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Exact Analysis of a Lost Sales Model under Stuttering Poisson Demand-Online Appendix

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We investigate the (S-1,S) inventory policy under stuttering Poisson demand and generally distributed lead time when the excess demand is lost. We correct results presented in Feeney and Sherbrooke's seminal paper (1966) and note that the stationary distribution of units on order for the general compound Poisson demand case is still an open question.

Key words: Lost Sales, Stuttering Poisson Process, Reversible Markov Chain

Glossary and Proofs

EC.1. Glossary

λ rate of customer arrivals

X customer order size

$k = 0, 1, \ldots$ order size

$p_k \equiv P\{X = k\}$

$\overline{P}_k \equiv P\{X > k\}$

I_t inventory on hand at time t

S order up to level

μ delivery rate

$\tau = \frac{1}{\mu}$ expected lead time

N_{kt} the number of replenishment orders size k at time t

$N_t = (N_{kt})_{k=1}^{S}$

$N = \{N_t, t \geq 0\}$

V state space of replenishment orders

$n_k(i)$ number of orders size k, $i \in V$

$n(i) = (n_1(i), n_2(i), ..., n_S(i))$

$n_0(i) \equiv S - \sum_{k=1}^{S} k n_k(i)$

$m(i)$ the number of orders outstanding

(i, j) transition

$||(i, j)|| \equiv \sum_{k=1}^{S} |n_k(i) - n_k(j)|$

$k_{ij} \equiv \sum_{k=1}^{S} k\, |n_k(i) - n_k(j)|$

V_C^2 customer order arrivals class

V_R^2 replenishment order arrivals class

A_{ij} infinitesimal generator for lost sale Markov process

$\tilde{X}(t)$ generic continuous Markov chain

$\tilde{V}$ generic state space

$\tilde{\xi}_i$ generic stationary distribution

$\tilde{A}$ generic generator for $\tilde{X}$

i_0 reference state

ν_i reversibility rates

$\pi = (\pi_s)$ stationary distribution of number of units on order

$\eta_{m,s} = \sum_{\substack{i \in V \\ S-n_0(i)=s \\ m(i)=m}} \xi_i$ stationary probability of m orders and s units on order

$\bar{\nu} = \frac{1}{1+\sum_{j \neq i_0} \nu_j}$ normalizing constant

$f_{NB}(\cdot; m, p)$ negative binomial probability distribution

$G(S)$ normalizing constant for (π_s) distribution

EC.2. Proofs

Proof of Proposition ??:

For any path chosen according to the largest subscript rule and for the generators (**??**),

$$A_{i_0 j_{m(i)-1}} A_{j_{m(i)-1} j_{m(i)-2}} \dots A_{j_1 i} = \prod_{l=m(i),m(i)-1,\dots,1} (\lambda p^{1\{n_0(j_{l-1})>0\}} (1-p)^{(k_{j_l,j_{l-1}}-1)})$$

$$= (\tfrac{\lambda p}{(1-p)})^{m(i)} (1-p)^{\sum_{l=1}^{m(i)} k_{j_l,j_{l-1}}} p^{-1\{n_0(i)=0\}}$$

$$= (\tfrac{\lambda p}{(1-p)})^{m(i)} (1-p)^{S-n_0(i)} p^{-1\{n_0(i)=0\}}.$$

Considering the path from i to i_0 and noting that if $n_k(i) = 0$, $n_k(i)! = 1$, we get

$$A_{i j_1} A_{j_1 j_2} \dots A_{j_{m(i)-1} i_0} = \prod_{l=0,1,\dots,m(i)-1} \mu n_{k_{j_l,j_{l+1}}}(j_l) = \mu^{m(i)} \prod_{k=1}^{S} (n_k(i)!).$$

Therefore, from (**??**)

$$\nu_i \equiv \frac{A_{i_0 j_{m(i)-1}} A_{j_{m(i)-1} j_{m(i)-2}} \dots A_{j_1 i}}{A_{i j_1} A_{j_1 j_2} \dots A_{j_{m(i)-1} i_0}} = \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)}}{\prod_{k=1}^{S} (n_k(i)!)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}}.$$

■

Proof of Theorem ??:

Consider any two distinct states $i, i' \in V$, with $n(i) = (n_1(i), n_2(i), \ldots, n_S(i))$ and $n(i') = (n_1(i'), n_2(i'), \ldots, n_S(i'))$, $i \neq i'$.

1. Since $i \neq i'$, $||(i,i')|| \neq 0$. Suppose $A_{ii'} = 0$, then by (**??**) $A_{i'i} = 0$, too. Hence, if $A_{ii'} = 0$, we have $\nu_i A_{ii'} = \nu_{i'} A_{i'i} \equiv 0$.

2. When $A_{ii'} \neq 0$ and $i \neq i'$, then, by (**??**), $||(i,i')|| = 1$, and either $(i,i') \in V_C^2$ or $(i,i') \in V_R^2$. Without loss of generality, we assume $(i,i') \in V_C^2$ and $(i',i) \in V_R^2$.

