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Capacity Allocation and Scheduling in Supply Chains

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ELECTRONIC COMPANION

Proof of Theorem 5

Given a payoff vector and a coalition, it can be verified in polynomial time that the payoff vector is not in the core. Therefore, the core membership test belongs to the class *co-NP*. The *co-NP*-completeness proof is by reduction from the following problem.

Partition (Garey and Johnson 1979): Given $2t$ elements with integer sizes a_1, \dots, a_{2t} , where $\sum_{j=1}^{2t} a_j = 2A$, does there exist a partition S_1 and S_2 of the index set $\{1, \dots, 2t\}$ such that $\sum_{j \in S_1} a_j = \sum_{j \in S_2} a_j = A$? Without loss of generality, we assume that $A \geq 3$.

Given an instance of Partition, we construct an instance of the distributors' LP game, where $m = 1$ and $n = 2t + 2$. This instance is shown in Table 9, where for each distributor j , \bar{v}_j is the profit of its order per unit of capacity requirement, \hat{p}_j is the capacity requirement of its resubmittable order, X_j is the capacity allocated by the manufacturer, and χ_j is the payoff received. Let $\varepsilon > 0$ be defined sufficiently small that $\varepsilon(2A^2 - A - 2) < 1$.

j	1	2	...	$2t$	$2t + 1$	$2t + 2$
\bar{v}_j	ε	ε	...	ε	$2A + 1$	$2A + 1$
X_j	$(2A - 1)a_1 - 2$	$(2A - 1)a_2$...	$(2A - 1)a_{2t}$	0	0
\hat{p}_j	$(2A - 1)a_1$	$(2A - 1)a_2$...	$(2A - 1)a_{2t}$	$(2A - 1)A$	$(2A - 1)A$
χ_j	$(4A - 2)a_1 - 4$	$(4A - 2)a_2$...	$(4A - 2)a_{2t}$	$(4A^2 - 1)(A - 1)$	$(4A^2 - 1)(A - 1)$

Table 9: Instance of Distributors' LP Game in Theorem 5.

First, note that $\sum_{j=1}^{2t+2} X_j = (2A - 1) \sum_{j=1}^{2t} a_j - 2 = (2A - 1)2A - 2 < (2A - 1)2A = \hat{p}_{2t+1} + \hat{p}_{2t+2}$. Also, $\bar{v}_{2t+1} = \bar{v}_{2t+2} = 2A + 1$ is the maximum profit per unit of production capacity. Therefore, the grand coalition generates a profit of $(2A + 1) \sum_{j=1}^{2t+2} X_j = (2A + 1)[(2A - 1)2A - 2] = 8A^3 - 6A - 2$. Let Y denote this value of total profit. Also, the total payoff is $\sum_{j=1}^{2t+2} \chi_j = (4A - 2)a_1 - 4 + (4A - 2)a_2 + \dots + (4A - 2)a_{2t} + 2(4A^2 - 1)(A - 1) = (4A - 2)2A - 4 + 2(4A^2 - 1)(A - 1) = 8A^3 - 6A - 2 = Y$. Hence, the payoff vector is feasible.

We now show that the given payoff vector (χ_1, \dots, χ_n) is not in the core if and only if the instance of Partition has a solution.

(\Rightarrow) If the instance of Partition has a solution, then there exists at least one coalition N' with $\sum_{j \in N'} X_j = (2A - 1)A$. Coalition N' includes exactly one of the two distributors $2t + 1$ and $2t + 2$, but does not include distributor 1. The total payoff received by coalition N' is $\sum_{j \in N'} \chi_j = (4A^2 - 1)(A - 1) + 2 \sum_{j \in N'} X_j = (4A^2 - 1)(A - 1) + (2A - 1)2A =$

$(4A^2 - 1)A - (2A - 1) < (4A^2 - 1)A = (2A + 1)(2A - 1)A$, which is the total profit generated by this coalition. Hence, the given payoff vector (χ_1, \dots, χ_n) is not in the core.

(\Leftarrow) If the instance of Partition has no solution, we consider the only possible three types of coalition.

