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e-companion

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Online Appendix

1. The MPEC Algorithm

1.1. Partition of X_g

We partition X_g into a set of polyhedra according to feasible complementary bases of (19). Let n and m be the dimensions of x_g and y (and w), respectively. Consider (19), given a partition $(\alpha, \bar{\alpha})$ of $\{1, 2, \ldots, m\}$, we define matrix $C_M(\alpha) \in R^{m\times m}$ as

$$C_M(\alpha)_{\bullet i} = \begin{cases} -M_{\bullet i} & \text{if } i \in \alpha \\ I_{\bullet i} & \text{if } i \in \bar{\alpha} \end{cases}.$$

$C_M(\alpha)$ is called a *complementary matrix* of $[-M, I]$ with respect to α; it is a *complementary basis* if nonsingular; it is a *feasible complementary basis* with respect to x_g if $C_M^{-1}(\alpha)(q + Ax_g) \geqslant 0$, where $q = t + A\bar{x}_{-g}$.

Now, given the partition $(\alpha, \bar{\alpha})$, let $w_\alpha = 0$ and $y_{\bar{\alpha}} = 0$, then (19) is reduced to

$$v^\alpha = C_M^{-1}(\alpha)(q + A^g x_g) \geqslant 0, \quad \text{where } v_i^\alpha = \begin{cases} y_i & \text{if } i \in \alpha \\ w_i & \text{if } i \in \bar{\alpha} \end{cases}.$$

Note that the orthogonality of w and y is guaranteed for $w_\alpha = 0$ and $y_{\bar{\alpha}} = 0$. The preceding is equivalent to

$$y_\alpha = M_{\alpha\alpha}^{-1}(q_\alpha + A_{\alpha\bullet} x_g) \geqslant 0$$
$$w_{\bar{\alpha}} = -M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1}(q_\alpha + A_{\alpha\bullet} x_g) + q_{\bar{\alpha}} + A_{\bar{\alpha}\bullet} x_g \geqslant 0.$$

When $C_M(\alpha)$ is a feasible complementary basis with respect to x_g, these two nonnegative constraints are satisfied.

As a result, the polyhedron

$$\tilde{P}_g(\alpha) = \{x_g \in R^n\colon C_M^{-1}(\alpha)(q + A^g x_g) \geqslant 0\}$$

defines a set of x_g with respect to which $C_M(\alpha)$ is feasible. Moreover, the state variables y and w are affine functions of $x_g \in \tilde{P}_g(\alpha)$:

$$y_\alpha = M_{\alpha\alpha}^{-1}(q_\alpha + A_{\alpha\bullet} x_g) \tag{EC1}$$

$$y_{\bar{\alpha}} = 0 \tag{EC2}$$

$$w_\alpha = 0 \tag{EC3}$$

$$w_{\bar{\alpha}} = -M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1}(q_\alpha + A_{\alpha\bullet} x_g) + q_{\bar{\alpha}} + A_{\bar{\alpha}\bullet} x_g \tag{EC4}$$

Equations (EC1)–(EC4) imply that $x_g \in Int(\tilde{P}_g(\alpha))$ if and only if $(y_\alpha, w_{\bar{\alpha}}) \in R^m_{++}$ (y and w are non-degenerate), and that $x_g \in Bd(\tilde{P}_g(\alpha))$ if and only if there exists some $i \in \alpha$ such that $y_i = w_i = 0$ (y and w are degenerate).

Enumerating all feasible complementary bases $C_M(\alpha)$, one can partition X_g into a set of polyhedra $P_g(\alpha) = X_g \cap \tilde{P}_g(\alpha)$. The uniqueness of the solution (y, w) to (19) guarantees that such partition is also unique (but with respect to a fixed $\bar{x}_{-g}$). Figure 2 illustrates a sample partition for the case of $n = 2$.

