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Appendices

Appendix A: Proofs

THEOREM EC.2. *k-PLEX is NP-complete for any fixed positive integer k.*

PROOF. Since the k-plex definition is *nontrivial*, *interesting* and *hereditary on induced subgraphs* the maximum k-plex problem is NP-hard by the result of Yannakakis (1979). Here, we provide a more direct proof by reducing CLIQUE (Garey and Johnson 1979), a well-known NP-complete problem, to k-PLEX. Given an instance $\langle G=(V,E),c\rangle$ of CLIQUE, we construct an instance $\langle G'=(V',E'),c'\rangle$ in polynomial time such that G has a clique of size c *if and only if* G' has a k-plex of size c'. To construct G', we expand G by adding $k-1$ copies of the complete graph of order $n=|V|$. Denote the vertex set of the r^{th} such copy by V_r, $r=1,\dots,k-1$, where $V_r=\{1_r,\dots,n_r\}$, and let $R=\bigcup_{r=1}^{k-1}V_r$. Put $V'=V\cup R$ and $E'=E\cup\hat{E}\cup\tilde{E}$, where $\hat{E}=\{(i,j_r): i\in V, j_r\in V_r, i\neq j, r=1,\dots,k-1\}$ and $\tilde{E}=\{(i_p,j_r): i_p\in V_p, j_r\in V_r, i\neq j, p,r=1,\dots,k-1\}$. The set $\hat{E}$ represents the edges between V and R, where every vertex $u\in V$ is connected to every vertex in every complete graph except its copies, *i.e.*, u is adjacent to every vertex in $R\setminus\{u_1,\dots,u_{k-1}\}$. The set $\tilde{E}$ includes the cross edges between distinct V_p and V_r, as well as all possible edges between vertices in V_p, $p=1,\dots,k-1$. In other words, every vertex $u_p\in V_p, p=1,\dots,k-1$ is adjacent to all the vertices in $V_r\setminus\{u_r\}$, $r=1,\dots,k-1$. Putting $c'=c+(k-1)n$ completes the reduction. Note that the instance $\langle G'=(V',E'),c'\rangle$ can be constructed in polynomial time.

We now show that if there exists a clique of size c in G then G' has a k-plex of size c'. Let $C\subseteq V$ induce a clique of size $c=|C|$ in G. We claim that the set $S=C\cup R$, where $|S|=c+n(k-1)=c'$, is a k-plex. For any $u\in C$, there exist $c-1$ neighbors inside C, and $(n-1)(k-1)$ neighbors in R. Thus, for $u\in C$, $deg_{G[S]}(u)=c-1+(n-1)(k-1)=c'-k$. For any $v_r\in R$, there exist $(n-1)(k-1)$ neighbors in R and c neighbors in C if $v\notin C$, and $c-1$ neighbors in C if $v\in C$. Again, for $v_r\in R$, $deg_{G[S]}(v_r)\geq c-1+(n-1)(k-1)=c'-k$. Hence, S induces a k-plex of size c'.

