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Appendix: Proofs of Consistency of the Second Stage Estimation

Proof of Proposition 1:

We wish to show that plim $\hat{\tilde{\beta}} = \beta$, where $\hat{\tilde{\beta}}$ is the OLS estimator of $\tilde{\beta}$ in (14). If the true productivity scores ln $\tilde{\theta}$ are available then the OLS estimation of

$$\ln \tilde{\theta} = \beta_0 - Z\beta + \delta \tag{A1}$$

yields a consistent estimator of β. Here, when the DEA estimator ln $\hat{\tilde{\theta}}$ is used in place of ln $\tilde{\theta}$ we need to show that the OLS estimator of $\tilde{\beta}$ in

$$\ln \hat{\tilde{\theta}} = \tilde{\beta}_0 - Z\,\tilde{\beta} + \tilde{\delta} \tag{A2}$$

also yields a consistent estimator of β. In finite samples, DEA estimators are biased (Banker 1993) since

$$\ln \hat{\tilde{\theta}}_j = \ln \tilde{\theta}_j + \eta_j \tag{A3}$$

where $\eta_j \geq 0$ for all j. From (A1), (A2) and (A3), we have

$$\begin{aligned}\hat{\tilde{\beta}} &= -(Z'Z)^{-1}Z' \ln \hat{\tilde{\theta}} \\ &= -(Z'Z)^{-1}Z' (\ln \tilde{\theta} + \eta) \\ &= -(Z'Z)^{-1}Z' (-Z\beta + \delta + \eta) \\ &= \beta - \mathbf{Q}^{-1}(Z'\delta/n) - \frac{1}{n}\mathbf{Q}^{-1} Z'\eta \end{aligned} \tag{A4}$$

First we show that the second term on the RHS converges in probability to zero. Since $\delta = -E(v-u) + (v-u)$ and since the random variables z_s, u and v are mutually independent, the random variables δ and z_s are mutually independent. Therefore, $\text{plim}(Z'\delta/n) = 0$ and the second term on the RHS of (A4) converges in probability to zero (Greene 1993, p. 353).

Next we show that the last term on the RHS in (A4) converges in probability to zero. From Banker (1993) we know that not only are DEA estimators consistent, but also $\text{plim}(\eta \mid Z) = 0$ for any Z. Therefore, $\text{plim}(Z'\eta) = 0$. Since both the second and third terms on the RHS of (A4) converge in probability to zero, plim $\hat{\tilde{\beta}} = \beta$.

Denote the variance of δ_j as σ^2. Since the limiting distribution of Z' η degenerates to a point located at the origin, the variance of the third term in (A4) converges to zero. Using this result, the limiting distribution of $\sqrt{n}$ ($\hat{\tilde{\beta}} - \beta$) can be derived as N[$\mathbf{0}$, $\sigma^2\, \mathbf{Q}^{-1}$] as in the standard case of OLS estimation (Greene 1993, p. 353). As a consequence, the asymptotic distribution of $\hat{\tilde{\beta}}$ can be derived as N[β, $(\sigma^2/n)\, \mathbf{Q}^{-1}$].

<u>Proof of Proposition 2</u>:

Our proof is based on Bierens (1994) and Gstach (1998). First stage DEA provides consistent estimators $\hat{\tilde{\theta}}$ for the productivity variable $\tilde{\theta}$. Let the p.d.f. of the error term ε be given by $f(\varepsilon;\alpha)$, and let $\xi = (\alpha,\beta)$. Let Θ and Ξ, respectively, denote the probability space of $\tilde{\theta}$ and ξ.

Recall from (13) that

$$\varepsilon = Z\beta + V^M + \ln \tilde{\theta}$$

The estimator $\hat{\varepsilon}$ for ε, therefore, is

$$\hat{\varepsilon} = Z\beta + V^M + \ln \hat{\tilde{\theta}}$$

By the assumed compactness of the input space *X*, its interior $\Im(X)$ is well defined. Similarly, the interior $\Im(Z)$ is also well defined for the contextual variables space *Z*. Let $\Omega \equiv X \times Z \times [0,1]$ denote the probability space under consideration.

Let $\hat{Q}_n(\xi) \equiv \ln\left[\prod_n f_\varepsilon(\hat{\varepsilon} \mid \xi)\right] \Big/ n$ and $Q_n(\xi) \equiv \ln\left[\prod_n f_\varepsilon(\varepsilon \mid \xi)\right] \Big/ n$

define, respectively, the mean log-likelihood function based on the estimator $\hat{\varepsilon}$ and the true value ε. Further, let

$$Q(\xi) \equiv \lim_{n\to\infty} E[Q_n(\xi)]$$

To prove that the ML estimator $\hat{\xi}_n$, implicitly defined by

$$\hat{Q}_n(\hat{\xi}_n) \equiv \sup_{\xi\in\Xi} \ln\left[\prod_n f_\varepsilon(\hat{\varepsilon} \mid \xi)\right] \Big/ n,$$

converges pseudo-uniformly in probability to ξ_0, where ξ_0 is a unique point in Ξ such that $Q(\xi_0) = \sup_{\xi\in\Xi} Q(\xi)$, it is sufficient to show that $\hat{Q}_n(\xi) \xrightarrow{Prob} Q(\xi)$ pseudo-uniformly $\forall \xi \in \Xi$ (Bierens 1994). We prove below that this is indeed the case and, therefore, the ML estimator of β is consistent.

