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ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—“Price-Directed Control of a Closed Logistics Queueing Network” by Daniel Adelman, *Operations Research* 2007, DOI 10.1287/opre.1070.0408.

Appendix A: Real-World Problem Instance

Here is a problem instance derived and camouflaged from real-world data obtained from the motivating firm for a representative portion of their network. The node set is

$$\mathcal{N} = \{\text{ATLA, LOSAN, PORTL, CHGO1, CHGO2, NY}\},$$

corresponding with city names. The arcs and other data are displayed in Table EC.1.

The λ_{ij} are measured in units per month, and the τ are measured in months. The number of service units (fleet boxes) in the system is $K = 400$.

The solution to (**NLP-**$\mathcal{M}$) using our iterative algorithm yields three prototypes with $K_1 = 289$, $K_2 = 92$, and $K_3 = 19$. The expression $\sum_{m\in\mathcal{M}} K_m P_i^{(m)}$ computes the expected inventory available for allocation at each node i. This yields Table EC.2, which shows a total of 264.9 units available in inventory, on average.

The recommended flow rates on the arcs are displayed in Table EC.3. We also give the average fill rate, which is the percentage of demand accepted, i.e. satisfied using fleet boxes.

Appendix B: Proof of Theorem 4

When $K = 1$ the model (**NLP-**$\mathcal{M}$) becomes a linear program, hence we assume $K > 1$. Denote the set of active nodes by $\mathcal{N}^{(m)} = \{i \in \mathcal{N}\colon P_i^{(m)} > 0\}$ for all m. Feasibility in (39) and non-negativity (41) implies that either there exists an $ij \in \mathcal{E}$ such that $X_{ij}^{(m)} > 0$ or there exists an $i \in \mathcal{N}$ such that $P_i^{(m)} > 0$. If the former, then by part 1 of Proposition 2 we have $P_i^{(m)} > 0$ and $P_j^{(m)} > 0$. Hence, $\mathcal{N}^{(m)}$ is non-empty for all m.

We now perform a series of perturbations to the solution (X, P) to obtain an alternative feasible solution $(\tilde{X}, \tilde{P})$ in the relative interior, i.e. which satisfies all equalities (38) and (39) and strictly satisfies the inequalities (40) and (41). This will give us a direction vector for use in the MFCQ.

Choose any $m \in \mathcal{M}$ for which not all of the constraints (40) are satisfied strictly. First consider an alternative solution

$$\begin{aligned} \hat{X}_{ij}^{(m)} &= \alpha^{(m)} X_{ij}^{(m)} \quad ij \in \mathcal{E} \\ \hat{P}_i^{(m)} &= \beta^{(m)} P_i^{(m)} \quad i \in \mathcal{N}, \end{aligned} \tag{EC1}$$

for constants $\alpha^{(m)}$ and $\beta^{(m)}$. Choose an $\alpha^{(m)} \in (0, 1)$ and set

$$\beta^{(m)} = \frac{1 - \alpha^{(m)} \sum_{ij} \tau_{ij} X_{ij}^{(m)}}{\sum_i P_i^{(m)}} \geqslant \frac{1 - \sum_{ij} \tau_{ij} X_{ij}^{(m)}}{\sum_i P_i^{(m)}} = 1, \tag{EC2}$$

where the denominator is strictly positive because $\mathcal{N}^{(m)}$ is non-empty. Then, by construction $(\hat{X}^{(m)}, \hat{P}^{(m)})$ satisfies (38), (39), and (41). Next, we verify (40) in three steps.

Step 1. We first check (40) for $m' \neq m$. The feasibility of (X, P) and $\hat{X}_{ij}^{(m)} \leqslant X_{ij}^{(m)}$ implies

$$\begin{aligned} X_{ij}^{(m')} &\leqslant \Big(\lambda_{ij} - \sum_{m''} (K_{m''} - \mathbb{1}\{m'' = m'\}) X_{ij}^{(m'')}\Big) P_i^{(m')} \\ &\leqslant \Big(\lambda_{ij} - \sum_{m'' \neq m} (K_{m''} - \mathbb{1}\{m'' = m'\}) X_{ij}^{(m'')} - K_m \hat{X}_{ij}^{(m)}\Big) P_i^{(m')} \end{aligned}$$

for every $ij \in \mathcal{E}$. Note that if the first inequality is already strict, then it remains so under the new solution.

