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Online Appendix: Optimal Control of an Assembly System with Multiple Stages and Multiple Demand Classes

Saif Benjaafar[1] • Mohsen ElHafsi[2] • Chung-Yee Lee[3] • Weihua Zhou[4]

[1]*Industrial & Systems Engineering, University of Minnesota, Minneapolis, MN 55455, saif@umn.edu*

[2]*The A. Gary Anderson Graduate School of Management, University of California, Riverside, CA 92521-0203, mohsen.elhafsi@ucr.edu*

[3]*Department of Industrial Engineering and Logistics, Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong, cylee@ust.hk*

[4]*Department of Management Science and Engineering, Zhejiang University, Hangzhou, China, larryzhou@zju.edu.cn*

The following notation is used throughout this appendix:

$\Delta_j v(\mathbf{x}) = v(\mathbf{x}+\mathbf{e}_j) - v(\mathbf{x}),$ for j = 1, …, m,

$\Delta_{i,j} v(\mathbf{x}) = \Delta_i \Delta_j v(\mathbf{x}) = \Delta_j v(\mathbf{x}+\mathbf{e}_i) - \Delta_j v(\mathbf{x}),$

$\Delta_{j-l} v(\mathbf{x}) = v(\mathbf{x}+\mathbf{e}_j - \mathbf{e}_l) - v(\mathbf{x}),$ and

$\Delta_{k-P(k)} v(\mathbf{x}) = v(\mathbf{x}+\mathbf{E}_k) - v(\mathbf{x}),$ where

$\mathbf{E}_k = \mathbf{e}_k - \mathbf{e}_{P(k)}.$

In order to prove Lemma 1, we introduce the following two properties:

B1: $\Delta_{i-P(i),1} v(\mathbf{x}) \geq 0,$ and

B2: $\Delta_{i-P(i),i-P(i)} v(\mathbf{x}) \geq 0.$

Note that B1 is a special case of A1 with $j=1.$ Next, we show that any cost function v that satisfies A5 and B1 also satisfies properties A1-A4. This result will allow us to significantly simplify the proof of Lemma 1 since it is sufficient to only show that the optimal cost function satisfies A5, A6 and B1. Property B2 is introduced here for convenience because it is used in intermediate steps in the proof. We formally do this through the following Proposition.

Proposition 1: *If v satisfies properties* A5 *and* B1, *then v satisfies properties* A1-A4 *and* B2.

Proof: In what follows, we show that each of properties A1-A4 and B2 are implied by properties A5 and B1. Specifically, we show that A5 implies A2; A5 and A2 together imply A3; B1 and A3 imply A1; A1 implies A4; and finally A1 and A2 imply B2.

<u>Property A2</u>

First, notice that $j \notin S(i)$ implies $i \neq j$. Now, consider any item j such that $P(j) = \varnothing$ (i.e., item j is a starting item), then by A5 we have $\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j) \leq \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{P(j)}) = \Delta_{i-P(i)} v(\mathbf{x}).$ Hence, A5 implies $\Delta_{i-P(i),j} x(\mathbf{x}) \leq 0$ when j is a starting item. Also, using A5, we can write $\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j) \leq \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{P(j)})$ for any item $j \notin S(i)$. Letting $P(j) = \{j_1, j_2, \cdots, j_{n_j}\}$ and since $j \notin S(i)$, for any $t \in P(j)$, we have $t \notin S(i).$ Therefore,

$$\begin{aligned}
\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j) &\leq \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_1}+\mathbf{e}_{j_2}+\cdots+\mathbf{e}_{j_{n_j}}) \\
&\leq \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{P(j_1)}+\mathbf{e}_{j_2}+\cdots+\mathbf{e}_{j_{n_j}}) && \text{(by applying A5)} \\
&\leq \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{P(j_1)}+\mathbf{e}_{P(j_2)}+\mathbf{e}_{j_3}+\cdots+\mathbf{e}_{j_{n_j}}) && \text{(by applying A5)} \\
&\;\;\vdots \\
&\leq \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{P(j_1)}+\mathbf{e}_{P(j_2)}+\cdots+\mathbf{e}_{P(j_{n_j})}) && \text{(by applying A5).}
\end{aligned}$$

Hence, by successively repeating the above process, we would eventually reach a set of starting items for which the *P* sets are empty. Let $\{k_1, k_2, \cdots, k_{n_k}\}$ be this set. Then, we have

$$\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j) \le \Delta_I v(\mathbf{x}+\mathbf{e}_{P(k_1)}+\mathbf{e}_{P(k_2)}+\cdots+\mathbf{e}_{P(k_{n_k})}).$$

Since for any $t \in \{k_1, k_2, \cdots, k_{n_k}\}$ we have $P(t)=\varnothing$, we also have

$$\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j) \le \Delta_{i-P(i)} v(\mathbf{x}) \text{ or } \Delta_{i-P(i),j} v(\mathbf{x}) \le 0 \text{ for } j \notin S(i).$$