We now establish the other direction stating that if there exists a k-plex of size c' in G' then G has a clique of size c. Let S be a k-plex of size $c'=c+n(k-1)$. Let $P=R\setminus S$ denote the set of vertices from R not included in the k-plex and let $|P|=p$. Then, the c' vertices in S consist of $n(k-1)-p$ vertices in $S\cap R$ and $c+p$ vertices in $S\cap V$. Without loss of generality, suppose that $S\cap V=\{1,\dots,c+p\}$ and further assume that for each $i\in S\cap V$ there exist q_i copies of i in P that are left out of the k-plex. Since every $i\in S\cap V$ has $p-q_i$ neighbors in P, we know that $|N(i)\cap(S\cap R)|=(n-1)(k-1)-(p-q_i)$. Since S is a k-plex, $\forall\ i\in S\cap V: deg_{G[S]}(i)=|N(i)\cap(S\cap R)|+|N(i)\cap(S\cap V)|\geq c+n(k-1)-k, \Rightarrow |N(i)\cap(S\cap V)|\geq c+p-1-q_i$. Recall that each q_i is a non-negative integer counting copies of vertex $i\in S\cap V$ in P and note that P can contain vertices that are not copies of any vertex in $S\cap V$. Thus, we have $\sum_{i=1}^{c+p}q_i\leq p$. Hence, there can exist at most p terms, q_i, in that sum that are strictly greater than 0, meaning that there exist at least c terms in that sum that are equal to 0. Without loss of generality, suppose that $q_i=0,\ i\in\{1,\dots,c\}$. Now, let $C=\{1,\dots,c\}$. We already know that for all $i\in C\subseteq S\cap V=\{1,\dots,c+p\}: |N(i)\cap(S\cap V)|\geq c+p-1-q_i=c+p-1$. But $|S\cap V|=c+p$, so for all $i\in C$, $|N(i)\cap(S\cap V)|=c+p-1$. Thus, every vertex in $C\subseteq S\cap V$ is adjacent to every vertex in $S\cap V$. Hence, every vertex in C is adjacent to every other vertex in C and $|C|\geq c$. □

THEOREM EC.3. *Let $P_k(G)$ denote the k-plex polytope of a given graph $G=(V,E)$, where $k>1$. Then, (1) $dim(P_k(G))=n$; (2) $x_i\geq 0$, and $x_i\leq 1$ induce facets of $P_k(G)$ for every $i\in V$.*

PROOF. Let e_i be the unit vector with i^{th} component 1 and the rest 0; $e_{ij}=e_i+e_j$. The points $\mathbf{0},e_1,e_2,\dots,e_n$ are clearly $n+1$ affinely independent points in $P_k(G)\subset\mathbb{R}^n$. Hence, $dim(P_k(G))=n$.

Let $F=\{x\in P_k(G): x_i=0\}$. Since an empty set or any vertex by itself is a k-plex, we have $\mathbf{0},e_j$ for all $j\in V\setminus\{i\}$ forming n affinely independent points in F. This shows that $dim(F)=n-1$ and it is a facet. Let $F'=\{x\in P_k(G): x_i=1\}$. We first observe that every vertex and any pair of vertices form a k-plex for any k such that $1 < k < n$. Then e_i and e_{ij} for all $j \in V \setminus \{i\}$ form n affinely independent points in F', indicating that $dim(F') = n-1$ and it is a facet. □

THEOREM EC.5. *For a subset $J \subseteq V$ such that $|J| \geq 3$, the inequality $\sum_{i \in J} x_i \leq 2$, induces a facet of $P_2(G)$ if and only if J is a maximal co-2-plex.*

PROOF. Let J be a maximal co-2-plex. First, recall that any 2 vertices from J form a 2-plex. Second, for every $v \in V \setminus J$, the above two conditions for a maximal co-2-plex imply the existence of two vertices $u, w \in J$ such that $\{v, u, w\}$ is a 2-plex. Indeed, if the first case holds, let $u \in J \cap N(v)$, then $N(u) \cap J = \{w\}$ and $\{v, u, w\}$ is a 2-plex. If the second case holds, $\{u, w\} \subseteq J \cap N(v)$ and again $\{v, u, w\}$ is a 2-plex. We use these observations to construct n affinely independent (a.i.) vectors that lie on the face defined by $F = \{x \in P_2(G) : \sum_{i \in J} x_i = 2\}$, so F is $(n-1)$-dimensional and hence, a facet. W.l.o.g. assume that $J = \{1, \ldots, r\}$ and $V \setminus J = \{r+1, \ldots, n\}$, where $r \geq 3$. Let $e_i \in \mathbb{R}^n$ denote the unit vector with i-th component one and all others zero. The a.i. vectors are constructed as: $x^v = e_v + e_r, \quad \forall\ v = 1, \ldots, r-1$; $x^r = e_1 + e_2$ (note that x^r is distinct from $x^1, \ldots, x^{r-1}$ as $r \geq 3$); $x^v = e_v + e_u + e_w, \quad \forall\ v = r+1, \ldots, n$, where for each $v \in V \setminus J$, $u, w \in J$ are particular vertices described before. Clearly, $x^v \in F$ and it is easy to verify that they are a.i.

