

**e - c o m p a n i o n**

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## Proofs of statements and outline of bounding procedure $H^2$

### EC.1. Proofs of statements

**THEOREM 1.** For a given vector  $\lambda$ , let  $\xi^*$  be an optimal  $\overline{LR}(\lambda)$  solution, and let  $D^* = \{i \in D : \xi_i^* = 1\}$ . Defining  $\mu_i = \lambda_i + \lambda_{n+i}$ ,  $\forall i \in P$ , and  $\sigma = \min\{0, \phi(i_m)\}$ , a feasible DLF solution  $\mathbf{u}$  of cost  $z(\overline{LR}(\lambda))$  is given by:

$$u_i = \begin{cases} \mu_i, & \text{if } n+i \notin D^* \\ \mu_i + \phi(n+i) - \sigma, & \text{if } n+i \in D^* \end{cases}, \quad \forall i \in P, \quad (15)$$

$$u_0 = \sigma. \quad (16)$$

*Proof.* As  $\sigma \leq 0$ , the DLF solution  $\mathbf{u}$  defined by expressions (15) and (16) satisfies  $u_0 \leq 0$ . We now show that the vector  $\mathbf{u}$ , given by expressions (15) and (16), satisfies the dual constraints associated with each variable  $x_\ell$ ,  $\ell \in \mathcal{R}$ , that is,  $\sum_{i \in S_\ell} u_i + u_0 \leq c_\ell$ .

Let  $S_\ell^* = \{i \in S_\ell : n+i \in D^*\}$ , and let  $\beta(\ell)$  be the last delivery vertex visited by route  $\ell$ . From expressions (15) and (16), we have:

$$\sum_{i \in S_\ell} u_i + u_0 = \sum_{i \in S_\ell} \mu_i + \sum_{i \in S_\ell^*} (\phi_{n+i} - \sigma) + \sigma. \quad (\text{EC.1})$$

We have two cases.

- A)  $S_\ell^* = \emptyset$ . In this case, we have to show that  $\sum_{i \in S_\ell} \mu_i + \sigma \leq c_\ell$ . Since  $\ell \in \mathcal{D}_{\beta(\ell)}$ , from expressions (9), we have  $\phi(\beta(\ell)) \leq (c_\ell - \sum_{i \in S_\ell} \mu_i)$ . Since  $\beta(\ell) \notin D^*$ , we have  $\sigma \leq \phi(\beta(\ell))$ . Combining these two inequalities, we obtain  $\sigma \leq (c_\ell - \sum_{i \in S_\ell} \mu_i)$ , or  $\sum_{i \in S_\ell} \mu_i + \sigma \leq c_\ell$ .
- B)  $S_\ell^* \neq \emptyset$ . Let  $k^* \in S_\ell^*$  be such that  $\phi(n+k^*) = \min\{\phi(n+i) : i \in S_\ell^*\}$ . Note that  $\phi(n+i) - \sigma \leq 0$  as  $\sigma \geq \phi(n+i)$ ,  $\forall n+i \in D^*$ . Therefore, we have  $\sum_{i \in S_\ell^*} (\phi(n+i) - \sigma) \leq \phi(n+k^*) - \sigma$ . Using expressions (EC.1), we derive:

$$\sum_{i \in S_\ell} u_i + u_0 \leq \sum_{i \in S_\ell} \mu_i + \sigma + \phi(n+k^*) - \sigma = \sum_{i \in S_\ell} \mu_i + \phi(n+k^*). \quad (\text{EC.2})$$

From the definition of  $k^*$ , we have  $\phi(n+k^*) \leq \phi(\beta(\ell))$ . Because of expression (9), we obtain  $\phi(n+k^*) \leq \phi(\beta(\ell)) \leq (c_\ell - \sum_{i \in S_\ell} \mu_i)$ . Therefore, from inequality (EC.2), we derive:

$$\sum_{i \in S_\ell} u_i + u_0 \leq \sum_{i \in S_\ell} \mu_i + (c_\ell - \sum_{i \in S_\ell} \mu_i) = c_\ell. \quad (\text{EC.3})$$

The cost of solution  $\mathbf{u}$  computed using expressions (15) and (16) is:

$$\sum_{i \in P: n+i \in D^*} (\mu_i + \phi(n+i) - \sigma) + \sum_{i \in P: n+i \notin D^*} \mu_i + m\sigma = \sum_{i \in P} \mu_i + \sum_{i \in D^*} \phi(i) + m\sigma - |D^*|\sigma. \quad (\text{EC.4})$$