1.2. Stationary Point

Equations (EC1)–(EC4) imply that, whenever $C_M(\alpha)$ is a feasible complementary basis, $f_g(x_g, y, w, \bar{x}_{-g})$ is reduced to a quadratic function with respect to $x_g \in P_g(\alpha)$. We denote this function as $f_{g,\alpha}(x_g, \bar{x}_{-g})$. Now, limiting $\mathcal{F}_g(\bar{x}_{-g})$ to $x_g \in P_g(\alpha)$ leads to the following program parameterized by $\bar{x}_{-g}$:

$$\begin{aligned} QP_g(\alpha): \quad & \min_{x_g} \; f_{g,\alpha}(x_g, \bar{x}_{-g}) \\ \text{subject to: } \; & x_g \in X_g \\ & M_{\alpha\alpha}^{-1}(q_\alpha + A_{\alpha\bullet} x_g) \geqslant 0 \\ & -M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1}(q_\alpha + A_{\alpha\bullet} x_g) + q_{\bar{\alpha}} + A_{\bar{\alpha}\bullet} x_g \geqslant 0 \end{aligned}$$

Notice that this problem does not involve y and w.

We call α the *associated* (*index*) *basis* of polyhedron $P_g(\alpha)$. Let $x_g \in X_g$ be given, and (y, w) be the corresponding solution to (19), Equations (EC1)–(EC4) hold for all associated bases in

$$B_g(x_g, \bar{x}_{-g}) = \{\alpha \subseteq \{1, 2, \ldots, m\}: \{i: y_i > 0\} \subseteq \alpha \subseteq \{i: w_i = 0\}\}.$$

We refer to this set as the *association* (*basis*) *set* at x_g. Clearly, $x_g \in P_g(\alpha)$ for all $\alpha \in B_g(x_g, \bar{x}_{-g})$.

We are now ready to characterize the B-stationary points of $\mathcal{F}_g(\bar{x}_{-g})$. Following Luo et al. (1996), a vector $(\bar{x}_g, y, w)$ is called a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$ if, for all feasible directions (with respect to (19)) $u \in R^{n+2m}$ at $(\bar{x}_g, y, w)$, the directional derivative $\nabla_u f_g(x_g, y, w, \bar{x}_{-g}) \geqslant 0$.

Thus, a point $\bar{x}_g \in X_g$ is a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$ if and only if, for all $\alpha \in B_g(\bar{x}_g, \bar{x}_{-g})$, either of the following holds

1. $P_g(\alpha)$ is a singleton containing only $\bar{x}_g$, i.e., $P_g(\alpha) = \{\bar{x}_g\}$;
2. for any unit-vector direction $u \in R^n$ such that there exists a sufficiently small scaler $\epsilon > 0$ satisfying $\bar{x}_g + \epsilon u \in P_g(\alpha)$, the directional derivative of $f_g(x_g, y, w, \bar{x}_{-g})$ at $\bar{x}_g$ with respect to u is non-negative, i.e.,

$$\nabla_u f_g(\bar{x}_g, y, w, \bar{x}_{-g}) = \left.\frac{\partial f_g}{\partial x_g}\right|_{\bar{x}_g} u + \left.\frac{\partial f_g}{\partial y}\frac{dy}{dx_g}\right|_{\bar{x}_g} u + \left.\frac{\partial f_g}{\partial w}\frac{dw}{dx_g}\right|_{\bar{x}_g} u \geqslant 0,$$

where y and w are as in (EC1)–(EC4).

The above B-stationary conditions suggest that, if a local minimum or stationary point $\bar{x}_g$ of $QP_g(\alpha)$ yields non-degenerate (y, w) in (19), it is a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$; otherwise, one should identify whether this point is a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$ by checking whether it is a local minimum or stationary point with respect to all polyhedra associated with $B_g(\bar{x}_g, \bar{x}_{-g})$.

1.3. The MPEC Algorithm

Let $\bar{x}_g \in X_g$ be a given starting point. If there exists an $\alpha \in B_g(\bar{x}_g, \bar{x}_{-g})$ such that $QP_g(\alpha)$ is unbounded, then $\mathcal{F}_g(\bar{x}_{-g})$ is also unbounded. If $\bar{x}_g$ is a local minimum or stationary point of problems $QP_g(\alpha)$ for all $\alpha \in B_g(\bar{x}_g, \bar{x}_{-g})$, it is a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$. Otherwise, there exists an $\alpha^* \in B_g(\bar{x}_g, \bar{x}_{-g})$ for which $QP_g(\alpha^*)$ yields a solution different than $\bar{x}_g$. Let this point be x_g^*, then its corresponding state variables y^* and w^* are as in (EC1)–(EC4) with α replaced with α^*. If y^* and w^* are non-degenerate, $x_g^* \in Int(\tilde{P}_g(\alpha^*))$ and hence a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$; otherwise, it serves as the starting point for the next (inner) iteration. The dashed lines in Figure 2 illustrate such a sample path.

