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e-companion

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Electronic Companion—"Optimal Inventory Policies when Purchase Price and Demand Are Stochastic" by Peter Berling and Victor Martínez-de-Albéniz, *Operations Research*, DOI 10.1287/opre.1100.0862.

Online Appendix for "Optimal Inventory Policies when Purchase Price and Demand are Stochastic"

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Proof of Theorem 1

Proof. Consider $J^{n,\Delta}$ be a finite, discrete version of the problem. Specifically, for a fixed time interval Δ, define

$$\begin{aligned}
&J^{0,\Delta} \equiv 0\\
&J^{n+1,\Delta}(0,c) = \min\Big\{\Pi(0) + e^c, b\Delta + e^{-r\Delta}\mathbb{E}_{\tilde{\epsilon}} J^{n,\Delta}\Big(0, c + \mu(c)\Delta + \sigma(c)\sqrt{\Delta}\tilde{\epsilon}\Big)\Big\}\\
&J^{n+1,\Delta}(k,c) = \min\Big\{\Pi(k) + e^c, e^{-r\Delta}\mathbb{E}_{\tilde{\epsilon},\tilde{P}} J^{n,\Delta}\Big((k-\tilde{P})^+, c + \mu(c)\Delta + \sigma(c)\sqrt{\Delta}\tilde{\epsilon}\Big)\Big\} \text{ for } k \geq 1
\end{aligned}$$

where $\tilde{\epsilon}$ is a random variable that takes values -1 and 1 with probability 1/2, and $\tilde{P}$ is a variable that takes the value 1 with probability $\lambda\Delta$ and 0 otherwise. Clearly, for all k, by value iteration, the limit when $n \to \infty$ exists, denote it J^{Δ}. J^{Δ} exhibits a number of properties. All the results above are preserved when we consider J, since it is the limit of J^{Δ} as $\Delta \to 0$.

$J^{\Delta}(k, \ln(C))$ **non-decreasing.** First, for Δ small enough, $c + \mu(c)\Delta + \sigma(c)\sqrt{\Delta}\tilde{\epsilon}$ is increasing in c, for all $c, \tilde{\epsilon}$. As a result, by a straightforward induction, $J^{\Delta}(k,c)$ must be increasing in c. Furthermore, it is clear that $J^{\Delta}(0,c) \leq \dfrac{b\Delta}{1-e^{-r\Delta}}$ (never placing an order at $k=0$), i.e., bounded from above.

Second, select Δ small enough, specifically such that, for all c,$\tilde{\epsilon} = +1, -1$,

$$1 + \frac{d\mu}{dc}\Delta + \frac{d\sigma}{dc}\sqrt{\Delta}\tilde{\epsilon} \geq 0.$$

$J^{\Delta}(k, \ln(C))$ **concave.** We claim that $J^{\Delta}(k, \ln(C))$ is concave in C for all k. This is shown after showing concavity of $J^{n,\Delta}(k, \ln(C))$, by induction on n. $J^{0,\Delta}(k, \ln(C))$ is indeed concave. Assume it is concave for $n \geq 0$. Define, given n, Δ, for each $k, \tilde{\epsilon}, \tilde{P}$, $\phi(C) = J^{n,\Delta}\Big((k-\tilde{P})^+, \ln(C) + \mu(\ln(C))\Delta + \sigma(\ln(C))\sqrt{\Delta}\tilde{\epsilon}\Big)$.

$$\phi'(C) = \frac{1}{C}\left(1 + \frac{d\mu}{dc}\Delta + \frac{d\sigma}{dc}\sqrt{\Delta}\tilde{\epsilon}\right)\frac{\partial J^{n,\Delta}}{\partial c}\Big((k-\tilde{P})^+, \ln(C) + \mu(\ln(C))\Delta + \sigma(\ln(C))\sqrt{\Delta}\tilde{\epsilon}\Big) \geq 0.$$

Since by induction $J^{n,\Delta}(k, \ln(C))$ is concave in C, we must have that

$$-\frac{1}{C^2}\frac{\partial J^{n,\Delta}}{\partial c}(k, \ln(C)) + \frac{1}{C^2}\frac{\partial^2 J^{n,\Delta}}{\partial c^2}(k, \ln(C)) \leq 0.$$