For the converse, suppose $\sum_{i \in J} x_i \leq 2$ induces a facet of $P_2(G)$. If J is not a co-2-plex, there exists some $v \in J$ with 2 neighbors in J which form a 2-plex. The incidence vector of this 2-plex violates the facet inducing inequality, leading to a contradiction. Hence, J must be a co-2-plex. If J is not maximal, then there exists a valid maximal co-2-plex inequality that dominates the given facet inducing inequality. Hence, J must be a maximal co-2-plex. □

THEOREM EC.7. *Let $G = (V, E)$ be a cycle on $n = t(k+1)$ vertices with $t \geq 2, k \geq 3$ such that $k+1$ is odd. Then the inequality $\sum_{i \in V} x_i \leq k+1$ induces a facet of $P_k(G)$.*

PROOF. Let $V = \{1, 2, \ldots, n\}$ and $E = \{(i, i+1 \mod n) : i \in V\}$. Suppose there exists a valid inequality $ax \leq b$ that contains the face induced by the hole inequality, we show that $a_i = \lambda, i \in V$ and $b = (k+1)\lambda$ for some scalar λ. Note that the union of a path on $k-1$ vertices ($k \geq 3$) and a path on 2 vertices forms a k-plex (this includes one path on $k+1$ vertices) that satisfies the hole inequality at equality. In the following arguments, we first *fix* a path on $k-1$ vertices and consider every 2-vertex path, with two consecutive k-plexes so constructed, differing by one vertex. That is, if S is such a k-plex with (v, u) as the 2-vertex path, the next k-plex is constructed as $T = S \cup \{w\} \setminus \{v\}$ where $v \in S$, $w \in V \setminus S$, (u, w) is the new 2-vertex path, and hence, $a_v = a_w$.

Case I: n is odd. Fix the path $(1, 2, \ldots, k-1)$ in every solution. We consider 2-vertex paths in the order $(k, k+1)$, then $(k+1, k+2)$ and so on. Thus we obtain $a_k = a_{k+2} = \cdots = a_{n-3} = a_{n-1}$ and $a_{k+1} = a_{k+3} = \cdots = a_{n-2} = a_n$. Fix the path $(n-k+2, \ldots, n)$ in the following solutions, and vary the 2-vertex path sequentially starting with $(1, 2)$. Here we obtain $a_1 = a_3 = \cdots = a_{k-1} = a_{k+1}$ and $a_2 = a_4 = \cdots = a_{k-2} = a_k$. Together we have all odd index coefficients to be equal and all even index coefficients to be equal. Consider paths $(1, 2, \ldots, k+1)$ and $(2, 3, \ldots, k+2)$ to obtain $a_1 = a_{k+2}$.

Case II: n is even. The fixed paths are chosen as in Case I to obtain $a_1 = a_3 = \cdots = a_{n-1}$ and $a_2 = a_4 = \cdots = a_n$. Considering paths $(1, 2, \ldots, k+1)$ and $(2, 3, \ldots, k+2)$ we obtain $a_1 = a_{k+2}$. □

THEOREM EC.8. *Let $G = (V, E)$ be a cycle on $n = t(k+1)$ vertices with $t \geq 2, k \geq 5$ such that $k+1$ is even. Then the inequality $\sum_{i \in V} x_i \leq k+1$ induces a facet of $P_k(G)$.*

PROOF. Note that when $k+1$ is even, so is n. The fixed paths are chosen as in Theorem 7 to obtain $a_1 = a_3 = \cdots = a_{n-3} = a_{n-1}$ and $a_2 = a_4 = \cdots = a_{n-2} = a_n$. In order to connect an odd coefficient to an even coefficient, we require $k \geq 5$. Then consider the paths $(1, 2, 3, 4, 5, \ldots, k+1)$ and $(1, 2, 3, 5, \ldots, k+2)$ (4 is deleted and $k+2$ is added). We obtain $a_4 = a_{k+2}$. □

Appendix B: Some Remarks on Valid Inequalities

B.1. Co-k-plex Inequalities for $k \geq 3$.