<u>Property A3</u>

Given $j \in S(i)$, $j \ne 1$, $l \in S(j)$, $l \ne j$, we have $i \ne l$. Hence, using A5, we have $\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_l) \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{P(l)})$. Letting $P(l)=\{j_{n_j}, l_2, \cdots, l_{n_l}\}$, we obtain

$$\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_l) \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_{n_j}}+\mathbf{e}_{l_2}+\cdots+\mathbf{e}_{l_{n_l}}).$$

Here, j_{n_j} is the item on the path from item *j* to item *l*, i.e., $j_{n_j} \in S(j)$. However, other items in $P(l)$ do not belong to $S(j)$. From A2, we further have

$$\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_{n_j}}+\mathbf{e}_{l_2}+\cdots+\mathbf{e}_{l_{n_l}}) \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_{n_j}}).$$

As a result,

$$\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_l) \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_{n_j}}).$$

Let $\{j, j_1, j_2, \cdots, j_{n_j}\} = S(j) \setminus l$ be the set of items on the path from *j* to *l* not including *l*. Repeating the above process all the way to item *j* gives

$$\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_l) \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_{n_j}}) \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_{n_j-1}}) \le \cdots \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_{j_1}) \le \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j).$$

Hence, we have $\Delta_{i-P(i),j-l} v(\mathbf{x}) \ge 0$ for $j \in S(i)$, $j \ne 1$, $l \in S(j)$, $l \ne j$.

<u>Property A1</u>

If $j=1$, then by B1 we have $\Delta_{i-P(i),1} v(\mathbf{x}) \ge 0$. Now, assume $j \ne 1$ and $j \in S(i)$. By A3, we have $\Delta_{i-P(i),j-1} v(\mathbf{x}) = \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j-\mathbf{e}_1) - \Delta_{i-P(i)} v(\mathbf{x}) \ge 0$, or equivalently $\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j) \ge \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_1)$. Using B1, we obtain $\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_j) \ge \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{e}_1) \ge \Delta_{i-P(i)} v(\mathbf{x})$. Hence, $\Delta_{i-P(i),j} v(\mathbf{x}) \ge 0$ for $j \in S(i)$.

<u>Property A4</u>

Given $j \in S(i)$, if item *i* is a starting item, i.e., $P(i)=\varnothing$, we have, by A1, $\Delta_j v(\mathbf{x}+\mathbf{e}_i) \ge \Delta_j v(\mathbf{x}+\mathbf{e}_{P(i)}) = \Delta_j v(\mathbf{x})$ for $j \in S(i)$. Hence, $\Delta_{j,i} v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x}) \ge 0$. If item *i* is not a starting item, let $P(i)=\{i_1, i_2, \cdots, i_{n_i}\}$. Then, for any $t \in P(i)$, we have $j \in S(t)$. Hence, by A1 we have

$$\begin{aligned}
\Delta_j v(\mathbf{x}+\mathbf{e}_i) \ge \Delta_j v(\mathbf{x}+\mathbf{e}_{P(i)}) &= \Delta_j v(\mathbf{x}+\mathbf{e}_{i_1}+\mathbf{e}_{i_2}+\cdots+\mathbf{e}_{i_{n_i}}) \\
&\ge \Delta_j v(\mathbf{x}+\mathbf{e}_{P(i_1)}+\mathbf{e}_{i_2}+\cdots+\mathbf{e}_{i_{n_i}}) && \text{(by applying A1)} \\
&\vdots \\
&\ge \Delta_j v(\mathbf{x}+\mathbf{e}_{P(i_1)}+\mathbf{e}_{P(i_2)}+\cdots+\mathbf{e}_{P(i_{n_i})}). && \text{(by applying A1)}
\end{aligned}$$

Applying this process repeatedly, we would eventually reach a set of starting items for which the P sets are empty. Let $\{k_1, k_2, \cdots, k_{n_k}\}$ be this set. Hence, we have

$$\Delta_j v(\mathbf{x}+\mathbf{e}_i) \geq \Delta_j v(\mathbf{x}+\mathbf{e}_{P(k_1)}+\mathbf{e}_{P(k_2)}+\cdots+\mathbf{e}_{P(k_{n_k})}).$$

Since for any $t \in \{k_1, k_2, \cdots, k_{n_k}\}$ $P(t)=\varnothing$, we have $\Delta_j v(\mathbf{x}+e_i) \geq \Delta_j v(\mathbf{x})$ or $\Delta_{j,i} v(\mathbf{x}) = \Delta_{i,j} v(\mathbf{x}) \geq 0$. Hence, $\Delta_{i,j} v(\mathbf{x}) \geq 0$ for $j \in S(i)$.