TABLE EC.1. Real-world 6 node, 15 arc problem instance.

Arcs $ij \in \mathscr{E}$	r_{ij}	λ_{ij}	τ_{ij}
LOSAN ATLA	−4.22	206	0.181667
LOSAN NY	−2.73	23.5	0.221667
CHGO1 ATLA	24.48	27	0.135667
CHGO1 NY	26.76	42	0.112667
CHGO2 ALTA	24.48	273	0.135667
CHGO2 NY	26.76	42	0.112667
PORTL ATLA	19.83	111.5	0.230667
PORTL NY	−0.50	14	0.191333
ATLA LOSAN	10.75	108	0.176
ATLA CHGO1	14.92	983.5	0.126333
ATLA PORTL	25.41	113.5	0.21
NY LOSAN	26.61	21	0.221333
NY CHGO1	15.60	169	0.121333
NY CHGO2	15.60	169	0.121333
NY PORTL	6.60	1	0.183333

Step 2. Now we check (40) for prototype m, for all arcs $ij \in \mathscr{E}$ such that $i \in \mathscr{N}^{(m)}$. At least one such arc exists due to $\mathscr{N}^{(m)} \neq \varnothing$, and we assume every node has at least one outgoing arc. From the feasibility of $(X^{(m)}, P^{(m)})$, we have

$$\hat{X}_{ij}^{(m)} = \alpha^{(m)} X_{ij}^{(m)} \leqslant \alpha^{(m)} \left(\lambda_{ij} - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m\}) X_{ij}^{(m')} \right) P_i^{(m)}$$
$$\leqslant \left(\lambda_{ij} - \sum_{m' \neq m} K_{m'} X_{ij}^{(m')} - (K_m - 1)\alpha^{(m)} X_{ij}^{(m)} \right) P_i^{(m)}$$

because $0 < \alpha^{(m)} < 1$ so that part 2 of Proposition 2 implies $\lambda_{ij} > \sum_{m'} K_{m'} X_{ij}^{(m')} \geqslant \sum_{m' \neq m} K_{m'} X_{ij}^{(m')} + (K_m - 1) X_{ij}^{(m)} \geqslant \sum_{m' \neq m} K_{m'} X_{ij}^{(m')} + (K_m - 1)\alpha^{(m)} X_{ij}^{(m)}$. If $\beta^{(m)} > 1$ then we have

$$\hat{X}_{ij}^{(m)} < \left(\lambda_{ij} - \sum_{m' \neq m} K_{m'} X_{ij}^{(m')} - (K_m - 1)\hat{X}_{ij}^{(m)} \right) \beta^{(m)} P_i^{(m)}$$
$$= \left(\lambda_{ij} - \sum_{m' \neq m} K_{m'} X_{ij}^{(m')} - (K_m - 1)\hat{X}_{ij}^{(m)} \right) \hat{P}_i^{(m)}$$

because $P_i^{(m)} > 0$. If $\beta^{(m)} = 1$ then $X_{ij}^{(m)} = 0$, and hence $\hat{X}_{ij}^{(m)} = 0$, for all ij. The right-hand side in the last display is strictly positive, and therefore the first inequality is still strict.

Step 3. Lastly, we check (40) for prototype m, for all $ij \in \mathscr{E}$ such that $i \in \mathscr{N} \setminus \mathscr{N}^{(m)}$. Observe that both sides of (40) equal 0, because $P_i^{(m)} = 0$ and thus the non-negativity of X in (41) implies $X_{ij}^{(m)} = 0$. To achieve strict inequality in (40), we further perturb $P_{i'}^{(m)}$ over all $i' \in \mathscr{N}$. Choose an $\epsilon^{(m)} \in (0, \min_{i' \in \mathscr{N}^{(m)}} \hat{P}_{i'}^{(m)})$ such that

$$\hat{X}_{i'j'}^{(m)} < \left(\lambda_{i'j'} - \sum_{m' \neq m} K_{m'} X_{i'j'}^{(m')} - (K_m - 1)\hat{X}_{i'j'}^{(m)} \right) (\hat{P}_{i'}^{(m)} - \epsilon^{(m)}) \tag{EC3}$$

TABLE EC.2. Average inventory in solution of real-world instance, aggregated over prototypes.