Although co-k-plex inequalities form facets of $P_k(G)$ for $k = 1, 2$, they do not in general for $k \geq 3$. Consider $G = (V, \emptyset)$ with at least k vertices. Note that G is a co-k-plex and the corresponding inequality $\sum_{i \in V} x_i \leq r_k$ is not supporting since $\omega_k(G) = k < r_k$ and there is no $x \in P_k(G)$ that satisfies it at equality. Hence, these inequalities do not form facets of $P_k(G)$ for all G. This is in contrast to the results known for $k = 1, 2$. The reason is $r_k = k$ for $k = 1, 2$ and every graph G with at least k vertices has a k-plex of size $r_k = k$. The next natural question, if they form facets when G is a co-k-plex with $\omega_k(G) = r_k, \quad k \geq 3$, is also settled in the negative by the following counterexamples. Assume that k is even. Construct graph G of arbitrary order $n \geq r_k$ as the union of $n - r_k$ clique components of size one and two clique components of size $k - 1 = r_k/2$. Then G is a co-k-plex with the two "large" clique components forming a k-plex of size r_k. Suppose $F = \{x \in P_k(G) : \sum_{i \in V} x_i = r_k\}$ is a facet of $P_k(G)$. Since $P_k(G)$ is an integral polytope, the extreme points of F are also integral. Consider one such binary vector $x^o \in F$. If $x^o_i = 1$ for some i that is a one-vertex clique component of G, for x^o to be feasible we have $\sum_{j \in V \setminus N[i]} x^o_j \leq k - 1$. Since $V \setminus N[i] = V \setminus \{i\}$, we have $\sum_{i \in V} x^o_i \leq k$, which contradicts the fact that $x^o \in F$ as $r_k > k$. Hence, the components of extreme points of F corresponding to one-vertex components of G are all zeros. Hence, there exists *exactly one extreme point* in $P_k(G)$ that satisfies $\sum_{i \in V} x_i \leq r_k$ at equality, which is the incidence vector of $K_{k-1} \cup K_{k-1}$. Thus, F is 0-dimensional and not a facet. For odd k, we can have arbitrarily large graphs by adding single vertex components to the antiweb $G_k[V']$ constructed before. By using similar arguments, we can again show that there exists only one point in the k-plex polytope that satisfies the co-k-plex inequality at equality.

From these observations we can conclude that $\omega_k(G) = r_k$ is only a necessary condition for the co-k-plex inequality to induce a facet of $P_k(G)$. Identifying graph classes for which the co-k-plex inequalities and rank inequalities $\sum_{i \in J} x_i \leq \omega_k(G[J])$ induce facets of the k-plex polytope when $k \geq 3$ is an important problem for future research. It is also a well-known fact that a graph is perfect if and only if its clique polytope is completely characterized by all the maximal independent set inequalities and non-negativity constraints (Cornuéjols 2001). Similarly, we could explore *k-plex perfectness* of graphs whose k-plex polytope can be completely described by the co-k-plex inequalities described here and the trivial facets. This is also an interesting topic for future research.

B.2. MIS and Co-k-plex Inequalities.