Proof of $\hat{Q}_n(\xi) \xrightarrow{\Pr ob} Q(\xi)$

The proof proceeds in two steps. First we show that $\hat{Q}_n(\xi) \xrightarrow{Prob} Q_n(\xi)$ and $Q_n(\xi) \xrightarrow{Prob} Q(\xi)$ and then use these results to show that $\hat{Q}_n(\xi) \xrightarrow{Prob} Q(\xi)$.

We prove by contradiction that $\hat{Q}_n(\xi) \xrightarrow{Prob} Q_n(\xi)$. Suppose this is not the case. Then $\lim_{n\to\infty} \Pr[\sup_{\xi\in\Xi} |\hat{Q}_n(\xi) - Q_n(\xi)| \le \varepsilon_0] < 1$ for some $\varepsilon_0 > 0$. Because of the continuity of the supremum function this implies that there exists a set $X' \times Z' \times \Theta' \subseteq \Omega$ and an infinite dimensional vector $\{\mathbf{x}_\infty, \mathbf{z}_\infty, \boldsymbol{\theta}_\infty\} \equiv \{x_i, z_i, \theta_i\}_{i=1}^{\infty}$ such that

$\{x_i, z_i, \theta_i\} \in X' \times Z' \times \Theta' \ \forall i = 1, 2, ...$ and $\sup_{\xi \in \Xi} |\hat{Q}_\infty(\xi) - Q_\infty(\xi)| > \varepsilon_0$. But when n→∞ the difference between $\tilde{\hat{\theta}}_i$ and $\tilde{\theta}_i$ vanishes for all i and, therefore, the difference between $\hat{Q}_n(\xi)$ and $Q_n(\xi)$ also vanishes for all $\xi \in \Xi$. Therefore, as n→∞, $\sup_{\xi \in \Xi} |\hat{Q}_\infty(\xi) - Q_\infty(\xi)| = 0$ and we then have 0 > ε_0 which contradicts the starting condition. So, $\hat{Q}_n(\xi) \xrightarrow{Prob} Q_n(\xi)$ pseudo-uniformly. If we define $\xi_1(\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n)$ as that value of ξ that maximizes the absolute difference $|\hat{Q}_n(\xi) - Q_n(\xi)|$, this is equivalent to saying that for all $\varepsilon_1 > 0$, there exists a set

$$\Omega_{1,n} \equiv \left\{\{x_n, z_n, \theta_n\} : |\hat{Q}_n(\xi_1(.)) - Q_n(\xi_1(.))| \le \varepsilon_1\right\}$$

$$\text{with } \lim_{n\to\infty} \Pr\left[(\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n) \in \Omega_{1,n}\right] = 1. \quad \text{(A5)}$$

Recall that $Q_n(\xi)$, the mean log-likelihood function based on the values of the random variable ε=v-u, is well defined for all $\xi \in \Xi$, since the density f(ε), given by (4), is continuous. Therefore, $Q_n(\xi) \xrightarrow{Prob} Q(\xi)$ pseudo-uniformly. If we define $\xi_2(\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n)$ as that value of ξ that maximizes the absolute difference $|Q_n(\xi) - Q(\xi)|$, this is equivalent to saying that for all $\varepsilon_2 > 0$, there exists a set

$$\Omega_{2,n} \equiv \left\{\{\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n\} : |Q_n(\xi_2(.)) - Q(\xi_2(.))| \le \varepsilon_2\right\}$$

$$\text{with } \lim_{n\to\infty} \Pr\left[(\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n) \in \Omega_{2,n}\right] = 1 \quad \text{(A6)}$$

Combining (A5) and (A6), defining $\Omega_n = \Omega_{1,n} \cap \Omega_{2,n}$ and utilizing the inequality

$$|\hat{Q}_n(\xi) - Q_n(\xi)| + |Q_n(\xi) - Q(\xi)| \ge |\hat{Q}_n(\xi) - Q(\xi)| \quad \text{(A7)}$$

we immediately get the following result:

$$\text{For all } \varepsilon_1, \varepsilon_2 > 0, \text{ there exists } \Omega_n \equiv \left\{ \{\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n\} : | \hat{Q}_n\ (\xi_3(.)) - Q(\xi_3(.)) | \le \varepsilon_1 + \varepsilon_2 \right\}$$

$$\text{with } \lim_{n \to \infty} \Pr\left[(\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n) \in \Omega_n \right] = 1 \qquad \text{(A8)}$$

where $\xi_3(\mathbf{x}_n, \mathbf{z}_n, \boldsymbol{\theta}_n)$ is defined as that value of ξ that maximizes the absolute difference $| \hat{Q}_n\ (\xi) - Q(\xi) |$. But (A8) can be true only if $\hat{Q}_n\ (\xi) \xrightarrow{Prob} Q(\xi)$ pseudo-uniformly leading to the desired result.

References

Banker, R.D., 1993. Maximum Likelihood, Consistency and Data Envelopment Analysis: A Statistical Foundation. *Management Science,* **39**(10), 1265-1273.

Bierens, H. J., 1994. Estimation, Testing, and Specification of Cross-section and Time Series Models. *Topics in Advanced Econometrics*, Cambridge University Press, Cambridge, UK.

Greene, W.H., 1993. *Econometric Analysis*. Prentice-Hall, Englewood Cliffs, NJ.

Gstach, D., 1998. Another Approach to Data Envelopment Analysis in Noisy Environments: DEA+. *Journal of Productivity Analysis,* **9**, 161-176.