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Electronic Companion—"Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices" by Xinmin Hu and Daniel Ralph, *Operations Research* 2007, DOI 10.1287/opre.1070.0431.

Online Appendix

PROOF OF PROPOSITION 1. Let $\mathscr{C}$ denote the (closed convex) feasible set of (1). Lemma 1 gives the existence of the solution $q^*(a)$ of (1). The usual optimality conditions for convex programs say that the gradient of the objective function of (1) at $q^*(a)$ satisfies

$$[a+2Qq^*(a)]^T(c-q^*(a))\geqslant 0,\quad \text{for all } c\in\mathscr{C}.$$

For another value of a, call it a', the optimal q is $q^*(a')\in\mathscr{C}$. Taking $c=q^*(a')$ above gives

$$[a+2Qq^*(a)]^T(q^*(a')-q^*(a))\geqslant 0.$$

Exchanging a and a' by symmetry,

$$[a'+2Qq^*(a')]^T(q^*(a)-q^*(a'))\geqslant 0.$$

Adding terms in the last two inequalities, we have

$$[a-a'+2Q(q^*(a)-q^*(a'))]^T(q^*(a')-q^*(a))\geqslant 0.$$

Hence

$$(a-a')^T(q^*(a')-q^*(a))\geqslant 2(q^*(a)-q^*(a'))^TQ(q^*(a)-q^*(a'))\geqslant 0 \tag{EC1}$$

where nonnegativity follows because Q is a diagonal matrix with positive diagonal entries (or, more generally, Q is positive definite).

If $a'_j=a_j$ for each $j\neq i$, then

$$(a-a')^T(q^*(a')-q^*(a))=(a_i-a'_i)(q^*_i(a')-q^*_i(a)). \tag{EC2}$$

So, (EC1) implies that $(a_i-a'_i)(q^*_i(a')-q^*_i(a))\geqslant 0$, which means that $q^*_i(\cdot,a_{-i})$ is a non-increasing function in a_i.

In addition, note that Q is a diagonal matrix with ith diagonal element $B_i>0$, hence $(q^*(a)-q^*(a'))^TQ(q^*(a)-q^*(a'))\geqslant(q^*_i(a)-q^*_i(a'))^2B_i$. From (EC1) and (EC2) we get

$$|q^*_i(a)-q^*_i(a')|\leqslant|a_i-a'_i|/(2B_i),$$

i.e., $q^*(\cdot,a_{-i})$ is Lipschitz of modulus $1/(2B_i)$. The required bound on the directional derivative follows. And the estimate of the set of generalized gradients follows directly from its definition. □

PROOF OF THEOREM 2. Define $X(a_{-i})=\{a_i\in\mathscr{A}_i\colon f_i(a_i,a_{-i})\geqslant 0\}$, $X^+(a_{-i})=\{a_i\in\mathscr{A}_i\colon f_i(a_i,a_{-i})>0\}$, and $X^-(a_{-i})=\{a_i\in\mathscr{A}_i\colon f_i(a_i,a_{-i})<0\}$. Let participant i be a generator in the proof; a similar proof holds for a consumer node. Let a_{-i} be fixed. Recall the payoff function $f_i(a_i)=[a_i-A_i+B_iq^*_i(a_i,a_{-i})]q^*_i(a_i,a_{-i})$ from (3). Using Proposition 1, we see that the directional derivative of the mapping $a_i\mapsto a_i-A_i+B_iq^*_i(a_i,a_{-i})$ in the direction $u=1$ is bounded below by $1/2$, i.e., the mapping is increasing in a_i. As $q^*_i(a_i,a_{-i})\geqslant 0$ is a non-increasing function in a_i by Proposition 1, the set $X^-(a_{-i})$ is an intersection of two open intervals in $\mathbb{R}$. So the set $X(a_{-i}) = \mathcal{A}_i \setminus X^-(a_{-i})$ is a closed interval of $\mathcal{A}_i$. It is nonempty since A_i is a feasible bid by assumption and $f_i(A_i) = B_i q_i^*(A_i, a_{-i})^2 \geqslant 0$. Similarly, the set $X^+(a_{-i})$ is an open, possibly empty, interval in $\mathcal{A}_i$.

