infty$ and $r_l(\Delta t) \to -\infty$ as $\Delta t \to 0$. Therefore, we can always find a small enough $\Delta t$ to ensure that $y(0) \in [r_l(\Delta t) - h, r_u(\Delta t) + h]$.

(3) We can prove this case by combining the results from the previous two cases. Because $\mu(y)$ is continuous, the set $\{|\mu(y)| \mid \hat{y} \leqslant y \leqslant \bar{y}\}$ is compact and therefore bounded. Let $M \equiv \max\{|\mu(y)| \mid \hat{y} \leqslant y \leqslant \bar{y}\}$ and let $L$ denote the constant so that $\mu(y)$ satisfies the Lipschitz continuity in (EC.B7). According to (EC.B6) and (EC.B9), we can conclude that for

$$\Delta t < \min\left\{ \left(\frac{c\sigma}{2M}\right)^2, \frac{1}{L} \right\},$$

if $\kappa = 0$ in one time period, in the next time period there will be only three possible values for $\kappa$, namely, $-1$, 0, and 1. The rest of the proof follows that in case 2. □

Proposition 1 shows that the convergence of the lattice relies on the property of the drift function. The lattice can converge for at least three types of drift function $\mu(y)$ given in the proposition, including that for an MR process. Note that $\mu(y)$ determines the branching factor $\kappa$ (23). An intuitive interpretation is that if $\mu(y)$ changes too quickly over a small interval of $y$ (e.g., not Lipschitz continuous), the movements of branches might become very large over a small time interval and cause the discrete distribution not to converge to the underlying, continuous one.

## Appendix C. Relieving the Impact of the Ramp Constraints

It is well known that the SDP (lattice) approach can only handle problems with constraints that are path-independent on the network of the state space. That is, each state should be independent of the paths that lead to it. The ramp constraints (6) are path-dependent unless we expand the state space of $\{x_t\}$ to include discrete levels of the generation $q_t$. Recall that in the proposed lattice model at each node $(y_1, y_2)$ at time $t$, there is an associated optimal generating level $g(y_1, y_2)$ (a special case with $y_1 = P_t^E$ and $y_2 = P_t^F$ is given in (12)). So, when a branch (an arc) moves from one node to another in the next time period, the difference between the two optimal generation levels associated with these two nodes determines the satisfaction of the corresponding ramp constraint. If this movement violates the ramp constraint, it means that the change of the prices over this time period (or equivalently, the time step $\Delta t$) is too big. Therefore, one may want to reduce the value of $\Delta t$ to reduce the price change over each time period. It can also be expected that reducing the value of $\Delta t$ can decrease the number of branches that violate the ramp constraints, although it may not be able to eliminate the violation of the ramp constraints completely. To model this effect, we present a continuous version of the ramp constraint (6) when the time step $\Delta t \leqslant 1$:

$$|q_t - q_{t-\Delta t}| \leqslant R\Delta t, \tag{EC.C10}$$

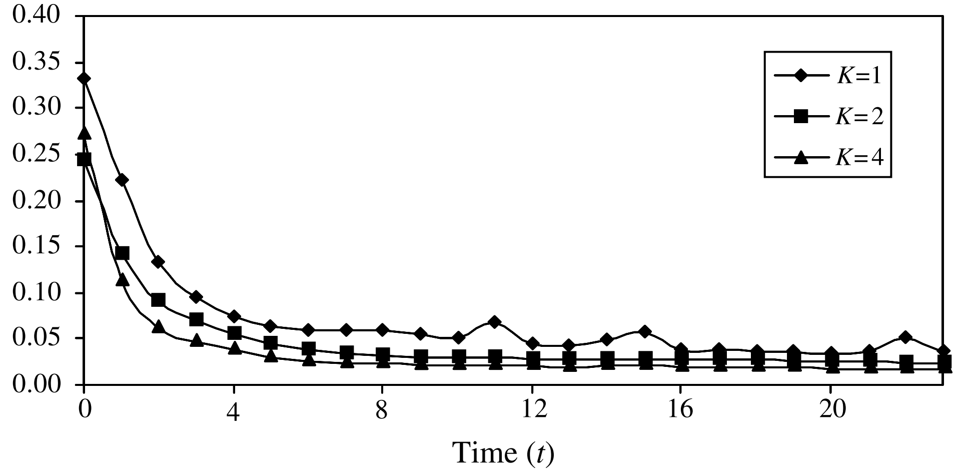
where we assume that the ramp rate is a linear function of $\Delta t$ for simplicity. Given a node $(y_1, y_2)$ at time $t$ and a branch incident from this node, if reducing $\Delta t$ can eventually satisfy the ramp constraint for this branch, that means