There are two subcases:

- $n_0(i') > 0$: In this case, a demand of size $k_{ii'}$ arrives which is strictly less than $n_0(i)$. Then $A_{ii'} = \lambda p(1-p)^{(k_{ii'}-1)}$, $A_{i'i} = n_{k_{ii'}}(i')\mu = (n_{k_{ii'}}(i)+1)\mu$ and $m(i') = m(i)+1$. Hence

$$\begin{aligned}
\nu_i A_{ii'} &= \frac{(\frac{\lambda p}{\mu(1-p)})^{m(i)}}{\prod_{k=1}^{S}(n_k(i)!)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} \cdot (\lambda p(1-p)^{(k_{ii'}-1)}) \\
&= \frac{(\frac{1}{\mu(1-p)})^{m(i)}}{\prod_{k=1}^{S}(n_k(i)!)} \cdot (1-p)^{S-n_0(i)+k_{ii'}-1}(\lambda p)^{1+m(i)} \\
&= \frac{\mu(\frac{1}{\mu(1-p)})^{m(i')}}{\prod_{k=1}^{S}(n_k(i')!)} \frac{n_{k_{i,i'}}(i')!}{n_{k_{i,i'}}(i)!} \cdot (1-p)^{S-n_0(i')}(\lambda p)^{m(i')} \\
&= \frac{(\frac{\lambda p}{\mu(1-p)})^{m(i')}}{\prod_{k=1}^{S}(n_k(i')!)} \cdot (1-p)^{S-n_0(i')} \cdot (\mu \frac{n_{k_{i,i'}}(i')!}{n_{k_{i,i'}}(i)!}) \\
&= \nu_{i'} \cdot (n_{k_{i,i'}}(i')\mu) \\
&= \nu_{i'} A_{i'i}.
\end{aligned}$$

- $n_0(i') = 0$: In this case, a demand arrives and the demand size is equal to or greater than $k_{ii'} = n_0(i)$, so $A_{ii'} = \lambda(1-p)^{(k_{ii'}-1)}$, and $A_{i'i} = n_{k_{ii'}}(i')\mu$. But

$$\nu_{i'} = \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i')}}{\prod_{k=1}^{S}(n_k(i')!)} \frac{(1-p)^{S-n_0(i')}}{p}.$$

Similarly,

$$\nu_i(A_{ii'}p) = \frac{(\frac{\lambda p}{\mu(1-p)})^{m(i)}}{\prod_{k=1}^{S}(n_k(i)!)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} \cdot (\lambda(1-p)^{(k_{ii'}-1)})p$$

$$= \frac{(\frac{1}{\mu(1-p)})^{m(i)}}{\prod_{k=1}^{S}(n_k(i)!)} \cdot (1-p)^{S-n_0(i)+k_{ii'}-1}(\lambda p)^{1+m(i)}$$

$$= \frac{\mu(\frac{1}{\mu(1-p)})^{m(i')}}{\prod_{k=1}^{S}(n_k(i')!)} \frac{n_{k_{i,i'}}(i')!}{n_{k_{i,i'}}(i)!} \cdot (1-p)^{S-n_0(i')}(\lambda p)^{m(i')}$$

$$= \frac{(\frac{\lambda p}{\mu(1-p)})^{m(i')}}{\prod_{k=1}^{S}(n_k(i')!)} \cdot (1-p)^{S-n_0(i')} \cdot (\mu \frac{n_{k_{i,i'}}(i')!}{n_{k_{i,i'}}(i)!})$$

$$= \nu_{i'}p \cdot (n_{k_{i,i'}}(i')\mu)$$

$$= (\nu_{i'}p)A_{i'i}.$$

Hence, $\nu_i A_{ii'} = \nu_{i'} A_{i'i}$.

Therefore, for any $i, i' \in V$, we have $\nu_i A_{ii'} = \nu_{i'} A_{i'i}$ and, after normalization, $\eta_i A_{ii'} = \eta_{i'} A_{i'i}$. By Proposition **??**, N is a reversible stochastic process whose stationary distribution is given by (**??**) and (**??**). ■

Proof of Theorem ??:

Let x and y be positive integers such that $S = x + y$ and $P(X = x) > 0$ and $P(X = y) > 0$. Consider the special states

$$n(i_0) = (0, 0, \ldots, 0)$$

and

$$\{n(i_2) : n_x(i_2) = 1, n_y(i_2) = 1, n_k(i_2) = 0, \text{ for } k \neq x, y\}.$$

Now pick the cyclic sequence: $i_0 \to i_1 \to i_2 \to i_1' \to i_0$, where

$$\{n(i_1) : n_x(i_1) = 1, n_k(i_1) = 0, \text{ for } k \neq x\},$$

and

$$\{n(i_1') : n_y(i_1') = 1, n_k(i_1') = 0, \text{ for } k \neq y\}.$$

If this is a reversible Markov process,

$$\nu_{i_0} A_{i_0,i_1} A_{i_1,i_2} A_{i_2,i_1'} A_{i_1',i_0} = \nu_{i_0} \lambda^2 p_x P(X \geq y) \mu^2$$

must equal

$$\nu_{i_0} A_{i_0,i_1'} A_{i_1',i_2} A_{i_2,i_1} A_{i_1,i_0} = \nu_{i_0} \lambda^2 p_y P(X \geq x) \mu^2.$$

i.e.

$$p_x P(X \geq y) = p_y P(X \geq x).$$

Now, if $\{p_k > 0, \text{ for } k = 1,2,\ldots\}$, an inductive proof easily establishes $p_k = p_1(1-p_1)^{k-1}$ by letting $x \equiv 1$. So X must be geometrically distributed with parameter $p = p_1$. Combined with Theorem (3.3), this is a sufficient and necessary condition for reversibility. ∎