If $X^+(a_{-i}) = \varnothing$, then $f_i(a_i, a_{-i}) = 0$ for any $a_i \in X(a_{-i})$. In this case, $f_i(a_i, a_{-i})$ is concave in $a_i \in X(a_{-i})$. So, assume that $X^+(a_{-i}) \neq \varnothing$.

Let $w_1, \ldots, w_W$ be the breakpoints of the derivative of $q_i^*(\cdot, a_{-i})$. The concavity of $f_i(\cdot, a_{-i})$ in (w_j, w_{j+1}) follows from the negative semidefiniteness of its Hessian, that is,

$$f_i'' = 2(1 + B_i q_i^{*\prime}) q_i^{*\prime} \leqslant 0, \tag{EC3}$$

where f_i'' is the second derivative of $f_i(\cdot, a_{-i})$, which follows from Proposition 1. Note that (EC3) implies that the derivative f_i' of $f_i(\cdot, a_{-i})$ is nonincreasing between any two consecutive breakpoints. Now we check the concavity of $f_i(\cdot, a_{-i})$ near a breakpoint w_j such that $f_i(w_j, a_{-i}) > 0$. By inequality (5), we have the following inequality:

$$\begin{aligned} \lim_{v \uparrow w_j} f_i'(v, a_{-i}) &= \lim_{v \uparrow w_j} [q_i^*(v, a_{-i}) + (v - A_i + 2B_i q_i^*(v, a_{-i})) q_i^{*\prime}(v, a_{-i})] \\ &\geqslant \lim_{v \downarrow w_j} [q_i^*(v, a_{-i}) + (v - A_i + 2B_i q_i^*(v, a_{-i})) q_i^{*\prime}(v, a_{-i})] = \lim_{v \downarrow w_j} f_i'(v, a_{-i}) \end{aligned} \tag{EC4}$$

since $w_j - A_i + 2B_i q_i^*(w_j, a_{-i}) > w_j - A_i + B_i q_i^*(w_j, a_{-i}) > 0$ by the positive profit condition.

Inequality (EC4) implies that the (generalized) gradient of $f_i(\cdot, a_{-i})$ is non-increasing near w_j. Therefore, $f_i(\cdot, a_{-i})$ is concave on $X^+(a_{-i})$ and is quasi-concave on the nonempty closed interval $X(a_{-i})$. By Theorem 1, the game $\{(4)\}_{i=1}^N$ has at least one Nash equilibrium. □

Proof of Proposition 5. Statement (i) ⇒ statement (ii): Write $\pi = (\mu, \lambda)$ and $\pi^* = (\mu^*, \lambda^*)$. Fix i and $x_{-i} = x_{-i}^*$. For any neighborhood U_i of (x_i^*, y^*) there exists a neighborhood V_i of (x_i^*, y^*, π^*) such that $(x_i, y, \pi) \in V_i$ implies $(x_i, y) \in U_i$, and the corresponding value of objective function of the MPEC (11) coincides with that of the bilevel game (9). It is therefore clear that (x^*, y^*, π^*) is a local Nash point of the EPEC if (x^*, y^*) is a local Nash point of the bilevel program.

Statement (ii) ⇒ statement (iii): This relies on a combination of standard MPEC stationary conditions using MPEC-LICQ, e.g., Scheel and Scholtes (2000), and work relating these to stationary conditions for NLP formulations of MPECs, Anitescu (2005). □

Proof of Theorem 3. Recall we are dealing with the implicit game $\{(4)\}_{i=1}^N$, the bid-a-only bilevel game $\{(2)\}_{i=1}^N$ with $b_i = B_i$ for all players, and the bid-a-only EPEC formulation $\{(15)\}_{i=1}^N$. As mentioned in §3.1, local Nash equilibria of the implicit game are equivalent to those of the bilevel game. Since the LICQ condition ensures that KKT multipliers of the lower-level problem are implicit functions of the bid vector a, then $\{(4)\}_{i=1}^N$ is also an implicit programming reformulation of the EPEC. Thus local Nash equilibria of the EPEC are equivalently local Nash equilibria of the former games, or, statements (i)–(iii) of the theorem are equivalent. The theorem will therefore hold if we can show that Nash stationary equilibria of the EPEC are local Nash equilibria of the implicit game, and further show that associated payoffs are nonnegative.