Node $i \in \mathscr{N}$	Average inventory available
LOSAN	33.7
ATLA	83.7
NY	2.4
CHGO1	81.8
CHGO2	45.9
PORTL	17.4

TABLE EC.3. X solution for real-world instance, aggregated over prototypes.

Arc $ij \in \mathscr{E}$	X_{ij}	Fill rate (%)
LOSAN ATLA	84.19	40.9
LOSAN NY	22.90	97.4
CHGO1 ATLA	26.91	99.7
CHGO1 NY	41.87	99.7
CHGO2 ATLA	78.05	28.6
CHGO2 NY	41.24	98.2
PORTL ATLA	99.26	89.0
PORTL NY	13.28	94.9
ATLA LOSA	107.09	99.2
ATLA CHGO1	68.78	7.0
ATLA PORTL	112.54	99.2
NY LOSAN	0.00	0.0
NY CHGO1	0.00	0.0
NY CHGO2	119.29	70.6
NY PORTL	0.00	0.0

for all $i'j' \in \mathscr{E}$ such that $i' \in \mathscr{N}^{(m)}$, i.e. from Step 2. Such an $\epsilon^{(m)}$ can be found for two reasons: (a) the first expression in parentheses is strictly positive from step 2 above, and (b) the inequality is already strict for $\epsilon^{(m)} = 0$. We now re-specify

$$\hat{P}_{i'}^{(m)} \leftarrow \begin{cases} \beta^{(m)} P_{i'}^{(m)} - \epsilon^{(m)} & \text{if } i' \in \mathscr{N}^{(m)} \\ \dfrac{|\mathscr{N}^{(m)}|}{|\mathscr{N} \backslash \mathscr{N}^{(m)}|} \epsilon^{(m)} & \text{if } i' \in \mathscr{N} \backslash \mathscr{N}^{(m)} \end{cases} \quad i' \in \mathscr{N}. \tag{EC4}$$

Equations (38) and the non-negativity of $\hat{X}$ are unaffected. By construction we have strict non-negativity of $\hat{P}^{(m)}$ and hence (41). Constraint (39) reads

$$\sum_{i'j'} \tau_{i'j'} \hat{X}_{i'j'}^{(m)} + \sum_{i'} \hat{P}_{i'}^{(m)} = \alpha^{(m)} \sum_{i'j'} \tau_{i'j'} X_{i'j'}^{(m)} + \beta^{(m)} \sum_{i'} P_{i'}^{(m)} - |\mathscr{N}^{(m)}| \epsilon^{(m)} + |\mathscr{N}^{(m)}| \epsilon^{(m)} = 1,$$

because of the way $\beta^{(m)}$ is chosen in (EC2). Furthermore, inequalities (40) are unaffected for prototypes $m' \neq m$, while (EC3) ensures they hold strictly for prototype m for all $ij \in \mathscr{E}$ considered in Step 2.

We now return to checking (40) for prototype m, for all $ij \in \mathscr{E}$ such that $i \in \mathscr{N} \backslash \mathscr{N}^{(m)}$. Because $X_{ij}^{(m)} = 0$ we have

$$\hat{X}_{ij}^{(m)} - \left(\lambda_{ij} - \sum_{m' \neq m} K_{m'} X_{ij}^{(m')} - (K_m - 1) \hat{X}_{ij}^{(m)} \right) \hat{P}_i^{(m)} = -\left(\lambda_{ij} - \sum_{m' \neq m} K_{m'} X_{ij}^{(m')} \right) \frac{|\mathscr{N}^{(m)}|}{|\mathscr{N} \backslash \mathscr{N}^{(m)}|} \epsilon^{(m)} < 0.$$

At this point (40) is satisfied with strict inequality for prototype m. Repeat the above procedure for every other prototype $m' \neq m$. The strictness of (40) for prototypes already seen is unaffected by the procedure, as shown in Step 1. For all m not considered above, i.e. under which (40) is already satisfied strictly by (X, P), let $\hat{X}^{(m)} = X^{(m)}$ and $\hat{P}^{(m)} = P^{(m)}$. Therefore, the resulting solution $(\hat{X}, \hat{P})$ strictly satisfies (40).