This can be used to bound $\phi''(C)$:

$$\begin{aligned}
\phi''(C) = {} & \frac{1}{C^2}\left(\frac{d^2\mu}{dc^2}\Delta + \frac{d^2\sigma}{dc^2}\sqrt{\Delta}\tilde{\epsilon} - 1 - \frac{d\mu}{dc}\Delta - \frac{d\sigma}{dc}\sqrt{\Delta}\tilde{\epsilon}\right) \\
& \times\frac{\partial J^{n,\Delta}}{\partial c}\Big((k-\tilde{P})^+, \ln(C) + \mu(\ln(C))\Delta + \sigma(\ln(C))\sqrt{\Delta}\tilde{\epsilon}\Big) \\
& +\frac{1}{C^2}\left(1 + \frac{d\mu}{dc}\Delta + \frac{d\sigma}{dc}\sqrt{\Delta}\tilde{\epsilon}\right)^2 \frac{\partial^2 J^{n,\Delta}}{\partial c^2}\Big((k-\tilde{P})^+, \ln(C) + \mu(\ln(C))\Delta + \sigma(\ln(C))\sqrt{\Delta}\tilde{\epsilon}\Big) \\
\leq {} & \frac{1}{C^2}\left[\frac{d^2\mu}{dc^2}\Delta + \frac{d^2\sigma}{dc^2}\sqrt{\Delta}\tilde{\epsilon} - 1 - \frac{d\mu}{dc}\Delta - \frac{d\sigma}{dc}\sqrt{\Delta}\tilde{\epsilon} + \left(1 + \frac{d\mu}{dc}\Delta + \frac{d\sigma}{dc}\sqrt{\Delta}\tilde{\epsilon}\right)^2\right] \\
& \times\frac{\partial J^{n,\Delta}}{\partial c}\Big((k-\tilde{P})^+, \ln(C) + \mu(\ln(C))\Delta + \sigma(\ln(C))\sqrt{\Delta}\tilde{\epsilon}\Big) \\
\leq {} & \frac{1}{C^2}\left[\left(\frac{d^2\sigma}{dc^2} + \frac{d\sigma}{dc}\right)\sqrt{\Delta}\tilde{\epsilon} + \left(\frac{d^2\mu}{dc^2} + \frac{d\mu}{dc} + \left(\frac{d\sigma}{dc}\right)^2\right)\Delta + 2\frac{d\mu}{dc}\frac{d\sigma}{dc}\Delta^{3/2}\tilde{\epsilon} + \left(\frac{d\mu}{dc}\right)^2\Delta^2\right] \\
& \times\frac{\partial J^{n,\Delta}}{\partial c}\Big((k-\tilde{P})^+, \ln(C) + \mu(\ln(C))\Delta + \sigma(\ln(C))\sqrt{\Delta}\tilde{\epsilon}\Big).
\end{aligned}$$

When Δ is sufficiently small, the right-hand side is indeed negative for all $\tilde{\epsilon}$ given that $\frac{d^2\sigma}{dc^2} + \frac{d\sigma}{dc} = 0$ and $\frac{d^2\mu}{dc^2} + \frac{d\mu}{dc} + \left(\frac{d\sigma}{dc}\right)^2 < 0$. In addition, if $\frac{d\sigma}{dc} = 0$ and $\frac{d\mu}{dc} = 0$, then $\phi(C) = J^{n,\Delta}\Big((k-\tilde{P})^+, \ln(C) + \mu\Delta + \sigma\sqrt{\Delta}\tilde{\epsilon}\Big)$ is directly concave. Thus, overall, $\phi(C)$ is concave when, as stated in the lemma, $\frac{d\sigma}{dc} = 0$ and $\frac{d^2\mu}{dc^2} + \frac{d\mu}{dc} < 0$ or $\frac{d\sigma}{dc} = 0$ and $\frac{d\mu}{dc} = 0$.

As a result, in the definition of $J^{n+1,\Delta}(k, \ln(C)) = \min\{\Pi(k) + C, e^{-r\Delta}\mathbb{E}_{\tilde{\epsilon}}\phi(C)\}$, the first term in brackets is linear, and the second term concave. Since the minimum of two concave functions is concave, then $J^{n+1,\Delta}(k, \ln(C))$ is concave, and hence, at the limit, $J^{\Delta}(k, \ln(C))$ is concave as well.