Property B2

Since $i \in S(i)$, by applying A1, we have $\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{E}_i) \geq \Delta_{i-P(i)}(\mathbf{x}-\mathbf{e}_{P(i)})$. Let $P(i)=\{i_1, i_2, \cdots, i_{n_i}\}$ and notice that for any $t \in P(i)$, $t \notin S(i)$. Hence by applying A2 successively to the elements of $P(i)$, we obtain

$$\begin{aligned}\Delta_{i-P(i)} v(\mathbf{x}+\mathbf{E}_i) &\geq \Delta_{i-P(i)} v(\mathbf{x}-\mathbf{e}_{P(i)}) && \text{(by A1)}\\ &= \Delta_{i-P(i)} v(\mathbf{x}-\mathbf{e}_{i_1}-\mathbf{e}_{i_2}-\cdots-\mathbf{e}_{i_{n_i}})\\ &\geq \Delta_{i-P(i)} v(\mathbf{x}). && \text{(by successive application of A2)}\end{aligned}$$

Hence, we have $\Delta_{i-P(i),i-P(i)} v(\mathbf{x}) \geq 0$. This completes the proof of Proposition 1. ■

We are now ready to proceed with the proof of Lemma 1. Given the result of Proposition 1, in order to show that $Tv \in \mathcal{V}$ if $v \in \mathcal{V}$ (i.e., Tv satisfies properties A1-A6 if v satisfies A1-A6), it is sufficient to show that Tv satisfies properties B1, A5 and A6. We divide the proof into three parts. In part 1, we show that if $v \in \mathcal{V}$ then $T_k v(\mathbf{x})$ satisfies properties B1, A5 and A6 for all k. In part 2, we show that if $v \in \mathcal{V}$, then $T_t^l v(\mathbf{x})$ satisfies properties B1, A5 and A6 for all l and t. In part 3, we use the results of parts 1 and 2 to complete the proof and show that if $v \in \mathcal{V}$ then Tv satisfies properties B1, A5 and A6 (i.e., $v \in \mathcal{V}$).

Proof Lemma 1

Operator $T_t^l v(\mathbf{x})$

We need to show that if $v \in \mathcal{V}$ then $T_t^l v(\mathbf{x})$ satisfies properties B1, A1 and A6. In order to do so, we need to develop Lemma OA-1. First, note that for $0 \leq u \leq \min(x_1, t)$, we have

$$\begin{aligned}T_t^l v(\mathbf{x}) &= \min_u \{v(\mathbf{x}-u\mathbf{e}_1)+(t-u)c_l\},\\ &= \min_u \left\{ v(\mathbf{x}+\mathbf{e}_1)+(t+1)c_l - \sum_{j=0}^{u} \left(\Delta_1 v(\mathbf{x}-j\mathbf{e}_1)+c_l\right) \right\},\\ &= v(\mathbf{x}+\mathbf{e}_1)+(t+1)c_l - \max_u \left\{ \sum_{j=0}^{u} \left(\Delta_1 v(\mathbf{x}-j\mathbf{e}_1)+c_l\right) \right\}.\end{aligned}$$

Let $u_l^*(\mathbf{x},t)$ be the value of u that minimizes $v(\mathbf{x}-u\mathbf{e}_1)+(t-u)c_l$, then we have:

$$u_l^*(\mathbf{x},t) = \underset{0\le u\le \min(t,x_1)}{\arg\min}\ \{v(\mathbf{x}-u\mathbf{e}_1)+(t-u)c_l$$

$$= \underset{0\le u\le \min(t,x_1)}{\arg\max} \left\{ \sum_{j=0}^{u} \left(\Delta_1 v(\mathbf{x}-j\mathbf{e}_1)+c_l \right) \right\},$$

and

$$u_l^*(\mathbf{x},t) = \begin{cases} 0 & \text{If } \Delta_1 v(\mathbf{x}) < -c_l, \\ \max\{u \mid \Delta_1 v(\mathbf{x}-u\mathbf{e}_1) \ge -c_l, 0\le u\le \min(t,x_1)\} & \text{Otherwise.} \end{cases} \tag{OA-1}$$

The last equality is due to Property A4 (i.e., $\Delta_{1,1}v(\mathbf{x})\ge 0$). For instance, if $\Delta_1 v(\mathbf{x}) < -c_l$, then by A4, we have $\Delta_1 v(\mathbf{x}-u\mathbf{e}_1) \le \Delta_1 v(\mathbf{x}-(u-1)\mathbf{e}_1) \le \cdots \le \Delta_1 v(\mathbf{x}) \le -c_l \le 0.$ As a result, we have

$$\sum_{j=0}^{u}\left(\Delta_1 v(\mathbf{x}-j\mathbf{e}_1)+c_l\right) \le \sum_{j=0}^{u-1}\left(\Delta_1 v(\mathbf{x}-j\mathbf{e}_1)+c_l\right) \le \cdots \le \sum_{j=0}^{0}\left(\Delta_1 v(\mathbf{x}-j\mathbf{e}_1)+c_l\right) = \Delta_1 v(\mathbf{x})+c_l$$

and $u_l^*(\mathbf{x},t)=0$ in this case. On the other hand, if $\Delta_1 v(\mathbf{x}) \ge -c_l$, then $u_l^*(\mathbf{x},t)$ must be the largest number such that $\Delta_1 v(\mathbf{x}-u\mathbf{e}_1)+c_l \ge 0$ for all $0\le u\le u_l^*(\mathbf{x},t)$.