Proof of Proposition ??:

First, let us show that

$$\sum_{\substack{i \in V \\ S - n_0(i) = s \\ m(i) = m}} \frac{m!}{\prod_{k=1}^{S} (n_k(i)!)} = \begin{pmatrix} s-1 \\ m-1 \end{pmatrix}. \tag{EC.1}$$

To better understand the combinatorial expressions, we recast the language from orders and order sizes into boxes and balls. We are considering placing s balls (i.e. units on order) into m boxes (i.e. orders). Suppose we have placed the s balls and have used exactly m boxes. Let $n_k \in \{0,1,2,...,s\}$ denote the number of boxes that contain exactly k balls, $k = 1,2,...,s$. We refer to n_k as the box size count for box size (equivalently, for ball count) k. Of the $m!$ permutations of boxes, we are interested only in sequences that are unique with respect to the number of balls in each box. Thus, for example, if k_j is the number of balls in box j, $j = 1,...,m$, the sequence $(k_1,k_2,k_3) = (0,1,1)$ corresponds to two equivalent permutations of the boxes since boxes numbered 2 and 3 can be reversed in sequence without changing the vector (k_1,k_2,k_3). For a given vector of box size counts, $n \equiv (n_1,n_2,...,n_s)$, the number of permutations of boxes that are unique with respect to box size (i.e. ball count), is given by:

$$\frac{m!}{\prod_{\substack{k \in \{1,\ldots,s\} \\ n_k > 0}} n_k!} = \frac{m!}{\prod_{k=1}^{s} n_k!},$$

where equality comes from the convention that $0! = 1$. From this, it follows that the number of ways of assigning s balls to exactly m boxes and sequencing the boxes so that the sequence is unique by ball count is given by

$$\sum_{\substack{n=(n_1,n_2,\ldots,n_s)\\ \sum_{k=1}^{s} kn_k=s\\ \sum_{k=1}^{s} n_k=m}} \frac{m!}{\prod_{k=1}^{s} n_k!} = \sum_{\substack{i\in V\\ S-n_0(i)=s\\ m(i)=m}} \frac{m!}{\prod_{k=1}^{S}(n_k(i)!)}.$$

This is the left hand side of (EC.1). Now, we consider the same combinatorial problem from a different perspective. If we take any sequence of balls and place dividers between some of them, we could then assign the balls between dividers to boxes in sequence. The placement of dividers would uniquely define a sequence of ball counts per box. To ensure that exactly m boxes were used (with positive ball counts in each) we would have to place exactly $m-1$ dividers into different positions between the s balls. (Placing two dividers between the same two balls would imply an empty box, which is not allowed.) Note that only $s-1$ positions are available in this partitioning process; therefore, it follows that the number of ways to place these dividers is given by

$$\binom{s-1}{m-1}.$$

From this we get (EC.1).

Therefore,

$$\begin{aligned}\eta_{m,s} &= \bar{\nu}\frac{\left(\frac{\lambda}{\mu}\right)^m}{p^{1\{s=S\}}m!}p^m(1-p)^{s-m}\binom{s-1}{m-1}\\ &= \bar{\nu}\frac{\left(\frac{\lambda}{\mu}\right)^m}{m!}p^{-1\{s=S\}}\binom{s-1}{s-m}p^m(1-p)^{s-m}\\ &= \bar{\nu}\frac{\left(\frac{\lambda}{\mu}\right)^m}{m!}p^{-1\{s=S\}}f_{\mathrm{NB}}(s-m;m,p).\end{aligned}$$

■

Proof of Theorem ??:

Theorem EC.2 shows that the stationary distribution of ν_i is unchanged if the lead time has the same mean $\frac{1}{\mu}$ but has a general distribution where the lead times are independently identically distributed. Therefore the stationary distribution of the number of units on order is still the same as that when lead times are exponentially distributed. ■

Proof of Theorem ??:

Recall that f is the pdf of a geometric distribution with parameter p. Then

$$f^{*m}(s) = f_{\mathrm{NB}}(s-m; m, p).$$

Thus, $\pi_s G(S) e^{-\frac{\lambda}{\mu}} = h(s)H(S)$ holds for $s = 0, 1, \ldots, S-1$.

When $s = S = 1$,

$$\pi_1 G(1) e^{-\frac{\lambda}{\mu}} = (\frac{\lambda}{\mu}) e^{-\frac{\lambda}{\mu}} \frac{f(1)}{p} = (\frac{\lambda}{\mu}) e^{-\frac{\lambda}{\mu}} = h(S)H(S).$$

Suppose $s = S > 1$ and $i > S$. When $m > 1$

$$\frac{f^{*m}(i+1)}{f^{*m}(i)} = \frac{\begin{pmatrix} (i+1)-1 \\ (i+1)-m \end{pmatrix} p^m (1-p)^{i+1-m}}{\begin{pmatrix} i-1 \\ i-m \end{pmatrix} p^m (1-p)^{i-m}} = \frac{i}{i+1-m}(1-p) > (1-p).$$