Existence of $c_k^{L,\Delta}, c_k^{H,\Delta}$. Thus, since $J^{\Delta}(k, \ln(C))$ is concave, and in the limit it is the maximum of $\Pi(k) + C$ and a concave function of C, it must be true that $J^{\Delta}(k, \ln(C)) = \Pi(k) + C$ if and only if $C_k^{L,\Delta} \leq C \leq C_k^{H,\Delta}$. Equivalently, we have existence of $c_k^{L,\Delta}, c_k^{H,\Delta}$ such that the order is placed if and only if $c_k^{L,\Delta} \leq c \leq c_k^{H,\Delta}$. Interestingly, c_k^L is non-decreasing in k, and c_k^H is non-increasing in k. Indeed, using the traditional approach for the problem, it is well-known that the optimal policy is a base-stock policy, dependent on c. This implies that at a given $C = e^c$, it is optimal to purchase up to some k, where k is the base-stock level. This automatically yields that c_k^H must be non-increasing, and c_k^L non-decreasing. ■

Proof of Proposition 1

Proof. Consider $J^{n,\Delta}$, J^{Δ} and $c_k^{L,\Delta}, c_k^{H,\Delta}$, defined in the previous proof. Interestingly, from Equation (3),

$$\begin{aligned}
\lambda\Pi(k-1) - & (r+\lambda)\Pi(k) \\
= & \frac{b}{r}\left[\frac{\lambda^k}{(\lambda+r)^{k-1}}G(L,k-1,\lambda+r) - e^{-rL}\lambda G(L,k-1,\lambda)\right] \\
& +\frac{h}{r}\left[e^{-rL}\lambda\overline{G}(L,k-1,\lambda) - \frac{\lambda^k}{(\lambda+r)^{k-1}}\overline{G}(L,k-1,\lambda+r)\right] \\
& -\frac{b}{r}\left[\frac{\lambda^k}{(\lambda+r)^{k-1}}G(L,k,\lambda+r) - e^{-rL}(\lambda+r)G(L,k,\lambda)\right] \\
& -\frac{h}{r}\left[e^{-rL}(\lambda+r)\overline{G}(L,k,\lambda) - \frac{\lambda^k}{(\lambda+r)^{k-1}}\overline{G}(L,k,\lambda+r)\right] \\
= & \frac{b}{r}\left[\frac{\lambda^k}{(\lambda+r)^{k-1}}\frac{[(\lambda+r)L]^{k-1}}{(k-1)!}e^{-(\lambda+r)L} - e^{-rL}\Big(\lambda\frac{(\lambda L)^{k-1}}{(k-1)!}e^{-\lambda L} - rG(L,k,\lambda)\Big)\right] \\
& +\frac{h}{r}\left[e^{-rL}\Big(-\lambda\frac{(\lambda L)^{k-1}}{(k-1)!}e^{-\lambda L} - r\overline{G}(L,k,\lambda)\Big) + \frac{\lambda^k}{(\lambda+r)^{k-1}}\frac{[(\lambda+r)L]^{k-1}}{(k-1)!}e^{-(\lambda+r)L}\right] \\
= & be^{-rL}G(L,k,\lambda) - he^{-rL}\overline{G}(L,k,\lambda)
\end{aligned}$$

Since G is decreasing in k and $\bar{G} = 1-G$, this expression is non-increasing in k. Thus $\lambda\Pi(k-1) - (r+\lambda)\Pi(k) \geq 0$ for $k < k^{\dagger}$ for some $k^{\dagger}$.

We can now show (i). For $k=0$, we have that

$$J^{\Delta}(0,c) = \min\left\{\Pi(0) + e^c, b\Delta + e^{-r\Delta}\mathbb{E}_{\tilde{\epsilon}}J^{\Delta}\Big(0, c+\mu(c)\Delta + \sigma(c)\sqrt{\Delta}\tilde{\epsilon}\Big)\right\}.$$

When $C=0$, it is clear that the optimal policy is to place the order: $J^{\Delta}(0,-\infty) = \Pi(0)$, the minimum possible since the lead-time is $L>0$. As a result, $c_0^{L,\Delta} = -\infty$.

For $k \geq 1$, we can use a similar argument. We show, by induction, that when $k < k^{\dagger}$, $c_k^{L,\Delta} = -\infty$. If this is true up to $k-1$, for $C \approx 0$, $J(i, \ln(C)) = \Pi(i) + C$ for $i \leq k-1$.

