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E-companion to: "Regulated random walks and the LCFS backlog probability: analysis and application"

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Abstract

Random walks have been used extensively within operations research models, such as inventory systems and single-server queues, to estimate performance measures. In this paper, we use sample-path analysis to express the steady-state probability of a one-sided regulated random walk to increase and be above a threshold, referred to as the last-come-first-serve (LCFS) backlog probability. We approximate the LCFS backlog probability under mild assumptions on the distribution of the random walk's steps and provide its exact expression when the steps are exponentially distributed, and a closed form approximation when the steps are normally distributed. In our numerical experiments, the average relative gap between the approximated LCFS backlog probabilities and their simulated values is 5.13%. We further show that the LCFS backlog probability is an upper bound on the loss probability—the probability that a two-sided regulated random walk is at a boundary. This bound is tighter than the backlog probability—the probability that a random walk ever crosses a threshold—that also bounds the loss probability. In an inventory application, we demonstrate that using the LCFS backlog probability rather than the backlog probability reduces the inventory level required to satisfy a service-level constraint on the percentage of orders backlogged. In our examples, this reduction leads to cost savings of 31% on average.

Subject classifications: Inventory/production, uncertainty, stochastic: base stock level for a single item. Probability, random walk: bounds on threshold related probabilities

Area of review: Stochastic Models.

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1 Detailed Numerical Results

1.1 Shifted-Exponential Steps

We consider shifted-exponential steps with $\mu = 1$ and a shift r of 1.25, 1.5, and 1.75. For any drift and service level we found d_{BLG} from (35) and d_{LCFS} from (36). The results are reported in Table 1.

We found that the average Backlog and Loss Ratios are 0.53 and 0.75, respectively, the average c Ratio is 0.974, which is translated to an average Error $P_{LCFS}(d)$ of 3.1%, because $P(Y > d)$ is exact. Observe that in the example of service level of 80% and a drift of 0.75, $d_{LCFS} = 0$ (and therefore $d_B = 0$); moreover, $d_{PL} = 0$ for drifts of 0.5 and 0.25 as well. Finally, the Cost Ratio ranged between 14.5% and 100% (100% for service level 80% and a drift of 0.75), averaged at 43.7%.

1.2 Normal(10, 2), With d From Normal(10, 1)

To compare the change in accuracy of our approximations with the standard deviation of X , we ran the case with normal steps with a standard deviation $\sigma = 2$ using the optimal d s for normal steps with $\sigma = 1$. That is we consider normal steps with a mean μ of -0.25 , -0.5 , and -0.75 and for any μ and service level we found different d values as if steps are normal(10, 1) distributed. (We chose d_{BLG} based on (39) and d_{LCFS} by multiplying c from (38) by $P(Y > d)$ from (39). We only evaluated the first 250 elements of the infinite series in (38) to express c .)

The detailed results are reported in Table 2. We see that the average Backlog and Loss Ratios were 0.59 and 0.78, respectively, the average c Ratio is 1.002, and the average Error $P_{LCFS}(d)$ is 13.%. Thus, in this scenario, with relatively small d 's and high backlog probabilities, our approximations for c are better than the approximation for $P(Y > d)$.