This means $f^{*m}(i+1) > (1-p) f^{*m}(i)$ and

$$f^{*m}(i) > (1-p)^{i-S} f^{*m}(S).$$

Therefore, when $S > 1$ and $m > 1$,

$$\sum_{i=S}^{\infty} f^{*m}(i) > \sum_{i=S}^{\infty} (1-p)^{i-S} f^{*m}(S) = f^{*m}(S) \sum_{i=0}^{\infty} (1-p)^i = \frac{f^{*m}(S)}{p}.$$

Since $f^{*0}(i) = 0$ for $i > 0$ and $f^{*1}(S)/p = f(S)/p = \sum_{i=S}^{\infty} f(i)$, we see that for $S > 1$

$$\pi_S G(S) e^{-\frac{\lambda}{\mu}} = \sum_{m=0}^{S} ((\frac{\lambda}{\mu})^m e^{-\frac{\lambda}{\mu}}/m!) \frac{f^{*m}(S)}{p} < \sum_{m=0}^{S} ((\frac{\lambda}{\mu})^m e^{-\frac{\lambda}{\mu}}/m!) \sum_{i=S}^{\infty} f^{*m}(i) = h(S)H(S).$$

After normalization, we have

$$h(S) > \pi_S \text{ and } \; h(s) < \pi_s, \text{ for } s = 0, 1, \ldots, S-1,$$

when $S > 1$. Furthermore,

$$\frac{h(s)}{\pi_s} = \frac{h(s')}{\pi_{s'}},$$

provided $s, s' < S$. ∎

EC.3. The Complete Fill Case

To this point we have considered only the partial fill case. Another possibility is that a customer order is rejected (all units lost) if there is insufficient stock on hand to fill it completely. We refer to this as the complete fill case. The analysis is very similar to the partial fill case. We have

$$\eta_{m,s} = \eta_{0,0}\left(\frac{\lambda}{\mu}\right)^m \frac{f_{NB}(s-m;m,p)}{m!}, \tag{EC.2}$$

where $\eta_{0,0}$ is the normalizer. The steady state distribution of units on order is given by the following:

PROPOSITION EC.1. *For the lost sales model with stuttering Poisson demand and complete fills, the stationary distribution of the number of units on order is given by:*

$$\pi_s = \frac{\sum_{m=0}^{s}(\frac{\lambda}{\mu})^m \frac{f_{NB}(s-m;m,p)}{m!}}{G(S)}, \tag{EC.3}$$

where $G(S) = \sum_{s=0}^{S}\sum_{m=0}^{s}\frac{(\frac{\lambda}{\mu})^m}{m!} f_{NB}(s-m;m,p)$, *and* $f_{NB}(s-m;0,p) = 1\{s=0\}$ *when* $m=0$. *i.e. the truncated compound Poisson distribution.*

Proof: In the case of complete fill the accepted demand is given by $X1_{X\le I}$, where X is the customer order size and I is inventory on hand at the time of the order, as before. The infinitesimal generator in the stuttering Poisson case becomes:

$$A_{ij} \equiv \begin{cases} n_{k_{ij}}(i)\mu & \text{if } (i,j)\in V_R^2, \\ \lambda p_{k_{ij}} & \text{if } (i,j)\in V_C^2, \\ -(m(i)\mu + \lambda(1-\overline{P}_{n_0(i)}))1\{n_0(i)\neq 0\} & \text{if } j=i, \\ 0 & \text{otherwise.} \end{cases} \tag{EC.4}$$

Following the notation and method of section 3, we get

$$\nu_i \equiv \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)}}{\prod_{k=1}^{S}(n_k(i)!)}(1-p)^{S-n_0(i)} \tag{EC.5}$$

as the complete fill counterpart to (**??**). Observe that the term $\frac{1}{p^{1\{n_0(i)=0\}}}$ is needed for the partial fill case (**??**).

In the analog of Theorem **??** for the complete fill case, simply replace (**??**) with (EC.5). The proof is identical except that the case $n_0(i)=0$ is no different from the $n_0(i)>0$ case with complete fills. In the analog of Proposition **??** and Corollary 1, omit the factor $\frac{1}{p^{1\{n_0(i)=0\}}}$ or $\frac{1}{p^{1\{s=S\}}}$. The analog to (**??**) for the complete fill case becomes (EC.3). ■

This is a bimodal distribution because the mode at $s = S$ disappears.

THEOREM EC.1. *Suppose in the lost sales model that demand occurs according to stuttering Poisson process and the replenishment order lead times are independent and identically distributed and have general distribution with finite mean $L = \frac{1}{\mu}$, where there is no point mass at zero. For the complete fill case, the stationary distribution of the number of units on order is given by*

$$\hat{\pi}_s = \pi_s,$$

where π_s, given by (EC.3), is the stationary distribution of the number of units on order in the lost sales model when lead times are exponentially distributed with mean $\frac{1}{\mu}$.

The proof is similar to that of Theorem **??**.

EC.4. A Lost Sales Model with Stuttering Poisson Demand and General Lead Times

We always assume that the lead times are independently identically distributed. The original process $X(t)$, with exponentially distributed lead times is a Markov process. When it is extended to the case of general time distributions, it becomes a generalized semi-Markov process (GSMP). By extending the state space, we can obtain a Markov process and derive the stationary distribution of the extended state space. Finally, we could prove the marginal stationary distribution of the number of units on order does not depend on the lead time distribution but only on its mean.