Both MIS and (maximal) co-k-plex inequalities generalize the MIS inequalities for the clique polytope. For $k = 1$ they are identical, and for $k = 2$, co-2-plex inequalities induce facets and dominate the MIS inequalities as the right-hand side bounds are equal. But based on Theorem 5 and the observation that every lifted MIS inequality has 0 or 1 variable coefficients, every lifted MIS facet of $P_2(G)$ is a co-2-plex facet. For $k \geq 3$ however, there could exist facets of $P_k(G)$ obtained by sequentially lifting an MIS inequality that are different from any inequality co-k-plexes produce. Similarly, although co-k-plex inequalities do not produce facets in general for $k \geq 3$, they could provide stronger inequalities compared to MIS in some cases. As a result, for $k \geq 3$ neither inequality dominates the other in general. Consider the graph G_3 in Figure EC.1 when $k = 3$. The vertex set is a co-3-plex and the co-3-plex inequality is $\sum_{i=1}^{7} x_i \leq 5$ (note that we could tighten the RHS bound as $\omega_3(G_3) = 4$). The MIS inequality on the other hand is $x_1 + x_3 + x_5 + x_7 \leq 3$, which cuts off the point $[1, 0, 1, 0, 1, 0, 1]^T$ that is not cut-off by the co-3-plex inequality. Lifting the MIS inequality following the sequence (x_2, x_4, x_6) yields $x_1 + x_2 + x_3 + x_5 + x_6 + x_7 \leq 3$, and following the sequence (x_4, x_2, x_6) yields $x_1 + x_3 + x_4 + x_5 + x_7 \leq 3$. Both are facets, as the variables were maximum lifted. Consider the graph G_4 in Figure EC.1 when $k = 3$. The independence number is 3 and the MIS inequalities are implied by variable bounds, while the co-3-plex inequality $\sum_{i=1}^{6} x_i \leq 5$ is not.

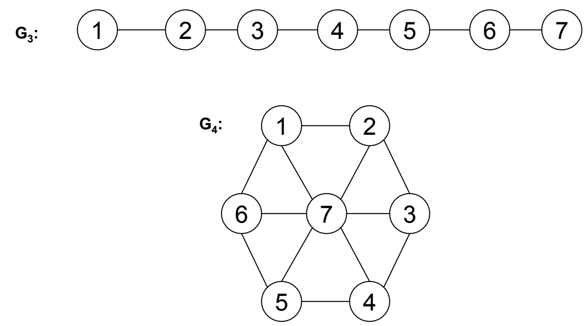

Figure EC.1 Example graphs for comparing MIS and co-3-plex inequalities.

Appendix C: Numerical Results: Sanchis Graphs

The largest order up to which optimal resolution was possible on Sanchis instances within the 3-hour time limit, using the specified algorithm, for each density is presented in Table EC.1. Note that "< 100" indicates that the smallest instance in our test bed with 100 vertices was not solved optimally. Tables EC.2 and EC.3 present the total running time (excluding read/write time) and number of BC nodes enumerated for solving maximum 1-plex problem on Sanchis graphs using BC(MIS) implementation. Non-optimal termination is indicated by the dagger symbol (†). Running times and number of BC nodes enumerated by BC(MIS) for $k = 2$ is presented in Tables EC.4 and EC.5. The size of the largest 2-plex found and an upper bound on the 2-plex numbers obtained from the BC(MIS) implementation is provided in Table EC.6. Running times and number of BC nodes enumerated by BC(co2plex) for $k = 2$ is presented in Tables EC.7 and EC.8 up to $n = 500$ and $d = 0.6$. Note that none of the other instances were solved optimally.

Table EC.1 Summary of results on Sanchis instances

k	Algorithm	$d=0.4$	$d=0.5$	$d=0.6$	$d=0.7$	$d=0.8$	$d=0.9$
1	CPLEX default	800	900	900	900	700	200
1	BC(MIS)	1000	1000	900	1000	800	300
2	CPLEX default	1000	600	200	100	< 100	< 100
2	BC(MIS)	1000	900	600	200	< 100	100
2	BC(co2plex)	400	300	200	< 100	< 100	< 100

Table EC.2 Running time (secs) of BC(MIS) for $k = 1$ on Sanchis instances

n	$d=0.4$	$d=0.5$	$d=0.6$	$d=0.7$	$d=0.8$	$d=0.9$
100	0.203	0.422	1.812	0.797	0.328	0.14
200	2.172	4.109	10.734	9.86	27.047	22.171
300	8.891	13.922	52.625	52.484	220.422	2.219
400	30.235	47.766	322.235	182.266	681.5	*TiLim*
500	69.86	90.5	807.296	322.953	1402.66	*TiLim*
600	169.171	226.813	2148.77	605.219	3570.42	*TiLim*
700	332.813	453	5594.47	1078.2	6591.23	*TiLim*
800	560.125	2041.39	5623.413252	1750.89	7028.73	*TiLim*
900	1057.34	1269.95	4995.91	3358.47	*TiLim*	*TiLim*
1000	1894.39	2349.67	*TiLim*	4260.28	*TiLim*	*TiLim*