Service Level	Drift	d	Values	Simulated Probabilities of			Analytic Probabilities of		Cost
				Backlog	LCFS Backlog	Loss	Backlog	LCFS Backlog	
80%	0.25	d_BLG	3.084	0.197	0.090	0.052	0.200	0.092	2.330
		d_LCFS	0.999	0.433	0.198	0.148	0.434	0.200	0.745
		d_PL	0.000	0.629	0.287	0.287	0.629	0.290	0.000
		d_B	0.623	0.501	0.228	0.187	0.499	0.230	0.461
	0.5	d_BLG	1.262	0.201	0.108	0.089	0.200	0.110	1.039
		d_LCFS	0.233	0.363	0.194	0.186	0.364	0.200	0.184
		d_PL	0.000	0.629	0.223	0.223	0.417	0.229	0.000
		d_B	0.143	0.384	0.205	0.200	0.384	0.211	0.112
	0.75	d_BLG	0.508	0.200	0.121	0.114	0.200	0.126	0.433
		d_LCFS	0.000	0.629	0.174	1.000	0.287	0.181	0.000
		d_PL	0.000	0.629	0.174	0.174	0.287	0.181	0.000
		d_B	0.000	0.629	0.174	0.174	0.287	0.181	0.000
90%	0.25	d_BLG	4.950	0.098	0.045	0.023	0.100	0.046	3.862
		d_LCFS	2.865	0.219	0.100	0.057	0.217	0.100	2.151
		d_PL	1.127	0.410	0.187	0.137	0.414	0.191	0.841
		d_B	2.070	0.296	0.134	0.084	0.291	0.134	1.542
	0.5	d_BLG	2.451	0.100	0.054	0.041	0.100	0.055	2.057
		d_LCFS	1.422	0.182	0.097	0.079	0.182	0.100	1.174
		d_PL	1.023	0.229	0.122	0.103	0.230	0.126	0.839
		d_B	1.165	0.210	0.113	0.094	0.212	0.116	0.959
	0.75	d_BLG	1.481	0.100	0.060	0.053	0.100	0.063	1.301
		d_LCFS	0.836	0.158	0.096	0.088	0.158	0.100	0.720
		d_PL	0.651	0.181	0.109	0.102	0.181	0.114	0.557
		d_B	0.712	0.172	0.104	0.097	0.173	0.109	0.612
95%	0.25	d_BLG	6.817	0.050	0.023	0.011	0.050	0.023	5.496
		d_LCFS	4.732	0.106	0.048	0.025	0.108	0.050	3.681
		d_PL	2.986	0.210	0.095	0.055	0.207	0.096	2.246
		d_B	3.342	0.184	0.084	0.047	0.182	0.084	2.527
	0.5	d_BLG	3.640	0.050	0.027	0.019	0.050	0.027	3.123
		d_LCFS	2.611	0.092	0.049	0.037	0.091	0.050	2.196
		d_PL	2.180	0.118	0.063	0.048	0.117	0.064	1.820
		d_B	2.218	0.115	0.062	0.047	0.115	0.063	1.854
	0.75	d_BLG	2.453	0.049	0.030	0.025	0.050	0.032	2.193
		d_LCFS	1.808	0.079	0.048	0.042	0.079	0.050	1.597
		d_PL	1.601	0.092	0.056	0.049	0.092	0.058	1.410
		d_B	1.605	0.093	0.056	0.049	0.092	0.058	1.411
99%	0.25	d_BLG	11.151	0.009	0.004	0.002	0.010	0.005	9.569
		d_LCFS	9.066	0.023	0.010	0.005	0.022	0.010	7.552
		d_PL	7.449	0.038	0.018	0.009	0.040	0.018	6.069
		d_B	7.097	0.046	0.021	0.010	0.045	0.021	5.737
	0.5	d_BLG	6.402	0.010	0.005	0.004	0.010	0.005	5.744
		d_LCF	5.373	0.019	0.010	0.007	0.018	0.010	4.749
		d_PL	4.781	0.027	0.014	0.010	0.026	0.014	4.188
		d_B	4.867	0.025	0.013	0.009	0.024	0.013	4.273
	0.75	d_BLG	4.712	0.010	0.006	0.005	0.010	0.006	4.357
		d_LCFS	4.067	0.015	0.009	0.008	0.016	0.010	3.724
		d_PL	3.856	0.019	0.011	0.010	0.018	0.012	3.518
		d_B	3.776	0.019	0.012	0.010	0.019	0.012	3.444

Table 1: Comparison of d_{BLG} , d_{LCFS} , d_{PL} , and d_{BA} ; the simulated backlog, LCFS backlog, and loss probabilities; the analytic backlog and LCFS backlog probabilities; and the cost for random walks with shifted-exponential steps.