$$|g(y_1, y_2) - g(y_1 + (\kappa_1 + 1)h_1, y_2 + (\kappa_2 - 1)h_2)| \leqslant R\Delta t \tag{EC.C11}$$

holds for some small $\Delta t$. Note that in (EC.C11), because we try to bound the change of the generation levels from above, we consider the transition that yields the largest change in $g$ (with high electricity price $(\kappa_1 + 1)$ and low fuel price $(\kappa_2 - 1)$). Dividing $\Delta t$ on both sides of (EC.C11) and letting $\Delta t \to 0$, we have

$$\left| \frac{\partial g}{\partial y_1}\mu_1(y_1) + \frac{\partial g}{\partial y_2}\mu_2(y_2) \right| \leqslant R, \tag{EC.C12}$$

**FIGURE EC.1.** The percentage of branches violating the ramp constraint at each time period.




if the above partial derivatives exist at the node. (Note that $g$ is continuous but may not be differentiable at all points.) In general, if the partial derivatives exist, $\partial g/\partial y_1 \geqslant 0$ and $\partial g/\partial y_2 \leqslant 0$. It is clear that (EC.C12) compares the increasing rate (or the total differential) of $g$ and the constant ramp rate $R$. Equation (EC.C12) may be viewed as a (necessary and sufficient) condition to verify whether reducing $\Delta t$ can eventually meet the ramp constraints.

It is worth noting that (EC.C12) may hold for *all* nodes under some special price processes. That means *all* the ramp constraints can be automatically satisfied by setting a $\Delta t$ small enough. Equivalently, the ramp constraints can be eliminated. This, unfortunately, is not the case for the MR processes used in our numerical tests. However, numerical tests indicate that by slightly reducing the value of $\Delta t$, the majority of the branches indeed satisfy the ramp constraints. Furthermore, the impact due to the branches that violate the ramp constraints is estimated to be insignificant. Therefore, reducing $\Delta t$ is an effective strategy to relieve the impact of the ramp constraints. Details of the numerical tests are presented next.

Consider the operation of the same unit used in §4.3 over a 24-hour period. Three different values of $\Delta t$ are tested: $\Delta t = 1$ ($K = 1$), $\Delta t = 0.5$ ($K = 2$), and $\Delta t = 0.25$ ($K = 4$). The number of branches that violate the ramp constraint are counted in each time period, and are then divided by the total number of branches in the same time period to yield a ratio, which is depicted in Figure EC.1. For example, when $\Delta t = 1$, there are nine branches incident from the initial node at $t = 0$, one-third (0.33) of them violate the ramp constraint. This ratio is reduced to 0.24 and 0.27 when $\Delta t = 0.5$ and 0.25, respectively. Note that when the value of $\Delta t$ is reduced to 0.5 and 0.25 from 1, the total number of the branches also increases from 9 to 90 and 756 within the first time period ($t = 0$), respectively. In general, the percentage of the branches that violate the ramp constraint is very low and decreases as $\Delta t$ decreases, except at $t = 0$.