Lemma OA-1:

I. $$u_l^*(x+e_1,t) = \begin{cases} u_l^*(x,t)=0 & \textit{if and only if } \Delta_1 v(x) < -c_l, \\ u_l^*(x,t)=t & \textit{if and only if } x_1 \ge t \textit{ and } \Delta_1 v(x-te_1) \ge -c_l, \\ u_l^*(x,t)+1 & \textit{otherwise.} \end{cases}$$

II. $u_l^*(\mathbf{x}+\mathbf{E}_i,t) \ge u_l^*(\mathbf{x},t)$ *for all i.*

Proof:

I. using (OA-1), we distinguish four cases:

1. $\Delta_1 v(\mathbf{x}) < -c_l \Rightarrow u_l^*(\mathbf{x}+\mathbf{e}_1,t) = u_l^*(\mathbf{x},t)=0.$ $\Delta_1 v(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_1) = \Delta_1 v(\mathbf{x}) < -c_l$ implies $u\ge 1$ cannot be optimal in state $x+\mathbf{e}_1$ since $\Delta_1 v(\mathbf{x}+\mathbf{e}_1-u\mathbf{e}_1) \le \cdots \le \Delta_1 v(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_1) < -c_l$. The inequalities are due to Property A4 (i.e., $\Delta_{11}v(\mathbf{x})\ge 0,$ since $1\in S(1)$). Hence, (OA-1) implies $u_l^*(\mathbf{x}+\mathbf{e}_1,t)=u_l^*(\mathbf{x},t)=0.$
2. $u_l^*(\mathbf{x},t)=t \Rightarrow u_l^*(\mathbf{x}+\mathbf{e}_1,t)=u_l^*(\mathbf{x},t)=t.$ $u_l^*(\mathbf{x},t)=t$ implies $\Delta_1 v(\mathbf{x}+\mathbf{e}_1-t\mathbf{e}_1) \ge \Delta_1 v(\mathbf{x}-t\mathbf{e}_1) \ge -c_l$. Hence, (OA-1) implies $u_l^*(\mathbf{x}+\mathbf{e}_1,t)=u_l^*(\mathbf{x},t)=t.$
3. $u_l^*(\mathbf{x},t)=0$ and $\Delta_1 v(\mathbf{x}) \ge -c_l \Rightarrow u_l^*(\mathbf{x}+\mathbf{e}_1,t)=u_l^*(\mathbf{x},t)+1.$ If $u_l^*(\mathbf{x},t)=0$ and $\Delta_1 v(\mathbf{x}) \ge -c_l$, then for $x_1\ge 1,$ (OA-1) implies $\Delta_1 v(\mathbf{x}-\mathbf{e}_1) = \Delta_1 v(\mathbf{x}+\mathbf{e}_1-2\mathbf{e}_1) < -c_l$ and $\Delta_1 v(\mathbf{x}) = \Delta_1 v(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_1) \ge -c_l$. Hence, $u_l^*(\mathbf{x}+\mathbf{e}_1,t)=1=u_l^*(\mathbf{x},t)+1.$
4. $t>1$ and $1\le u_l^*(\mathbf{x},t)<t \Rightarrow u_l^*(\mathbf{x}+\mathbf{e}_1,t)=u_l^*(\mathbf{x},t)+1.$ (OA-1) implies $\Delta_1 v\left(\mathbf{x}-u_l^*(\mathbf{x},t)\mathbf{e}_1\right) \ge -c_l > \Delta_1 v\left(\mathbf{x}-(u_l^*(\mathbf{x},t)+1)\mathbf{e}_1\right)$ which is equivalent to $\Delta_1 v\left(\mathbf{x}+\mathbf{e}_1-(u_l^*(\mathbf{x},t)+1)\mathbf{e}_1\right) \ge -c_l > \Delta_1 v\left(\mathbf{x}+\mathbf{e}_1-(u_l^*(\mathbf{x},t)+2)\mathbf{e}_1\right).$ Hence, using (OA-1), we have $u_l^*(\mathbf{x}+\mathbf{e}_1,t)=u_l^*(\mathbf{x},t)+1.$

We can show that the reverse of cases 1- 4 is true using a contradiction argument. For instance, for case 1, assume $\Delta_1 v(\mathbf{x}) < -c_l$ and $u_l^*(\mathbf{x}+\mathbf{e}_1,t)=1$. Hence, (OA-1) implies $\Delta_1 v(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_1) \geq -c_l > \Delta_1 v(\mathbf{x}+\mathbf{e}_1-2\mathbf{e}_1)$. Equivalently, $\Delta_1 v(\mathbf{x}) \geq -c_l > \Delta_1 v(\mathbf{x}-\mathbf{e}_1)$ which contradict the assumption. The other cases can be shown in a similar way.

II. Using (OA-1) and B1, we have $\Delta_1 v(\mathbf{x}+\mathbf{E}_i-u_l^*(\mathbf{x},t)\mathbf{e}_1) \geq \Delta_1 v(\mathbf{x}-u_l^*(\mathbf{x},t)\mathbf{e}_1) \geq -c_l$. Hence, $u_l^*(t,\mathbf{x}+\mathbf{E}_i)$ is at least equal to $u_l^*(t,\mathbf{x})$. ■

We are now ready to prove Conditions B1, A5 and A6.