Let f_i be player i's objective function, (3). From Proposition 6 we have that a_i^* is B-stationary for Player i's problem (4). Furthermore, since $f_i(\cdot, a_{-i}^*)$ is piecewise quadratic, from previous remarks we know a sufficient condition for local optimality of a_i^* is that for any feasible direction α_i with $f_i'(a_i^*, a_{-i}^*; \alpha_i) = 0$, we must have $f_i''(a_i^*, a_{-i}^*; \alpha_i, \alpha_i) \leqslant 0$. We will prove that this sufficient condition holds; this yields statement (ii). Then we'll show that B-stationary points have nonnegative payoffs, statement (iii). We'll refer to $q_i(a_i^*, a_{-i}^*)$ as q_i^* but will drop the $*$ superscript for a_i and a_{-i}.

Let a_{-i} be given. Using piecewise linearity of $q_i(\cdot, a_{-i})$, we can show (details given after this proof) that

$$f_i''(a_i, a_{-i}; \alpha_i, \alpha_i) = f_i'(a_i, a_{-i}; \alpha_i) + q_i'(a_i, a_{-i}; \alpha_i)\alpha_i + B_i(q_i'(a_i, a_{-i}; \alpha_i))^2.$$

Let α_i be a feasible direction at a_i such that $f_i'(a_i, a_{-i}; \alpha_i) = 0$; assume $\alpha_i \neq 0$ without loss of generality. We have

$$f_i''(a_i, a_{-i}; \alpha_i, \alpha_i) = [\alpha_i + B_i q_i'(a_i, a_{-i}; \alpha_i)] q_i'(a_i, a_{-i}; \alpha_i)$$

and by Proposition 1,

$$-\frac{\alpha_i}{2B_i} \leqslant q_i'(a_i, a_{-i}; \alpha_i) \leqslant 0 \quad \text{if } \alpha_i > 0, \qquad -\frac{\alpha_i}{2B_i} \geqslant q_i'(a_i, a_{-i}; \alpha_i) \geqslant 0 \quad \text{if } \alpha_i < 0.$$

Therefore, in both cases, we have $f_i''(a_i, a_{-i}; \alpha_i, \alpha_i) \leqslant 0$, hence a_i is a local maximum for Player i.

Finally, we shall show that $f_i(a_i, a_{-i}) \geqslant 0$. We assume without loss of generality that $q_i^* > 0$ and $a_i < \bar{A}_i$. Therefore $\alpha_i = 1$ is a feasible direction, hence B-stationarity of a_i yields

$$0 \geqslant f_i'(a_i, a_{-i}; 1) = q_i^*[1 + 2B_i q_i'(a_i, a_{-i}; 1)] + (a_i - A_i) q_i'(a_i, a_{-i}; 1).$$

Proposition 1 tells us that the first term is nonnegative, whence $(a_i - A_i)q_i'(a_i, a_{-i}; 1) \leqslant 0$; and that $q_i'(a_i, a_{-i}; 1) \leqslant 0$, whence either $a_i \geqslant A_i$ or $q_i'(a_i, a_{-i}; 1) = 0$. If $q_i'(a_i, a_{-i}; 1) = 0$ then $f_i'(a_i, a_{-i}; 1) = q_i^* > 0$, which is not possible. Instead, we have $a_i \geqslant A_i$ and it follows from the definition of f_i that $f_i(a_i, a_{-i}) > 0$. □

Derivation of the Second Order Directional Derivative in the Proof of Theorem 3.
We derive the formula for an arbitrary player. Its index, strategies of all other players and all subscripts and superscripts are dropped here to simplify notations. First for a piecewise linear function $g(x)$ of the form $g(x) = g_0 + k(x - x_0)$ with $x \geqslant x_0$, we have for $h > 0$

$$g'(x_0; h) = kh, \qquad g''(x_0; h, h) = kh.$$

Now for any player, its profit function takes the form of $f(a) = (a - A)q(a) + Bq(a)^2$ and noting that $q(a)$ is piecewise linear in a as shown before, we have

$$\begin{aligned}
f'(a; h) &= \lim_{t \downarrow 0^+} [f(a + th) - f(a)]/t \\
&= \lim_{t \downarrow 0^+} \left[(a - A)\frac{q(a + th) - q(a)}{t} + hq(a + th) + B(q(a + th) + q(a))\frac{q(a + th) - q(a)}{t} \right] \\
&= (a - A)q'(a; h) + hq(a) + 2Bq'(a; h).
\end{aligned}$$