To complete the construction of a point in the relative interior, we must perturb the solution again to ensure $\hat{X} > 0$. By construction, we already have $\hat{P}_i^{(m)} > 0$ for all i for prototypes m perturbed above. For m in which (40) is satisfied strictly by (X, P) for all ij, we also have $\hat{P}_i^{(m)} = P_i^{(m)} > 0$ for all i. Otherwise, there exists an i such that $P_i^{(m)} = 0$, and then by (40) and non-negativity of X we have $X_{ij}^{(m)} = 0$ for all j, contradicting the fact that (40) is satisfied strictly.

Our final perturbation is as follows. Under the assumption stated in the theorem, for every arc $(i_0, i_1) \in \mathscr{E}$ let $\mathscr{D}_{i_0, i_1}$ denote the set of arcs on a directed path from node i_1 leading back to node i_0, plus the arc (i_0, i_1). Then it is straightforward to see that $\gamma_{ij} = \sum_{i'j' \in \mathscr{E}} \mathbb{1}\{ij \in \mathscr{D}_{i'j'}\} > 0$ for all $ij \in \mathscr{E}$ satisfies flow balance (38). It puts positive flow onto every arc in the network. Now set

$$\tilde{X}_{ij}^{(m)} = \hat{X}_{ij}^{(m)} + \gamma_{ij} \tilde{\epsilon} \quad ij \in \mathscr{E}, \quad m \in \mathscr{M} \tag{EC5}$$

$$\tilde{P}_i^{(m)} = \hat{P}_i^{(m)} - \tilde{\delta} / |\mathscr{N}| \quad i \in \mathscr{N}, \quad m \in \mathscr{M} \tag{EC6}$$

for some $\tilde{\delta} > 0$ and $\tilde{\epsilon} > 0$. This solution clearly satisfies (38). Constraints (39) read, for every m,

$$\begin{aligned}\sum_{ij} \tau_{ij} \tilde{X}_{ij}^{(m)} + \sum_i \tilde{P}_i^{(m)} - 1 &= \left(\sum_{ij} \tau_{ij} \hat{X}_{ij}^{(m)} + \sum_i \hat{P}_i^{(m)}\right) + \tilde{\epsilon} \sum_{ij} \tau_{ij} \gamma_{ij} - \tilde{\delta} - 1 \\ &= \tilde{\epsilon} \sum_{ij} \tau_{ij} \gamma_{ij} - \tilde{\delta}.\end{aligned}$$

Therefore, we satisfy (39) for every m when

$$\tilde{\epsilon} = \frac{\tilde{\delta}}{\sum_{ij} \tau_{ij} \gamma_{ij}}. \tag{EC7}$$

Next we show that $\tilde{\delta}$ can be selected so that we still strictly satisfy (40). For a given prototype m and arc ij, collecting terms in (40) yields

$$\begin{aligned}&\left(\lambda_{ij} - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m\}) \tilde{X}_{ij}^{(m')}\right) \tilde{P}_i^{(m)} - \tilde{X}_{ij}^{(m)} \\ &\quad = \left(\lambda_{ij} - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m\}) (\hat{X}_{ij}^{(m')} + \gamma_{ij} \tilde{\epsilon})\right) (\hat{P}_i^{(m)} - \tilde{\delta}/|\mathcal{N}|) - (\hat{X}_{ij}^{(m)} + \gamma_{ij} \tilde{\epsilon}) \\ &\quad = \left[\left(\lambda_{ij} - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m\}) \hat{X}_{ij}^{(m')}\right) \hat{P}_i^{(m)} - \hat{X}_{ij}^{(m)}\right] - \frac{\tilde{\delta}}{|\mathcal{N}|} \left(\lambda_{ij} - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m\}) (\hat{X}_{ij}^{(m')} + \gamma_{ij} \tilde{\epsilon})\right) \\ &\qquad - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m\}) \gamma_{ij} \tilde{\epsilon} \hat{P}_i^{(m)} - \gamma_{ij} \tilde{\epsilon}.\end{aligned}$$