<u>Property B1</u>

In order to show that $T_t^l v(\mathbf{x})$ satisfies B1, we must show that $\Delta_{i-P(i),1} T_t^l v(\mathbf{x}) \geq 0$ for all *l*, *t* and *i*. First note that

$$\begin{aligned}\Delta_{i-P(i),1} T_t^l v(\mathbf{x}) &= T_t^l v(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) - T_t^l v(\mathbf{x}+\mathbf{E}_i) - T_t^l v(\mathbf{x}+\mathbf{e}_1) + T_t^l v(x)\\ &= v(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1-u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{e}_1) - v(\mathbf{x}+\mathbf{E}_i-u_l^*(t,\mathbf{x}+\mathbf{E}_i)\mathbf{e}_1) - v(\mathbf{x}+\mathbf{e}_1-u_l^*(t,\mathbf{x}+\mathbf{e}_1)\mathbf{e}_1)\\ &\quad + v(\mathbf{x}-u_l^*(t,\mathbf{x})\mathbf{e}_1) - \left\{u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) - u_l^*(t,\mathbf{x}+\mathbf{E}_i) - u_l^*(t,\mathbf{x}+\mathbf{e}_1) + u_l^*(t,\mathbf{x})\right\} c_l.\end{aligned}$$

For simplicity, let $u_l^*(t,\mathbf{x}) = u^*$. Using Lemma OA-1, we have $u^* \leq u_l^*(t,\mathbf{x}+\mathbf{e}_1) \leq u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)$, $u^* \leq u_l^*(t,\mathbf{x}+\mathbf{E}_i) \leq u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)$.

There are three possible cases.

1. $u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)=0 \Rightarrow u_l^*(t,\mathbf{x}+\mathbf{E}_i)=u_l^*(t,\mathbf{x}+\mathbf{e}_1)=u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)=0$. Hence, $\Delta_{i-P(i),1} T_t^l v(\mathbf{x}) = \Delta_{i-P(i),1} v(\mathbf{x}) \geq 0$ (by B1).
2. $u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \geq 1$ and $u_l^*(t,\mathbf{x}+\mathbf{e}_1)=u^*=t \Rightarrow u^*=u_l^*(t,\mathbf{x}+\mathbf{e}_1)=u_l^*(t,\mathbf{x}+\mathbf{E}_i)=u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)=t$. Hence, $\Delta_{i-P(i),1} T_t^l v(\mathbf{x}) = \Delta_{i-P(i),1} v(\mathbf{x}-t\mathbf{e}_1) \geq 0$.
3. $u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \geq 1$ and $u_l^*(t,\mathbf{x}+\mathbf{e}_1)=u^*=0 \Rightarrow$
$$\begin{aligned}\Delta_{i-P(i),1} T_t^l v(\mathbf{x}) &\geq v(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1-u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{e}_1) - v(\mathbf{x}+\mathbf{E}_i-\left(u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)-1\right)\mathbf{e}_1)\\ &\quad - v(\mathbf{x}+\mathbf{e}_1) + v(\mathbf{x}) - c_l\\ &= -\Delta_1 v(\mathbf{x}) - c_l.\end{aligned}$$
By Lemma OA-1, $u_l^*(t,\mathbf{x}+\mathbf{e}_1)=u_l^*(t,\mathbf{x})=0$ happens only when $-c_l > \Delta_1 v(\mathbf{x})$. Hence, $\Delta_{i-P(i),1} T_t^l v(\mathbf{x}) \geq 0$.
4. $u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \geq 1$ and $u_l^*(t,\mathbf{x}+\mathbf{e}_1)=u^*+1 \Rightarrow$
$$\begin{aligned}\Delta_{i-P(i),1} T_t^l v(\mathbf{x}) &\geq v(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1-u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{e}_1) - v(\mathbf{x}+\mathbf{E}_i-\left(u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)-1\right)\mathbf{e}_1)\\ &\quad - v(\mathbf{x}+\mathbf{e}_1-(u^*+\mathbf{1})\mathbf{e}_1) + v(\mathbf{x}-u^*\mathbf{e}_1)\\ &= 0.\end{aligned}$$

It follows that operator $T_t^l v(\mathbf{x})$ satisfies condition B1.

<u>Property A5</u>

In order to show that $T_t^l v(\mathbf{x})$ satisfies A5, we need to show that $\Delta_{i-P(i),j-P(j)} T_t^l v(\mathbf{x}) \le 0$ for all *l*, *t*, *i*, and $j \neq i$. First note that

$$\begin{aligned}\Delta_{i-P(i),j-P(j)} T_t^l v(\mathbf{x}) &= T_t^l v(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j) - T_t^l v(\mathbf{x}+\mathbf{E}_i) - T_t^l v(\mathbf{x}+\mathbf{E}_j) + T_t^l v(\mathbf{x}) \\ &= v(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j - u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)\mathbf{e}_1) - v(\mathbf{x}+\mathbf{E}_i - u_l^*(t,\mathbf{x}+\mathbf{E}_i)\mathbf{e}_1) - v(\mathbf{x}+\mathbf{E}_j - u_l^*(t,\mathbf{x}+\mathbf{E}_j)\mathbf{e}_1) \\ &\quad + v(\mathbf{x} - u_l^*(t,\mathbf{x})\mathbf{e}_1) - \left\{u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j) - u_l^*(t,\mathbf{x}+\mathbf{E}_i) - u_l^*(t,\mathbf{x}+\mathbf{E}_j) + u_l^*(t,\mathbf{x})\right\} c_l.\end{aligned}$$

Lemma OA-1 implies $u^* \le u_l^*(t,\mathbf{x}+\mathbf{E}_i) \le u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)$ and $u^* \le u_l^*(t,\mathbf{x}+\mathbf{E}_j) \le u_l^*(t,\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)$. Hence, there are two possible cases.