Therefore the second order directional derivative is given by

$$\begin{aligned}
f''(a; h, h) &= \lim_{t \downarrow 0^+} [f(a + th + t^2 h) - f(a) - tf'(a; h)]/t^2 \\
&= \lim_{t \downarrow 0^+} \Bigg[(a - A)\frac{q(a + th + t^2 h) - q(a) - tq'(a; h)}{t^2} + h\frac{q(a + th) - q(a)}{t} + hq(a + th + t^2 h) \\
&\qquad + B\frac{q(a + th + t^2 h)^2 - q(a)^2 - 2tq(a)q'(a; h)}{t^2} \Bigg] \\
&= (a - A)q'(a; h) + hq'(a; h) + hq(a) \\
&\qquad + B \lim_{t \downarrow 0^+} (q(a + th + t^2 h) + q(a))\frac{q(a + th + t^2 h) - q(a) - tq'(a; h)}{t^2} \\
&\qquad + Bq'(a; h) \lim_{t \downarrow 0^+} \frac{q(a + th + t^2 h) - q(a)}{t} \\
&= (a - A + 2B)q'(a; h) + q(a)h + q'(a; h)h + Bq'(a; h)^2 \\
&= f'(a; h) + q'(a; h)h + Bq'(a; h)^2
\end{aligned}$$

which establishes the formula for the second order parabolic directional derivative of f_i used in the proof of Theorem 3. □

Proof of Proposition 6. Let $(x^*, y^*, \lambda^*, \mu^*)$ be Nash stationary with the LICQ holding as described above. Consider player i's MPEC with $x_{-i} = x_{-i}^*$. According to Anitescu (2005), since $(x_i^*, y^*, \lambda^*, \mu^*)$ is stationary for the corresponding NLP (13), then it is strongly stationary for the MPEC (11) in the terminology of Scheel and Scholtes (2000). Strong stationarity implies what is called B-stationarity of the MPEC which is expressed in terms of a linearization of the problem about the point $(x_i^*, y^*, \lambda^*, \mu^*)$; again see Scheel and Scholtes (2000). Since the constraint functions of the MPEC are already affine, linearizing these functions is redundant and we can express B-stationarity using the tangent cone T to the feasible set at $(x_i^*, y^*, \lambda^*, \mu^*)$, that is

$$\nabla_{x_i, y} \phi_i(x^*, y^*)(d_i, d_y) \leqslant 0 \quad \text{for each } d = (d_i, d_y, d_\lambda, d_\mu) \in T. \tag{EC5}$$

This is a basic stationarity condition for the MPEC discussed in generality in Luo et al. (1996, §3.1); it is implied by B-stationarity under smooth rather than affine constraint functions, and is equivalent to B-stationarity under linear constraint functions or other constraint qualifications.

Next suppose d_i lies in the tangent cone of the feasible set $\{x_i\colon\ g_i(x_i) \geqslant 0, h_i(x_i) = 0\}$ of the implicit program (15) at x_i^*. This feasible set is exactly the upper-level feasible set for (11) and is polyhedral. This means $x_i^* + \tau d_i$ is upper-level feasible for all small $\tau > 0$. Now use piecewise smoothness of $(y(x), \lambda(x), \mu(x))$ to obtain the following partial directional derivatives:

$$(y_i'(x^*; d_i), \lambda_i'(x^*; d_i), \mu_i'(x^*; d_i)) = \lim_{\tau \downarrow 0} \frac{(y(x_i^* + \tau d_i), \lambda(x_i^* + \tau d_i), \mu(x_i^* + \tau d_i)) - (y^*, \lambda^*, \mu^*)}{\tau}.$$

That is, $(d_i, y_i'(x^*; d_i), \lambda_i'(x^*; d_i), \mu_i'(x^*; d_i))$ belongs to T.

Finally we have for any d_i, from first principles or by using a chain rule Shapiro (1990), that

$$f_i'(x_i^*, x_{-i}^*; d_i) = \nabla_{x_i, y} \phi_i(x^*, y^*)(d_i, y_i'(x^*; d_i)).$$

With this, the above tangent cone relationship and the MPEC B-stationary condition (EC5), the result follows. □