The MPEC algorithm

Input: $\bar{x}_g$, $\bar{x}_{-g}$

0. (Initialization) Set $\alpha^* := \varnothing$.
1. (Subroutine call) Call the *search subroutine*.
2. (Termination check)
   If the subroutine reports unboundedness,
      report the problem $\mathcal{F}_g(\bar{x}_{-g})$ as unbounded, stop.
   else if the subroutine reports $\bar{x}_g$ as a B-stationary point,
      $\bar{x}_g$ is a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$, stop.
   else
      let x_g^* and α^* be the returned point and the associated basis, respectively.
      let y^* and w^* solve (19) with x_g^*.

If $(y^*_{\alpha^*}, w^*_{\bar{\alpha}^*}) \in R^m_{++}$,
x^*_g is a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$, stop.
else
set $\bar{x}_g := x^*_g$, go to step 1.

The *search subroutine* (input: α^*, $\bar{x}_g$, $\bar{x}_{-g}$)
0. (Initial pivoting) Pivot to an associated basis $\alpha \in B_g(\bar{x}_g, \bar{x}_{-g})\backslash\{\alpha^*\}$ at $\bar{x}_g$.
1. (Search)
Call a *quadratic programming subroutine* to solve $QP_g(\alpha)$.
If $QP_g(\alpha)$ has an unbounded direction,
report unboundedness.
If $QP_g(\alpha)$ yields a point $x^*_g \neq \bar{x}_g$ with a decreased objective value,
set $\alpha^* := \alpha$, return x^*_g and α^*.
2. (Termination Check)
If all bases in $B_g(\bar{x}_g, \bar{x}_{-g})\backslash\{\alpha^*\}$ have been visited,
return $\bar{x}_g$ as a B-stationary point.
else
pivot (at $\bar{x}_g$) to the next $\alpha \in B_g(\bar{x}_g, \bar{x}_{-g})\backslash\{\alpha^*\}$, go to step 1.

Here, one can use any available quadratic programming solver as the *quadratic programming subroutine*.

1.4. Remarks

- Because the number of zones is typically much smaller than the number of nodes, the dimension of $QP_g(\alpha)$, $|x_g|$, is usually much smaller than that of $\mathcal{F}_g(\bar{x}_{-g})$. Therefore, the proposed MPEC algorithm improves the performance of the general PSQP method (Luo et al. 1996).
- The MPEC algorithm maintains feasibility with respect to all constraints (including the complementarity constraint) in (19).
- If $|B_g(\bar{x}_g, \bar{x}_{-g})| \leqslant 2$ throughout the course of the MPEC algorithm, then no basis will be repeated. This, combined with the fact that there exists a finite number of partition of X_g (bounded by the number of feasible complementary bases in (19)), establishes the finite global convergence of the MPEC algorithm. If the preceding condition is violated (that is, if $|B_g(\bar{x}_g, \bar{x}_{-g})| > 2$ for some $\bar{x}_g$), one can use any of the standard lexicographic schemes (in the context of LCP pivoting; see, for example, Cottle et al. 1992) to avoid cycling. It should be noted that different lexicographic schemes might lead to different search paths of the MPEC algorithm and thus possibly to different B-stationary points. For example, if the lexicographic scheme selects the basis α_2, instead of α_1, at point A in Figure 2, the algorithm terminates immediately.
- Note that, to solve $QP_g(\alpha)$, we need to compute $M^{-1}_{\alpha\alpha}$ and possibly, depending on the quadratic programming subroutine in use, a starting point. $M^{-1}_{\alpha\alpha}$ can be computed efficiently from the corresponding matrix of the previously-visited basis, which differs from α by one index. The solution to the quadratic program with respect to the previous-visited basis can be used as the starting point.

2. The EPEC Scheme

2.1. B-Stationary Equilibrium

To define the B-stationary equilibrium for $\{\mathcal{F}_g(\cdot)\}_{g\in G}$, we extend the definition of association set for the MPECs to the EPEC. Because the association set depends on (y, w), which is determined through (19) jointly by all MPECs' design variables, we state the following equivalence property.