Case 1. $u_l^*(t,\mathbf{x}+\mathbf{E}_i) \le u_l^*(t,\mathbf{x}+\mathbf{E}_j)$. Hence,

$$\begin{aligned}\Delta_{i-P(i),j-P(j)} T_t^l v(\mathbf{x}) &\le v(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j - u_l^*(\mathbf{x}+\mathbf{E}_j)\mathbf{e}_1) - v(\mathbf{x}+\mathbf{E}_i - u_l^*(\mathbf{x}+\mathbf{E}_i)\mathbf{e}_1) \\ &\quad - v(\mathbf{x}+\mathbf{E}_j - u_l^*(\mathbf{x}+\mathbf{E}_j)\mathbf{e}_1) + v(\mathbf{x} - u_l^*(\mathbf{x}+\mathbf{E}_i)\mathbf{e}_1) \\ &= \Delta_{i-P(i)} v(\mathbf{x}+\mathbf{E}_j - u_l^*(\mathbf{x}+\mathbf{E}_j)\mathbf{e}_1) - \Delta_{i-P(i)} v(\mathbf{x} - u_l^*(\mathbf{x}+\mathbf{E}_i)\mathbf{e}_1) \\ &\le \Delta_{i-P(i)} v(\mathbf{x} - u_l^*(\mathbf{x}+\mathbf{E}_j)\mathbf{e}_1) - \Delta_{i-P(i)} v(\mathbf{x} - u_l^*(\mathbf{x}+\mathbf{E}_i)\mathbf{e}_1) \quad \text{(By A5)} \\ &\le 0 \quad \text{(By B1 and } u_l^*(\mathbf{x}+\mathbf{E}_i) \le u_l^*(\mathbf{x}+\mathbf{E}_j)\text{)}.\end{aligned}$$

Case 2. $u_l^*(t,\mathbf{x}+\mathbf{E}_i) \ge u_l^*(t,\mathbf{x}+\mathbf{E}_j)$. The proof is similar to the above case and is omitted for brevity.

It follows from the above that operator $T_t^l v(\mathbf{x})$ satisfies property A1.

Property A6

In order to show that operator $T_t^l v(\mathbf{x})$ satisfies A6, we need to show that $\Delta_1 T_t^l v(\mathbf{x}) \ge -c_1$ for all *l* and *t*. First, note that $\Delta_1 T_t^l v(\mathbf{x}) = T_t^l v(\mathbf{x}+\mathbf{e}_1) - T_t^l v(\mathbf{x})$. Using Lemma OA-1, we distinguish two cases.

Case I. $u_l^*(t,\mathbf{x}+\mathbf{e}_1) = u^*$. In this case, $\Delta_1 T_t^l v(\mathbf{x}) = \Delta_1 v(\mathbf{x} - u^*\mathbf{e}_1) \ge -c_l$ (by A6).

Case II. $u_l^*(t,\mathbf{x}+\mathbf{e}_1) = u^* + 1$. In this case, $\Delta_1 T_t^l v(\mathbf{x}) = -c_l$.

It follows that operator $T_t^l v(\mathbf{x})$ satisfies condition A6.

Operator $T_k v(\mathbf{x})$

We need to show that if $v \in \mathcal{V}$ then $T_k v(\mathbf{x})$ satisfies properties B1, A5 and A6. First, we denote by $\bar{q}_k(\mathbf{x})$ the maximum number of units of item *k* that could be produced in state **x**. Since $\bar{q}_k(\mathbf{x})$ is restricted by the number of units of each of the predecessors of item *k*, we have

$$\bar{q}_k(\mathbf{x}) = \min\{x_t, t \in P(k)\}.$$

Note that for $0 \le q \le \bar{q}_k(\mathbf{x})$, we have

$$T_k v(\mathbf{x}) = \min_q \left\{ v\left(\mathbf{x} + q\mathbf{E}_k\right)\right\},$$
$$= v\left(\mathbf{x}\right) + \min_q \left\{ v\left(\mathbf{x} + q\mathbf{E}_k\right) - v\left(\mathbf{x}\right)\right\},$$
$$= v\left(\mathbf{x}\right) + \min_q \left\{ \sum_{j=0}^{q-1} \Delta_{k-P(k)} v\left(\mathbf{x} + j\mathbf{E}_k\right)\right\}.$$