Let $F(\cdot)$ denote the general cumulative distribution function(CDF) of order lead times with no point mass at zero. Expand the underlying state space from V to $V \times \Re^+_{S,S}$. Here $U = (u_{s,r}) \in \Re^+_{S,S}$ is an S by S matrix with non-negative elements. We construct the lost sales model with generally distributed lead times as a stochastic process $Z(t)$, with state space $V \times \Re^+_{S,S}$:

$$Z(t) = (i, U) = (n_1(i), n_2(i), \dots, n_S(i); U) = \begin{pmatrix} n_1(i) & n_2(i) & \dots & \dots & n_S(i) \\ u_1^{(1)} & u_2^{(1)} & \dots & \dots & u_S^{(1)} \\ u_1^{(2)} & u_2^{(2)} & \dots & \dots & u_S^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(S)} & u_2^{(S)} & \dots & \dots & u_S^{(S)} \end{pmatrix}.$$

Here, $n_s(i)$ is the number of outstanding orders with size s and $u_s^{(1)} \geq u_s^{(2)} \geq \ldots \geq u_s^{(S)} \geq 0$ stand for the ordered replenishment ages for orders with size s. That is $u_s^{(r)}$ is the age of the rth oldest replenishment order of size s. The new process $Z(t)$ is a Markov process on $V \times \Re^{+}_{S,S}$.

Define $R(i) = \{(s,r) : r \leq n_s(i)\}$ as the replenishment order index set. So we have $u_s^{(r)} = 0$ if $(s,r) \notin R(i)$. Define

$$\Re_{S,S}(i) \equiv \{U \in \Re^{+}_{S,S} : u_s^{(1)} \geq u_s^{(2)} \geq \ldots \geq u_s^{(S)} \geq 0, \text{ and } u_s^{(r)} \equiv 0 \text{ if } (s,r) \notin R(i)\}.$$

Each state (i,U) in this system satisfies the condition $U \in \Re_{S,S}(i)$ and therefore we have $(i,U) \in V \times \Re_{S,S}(i) \subseteq V \times \Re^{+}_{S,S}$.

Our intent is to show that the stationary distribution of $Z(t)$ is insensitive to the lead time distribution for a given mean, $\frac{1}{\mu}$, under the partial fill case. The proof for the complete fill case is nearly the same.

LEMMA EC.1. *Given state i $((n_1(i), n_2(i), \ldots\ldots n_S(i)))$,*

$$\int_{U \in \Re_{S,S}(i)} \prod_{s,r} [1 - F(u_s^{(r)})] du_1^{(1)} \ldots u_S^{(S)} = \frac{1}{\prod_{s=1}^{S} n_s(i)!} (\frac{1}{\mu})^{m(i)},$$

where $m(i) = \sum_{s=1}^{S} n_s(i)$.

Proof: Since $\prod_{s,r}[1 - F(u_s^{(r)})]$ does not depend on the order of $u_s^{(r)}$, we could integrate it on the whole space and divide the results by $n_s(i)!$ for each s fixed. Therefore,

$$\begin{aligned} \int_{U \in \Re_{S,S}(i)} \prod_{s,r}[1 - F(u_s^{(r)})] du_1^{(1)} \ldots u_S^{(S)} &= \prod_{s=1}^{S} [\int_{u_s^{(1)} \geq u_s^{(2)} \geq \ldots \geq u_s^{(S)} \geq 0} \prod_r [1 - F(u_s^{(r)})] du_s^{(1)} \ldots u_s^{(S)}] \\ &= \prod_{s=1}^{S} [\int_{t_{s,1}=0}^{\infty} \cdots \int_{t_{s,S}=0}^{\infty} [1 - F(t_{s,r})] \tfrac{1}{n_s(i)!} dt_{s,1} \ldots dt_{s,S}] \\ &= \prod_{s=1}^{S} \{ \tfrac{1}{n_s(i)!} [\int_{t=0}^{\infty} [1 - F(t)] dt]^{n_s(i)} \} \\ &= \prod_{s=1}^{S} \{ \tfrac{1}{n_s(i)!} [\tfrac{1}{\mu}]^{n_s(i)} \} \\ &= \tfrac{1}{\prod_{s=1}^{S} n_s(i)!} (\tfrac{1}{\mu})^{m(i)}. \end{aligned}$$

■

The proof of uniqueness and ergodicity of the stationary distribution of this Markov process $Z(t)$ is a consequence of Theorem 1 in Sevastyanov (1957). The proof just follows the routine of proving the results for a telephone system with refusals (Sevastyanov, 1957, section 3). The stationary distribution of $Z(t)$ is given by the following theorem. The marginal distribution of $X(t)$ is seen to be invariant to the form of the lead time distribution.

THEOREM EC.2. *The steady state distribution of this Markov process $Z(t)$, $\zeta_{(i,U)}$ is*

$$\zeta_{(i,U)} = C\left(\frac{\lambda p}{1-p}\right)^{m(i)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} \prod_{(s,r)\in R(i)} [1-F(u_s^{(r)})], \tag{EC.6}$$

*where $C = \frac{1}{G(S)}$ is the same normalizer as in Corollary **??**. Therefore, the steady state distribution of the original GSMP,*

$$\tilde{\xi}_i = \int_{U\in\Re^+_{S,S}} \zeta_{(i,U)} dU = C\frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)}}{\prod_{r=1}^{S}(n_r(i)!)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}},$$

which is the same stationary distribution that is obtained when the lead times are exponentially distributed with mean $\frac{1}{\mu}$.