The first term is strictly greater than 0. By substituting (EC7) we see that the last three terms are equal to zero when $\tilde{\delta} = 0$. A simple calculation shows that the second term is positive (and then negated) when

$$\tilde{\delta} < \min_{i'j' \in \mathcal{E},\, m'' \in \mathcal{M}} \left(\frac{\lambda_{i'j'} - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m''\}) \hat{X}_{i'j'}^{(m')}}{\sum_{m'} (K_{m'} - \mathbb{1}\{m' = m''\}) \gamma_{i'j'}}\right) \Psi$$

where

$$\Psi = \sum_{i'j' \in \mathcal{E}} \tau_{i'j'} \gamma_{i'j'},$$

and the right-hand side is defined and positive. The last two terms are positive (and then negated) for all $\tilde{\delta} > 0$. Therefore, because the last three terms are a continuous function of $\tilde{\delta}$, we can make them arbitrary small, and thereby make the entire calculation strictly greater than zero. Furthermore, because $\hat{P}_i^{(m)} > 0$ for all i and m, we can choose a $\tilde{\delta} > 0$ such that $\tilde{\delta} < |\mathcal{N}| \hat{P}_i^{(m)}$, so that $\tilde{P}_i^{(m)} > 0$, for all i and m. Thus, the solution given by (EC5) and (EC6) is feasible and can be made to satisfy (40) and (41) with strict inequality.

We now verify the MFCQ condition, i.e. Proposition 3.3.8 of Bertsekas (1995). Consider a direction d defined by the column vector

$$d = \begin{pmatrix} \tilde{X} - X \\ \tilde{P} - P \end{pmatrix}.$$

Denote the left-hand side minus the right-hand side of (40) by $g_{ij}^{(m)}(X, P)$. The non-zero partial derivatives of $g_{ij}^{(m)}$ are

$$\frac{\partial g_{ij}^{(m)}(X, P)}{\partial X_{ij}^{(m')}} = \begin{cases} 1 + (K_m - 1) P_i^{(m)} & \text{if } m' = m \\ K_{m'} P_i^{(m)} & \text{if } m' \neq m \end{cases} \quad \forall\, m'$$

and

$$\frac{\partial g_{ij}^{(m)}(X, P)}{\partial P_i^{(m)}} = -\left(\lambda_{ij} - \sum_{m'} (K_{m'} - \mathbb{1}\{m' = m\}) X_{ij}^{(m')}\right).$$

Thus, for every ij and m such that (40) is tight under solution (X, P), the dot product of the gradient of g with the direction d equals

$$\begin{aligned}(\nabla g_{ij}^{(m)}(X,P))^t d &= \sum_{m'\neq m}(\tilde{X}_{ij}^{(m')} - X_{ij}^{(m')})K_{m'}P_i^{(m)} + (\tilde{X}_{ij}^{(m)} - X_{ij}^{(m)})(1+(K_m-1)P_i^{(m)}) \\ &\quad - (\tilde{P}_i^{(m)} - P_i^{(m)})\left(\lambda_{ij} - \sum_{m'}(K_{m'} - \mathbb{1}\{m'=m\})X_{ij}^{(m')}\right) \\ &= \left(\sum_{m'}(K_{m'} - \mathbb{1}\{m'=m\})(\tilde{X}_{ij}^{(m')} - X_{ij}^{(m')})P_i^{(m)}\right) \\ &\quad + \left(\tilde{X}_{ij}^{(m)} - \left(\lambda_{ij} - \sum_{m'}(K_{m'} - \mathbb{1}\{m'=m\})X_{ij}^{(m')}\right)\tilde{P}_i^{(m)}\right), \end{aligned} \tag{EC8}$$

where the second equality follows from the fact that (40) is tight. We need to show that this is strictly negative.

Let $\Lambda_{ij}^{(m)} = (\lambda_{ij} - \sum_{m'}(K_{m'} - \mathbb{1}\{m'=m\})X_{ij}^{(m')})$, so that the last line can be written as $\tilde{X}_{ij}^{(m)} - \Lambda_{ij}^{(m)}\tilde{P}_i^{(m)}$. Substituting (EC5), (EC6), and (EC7), we see that this expression is strictly negative when

$$\tilde{\delta} < \left(\frac{\hat{P}_i^{(m)}\Lambda_{ij}^{(m)} - \hat{X}_{ij}^{(m)}}{\gamma_{ij}/\Psi + \Lambda_{ij}^{(m)}/|\mathcal{N}|}\right). \tag{EC9}$$

The denominator is strictly positive, and so we need to show that the numerator is strictly positive in order to ensure that we can find a $\tilde{\delta} > 0$ satisfying this condition. We consider two cases.