Using the definition of  $\boldsymbol{\mu}$  and  $\boldsymbol{\xi}^*$ , and  $D^*$ , equation (EC.4) rewrites as:

$$\sum_{i \in P \cup D} \lambda_i + \sum_{i \in D} \xi_i^* \phi(i) + m\sigma - |D^*|\sigma. \quad (\text{EC.5})$$

By definition of  $\boldsymbol{\xi}^*$ , we have  $|D^*| \leq m$ . Therefore, there are two cases: (i)  $|D^*| = m$ : then we have  $m\sigma - |D^*|\sigma = 0$ ; (ii)  $|D^*| < m$ : then definition of  $\sigma$  together with expressions (13) implies  $\sigma = 0$ , that is,  $|D^*|\sigma + m\sigma = 0$ ; In both cases we have  $m\sigma - |D^*|\sigma = 0$ , and since  $\boldsymbol{\xi}^*$  is an optimal  $\overline{LR}(\boldsymbol{\lambda})$  solution, expression (EC.5) equals  $z(\overline{LR}(\boldsymbol{\lambda}))$ .  $\square$

**THEOREM 2.** *Associate penalties  $\mu_i \in \mathbb{R}$ ,  $\forall i \in P$ , and  $\mu_0 \leq 0$  with constraints (2) and (3), respectively. Let  $w_i$ ,  $\forall i \in P$ , be a non-negative weight associated with each pickup vertex  $i$ . A feasible DLF solution  $\mathbf{u}$  can be obtained by means of the following expressions:*

$$u_i = w_i \min_{\ell \in \mathcal{R}_i} \left\{ \frac{(c_\ell - \mu(S_\ell) - \mu_0)}{w(S_\ell)} \right\} + \mu_i, \quad \forall i \in P, \quad (17)$$

$$u_0 = \mu_0, \quad (18)$$

where  $w(S_\ell) = \sum_{i \in S_\ell} w_i$  and  $\mu(S_\ell) = \sum_{i \in S_\ell} \mu_i$ .

*Proof.* Consider a route  $\ell \in \mathcal{R}$ . Since  $\ell \in \mathcal{R}_i$ ,  $\forall i \in S_\ell$ , we have:

$$w_i \min_{\ell' \in \mathcal{R}_i} \left\{ \frac{(c_{\ell'} - \mu(S_{\ell'}) - \mu_0)}{w(S_{\ell'})} \right\} + \mu_i \leq w_i \frac{(c_\ell - \mu(S_\ell) - \mu_0)}{w(S_\ell)} + \mu_i, \quad \forall i \in S_\ell. \quad (\text{EC.6})$$

From expression (17) and inequalities (EC.6) we derive:

$$\sum_{i \in S_\ell} u_i \leq \sum_{i \in S_\ell} w_i \frac{(c_\ell - \mu(S_\ell) - \mu_0)}{w(S_\ell)} + \sum_{i \in S_\ell} \mu_i = c_\ell - \mu_0, \quad (\text{EC.7})$$

Therefore, from inequalities (EC.7) and expression (18), we derive:

$$\sum_{i \in S_\ell} u_i + u_0 \leq c_\ell. \quad \square \quad (\text{EC.8})$$

**Dominance Rule 3.** Let  $L_1$  and  $L_2$  be two forward paths, and let  $\ell_1$  and  $\ell_2$  be the routes of minimum reduced cost  $c'_{\ell_1}$  and  $c'_{\ell_2}$  containing  $L_1$  and  $L_2$ , respectively. If arc costs  $\{d'_{ij}\}$  satisfy the condition  $d'_{ij} + d'_{jk} \geq d'_{ik}$ ,  $\forall i, k \in V$ ,  $\forall j \in D$ , and  $e(L_1) = e(L_2)$ ,  $\tilde{P}(L_1) \subseteq \tilde{P}(L_2)$ ,  $P(L_1) \subseteq P(L_2)$ ,  $\tau(L_1) \leq \tau(L_2)$  and  $d'(L_1) \leq d'(L_2)$ ; then  $L_1$  dominates  $L_2$ , because  $c'_{\ell_1} \leq c'_{\ell_2}$ .