Table EC.3 Number of nodes enumerated by BC(MIS) for $k = 1$ on Sanchis instances

n	$d = 0.4$	$d = 0.5$	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$
100	29	199	1126	86	39	4
200	354	1152	4022	170	1440	4065
300	511	2041	13547	303	3554	5
400	1613	5474	50350	442	4794	250409†
500	1626	4910	94790	373	4575	104917†
600	4199	10788	183446	402	6616	60535†
700	6979	15313	314525	406	7384	38725†
800	6774	83045	423327	458	5377	25228†
900	14221	26694	158301	654	5459†	17790†
1000	21789	40319	276913†	570	3684†	11660†

Table EC.4 Running time (secs) of BC(MIS) for $k = 2$ on Sanchis instances

n	$d = 0.4$	$d = 0.5$	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$
100	1.266	2.562	12.203	320.125	*TiLim*	18.312
200	10.297	17.859	61.797	4075.13	*TiLim*	*TiLim*
300	47.844	76.36	286.235	*TiLim*	*TiLim*	*TiLim*
400	141.61	248.125	999.922	*TiLim*	*TiLim*	*TiLim*
500	377.422	688.25	2495.25	*TiLim*	*TiLim*	*TiLim*
600	820.344	1466.3	5525.61	*TiLim*	*TiLim*	*TiLim*
700	1610.06	2981.74	*TiLim*	*TiLim*	*TiLim*	*TiLim*
800	3083.42	5478.7	*TiLim*	*TiLim*	*TiLim*	*TiLim*
900	6058.16	9204.52	*TiLim*	*TiLim*	*TiLim*	*TiLim*
1000	9926.52	*TiLim*	*TiLim*	*TiLim*	*TiLim*	*TiLim*

Table EC.5 Number of nodes enumerated by BC-MIS for $k = 2$ on Sanchis instances

n	$d = 0.4$	$d = 0.5$	$d = 0.6$	$d = 0.7$	$d = 0.8$	$d = 0.9$
100	523	1467	7946	169319	1851177†	10850
200	1574	4762	24859	283774	753120†	699101†
300	2910	9525	59309	106882†	238681†	597044†
400	3976	15987	119868	38867†	81804†	242840†
500	5249	23613	174602	16181†	34463†	101949†
600	6411	29417	247807	7115†	17901†	52399†
700	8918	41936	269150†	4022†	9490†	30193†
800	10095	45481	141117†	2426†	5713†	19134†
900	15997	50779	92591†	1622†	3693†	12578†
1000	19197	24291†	58027†	1109†	2528†	8579†

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Table EC.6 2-plex numbers found by BC(MIS) on Sanchis instances

n	$d=0.4$	$d=0.5$	$d=0.6$	$d=0.7$	$d=0.8$	$d=0.9$
100	20	20	20	20	[24, 25]	38
200	40	40	40	40	[40, 60]	[50, 75]
300	60	60	60	[60, 90]	[60, 103]	[59, 116]
400	80	80	80	[80, 130]	[80, 148]	[76, 159]
500	100	100	100	[100, 176]	[100, 191]	[99, 201]
600	120	120	120	[120, 222]	[120, 231]	[117, 245]
700	140	140	[140, 146]	[140, 266]	[140, 280]	[136, 297]
800	160	160	[160, 236]	[160, 306]	[160, 324]	[155, 339]
900	180	180	[180, 292]	[180, 349]	[180, 370]	[180, 388]
1000	200	[200, 333]	[200, 352]	[200, 386]	[200, 414]	[196, 440]