Consider the decision at c close to $-\infty$, i.e., $C=0$. If $c_k^{L,\Delta} > -\infty$, then the order should not be placed now. In that case, the order is placed either when the price reaches $c = c_k^{L,\Delta}$, which occurs after a stochastic time T, or when an arrival occurs, after a stochastic time E_1. The realized discounted cost is thus $e^{-rE_1}(\Pi(k-1) + C(E_1))1_{E_1<T} + e^{-rT}(\Pi(k) + c_k^{\Delta,L})1_{E_1<T}$. Since E_1 and T are independent (one is related to arrivals, the other to price evolution), the expectation with respect to E_1 is

$$\begin{aligned}
& \int_0^T \lambda e^{-\lambda s}e^{-rs}(\Pi(k-1) + C(s))ds + \int_T^{\infty} \lambda e^{-\lambda s}e^{-rT}(\Pi(k) + c_k^{L,\Delta})ds \\
\geq & \int_0^T \lambda e^{-\lambda s}e^{-rs}\Pi(k-1)ds + \int_T^{\infty} \lambda e^{-\lambda s}e^{-rT}\Pi(k)ds \\
= & \frac{\lambda(1-e^{-(r+\lambda)T})}{r+\lambda}\Pi(k-1) + e^{-(r+\lambda)T}\Pi(k) \\
\geq & \Pi(k)
\end{aligned}$$

where the last inequality is found using that $\lambda\Pi(k-1) - (r+\lambda)\Pi(k) \geq 0$. Thus, the cost of such strategy is higher than placing the order at $C=0$. Hence, $c_k^{L,\Delta} = -\infty$. This completes the induction.

We prove now (ii). Consider now the first k such that $\lambda\Pi(k-1)-(r+\lambda)\Pi(k)=-(r+\lambda)M<0$. This is $k=k^\dagger$. Consider the policy of never ordering if no arrival occurs. The expected discounted cost is then

$$\begin{aligned}
&\mathbb{E}\left\{\int_0^\infty \lambda e^{-\lambda s}e^{-rs}J\Big(k-1,\ln(C(s))\Big)ds\right\}\\
\leq\;& \int_0^\infty \lambda e^{-\lambda s}e^{-rs}\Big(\Pi(k-1)+\mathbb{E}C(s)\Big)ds\\
=\;& \frac{\lambda}{r+\lambda}\Pi(k-1)+\int_0^\infty \lambda e^{-\lambda s}e^{-rs}\mathbb{E}C(s)ds\\
<\;& \Pi(k)+C(0)-M+\int_0^\infty \lambda e^{-\lambda s}e^{-rs}\mathbb{E}C(s)ds-C(0)
\end{aligned}$$

If for all $C(0)\leq b/r$, $\displaystyle\int_0^\infty \lambda e^{-\lambda s}e^{-rs}\mathbb{E}C(s)ds-C(0)\leq M$, then it is clear that not placing an order outperforms placing one. Also, if $C(0)>b/r$ it is better not to place an order, as it is cheaper to keep paying the back-ordering cost. Thus, $c_k^{H,\Delta}=-\infty$.

Hence, the optimal policy can be defined such that an order is placed if and only if $c\leq c_k^{H,\Delta}$. $c_k^{H,\Delta}$ is finite for $k<k^\dagger$, and $-\infty$ for $k\geq k^\dagger$. The same holds true as we let $\Delta\to 0$. ■

Proof of Proposition 2

Proof. Let us proof $\int_0^\infty \lambda e^{-\lambda s}e^{-rs}\mathbb{E}C(s)ds-C(0)\leq M$ for all $C(0)\leq b/r$. Furthermore, to simplify notation we denote $C(0)$ with just C in the remainder of the proof.

Let us first consider case (a), the geometric Brownian motion model. Define

$$\begin{aligned}
m=\;& \int_0^\infty \lambda e^{-\lambda s}e^{-rs}\mathbb{E}C(s)ds-C\\
=\;& C\cdot\left(\int_0^\infty \lambda e^{-\lambda s}e^{-rs}e^{(\mu+\sigma^2/2)s}ds-1\right)\\
=\;& C\cdot\left(\frac{\lambda}{r+\lambda-(\mu+\sigma^2/2)}-1\right)
\end{aligned}$$

The expression above is negative for any C if $\mu+\sigma^2/2\leq r$ (and thus is smaller than M). If $\mu+\sigma^2/2>r$, then it is increasing in C. We must make sure that its value at $C=b/r$ is less or equal to M, i.e.,

$$\frac{\lambda}{r+\lambda-(\mu+\sigma^2/2)}-1\leq\frac{rM}{b},$$

which is equivalent to the condition in the proposition.