Proof: Notice that $\prod_{(s,r)}[1-F(u_s^{(r)})] = \prod_{(s,r)\in R(i)}[1-F(u_s^{(r)})]$ since $1-F(u_s^{(r)}) = 1$ for (s,r) outside of $R(i)$. Integrating $\zeta_{(i,U)}$ with respect to $U \in \Re_{S,S}(i)$ and use LemmaEC.1, we have

$$\begin{aligned}\int_{U\in\Re_{S,S}(i)} \zeta_{(i,U)} dU &= C\left(\frac{\lambda p}{1-p}\right)^{m(i)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} \int_{U\in\Re_{S,S}(i)} \prod_{s,r}[1-F(u_s^{(r)})]dU \\ &= C\frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)}}{\prod_{r=1}^{S}(n_r(i)!)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}}\end{aligned}$$

Let $U+\Delta t$(or $U-\Delta t$) denote adding (or subtracting) small Δt (or $\min(\Delta t, u_s^{(r)})$) to U's each entry $u_s^{(r)}$ if $(s,r)\in R(i)$.

We claim that for Δt sufficiently small, there will occur at most one event (customer arrival or order replenishment delivery) in the interval $(t, t+\Delta t]$ for any t. This follows because the delivery process is simply a shifted, filtered version of the arrival process. Consequently, the combined process is a filtered version of a Poisson process (refer to Resnick 2005, section 4.4 page 316). So now we choose a Δt sufficiently small so that at most one event happens within the interval $(t, t+\Delta t]$.

Define $Q_{(i,U),(j,U')}(\Delta t)$ as the transition probability from state (i,U) to state (j,U') during time Δt. Since $Z(t)$ is a Markov process, Q has no dependence on t. For sufficiently small Δt, we have the following transition probabilities:

Case 1 : If no customer arrives, $n_0(i) = S$, and $(j,U') = (i_0, O)$, where O is the matrix with zeros entries,

$$Q_{(i_0,O),(i_0,O)}(\Delta t) = 1 - \lambda \Delta t + o(\Delta t).$$

Case 2 : If no replenishment order arrives when (i,U) has $n_0(i) = 0$ (any arrival is lost), we have

$$Q_{(i,U),(i,U+\Delta t)}(\Delta t) = \prod_{(s,r)\in R(i)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})}.$$

Case 3 : If no customer arrives for general state (i,U) with $0 < n_0(i) < S$,

- (Case 3a) When no customer arrives, or no replenishment order arrives during time Δt case,

$$Q_{(i,U),(i,U+\Delta t)}(\Delta t) = \prod_{(s,r)\in R(i)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})}(1 - \lambda \Delta t + o(\Delta t))$$

- (Case 3b) Now suppose no customer arrives but one replenishment order of size k_{ij} $((i,j) \in V_R^2)$

arrives. Suppose that order is the lth oldest order, $1 \le l \le n_{k_{ij}}(i)$. Let $U_{i,j}^{l-}$ be the same as U except that the element $u_{k_{ij}}^{(l)}$ is deleted so that the lth column changes from

$$(u_{k_{ij}}^{(1)}, \ldots, u_{k_{ij}}^{(n_{k_{ij}}(i))}, 0, \ldots, 0)',$$

to

$$(u_{k_{ij}}^{(1)}, \ldots, u_{k_{ij}}^{(l-1)}, u_{k_{ij}}^{(l+1)}, \ldots, u_{k_{ij}}^{(n_{k_{ij}}(i))}, 0, 0, \ldots, 0)'.$$

Actually, U_{ij}^{l-} is U after recording delivery of lth oldest order of size k_{ij}. Thus,

$$Q_{(i,U),(j,U_{i,j}^{l-}+\Delta t)}(\Delta t) = \frac{F(u_{k_{ij}}^{(l)} + \Delta t) - F(u_{k_{ij}}^{(l)})}{1 - F(u_{k_{ij}}^{(l)}))} \prod_{(s,r)\in R(i)/(k_{ij},l)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})}(1 - \lambda \Delta t + o(\Delta t)).$$

Case 4 : When (i,U) satisfies $n_0(i)>0$, and one customer arrives with accepted order size k_{ij} ($(i,j)\in V_C^2$) and has age u ($0<u\le \Delta t$) at the end of the interval and no replenishment order arrives, we have

$$Q_{(i,U),(j,U_{i,j})}(\Delta t)=\prod_{(s,r)\in R(i)}\frac{1-F(u_s^{(r)}+\Delta t)}{1-F(u_s^{(r)})}(A_{ij}\Delta t+o(\Delta t))(\frac{1}{\Delta t})(1-F(u)).$$

Here $U_{i,j}=U+\Delta t$ except that the new replenishment order caused by the new arrival has age $u_{k_{ij}}^{(n_{k_{ij}}(j))}=u$. Notice that $(\frac{1}{\Delta t})$ is the conditional density of the new replenishment order with u being the age at the end of the interval $(0,\Delta t]$. This is because of the uniformly distributed arrival time of the Poisson process conditioned on one arrival occurring during an interval of length Δt. A special case when $(i,U)=(i_0,O)$,$(i_0,j)\in V_C^2$, we have

$$Q_{(i_0,O),(j,U_{i_0,j})}(\Delta t)=\frac{A_{i_0,j}\Delta t+o(\Delta t)}{\Delta t}(1-F(u)),$$

where $U_{i_0,j}=O$ except $u_{k_{ij}}^{(1)}=u$.