Table EC.7 Running time (secs) of BC(co2plex) for $k=2$ on Sanchis instances

n	$d=0.4$	$d=0.5$	$d=0.6$
100	6.781	7.985	35.5
200	189.797	485.625	1667.7
300	1658.7	4966.11	*TiLim*
400	9486.86	*TiLim*	*TiLim*
500	*TiLim*	*TiLim*	*TiLim*

Table EC.8 Number of nodes enumerated by BC(co2plex) for $k=2$ on Sanchis instances

n	$d=0.4$	$d=0.5$	$d=0.6$
100	607	1515	8445
200	1779	5780	27896
300	3754	12932	$13899^{\dagger}$
400	7492	$3541^{\dagger}$	$3490^{\dagger}$
500	$1109^{\dagger}$	$1452^{\dagger}$	$762^{\dagger}$

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Table EC.9 DIMACS benchmarks.

Graphs	$\|V\|$	$\|E\|$	d	$\omega(G)$	BC(MIS) (secs)	$\omega_2(G)$	BC(MIS) (secs)
c-fat200-1.clq	200	1534	0.077	12	17.1	12	148.9
c-fat200-2.clq	200	3235	0.163	24	10.4	24	19.1
c-fat200-5.clq	200	8473	0.426	58	2.1	58	2.1
c-fat500-1.clq	500	4459	0.036	14	1334.4	14	1356.1
c-fat500-2.clq	500	9139	0.073	26	535.7	26	605.3
c-fat500-5.clq	500	23191	0.186	64	141.6	64	141.5
c-fat500-10.clq	500	46627	0.374	126	39.3	126	76.5
hamming6-2.clq	64	1824	0.905	32	0.0	32	0.0
hamming6-4.clq	64	704	0.349	4	0.2	6	0.3
hamming8-2.clq	256	31616	0.969	128	0.0	128	189.5
hamming8-4.clq	256	20864	0.639	16	52.2	16	8115.2
hamming10-2.clq	1024	518656	0.990	512	0.8	[512,530]	*TiLim*
hamming10-4.clq	1024	434176	0.829	[36,234]	*TiLim*	[41,153]	*TiLim*
johnson8-2-4.clq	28	210	0.556	4	0.0	5	0.0
johnson8-4-4.clq	70	1855	0.768	14	0.1	14	4.4
MANN_a9.clq	45	918	0.927	16	0.0	26	0.0
MANN_a27.clq	378	70551	0.990	126	430.3	236	79.8
MANN_a45.clq	1035	533115	0.996	[344,347]	*TiLim*	[662,668]	*TiLim*
keller4.clq	171	9435	0.649	11	129.8	15	365.4
brock200_1.clq	200	14834	0.745	[20,31]	*TiLim*	[25,53]	*TiLim*
brock200_2.clq	200	9876	0.496	12	152.5	[13,24]	*TiLim*
brock200_4.clq	200	13089	0.658	17	6617.5	[19,41]	*TiLim*
brock400_2.clq	400	59786	0.749	[24,68]	*TiLim*	[27,133]	*TiLim*
brock400_4.clq	400	59765	0.749	[23,69]	*TiLim*	[27,133]	*TiLim*
brock800_2.clq	800	208166	0.651	[19,116]	*TiLim*	[23,253]	*TiLim*
brock800_4.clq	800	207643	0.650	[19,108]	*TiLim*	[23,252]	*TiLim*
p_hat300-1.clq	300	10933	0.244	8	127.0	[9,66]	*TiLim*
p_hat300-2.clq	300	21928	0.489	[25,51]	*TiLim*	[28,85]	*TiLim*
p_hat300-3.clq	300	33390	0.744	[35,71]	*TiLim*	[43,108]	*TiLim*
p_hat700-1.clq	700	60999	0.249	[11,40]	*TiLim*	[10,291]	*TiLim*
p_hat700-2.clq	700	121728	0.498	[44,208]	*TiLim*	[50,298]	*TiLim*
p_hat700-3.clq	700	183010	0.748	[62,201]	*TiLim*	[73,311]	*TiLim*

Table EC.10 Runtime in seconds for Erdös networks using IPBC algorithm.