*Proof.* Ropke and Cordeau (2009) prove Dominance Rule 3 in case  $\mathcal{C}'_{L_1} = \mathcal{C}'_{L_2} = \emptyset$ , that is  $c'(L) = d'(L)$ . Assume  $\mathcal{C}'_{L_1} \neq \emptyset$  and  $\mathcal{C}'_{L_2} \neq \emptyset$ . The reduced cost  $c'_{\ell_1}$  of route  $\ell_1$  can be written as  $c'_{\ell_1} = d'(R_{\ell_1}) - \sum_{C \in \mathcal{C}'_{\ell_1}} v'_C$ . Because  $P(L_1) \subseteq P(L_2)$ , we have  $\mathcal{C}'_{\ell_1} \subseteq \mathcal{C}'_{\ell_2}$ . Since  $v'_C \leq 0$ ,  $\forall C \in \mathcal{C}'$ , we have  $\sum_{C \in \mathcal{C}'_{\ell_1}} v'_C \geq \sum_{C \in \mathcal{C}'_{\ell_2}} v'_C$ . Moreover, because  $d'(R_{\ell_1}) \leq d'(R_{\ell_2})$  (see Ropke and Cordeau 2009), we have  $d'(R_{\ell_1}) - \sum_{C \in \mathcal{C}'_{\ell_1}} v'_C \leq d'(R_{\ell_2}) - \sum_{C \in \mathcal{C}'_{\ell_2}} v'_C$ . We thus conclude that  $c'_{\ell_1} \leq c'_{\ell_2}$ .  $\square$

## EC.2. Outline of bounding procedure $H^2$

Let *Maxit2* be an a-priori defined number that is used to limit the maximum number of iterations executed by  $H^2$  after finding a feasible dual solution.  $H^2$  executes the following steps.

### Procedure $H^2$

Step 1. *Initialization.* A route set  $\overline{\mathcal{R}} \subset \mathcal{R}$  containing at most  $\Delta^{min}$  routes of minimum reduced cost with respect to the *DLF* solution  $\mathbf{u}^1$  ( $\Delta^{min}$  is an a-priori defined parameter) is generated and used to initialize the master problem which corresponds to *LF*, where  $\mathcal{R}$  is replaced with  $\overline{\mathcal{R}}$ . Set  $LH2 = LH1$ ,  $\mathbf{u}^2 = \mathbf{u}^1$ ,  $\boldsymbol{\mu} = \mathbf{u}^1$  and  $iter = 1$ .

Step 2. *Find a master dual solution  $\bar{\mathbf{u}}$  of cost  $\bar{z}$ .* A near-optimal dual solution  $\bar{\mathbf{u}}$  of cost  $\bar{z}$  of the master is computed by an iterative procedure which initialize  $\bar{z} = 0$  and performs *Maxit3* iterations of the following operations:

- (i) Using the current vector  $\boldsymbol{\mu}$ , it computes a dual solution  $\mathbf{u}$  of the master of cost  $z$  by means of expressions (17) and (18) where  $\mathcal{R}$  is replaced with  $\overline{\mathcal{R}}$ . If  $z < \bar{z}$  then it updates  $\bar{z} = z$  and  $\bar{\mathbf{u}} = \mathbf{u}$ .
- (ii) Using the subgradient vector  $\boldsymbol{\theta}$  given by expressions (20) it modifies the penalty vector  $\boldsymbol{\mu}$  using subgradient optimization.

- Step 3. *Check if  $\bar{\mathbf{u}}$  is a feasible DLF solution.* Generate the largest subset  $\mathcal{N}$  of routes having negative reduced cost with respect to the current dual master solution  $\bar{\mathbf{u}}$  and such that  $|\mathcal{N}| \leq \Delta^a$  ( $\Delta^a$  is a parameter defined a-priori). If  $\mathcal{N} = \emptyset$  and  $\bar{z}$  is greater than  $LH2$ , then update  $LH2 = \bar{z}$ ,  $\mathbf{u}^2 = \bar{\mathbf{u}}$  and  $\boldsymbol{\mu}^2 = \boldsymbol{\mu}$ ; otherwise update  $\bar{\mathcal{R}} = \bar{\mathcal{R}} \cup \mathcal{N}$ .
- Step 4. *Termination criteria.* Set  $iter = iter + 1$ . If  $iter = Maxit2$  then stop; otherwise, return to Step 2.

At the end,  $H^2$  provides a feasible *DLF* solution  $\mathbf{u}^2$  of cost  $LH2$  and a corresponding vector  $\boldsymbol{\mu}^2$ .

The initial route subset  $\bar{\mathcal{R}}$  and the subsets  $\mathcal{N}$  are generated using procedure GENR described in Section 5.