If the corresponding branching probability of each branch that violates the ramp constraint is also considered, the violation of the ramp constraint overall accounts for 4.4% of the branching probability when $\Delta t = 1$. This probability is reduced to 2.75% and 2.0% when $\Delta t$ is reduced to 0.5 and 0.25, respectively. On the other hand, from Table 2, when $\Delta t = 1$ is reduced to 0.5 and 0.25, the plant value increases 1.6% and 2.3%, respectively. Roughly, every 1% of improvement of the branching probability that violates the ramp constraint is associated with a 1% increase in power plant value.

While indeed the ramp constraint may not be satisfied at all branches in the proposed lattice model, this test result suggests that its impact is very limited, which can be further relieved by reducing $\Delta t$.

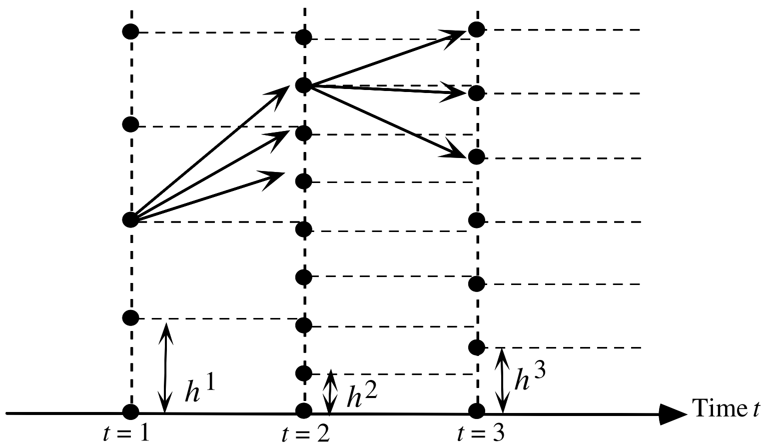
## Appendix D. Extensions of the Proposed Valuation Model

This section discusses potential extensions of the proposed valuation model, including time-dependent drift and volatility, long-term valuation, and incorporation of network constraints.

### EC.D1. Time-Dependent Drift and Volatility

Drift function and volatility may be time dependent. For example, the mean level of the electricity price at each hour may be different following some hourly pattern, such as peak hours and off-peaks hours. The volatility in

**FIGURE EC.2.** A one-factor lattice with time-dependent volatility.




the peak hours may be different from that in the off-peak hours. The lattice model developed in the preceding sections can be easily extended to accommodate time-dependent drift and volatility.

Recall the definition of branching factor $\kappa$ in (23). If the drift function $\mu(y)$ is now time-dependent, say $\mu_t(y)$, one can simply replace $\mu(y)$ in (23) by $\mu_t(y)$ for determining $\kappa$. Given a lattice node $y_t$ at time $t$, (23) also implies that

$$y_t + \mu(y_t)\Delta t \in \left[y_t + \left(\kappa - \tfrac{1}{2}\right)h,\, y_t + \left(\kappa + \tfrac{1}{2}\right)h\right). \qquad \text{(EC.D13)}$$

Because $y_t$ is also a lattice node at $t+1$, the branching factor $\kappa$, defined by (EC.D13), shows the relative positions of the three nodes to be branched into with respect to (w.r.t.) $y_t$.

Consider the following one-factor process with time-dependent drift and volatility:

$$dy = \mu_t(y)dt + \sigma_t dB. \qquad \text{(EC.D14)}$$

Because the volatility determines the (discrete) price increment $h$ of the lattice, with a time-dependent volatility $\sigma_t$ the price increment may be different from hour to hour (see Figure EC.2).