Define $P_{(i,U)}(t)=P[Z(t)=(i,U)]$. Making use of the Markov property and $Q_{(i,U)(j,U')}(\Delta t)$, we obtain:

Case 1 For $(i,U)=(i_0,O)$,

$$P_{(i_0,O)}(t+\Delta t)=P_{(i_0,O)}(t)(1-\lambda\Delta t)+\sum_{\{j:(j,i_0)\in V_R^2\}}\int_0^\infty P_{(j,U)}(t)\frac{F(u_{k_{ji_0}}^{(1)}+\Delta t)-F(u_{k_{ji}}^{(1)})}{1-F(u_{k_{ji_0}}^{(1)})}du_{k_{ji_0}}^{(1)}+o(\Delta t), \tag{EC.7}$$

except $u_{k_{ji_0}}^{(1)}$, the other entries of U are zeros.

Case 2 For (i,U) with $n_0(i)=0$,

$$P_{(i,U)}(t+\Delta t)=P_{(i,U-\Delta t)}(t)\prod_{(s,r)\in R(i)}\frac{1-F(u_s^{(r)})}{1-F(u_s^{(r)}-\Delta t)}+o(\Delta t). \tag{EC.8}$$

Case 3 For general (i,U) with $0<n_0(i)<S$, and $u_s^{(r)}>0$ for all $(s,r)\in R(i)$,

$$\begin{aligned}&P_{(i,U)}(t+\Delta t)\\ =\;&P_{(i,U-\Delta t)}(t)\textstyle\prod_{(s,r)\in R(i)}\frac{1-F(u_s^{(r)})}{1-F(u_s^{(r)}-\Delta t)}(1-\lambda\Delta t)\\ &+\textstyle\sum_{\{(j,U'):(j,i)\in V_R^2,U'=(U-\Delta t)_{ji}^+\}}\int_0^\infty P_{(j,U')}(t)\prod_{(s,r)\in R(i)}\frac{1-F(u_s^{(r)})}{1-F(u_s^{(r)}-\Delta t)}\frac{F(u)-F(u-\Delta t)}{1-F(u-\Delta t)}du(1-\lambda\Delta t)\\ &+o(\Delta t),\end{aligned} \tag{EC.9}$$

where $(U-\Delta t)_{ji}^+$ is the $U-\Delta t$ inserting $u_{k_{ji}}^{(l)}=u$ for some $l\le n_{k_{ji}}(j)$.

Case 4 For general (i,U), with one $u_s^{(r)} = u$ with $0 < u \le \Delta t$ for $(s,r) \in R(i)$,

$$P_{(i,U)}(t+\Delta t) = P_{(j,U-\Delta t)}(t) \prod_{(s,r)\in R(j)} \frac{1-F(u_s^{(r)})}{1-F(u_s^{(r)}-\Delta t)} (A_{ji}\Delta t + o(\Delta t)) \frac{1}{\Delta t}(1-F(u)), \qquad \text{(EC.10)}$$

where $(j,i) \in V_C^2$.

Define $P^*_{(i,U)}(t) = \frac{P_{(i,U)}(t)}{\prod_{(s,r)\in R(i)}[1-F(u_s^{(r)})]}$, which is the conditional probability in state i given the ages of replenishment orders at time t. Assume the existence of $\frac{\partial P^*_{(i,U)}(t)}{\partial t}$ and $\frac{\partial P^*_{(i,U)}(t)}{\partial u_s^{(r)}}$. Dividing equations (EC.7)-(EC.9) by Δt and letting $\Delta t \to 0$ in equation (EC.7)-(EC.10), we obtain the following system of integro-differential equations

Case 1

$$\frac{\partial P^*_{(i_0,O)}(t)}{\partial t} + \lambda P^*_{(i_0,O)}(t) = \sum_{\{j:(j,i_0)\in V_R^2\}} \int_0^\infty P^*_{(j,U)}(t) dF(u^{(1)}_{k_{ji_0}}),$$

$u^{(1)}_{k_{ji_0}}$ is the only positive entry of U.

Case 2 For (i,U) with $n_0(i) = 0$,

$$\frac{\partial P^*_{(i,U)}(t)}{\partial t} + \sum_{(s,r)\in R(i)} \frac{\partial P^*_{(i,U)}(t)}{\partial u_s^{(r)}} = 0.$$

Case 3 For general (i,U) with $i \ne i_0$ and $0 < n_0(i) < S$,

$$\frac{\partial P^*_{(i,U)}(t)}{\partial t} + \sum_{(s,r)\in R(i)} \frac{\partial P^*_{(i,U)}(t)}{\partial u_s^{(r)}} + \lambda P^*_{(i,U)}(t) = \sum_{\{(j,U'):(j,i)\in V_R^2, U'=(U)^+_{ji}\}} \int_0^\infty P^*_{(j,U')}(t) dF(u) \qquad \text{(EC.11)}$$

where $(U)^+_{ji}$ is the U inserting $u^{(l)}_{k_{ji}} = u$ for some $l \le n_{k_{ji}}(j)$.