Let (y, w) solve (19) with given $\{\bar{x}_g \in X_g\}_{g\in G}$ and consider any two firms g and g', then the association set at $\bar{x}_g$ for $\mathcal{F}_g(\bar{x}_{-g})$ and the association set at $\bar{x}_{g'}$ for $\mathcal{F}_{g'}(\bar{x}_{-g'})$ are equivalent, i.e.,

$$B_g(\bar{x}_g, \bar{x}_{-g}) = B_{g'}(\bar{x}_{g'}, \bar{x}_{-g'}), \quad g, g' \in G.$$

The above equivalence of the association sets among all MPECs implies that

- if $\bar{x}_g \in Int(\tilde{P}_g(\alpha))$ for some α, then $\bar{x}_{g'} \in Int(\tilde{P}_{g'}(\alpha))$;
- if $\bar{x}_g$ is in the boundaries of polyhedra $\tilde{P}_g(\alpha_1), \tilde{P}_g(\alpha_2), \ldots, \tilde{P}_g(\alpha_k)$, $\bar{x}_{g'}$ is also in the boundaries of polyhedra $\tilde{P}_{g'}(\alpha_1), \tilde{P}_{g'}(\alpha_2), \ldots, \tilde{P}_{g'}(\alpha_k)$.

We define the association set of $\{\mathcal{F}_g(\cdot)\}_{g\in G}$ as follows. Given $\{\bar{x}_g\}_{g\in G}$, let (y, w) solve (19), then the association set for the EPEC $\{\mathcal{F}_g(\cdot)\}_{g\in G}$ is

$$B(\{\bar{x}_g\}_{g\in G}) = \{\alpha \subseteq \{1, 2, \ldots, m\}: \{i: y_i > 0\} \subseteq \alpha \subseteq \{i: w_i = 0\}\}.$$

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The association set of $\{\mathcal{F}_g(\cdot)\}_{g\in G}$ suggests a characterization of the B-stationary equilibria of $\{\mathcal{F}_g(\cdot)\}_{g\in G}$ as follows. A set $\{\bar{x}_g\}_{g\in G}$ is a *B-stationary equilibrium* of $\{\mathcal{F}_g(\cdot)\}_{g\in G}$ if $\bar{x}_g$ is a B-stationary point of $\mathcal{F}_g(\bar{x}_{-g})$ for all $g\in G$, i.e., $\bar{x}_g$ is a local minimum or stationary point of $QP_g(\alpha)$ for all $\alpha\in B(\{\bar{x}_g\}_{g\in G})$.

2.2. The Scheme

The EPEC Scheme

0. (Initialization) Select an arbitrary $\{\bar{x}_g^0\in X_g\}_{g\in G}$. Let $k:=1$.
1. (Loop) Let $\{\bar{x}_g^k\}_{g\in G}:=\{\bar{x}_g^{k-1}\}_{g\in G}$.
   For each $g\in G$,
      apply the MPEC algorithm to $\mathcal{F}_g(\bar{x}_{-g}^k)$.
      if $\mathcal{F}_g(\bar{x}_{-g}^k)$ is unbounded,
         report the failure of finding an equilibrium, stop.
      else
         let $\bar{x}_g^k$ and (y, w) be the returned decision and state variables.
2. (Termination check)
   If $\|\{\bar{x}_g^k-\bar{x}_g^{k-1}\}_{g\in G}\|$ is within a given error bound,
      report $(\{\bar{x}_g^k\}_{g\in G}, y, w)$ as a B-stationary equilibrium, stop.
   else if the predetermined bound of the number of iterations is reached,
      stop.
   else
      go to step 1 with $k:=k+1$.

2.3. Remarks

- The termination basis of an MPEC problem can be used as the starting basis for the next MPEC problem.
- The termination point for an MPEC problem can be used as the starting point for solving the next MPEC problem.
- Since the LCP constraints (19) are satisfied throughout the course, this allows one to terminate the MPEC algorithm before it reaches a B-stationary point, and to trade the accuracy of the MPEC solution for the speed of the overall EPEC scheme.