Let us now consider case (b), the Ornstein-Uhlenbeck process. Let

$$\begin{aligned}
m=\;& \int_0^\infty \lambda e^{-\lambda s}e^{-rs}\mathbb{E}C(s)ds-C\\
=\;& \int_0^\infty \lambda e^{-\lambda s}e^{-rs}\cdot C\cdot exp\left[-(\ln(C)-\bar{c})\left(1-e^{-\kappa s}\right)+\frac{\sigma^2}{4\kappa}\left(1-e^{-2\kappa s}\right)\right]ds-C
\end{aligned}$$

If $\ln(C)\geq\bar{c}$ then $\mathbb{E}C(s)\leq C$ for all s, and it follows directly that $m<0<M$.

If $\ln(C) < \bar{c}$ then the inequality $m < M$ is satisfied depending on the relationship between λ, r, κ and σ.

We can obtain a first bound by using that $1 - e^{-\kappa s} \leq 1$, $1 - e^{-2\kappa s} \leq 1$ and $-C \leq 0$:

$$\begin{aligned} m \leq & \int_0^{\infty} \lambda e^{-\lambda s} e^{-rs} e^{\bar{c} + \frac{\sigma^2}{4\kappa}} ds \\ = & \frac{\lambda}{\lambda + r} e^{\bar{c} + \frac{\sigma^2}{4\kappa}} \end{aligned}$$

If condition (b1) is true, then the above inequality proves that $m \leq M$.

A second bound can be found by noting that $1 - e^{-\kappa s} \leq 1$, $1 - e^{-2\kappa s} \leq 2\kappa s$ and $-C \leq 0$:

$$\begin{aligned} m \leq & \int_0^{\infty} \lambda e^{-\lambda s} e^{-rs} e^{\bar{c}} e^{\frac{\sigma^2}{2} s} ds \\ = & \frac{\lambda}{\lambda + r - \frac{\sigma^2}{2}} e^{\bar{c}} \end{aligned}$$

Condition (b2) implies that $m \leq M$. Note that obtaining tighter bounds is possible by maximizing m over $C \leq b/r$. ■

Proof of Theorem 2

Proof. We show the theorem by recursion on k. The structure is true for $k = 0$, as presented in Equation (11). For $k \geq 1$, first, Theorem 1 establishes the existence of c_k^H. For $c_k^L \leq c \leq c_k^H$, $J(k, c) = \Pi(k) + e^c$. For $c \geq c_k^H$, $J(k, c)$ satisfies Equation (7). We verify that the proposed formulation, in interval $[c_i^H, c_{i-1}^H]$, $i \geq 1$, i.e.,

$$J(k, c) = M_{k,i} + N_{k,i} e^c + \left(A_{k,i}^0 + A_{k,i}^1 c + \ldots + A_{k,i}^{k-1} c^{k-1}\right) e^{-\alpha c} + \left(B_{k,i}^0 + B_{k,i}^1 c + \ldots + B_{k,i}^{k-1} c^{k-1}\right) e^{\beta c}$$

satisfies the differential equation:

$$\begin{aligned} 0 = \; & -(r+\lambda) J(k,c) + \lambda J(k-1,c) + \mu \frac{dJ}{dc}(k,c) + \frac{\sigma^2}{2} \frac{d^2 J}{dc^2}(k,c) \\ = \; & \lambda \left[\begin{array}{c} M_{k-1,i} + N_{k-1,i} e^c + \left(A_{k-1,i}^0 + A_{k-1,i}^1 c + \ldots + A_{k-1,i}^{k-2} c^{k-2}\right) e^{-\alpha c} \\ + \left(B_{k-1,i}^0 + B_{k-1,i}^1 c + \ldots + B_{k-1,i}^{k-2} c^{k-2}\right) e^{\beta c} \end{array} \right] \\ & -(r+\lambda) \left[\begin{array}{c} M_{k,i} + N_{k,i} e^c + \left(A_{k,i}^0 + A_{k,i}^1 c + \ldots + A_{k,i}^{k-1} c^{k-1}\right) e^{-\alpha c} \\ + \left(B_{k,i}^0 + B_{k,i}^1 c + \ldots + B_{k,i}^{k-1} c^{k-1}\right) e^{\beta c} \end{array} \right] \\ & + \mu \left[\begin{array}{c} N_{k,i} e^c - \alpha \left(A_{k,i}^0 + A_{k,i}^1 c + \ldots + A_{k,i}^{k-1} c^{k-1}\right) e^{-\alpha c} + \left(A_{k,i}^1 + \ldots + (k-1) A_{k,i}^{k-1} c^{k-2}\right) e^{-\alpha c} \\ + \beta \left(B_{k,i}^0 + B_{k,i}^1 c + \ldots + B_{k,i}^{k-1} c^{k-1}\right) e^{\beta c} + \left(B_{k,i}^1 + \ldots + (k-1) B_{k,i}^{k-1} c^{k-2}\right) e^{\beta c} \end{array} \right] \\ & + \frac{\sigma^2}{2} \left[\begin{array}{c} N_{k,i} e^c + \alpha^2 \left(A_{k,i}^0 + A_{k,i}^1 c + \ldots + A_{k,i}^{k-1} c^{k-1}\right) e^{-\alpha c} \\ -2\alpha \left(A_{k,i}^1 + \ldots + (k-1) A_{k,i}^{k-1} c^{k-2}\right) e^{-\alpha c} + \left(2A_{k,i}^2 + \ldots + (k-2)(k-1) A_{k,i}^{k-1} c^{k-3}\right) e^{-\alpha c} \\ + \beta^2 \left(B_{k,i}^0 + B_{k,i}^1 c + \ldots + B_{k,i}^{k-1} c^{k-1}\right) e^{\beta c} + 2\beta \left(B_{k,i}^1 + \ldots + (k-1) B_{k,i}^{k-1} c^{k-2}\right) e^{\beta c} \\ + \left(2B_{k,i}^2 + \ldots + (k-2)(k-1) B_{k,i}^{k-1} c^{k-3}\right) e^{\beta c} \end{array} \right] \end{aligned}$$

Simple identification of terms yields that

$$M_{k,i} = \frac{\lambda M_{k-1,i}}{r+\lambda} \text{ and } N_{k,i} = \frac{\lambda N_{k-1,i}}{(r+\lambda) - \mu - \dfrac{\sigma^2}{2}},$$

and in addition, since $(r+\lambda) + \mu\alpha - \frac{\sigma^2}{2}\alpha^2 = 0$ and $(r+\lambda) - \mu\beta - \frac{\sigma^2}{2}\beta^2 = 0$,

$$\begin{aligned}
\lambda A^j_{k-1,i} = {} & \left[(r+\lambda) + \mu\alpha - \frac{\sigma^2}{2}\alpha^2\right] A^j_{k,i} + \left[-\mu(j+1) + \frac{\sigma^2}{2}2\alpha(j+1)\right] A^{j+1}_{k,i} + \left[-\frac{\sigma^2}{2}(j+1)(j+2)\right] A^{j+2}_{k,i} \\
= {} & \left[-\mu(j+1) + \frac{\sigma^2}{2}2\alpha(j+1)\right] A^{j+1}_{k,i} + \left[-\frac{\sigma^2}{2}(j+1)(j+2)\right] A^{j+2}_{k,i}, \\
\lambda B^j_{k-1,i} = {} & \left[(r+\lambda) - \mu\beta - \frac{\sigma^2}{2}\beta^2\right] B^j_{k,i} + \left[-\mu(j+1) - \frac{\sigma^2}{2}2\beta(j+1)\right] B^{j+1}_{k,i} + \left[-\frac{\sigma^2}{2}(j+1)(j+2)\right] B^{j+2}_{k,i} \\
= {} & \left[-\mu(j+1) - \frac{\sigma^2}{2}2\beta(j+1)\right] B^{j+1}_{k,i} + \left[-\frac{\sigma^2}{2}(j+1)(j+2)\right] B^{j+2}_{k,i}.
\end{aligned}$$

Solving these equations provides an expression of $A^j_{k,i}, B^j_{k,i}$ for $j \geq 1$. One can see that the expression provided in the theorem satisfies the differential equation for all $A^0_{k,i}, B^0_{k,i}$.