We use the convention $\sum_{j=0}^{-1} \Delta_{k-P(k)} v\left(\mathbf{x} + j\mathbf{E}_k\right) = 0.$ Let $q_k^*(\mathbf{x})$ be the value of q that minimizes $v\left(\mathbf{x} + q\mathbf{E}_k\right)$, then we have:

$$q_k^*(\mathbf{x}) = \begin{cases} \bar{q}_k(\mathbf{x}) & \text{If } \Delta_{k-P(k)} v\left(\mathbf{x} + \left(\bar{q}_k(\mathbf{x}) - 1\right)\mathbf{E}_k\right) < 0, \\ \min\left\{ q \mid \Delta_{k-P(k)} v(\mathbf{x} + q\mathbf{E}_k) \geq 0, 0 \leq q \leq \bar{q}_k(\mathbf{x})\right\} & \text{Otherwise.} \end{cases} \quad \text{(OA-2)}$$

The last equality is due to Property B2 (i.e., $\Delta_{k-P(k),k-P(k)} v(\mathbf{x}) \geq 0$). For instance, if $\Delta_{k-P(k)} v\left(\mathbf{x} + \left(\bar{q}_k(\mathbf{x}) - 1\right)\mathbf{E}_k\right) < 0,$ then by B2, we have $\Delta_{k-P(k)} v\left(\mathbf{x}\right) \leq \cdots \leq \Delta_{k-P(k)} v\left(\mathbf{x} + \left(\bar{q}_k(\mathbf{x}) - 1\right)\mathbf{E}_k\right) < 0.$
As a result, we have

$$\sum_{j=0}^{q-1} \Delta_{k-P(k)} v\left(\mathbf{x} + j\mathbf{E}_k\right) \leq \sum_{j=0}^{q-2} \Delta_{k-P(k)} v\left(\mathbf{x} + j\mathbf{E}_k\right) \leq \cdots \leq \sum_{j=0}^{-1} \Delta_{k-P(k)} v\left(\mathbf{x} + j\mathbf{E}_k\right) = 0$$

and $q_k^*(\mathbf{x}) = \bar{q}_k(\mathbf{x})$ in this case. On the other hand, if $\Delta_{k-P(k)} v\left(\mathbf{x} + \left(\bar{q}_k(\mathbf{x}) - 1\right)\mathbf{E}_k\right) \geq 0,$ then $q_k^*(\mathbf{x})$ must be the smallest number such that $\Delta_{k-P(k)} v(\mathbf{x} + q\mathbf{E}_k) \geq 0$ for all $0 \leq q \leq \bar{q}_K(\mathbf{x}).$ As a result of OA-2 we have

Lemma OA-2:

1. $q_k^*(\mathbf{x} + \mathbf{e}_1) \leq q_k^*(\mathbf{x})$ *and* $\bar{q}_k(\mathbf{x} + \mathbf{e}_1) = \bar{q}_k(\mathbf{x})$;
2. *for* $i \neq k$ *,* $q_k^*(\mathbf{x} + \mathbf{E}_i) \geq q_k^*(\mathbf{x})$ *and* $\bar{q}_k(\mathbf{x} + \mathbf{E}_i) \geq \bar{q}_k(\mathbf{x})$;
3. *for* $i = k$ *,* $\bar{q}_k(\mathbf{x} + \mathbf{E}_k) = \max\left\{0, \bar{q}_k(\mathbf{x}) - 1\right\}$ *and*
$$q_k^*(x + E_i) = \begin{cases} q_k^*(x) - 1 & \textit{if } q_k^*(x) \geq 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof:

1. Using OA-2, we have $\Delta_{k-P(k)} v\left(\mathbf{x} + q_k^*(\mathbf{x})\mathbf{E}_k\right) \geq 0 > \Delta_{k-P(k)} v\left(\mathbf{x} + (q_k^*(\mathbf{x}) - 1)\mathbf{E}_k\right).$ Using B1, we have $\Delta_{k-P(k)} v\left(\mathbf{x} + \mathbf{e}_1 + q_k^*(\mathbf{x})\mathbf{E}_k\right) \geq \Delta_{k-P(k)} v\left(\mathbf{x} + q_k^*(\mathbf{x})\mathbf{E}_k\right) \geq 0.$ Hence, $q_k^*(\mathbf{x} + \mathbf{e}_1)$ is at most equal to $q_k^*(\mathbf{x}).$
2. Using OA-2 and A5, we have $0 > \Delta_{k-P(k)} v\left(\mathbf{x} + (q_k^*(\mathbf{x}) - 1)\mathbf{E}_k\right) > \Delta_{k-P(k)} v\left(\mathbf{x} + \mathbf{E}_i + (q_k^*(\mathbf{x}) - 1)\mathbf{E}_k\right).$ Hence, $q_k^*(\mathbf{x} + \mathbf{E}_i)$ is at least equal to $q_k^*(\mathbf{x}).$
3. If $q_k^*(\mathbf{x}) = 0,$ then B2 and OA-2 implies $\Delta_{k-P(k)} v\left(\mathbf{x} + \mathbf{E}_k\right) \geq \Delta_{k-P(k)} v\left(\mathbf{x}\right) \geq 0.$ Hence, $q_k^*(\mathbf{x} + \mathbf{E}_k) = 0.$ On the other hand if $q_k^*(\mathbf{x}) \geq 1,$ then OA-2 implies
$$\Delta_{k-P(k)} v\left(\mathbf{x} + \mathbf{E}_k + (q_k^*(\mathbf{x}) - 1)\mathbf{E}_k\right) = \Delta_{k-P(k)} v\left(\mathbf{x} + q_k^*(\mathbf{x})\mathbf{E}_k\right) \geq 0 > \Delta_{k-P(k)} v\left(\mathbf{x} + (q_k^*(\mathbf{x}) - 1)\mathbf{E}_k\right)$$
$$= \Delta_{k-P(k)} v\left(\mathbf{x} + \mathbf{E}_k + (q_k^*(\mathbf{x}) - 2)\mathbf{E}_k\right).$$