Define the price increment at hour $t$ as

$$h^t = c\sigma_t\sqrt{\Delta t}. \qquad \text{(EC.D15)}$$

A superscript $t$ is used to distinguish $h^t$ from $h_1$ and $h_2$ previously defined in the two-factor lattice. Because now a lattice node $y_t$ at time $t$ may not be a lattice node at time $t+1$, it is better to change the notion of $\kappa$ from a *relative* position w.r.t. $y_t$ to an *absolute* position w.r.t. a common reference point such as zero. We thus define the time-dependent branching factor $\kappa_{t+1}$ as follows:

$$\kappa_{t+1} \equiv \left\lfloor \frac{y_t + \mu_t(y_t)\Delta t}{h^{t+1}} + \frac{1}{2} \right\rfloor. \qquad \text{(EC.D16)}$$

With (EC.D16), a lattice node $y_t$ at time $t$ now branches into three nodes $(\kappa_{t+1} - 1)h^{t+1}$, $\kappa_{t+1}h^{t+1}$, and $(\kappa_{t+1} + 1)h^{t+1}$ at time $t+1$. Accordingly, the definition of $\epsilon$ in (26) is modified by

$$\epsilon_{t+1} \equiv \frac{y_t + \mu_t(y_t)\Delta t}{h^{t+1}} - \kappa_{t+1} \qquad \text{(EC.D17)}$$

and the following property of $\epsilon_t$ remains valid:

$$|\epsilon_t| \leqslant \tfrac{1}{2} \quad \forall\, t. \qquad \text{(EC.D18)}$$

Therefore, all properties of the lattice (both one-factor and two-factor) derived previously follow.

## EC.D2. Long-Term Valuation Model

There is no doubt that the generation asset valuation must account for the operational constraints, whether it be short term (weeks) or long term (years). However, because the operational constraints, such as the minimum uptime and downtime constraints, can only be modeled via hour-by-hour unit commitment (UC), it is not a trivial task to integrate a long-term valuation model with physical constraints. We believe that the proposed model can be directly applied to long-term asset valuation, say over a 10-year period. The only concern is that whether the underlying price model can capture longer term price signals (such as those provided by forward price curve). This can be handled by converting the price information observed in the forward curves to the "mean" price levels of the MR process. For example, if the MR process follows the standard Ornstein-Uhlenbeck process:

$$\mu_i(y_i) = \eta_i(m_{it} - y_i), \quad i = 1, 2, \tag{EC.D19}$$

where $\eta_i$ is the speed of reversion and $m_{it}$ is the mean level of $y_i$ at time $t$, one can set the (hourly) profile of the mean level $\{m_{it}\}$ such that it reflects the forward curve. Note that a monthly forward price can be viewed as an average of all hourly prices over a month. Therefore, some scaling may be necessary to convert forward prices to hourly prices. In addition to setting the drift function of the price model, one can also set the volatility that varies from time to time over the entire planning period, which has been covered in §EC.D1. Because we have shown that the size of the proposed two-factor lattice will be capped for the MR underlying processes, the computational complexity for the version of long-term valuation will only increase linearly, as does the CPU time. Therefore, directly applying the proposed valuation to a long-term period is technically feasible.

## EC.D3. Incorporating Network Constraints

In Valenzuela and Mazumdar (2003), the authors showed that with the existence of a competitive spot market, the UC problem may become decomposable if the spot market has an infinite capacity and can trade (including buying and selling) electricity instantaneously. In this situation, the UC problem can be decomposed to independent unit subproblems. An intuitive explanation is that the power imbalance of the demand constraint can be offset by the market through trading instantaneously. This coincides with our rationale to discuss single unit asset valuation in a competitive market with spot markets. When the network constraints exist, one can no longer assume that the spot market (presumably near the power plant of interest) has an infinite capacity due to transmission constraints. In this case, it makes better sense to consider the valuation of multiple generating units over various locations as a portfolio, which is beyond the scope of this paper. However, to consider the impact of the network constraints on a specific unit at a specific location, one can model the network congestion in terms of the capacity of the unit. For example, the maximal and minimal capacities of the unit may be time dependent and/or even random:

$$q_t^{\min} \leqslant q_t \leqslant q_t^{\max}. \tag{EC.D20}$$

This can be easily incorporated in the proposed lattice model.

## References

*See references list in the main paper.*

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