1. Coalitions not containing distributor $2t + 1$ or $2t + 2$. In this case, every coalition with total allocated capacity \bar{X} has a profit of $\varepsilon\bar{X}$, which is less than the payoff $2\bar{X}$ received from the payoff vector.
2. Coalitions containing both distributors $2t + 1$ and $2t + 2$. In this case, for every coalition N' , the profit generated is $Y - (2A + 1) \sum_{j \notin N'} X_j \leq Y - 2 \sum_{j \notin N'} X_j = Y - \sum_{j \notin N'} \chi_j$, where the last term is the total payoff received from the payoff vector.
3. Coalitions containing exactly one of distributors $2t + 1$ and $2t + 2$. In this case, the profit generated by a coalition is no greater than that of a coalition containing all orders except for order $2t + 2$: $\bar{v}_{2t+1}\hat{p}_{2t+1} + \varepsilon(\sum_{j=1}^{2t} X_j - \hat{p}_{2t+2}) = (2A + 1)(2A - 1)A + \varepsilon[(2A - 1)A - 2] = 4A^3 - A + \varepsilon(2A^2 - A - 2)$. Since the instance of Partition has no solution, the total capacity allocated to such a coalition cannot fall in the interval $((2A - 1)(A - 1), (2A - 1)[(A + 1) - 2])$. If the total capacity is no greater than $(2A - 1)(A - 1)$, then the total profit generated is no greater than $(2A + 1)(2A - 1)(A - 1) = (4A^2 - 1)(A - 1)$, which is the payoff received by distributor $2t + 1$ or $2t + 2$ alone. Alternatively, if the total capacity is no less than $(2A - 1)(A + 1) - 2$, then the total payoff received is at least $(4A^2 - 1)(A - 1) + 2[(2A - 1)(A + 1) - 2] = 4A^3 + A - 5$. Since $A \geq 3$ and $\varepsilon(2A^2 - A - 2) < 1$, the difference between the total payoff received by the distributors and their total profit is $(4A^3 + A - 5) - [4A^3 - A + \varepsilon(2A^2 - A - 2)] > (4A^3 + A - 5) - (4A^3 - A + 1) = 2A - 6 \geq 0$.

Therefore, in all cases, the total payoff received by any coalition is no less than the profit generated by the coalition. Hence, the given payoff vector is a core member. \square

Proof of Theorem 7

The proof is by reduction from Partition, which is defined in the proof of Theorem 5. Given an arbitrary instance of Partition, we construct an instance of the knapsack game, where $m = 1$ and $n = 2t + 1$. The detailed information is shown in Table 10, using the same notation as in Table 9. The last row of Table 10 contains an example payoff vector.

j	1	2	...	$2t$	$2t + 1$
\bar{v}_j	A	A	...	A	$A - 1$
\hat{p}_j	$2A$	$2A$...	$2A$	A
X_j	a_1	a_2	...	a_{2t}	A
χ_j	a_1A	a_2A	...	$a_{2t}A$	$A(A - 1)$

Table 10: Knapsack Game of Theorem 7.

Next we show that the knapsack game instance has an empty core if and only if the instance of Partition has a solution.

(\Rightarrow) If the instance of Partition has a solution, then consider three coalitions 1, 2 and 3, where coalitions 1 and 2 are disjoint and each contains distributors from $\{1, \dots, 2t\}$ with total capacity allocation A , and coalition 3 contains only distributor $2t + 1$. Any two of these three coalitions can jointly achieve a profit of $2A^2$. Let χ_1 , χ_2 and χ_3 be the payoffs of coalitions 1, 2 and 3, respectively. Then any fair payoff division (χ_1, χ_2, χ_3) satisfies

$$\begin{aligned}\chi_1 + \chi_2 &\geq 2A^2 \\ \chi_1 + \chi_3 &\geq 2A^2 \\ \chi_2 + \chi_3 &\geq 2A^2.\end{aligned}$$

Hence, $\chi_1 + \chi_2 + \chi_3 \geq 3A^2$. However, the grand coalition can generate profit of at most $2A^2 + A(A - 1)$, which implies that $\chi_1 + \chi_2 + \chi_3 = 3A^2 - A$. Therefore, no fair payoff division exists and the instance of the knapsack game has an empty core.

(\Leftarrow) If the instance of Partition has no solution, we show that the payoff vector in the last row of Table 10 is a core member. We consider the only possible two types of coalition.

1. Coalitions not containing distributor $2t + 1$. Here, if a coalition has allocated capacity $2A$, then it can generate a profit of at most $2A^2$. This profit is equal to the total payoff received. Otherwise, since any single order requires capacity $2A$ which exceeds the allocated capacity, the maximum profit is 0.
2. Coalitions containing distributor $2t + 1$. Here, no coalition has allocated capacity of exactly $2A$, otherwise a solution exists for the instance of Partition. First, the grand coalition has capacity $3A$, and a profit of $2A^2 + A(A - 1) = 3A^2 - A$. Since $A \sum_{j=1}^{2t} a_j + A(A - 1) = 3A^2 - A$, this profit is equal to the total payoff received. Second, for coalitions having capacity less than $2A$, since only distributor $2t + 1$'s order is feasible, the maximum profit is $A(A - 1)$. However, since this payoff is received by

distributor $2t + 1$ alone, the total payoff received is at least $A(A - 1)$. Finally, for coalitions having capacity less than $3A$ but greater than $2A$, only one order requiring capacity $2A$ can be selected. This order generates a profit of $2A^2$. However, the total allocated capacity is at least $A + 1$, and the payoff received by distributor $2t + 1$ is $A(A - 1)$. Therefore, the total payoff received by the distributors from $1, \dots, 2t$ is at least $(A + 1)A$. Thus, the total payoff received is at least $(A + 1)A + (A - 1)A = 2A^2$. Therefore in all cases, no coalition can generate more profit than the payoff received from the given payoff vector. It follows that the given payoff vector is in the core. \square