Case 1. Suppose $i \in \mathcal{N}\backslash\mathcal{N}^{(m)}$. Then $P_i^{(m)} = 0$ implies from (40) and non-negativity of X that $\hat{X}_{ij}^{(m)} = X_{ij}^{(m)} = 0$. Therefore, the numerator of (EC9) equals $\hat{P}_i^{(m)}\Lambda_{ij}^{(m)} > 0$, and so we can choose a $\tilde{\delta} > 0$ that satisfies (EC9). Furthermore, the term (EC8) equals 0 because $P_i^{(m)} = 0$. Therefore, we can make $(\nabla g_{ij}^{(m)}(X,P))^t d < 0$.

Case 2. Suppose $i \in \mathcal{N}^{(m)}$. Substituting (EC4) into the numerator of (EC9), we see that it is positive if

$$\epsilon^{(m)} < (\beta^{(m)}P_i^{(m)}\Lambda_{ij}^{(m)} - \hat{X}_{ij}^{(m)})(1/\Lambda_{ij}^{(m)}).$$

We therefore may need to adjust $\epsilon^{(m)}$ in Step 3 above (and the subsequent calculation of $\tilde{\delta}$). Next we confirm that we can choose $\epsilon^{(m)} > 0$ and satisfy this last display. Because $P_i^{(m)} > 0$, from the tightness of (40) and $\Lambda_{ij}^{(m)} > 0$ we have that $X_{ij}^{(m)} > 0$. Thus, $0 < \alpha^{(m)} < 1$ implies that $\hat{X}_{ij}^{(m)} = \alpha^{(m)}X_{ij}^{(m)} < X_{ij}^{(m)}$. Consequently, because $\beta^{(m)} \geqslant 1$ we have

$$\beta^{(m)}P_i^{(m)}\Lambda_{ij}^{(m)} - \hat{X}_{ij}^{(m)} > \beta^{(m)}P_i^{(m)}\Lambda_{ij}^{(m)} - X_{ij}^{(m)} \geqslant P_i^{(m)}\Lambda_{ij}^{(m)} - X_{ij}^{(m)} = 0.$$

Substituting (EC1) and (EC5), the expression (EC8) equals

$$\begin{aligned}&\sum_{m'}((\alpha^{(m')}-1)X_{ij}^{(m')} + \gamma_{ij}\tilde{\epsilon})(K_{m'} - \mathbb{1}\{m'=m\})P_i^{(m)} \\ &\quad = \left((K-1)\gamma_{ij}\tilde{\epsilon} + \sum_{m'}((\alpha^{(m')}-1)X_{ij}^{(m')})(K_{m'} - \mathbb{1}\{m'=m\})\right)P_i^{(m)}.\end{aligned}$$

Substituting (EC7) for $\tilde{\epsilon}$, we see that this term is strictly negative when

$$\tilde{\delta} < \left(\frac{\sum_{m'}((1-\alpha^{(m')})X_{ij}^{(m')})(K_{m'} - \mathbb{1}\{m'=m\})}{(K-1)\gamma_{ij}}\right)\Psi.$$

Because $X_{ij}^{(m)} > 0$, the right-hand side is strictly positive, and therefore we can choose $\tilde{\delta} > 0$. Therefore, we can make $(\nabla g_{ij}^{(m)}(X,P))^t d < 0$.

Now consider ij and m such that $X_{ij}^{(m)} = 0$. Let $f_{ij}^{(m)}(X,P) = -X_{ij}^{(m)}$, so that the non-negativity constraint (41) can be expressed as $f_{ij}^{(m)}(X,P) \leqslant 0$. Then

$$\frac{\partial f_{ij}^{(m)}(X,P)}{\partial X_{ij}^{(m)}} = -1,$$

and all other partial derivatives equal 0. Thus,

$$(\nabla f_{ij}^{(m)}(X, P))^t d = -1(\tilde{X}_{ij}^{(m)} - X_{ij}^{(m)}) = -\tilde{X}_{ij}^{(m)} < 0$$

because $\tilde{\epsilon} > 0$ and (EC5). A similar argument holds for the non-negativity constraint of P, i.e. i and m such that $P_i^{(m)} = 0$.

All other constraints are linear, and without loss of generality we can assume they are linearly independent. Therefore, the conditions of Proposition 3.3.8 of Bertsekas (1995) hold, i.e. the MFCQ. □

Reference

See references list in the main paper.

Bertsekas, D. P. 1995. *Nonlinear Programming*, 2nd ed. Athena Scientific, Belmont, MA.