Case 4 for $(j,i) \in V_C^2$,

$$P^*_{(i,U)}(t) = A_{ji} P^*_{(j,U)}(t).$$

If we start with the stationary distribution, then all the derivatives with respect to time t vanish. Dropping the dependence on t, we have

Case 1

$$\lambda P^*_{(i_0,O)} = \sum_{\{j:(j,i_0)\in V_R^2\}} \int_0^\infty P^*_{(j,U)} dF(u^{(1)}_{k_{ji_0}}), \qquad \text{(EC.12)}$$

$u^{(1)}_{k_{ji_0}}$ is the only positive entry of U.

Case 2 For (i,U) with $n_0(i)=0$,

$$\sum_{(s,r)\in R(i)} \frac{\partial P^*_{(i,U)}}{\partial u_s^{(r)}} = 0. \tag{EC.13}$$

Case 3 For general (i,U) with $i \neq i_0$ and $0 < n_0(i) < S$,

$$\sum_{(s,r)\in R(i)} \frac{\partial P^*_{(i,U)}}{\partial u_s^{(r)}} + \lambda P^*_{(i,U)} = \sum_{\{(j,U'):(j,i)\in V_R^2, U'=(U)^+_{ji}\}} \int_0^\infty P^*_{(j,U')} dF(u) \tag{EC.14}$$

where $(U)^+_{ji}$ is the U inserting $u_{k_{ji}}^{(l)} = u$ for some $l \leq n_{k_{ji}}(j)$.

Case 4 For $(j,i)\in V_C^2$,

$$P^*_{(i,U)}(t) = A_{ji} P^*_{(j,U)}(t). \tag{EC.15}$$

Let $\zeta_{(i,U)}$ be given by (EC.6). It is straightforward to verify that the substitution $P^*_{(i,U)}$ by $\frac{\zeta_{(i,U)}}{\prod_{(s,r)\in R(i)}[1-F(u_s^{(r)})]}$ satisfies equations (EC.12)-(EC.15). We show only the proof of Case 3 here. From (EC.6), we know

$$\frac{\zeta_{(i,U)}}{\prod_{(s,r)\in R(i)}[1-F(u_s^{(r)})]} = C\left(\frac{\lambda p}{1-p}\right)^{m(i)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}}, \tag{EC.16}$$

where C is the normalizer. Now the substitution $P^*_{(i,U)}$ in (EC.14) by (EC.16), we have the left hand side (LHS) as

$$LHS = \lambda C\left(\frac{\lambda p}{1-p}\right)^{m(i)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} = \lambda C\left(\frac{\lambda p}{1-p}\right)^{m(i)} (1-p)^{S-n_0(i)}.$$

The right hand side (RHS) becomes

$$\begin{aligned}
RHS &= \textstyle\sum_{\{(j,U'):(j,i)\in V_R^2, U'=(U)^+_{ji}\}} \int_0^\infty C\left(\frac{\lambda p}{1-p}\right)^{m(j)} \frac{(1-p)^{S-n_0(j)}}{p^{1\{n_0(j)=0\}}} dF(u)\\
&= \textstyle\sum_{\{j:(j,i)\in V_R^2\}} C\left(\frac{\lambda p}{1-p}\right)^{m(j)} \frac{(1-p)^{S-n_0(j)}}{p^{1\{n_0(j)=0\}}}\\
&= \textstyle\sum_{\{j:(j,i)\in V_R^2\}} C\left(\frac{\lambda p}{1-p}\right)^{m(i)+1} \frac{(1-p)^{S-n_0(i)+n_{ij}}}{p^{1\{n_0(j)=0\}}}\\
&= \lambda C\left(\frac{\lambda p}{1-p}\right)^{m(i)} (1-p)^{S-n_0(i)} \left(\frac{p}{1-p}\right) \textstyle\sum_{\{j:(j,i)\in V_R^2\}} \frac{(1-p)^{n_{ij}}}{p^{1\{n_0(j)=0\}}}\\
&= \lambda C\left(\frac{\lambda p}{1-p}\right)^{m(i)} (1-p)^{S-n_0(i)} \left(\frac{p}{1-p}\right) \textstyle\sum_{n_{ij}=1}^{n_0(i)} \frac{(1-p)^{n_{ij}}}{p^{1\{n_0(j)=0\}}}\\
&= \lambda C\left(\frac{\lambda p}{1-p}\right)^{m(i)} (1-p)^{S-n_0(i)} \left(\frac{p}{1-p}\right) \left(\textstyle\sum_{k=1}^{n_0(i)-1} (1-p)^k + \frac{(1-p)^{n_0(i)}}{p}\right)\\
&= \lambda C\left(\frac{\lambda p}{1-p}\right)^{m(i)} (1-p)^{S-n_0(i)} \left(\frac{p}{1-p}\right) \left(\frac{1-p}{p}\right)\\
&= \lambda C\left(\frac{\lambda p}{1-p}\right)^{m(i)} (1-p)^{S-n_0(i)}\\
&= LHS.
\end{aligned}$$

Therefore, $\zeta_{(i,u)}$ is the stationary distribution of $Z(t)$.

■