k	Graph	IPBC Time	BC Time	#BC Calls
1	ERDOS-97-1	1.4	0.0	5
	ERDOS-98-1	1.4	0.1	6
	ERDOS-99-1	1.5	0.0	6
	ERDOS-97-2	367.5	0.1	7
	ERDOS-98-2	445.4	0.0	7
	ERDOS-99-2	491.9	0.0	4
2	ERDOS-97-1	1.5	0.2	5
	ERDOS-98-1	1.7	0.2	6
	ERDOS-99-1	1.8	0.3	6
	ERDOS-97-2	392.9	0.5	7
	ERDOS-98-2	464.3	0.6	7
	ERDOS-99-2	526.5	0.4	9
3	ERDOS-97-1	1.8	0.4	5
	ERDOS-98-1	1.8	0.3	6
	ERDOS-99-1	1.8	0.3	6
	ERDOS-97-2	394.1	8.7	7
	ERDOS-98-2	457.1	1.1	7
	ERDOS-99-2	520.0	3.2	9
4	ERDOS-97-1	2.2	1.0	4
	ERDOS-98-1	2.8	1.5	4
	ERDOS-99-1	1.8	0.3	4
	ERDOS-97-2	424.0	39.6	3
	ERDOS-98-2	614.7	159.8	3
	ERDOS-99-2	526.3	10.6	4
5	ERDOS-97-1	5.7	4.5	4
	ERDOS-98-1	7.9	6.6	4
	ERDOS-99-1	9.9	8.5	4
	ERDOS-97-2	1042.8	688.2	3
	ERDOS-98-2	1664.6	1244.6	3
	ERDOS-99-2	653.5	178.1	4

Table EC.11 Runtime in seconds for biological networks using IPBC algorithm.

k	Graph	IPBC Time	BC Time	#BC Calls
1	H. Pylori	11.5	0.0	13
	S. Cerevisiae	44.1	0.0	0
2	H. Pylori	12.6	1.3	2
	S. Cerevisiae	46.4	0.0	0
3	H. Pylori	37.8	26.6	2
	S. Cerevisiae	45.1	0.0	1
4	H. Pylori	29.3	18.3	2
	S. Cerevisiae	45.0	0.0	1
5	H. Pylori	133.1	123.0	3
	S. Cerevisiae	41.8	0.0	3

Table EC.12 Runtime in seconds for computational geometers collaboration networks using IPBC algorithm.

k	Graph	IPBC Time	BC Time	#BC Calls
1	GEOM-0	2287.6	0.0	0
	GEOM-1	701.2	0.0	0
	GEOM-2	479.6	0.0	0
2	GEOM-0	2384.4	0.0	0
	GEOM-1	753.2	0.1	2
	GEOM-2	530.6	0.1	4
3	GEOM-0	2387.1	0.0	0
	GEOM-1	747.7	0.1	2
	GEOM-2	524.3	0.0	1
4	GEOM-0	2383.7	0.0	0
	GEOM-1	743.7	0.4	2
	GEOM-2	522.2	0.1	1
5	GEOM-0	2298.1	0.0	0
	GEOM-1	691.6	1.8	2
	GEOM-2	472.6	0.5	4

Table EC.13 Runtime in seconds for the Reuters terror news networks using IPBC algorithm.

k	Graph	IPBC Time	BC Time	#BC Calls
1	DAYS-3	3110.8	0.1	3
	DAYS-4	2940.8	0.2	1
	DAYS-5	2758.0	0.0	0
2	DAYS-3	3367.8	4.1	1
	DAYS-4	2635.7	0.3	1
	DAYS-5	2462.9	0.1	2
3	DAYS-3	3395.4	45.5	1
	DAYS-4	2625.1	4.7	1
	DAYS-5	2445.5	0.2	2
4	DAYS-3	3489.8	203.0	1
	DAYS-4	2642.3	51.4	1
	DAYS-5	2426.3	2.7	1
5	DAYS-3	15336.9	12329.1	1
	DAYS-4	6201.4	3316.8	1
	DAYS-5	2820.8	113.1	1