Service Level	Draft	d	Values	Simulated Probabilities of			Analytic Probabilities of		Cost
				Backlog	LCFS Backlog	Loss	Backlog	LCFS Backlog	
80%	0.25	d_BLG	2.636	0.628	0.344	0.22230	0.622	0.342	1.513
		d_LCFS	1.595	0.704	0.387	0.29360	0.708	0.389	0.901
		d_PL	0.912	0.760	0.415	0.35440	0.771	0.424	0.510
		d_BA	1.142	0.746	0.407	0.33400	0.749	0.412	0.639
	0.5	d_BLG	1.026	0.567	0.335	0.28990	0.578	0.344	0.635
		d_LCFS	0.606	0.619	0.363	0.33370	0.642	0.382	0.370
		d_PL	0.496	0.632	0.370	0.34520	0.660	0.392	0.301
	0.75	d_BA	0.505	0.629	0.369	0.34370	0.658	0.391	0.307
		d_BLG	0.490	0.498	0.313	0.29590	0.537	0.338	0.324
		d_LCFS	0.252	0.536	0.334	0.32450	0.588	0.369	0.164
		d_PL	0.103	0.558	0.346	0.34140	0.621	0.391	0.067
		d_BA	0.121	0.554	0.344	0.33900	0.617	0.388	0.078
90%	0.25	d_BLG	4.022	0.518	0.285	0.15550	0.523	0.287	2.377
		d_LCFS	2.981	0.593	0.326	0.20150	0.595	0.327	1.730
		d_PL	2.136	0.658	0.362	0.25250	0.662	0.364	1.219
		d_BA	2.163	0.662	0.363	0.25270	0.660	0.362	1.232
	0.5	d_BLG	1.720	0.483	0.288	0.22870	0.486	0.289	1.085
		d_LCFS	1.299	0.535	0.318	0.26530	0.540	0.321	0.808
		d_PL	1.201	0.542	0.322	0.27300	0.553	0.329	0.747
	0.75	d_BA	1.195	0.544	0.322	0.27360	0.554	0.329	0.743
		d_BLG	0.952	0.432	0.275	0.24720	0.452	0.284	0.640
		d_LCFS	0.714	0.467	0.295	0.27210	0.494	0.311	0.475
		d_PL	0.694	0.470	0.297	0.27530	0.498	0.313	0.461
		d_BA	0.694	0.469	0.296	0.27420	0.498	0.313	0.462
95%	0.25	d_BLG	5.409	0.440	0.242	0.11620	0.440	0.242	3.264
		d_LCFS	4.368	0.500	0.276	0.14590	0.501	0.275	2.584
		d_PL	3.404	0.566	0.311	0.18230	0.565	0.310	1.984
		d_BA	3.231	0.577	0.317	0.18970	0.577	0.317	1.875
	0.5	d_BLG	2.413	0.408	0.244	0.17920	0.409	0.243	1.552
		d_LCFS	1.992	0.449	0.268	0.20640	0.454	0.270	1.269
		d_PL	1.850	0.464	0.278	0.21720	0.471	0.280	1.174
	0.75	d_BA	1.854	0.466	0.279	0.21730	0.470	0.279	1.176
		d_BLG	1.414	0.373	0.238	0.20480	0.380	0.239	0.966
		d_LCFS	1.176	0.403	0.257	0.22650	0.419	0.263	0.797
		d_PL	1.207	0.401	0.255	0.22450	0.411	0.258	0.817
		d_BA	1.176	0.403	0.257	0.22670	0.416	0.261	0.796
99%	0.25	d_BLG	8.627	0.291	0.160	0.06390	0.294	0.162	5.479
		d_LCFS	7.587	0.331	0.183	0.07750	0.335	0.184	4.735
		d_PL	6.381	0.382	0.211	0.09620	0.389	0.214	3.909
		d_BA	6.070	0.393	0.216	0.10020	0.405	0.222	3.717
	0.5	d_BLG	4.022	0.271	0.162	0.10460	0.273	0.162	2.711
		d_LCFS	3.602	0.303	0.182	0.12070	0.337	0.200	2.396
		d_PL	3.530	0.309	0.185	0.12350	0.309	0.184	2.338
	0.75	d_BA	3.430	0.318	0.191	0.12820	0.317	0.188	2.266
		d_BLG	2.487	0.255	0.165	0.13030	0.254	0.160	1.762
		d_LCFS	2.249	0.280	0.180	0.14410	0.278	0.175	1.580
		d_PL	2.333	0.268	0.173	0.13810	0.269	0.169	1.649
		d_BA	2.249	0.279	0.180	0.14410	0.278	0.175	1.582

Table 2: Comparison of d_{BLG} , d_{LCFS} , d_{PL} , and d_{BA} ; the simulated backlog, LCFS backlog, and loss probabilities; the analytic backlog and LCFS backlog probabilities; and the cost for random walks with Normal(10,2) steps with d corresponding to the service level constraints for Normal(10,1) steps.