Furthermore, when $c \geq c^H_0$, the same argument yields

$$M_{k,0} = \frac{\lambda M_{k-1,0}}{r+\lambda} \text{ and } N_{k,0} = \frac{\lambda N_{k-1,0}}{(r+\lambda) + \mu\alpha_0 - \dfrac{\sigma^2}{2}\alpha_0^2},$$

and

$$\begin{aligned}
\lambda A^j_{k-1,0} = {} & \left[(r+\lambda) + \mu\alpha - \frac{\sigma^2}{2}\alpha^2\right] A^j_{k,0} + \left[-\mu(j+1) + \frac{\sigma^2}{2}2\alpha(j+1)\right] A^{j+1}_{k,0} + \left[-\frac{\sigma^2}{2}(j+1)(j+2)\right] A^{j+2}_{k,0} \\
= {} & \left[-\mu(j+1) + \frac{\sigma^2}{2}2\alpha(j+1)\right] A^{j+1}_{k,0} + \left[-\frac{\sigma^2}{2}(j+1)(j+2)\right] A^{j+2}_{k,0}.
\end{aligned}$$

Finally, in order to have $J(k,c)$ continuously differentiable at $c^H_k, \ldots, c^H_0$, we must satisfy $2(k+1)$ equations (twice for each break-point). We can do so by adjusting $A^0_{k,0}, A^0_{k,1}, B^0_{k,1}, \ldots, A^0_{k,k}, B^0_{k,k}$ and c_k, $2(k+1)$ variables. ■

Proof of Theorem 3

Proof. We prove the theorem by induction.

First of all, we are interested in solving Equation (15) to characterize $J(k,c)$. Letting $\phi(x) = J(k, c-\bar{c}) - \frac{b}{r}\left(\frac{\lambda}{r+\lambda}\right)^k$, the differential equation can be reformulated into (17). Since this is a second-order differential equation, the solution to it can be expressed as the sum of a particular solution, plus a combination of solutions to (16). We are only interested in solutions that are not divergent, and hence the only candidate solution to (16) is a constant times ψ_1.

For $k=1$, we know the expression of $J(0,c)$, provided in (14). For $x \geq c^H_0 - \bar{c}$, $J(0, x+\bar{c}) - \frac{b}{r} = A_0\psi_0(x)$. As a result, since ψ_0 solves Equation (13), $A_0\psi_0(x)$ is a convergent solution of (17) for

$x \geq c_0^H - \bar{c}$. Expanding the solution of the differential equation to $x \leq c_0^H - \bar{c}$ provides θ_1, a particular solution to (17). Thus, $J(1, c)$ can be characterized as

$$\frac{b}{r}\left(\frac{\lambda}{r+\lambda}\right) + \theta_1(c - \bar{c}) + A_1\psi_1(c - \bar{c})$$

for $c \geq c_1^H$, and $\Pi(1) + e^c$ for $c_1^L \leq c \leq c_1^H$. A_1, c_1^H are determined so that $J(1, c)$ is continuously differentiable at c_1^H.

Assuming that the theorem is true for $k - 1 \geq 1$, then we claim that it also holds for k. We need a particular convergent solution to (17):

$$-(r+\lambda)\theta - \kappa x\theta' + \frac{\sigma^2}{2}\theta'' = -\lambda\phi_{k-1}$$

where $\phi_{k-1}(x) = J(k-1, x+\bar{c}) - \frac{b}{r}\left(\frac{\lambda}{r+\lambda}\right)^{k-1} = \theta_{k-1} + A_{k-1}\psi_1$. Since $\lim_{x\to\infty}\phi_{k-1}(x) = 0$ by construction, we can find a particular solution that is convergent, as for $\theta'(0)$ higher than a threshold, the solution diverges to $+\infty$ and for $\theta'(0)$ smaller than that threshold, it diverges to $-\infty$. Thus, we can express the solution to Equation (15) as

$$\frac{b}{r}\left(\frac{\lambda}{r+\lambda}\right)^k + \theta_k(c - \bar{c}) + A_k\psi_1(c - \bar{c}),$$

and yields (18). This completes the induction. ■