   Hence, $q_k^*(\mathbf{x} + \mathbf{E}_k) = q_k^*(\mathbf{x}) - 1.$

The proof is trivial for the maximum quantities at the corresponding states. ■

We are now ready to prove properties B1, A5, and A6.

Property B1

In order to show that $T_k v(\mathbf{x})$ satisfies B1, we must show that $\Delta_{i-P(i),1} T_k v(\mathbf{x}) \geq 0$ for all *k*, and *i*. First note that

$$\begin{aligned}\Delta_{i-P(i),1} T_k v(\mathbf{x}) &= T_k v(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) - T_k v(\mathbf{x}+\mathbf{E}_i) - T_k v(\mathbf{x}+\mathbf{e}_1) + T_k v(\mathbf{x}) \\ &= v(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k) - v(\mathbf{x}+\mathbf{E}_i+q_k^*(\mathbf{x}+\mathbf{E}_i)\mathbf{E}_k) \\ &\quad - v(\mathbf{x}+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{e}_1)\mathbf{E}_k) + v(\mathbf{x}+q_k^*(\mathbf{x})\mathbf{E}_k).\end{aligned}$$

For simplicity, let $q_k^*(\mathbf{x}) = q^*$. Hence there are two possible cases.

Case I. $i \neq k$

Using Lemma OA-2, we have $\bar{q}_k(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \geq \bar{q}_k(\mathbf{x}+\mathbf{e}_1) = \bar{q}_k(\mathbf{x})$,
$\bar{q}_k(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) = \bar{q}_k(\mathbf{x}+\mathbf{E}_i) \geq \bar{q}_k(\mathbf{x})$, $q_k^*(\mathbf{x}+\mathbf{e}_1) \leq q^* \leq q_k^*(\mathbf{x}+\mathbf{E}_i)$, and
$q_k^*(\mathbf{x}+\mathbf{e}_1) \leq q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \leq q_k^*(\mathbf{x}+\mathbf{E}_i)$. This leads to two sub-cases:

1. $q^* \leq q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)$. In this case, we have

$$\begin{aligned}\Delta_{i-P(i),1} T_k v(\mathbf{x}) &\geq v\left(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) - v\left(\mathbf{x}+\mathbf{E}_i+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) \\ &\quad - v\left(\mathbf{x}+\mathbf{e}_1+q^*\mathbf{E}_k\right) + v\left(\mathbf{x}+q^*\mathbf{E}_k\right) \\ &= \Delta_1\left(\mathbf{x}+\mathbf{E}_i+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) - \Delta_1\left(\mathbf{x}+q^*\mathbf{E}_k\right) \\ &\geq \Delta_1\left(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) - \Delta_1\left(\mathbf{x}+q^*\mathbf{E}_k\right) \geq 0.\end{aligned}$$

Here, the first inequality is due to the definitions of $T_k v(\mathbf{x}+\mathbf{E}_i)$ and $T_k v(\mathbf{x}+\mathbf{e}_1)$. The last inequality is due to the fact that $q^* \leq q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)$ and B1.

2. $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \leq q^*$. In this case, we have

$$\begin{aligned}\Delta_{i-P(i),1} T_k v(\mathbf{x}) &\geq v\left(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) - v\left(\mathbf{x}+\mathbf{E}_i+q^*\mathbf{E}_k\right) - v\left(\mathbf{x}+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) \\ &\quad + v\left(\mathbf{x}+q^*\mathbf{E}_k\right) \\ &= \Delta_{i-P(i)}\left(\mathbf{x}+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) - \Delta_{i-P(i)}\left(\mathbf{x}+q^*\mathbf{E}_k\right) \\ &\geq \Delta_{i-P(i)}\left(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)\mathbf{E}_k\right) - \Delta_{i-P(i)}\left(\mathbf{x}+q^*\mathbf{E}_k\right) \geq 0.\end{aligned}$$

Here, the first inequality is due to the definitions of $T_k v(\mathbf{x}+\mathbf{E}_i)$ and $T_k v(\mathbf{x}+\mathbf{e}_1)$. The last inequality is due to the fact that $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \leq q^*$ and A5.

Case II. $i = k$

Using Lemma OA-2, we have $\bar{q}_k(\mathbf{x}+\mathbf{e}_1) = \bar{q}_k(\mathbf{x})$, $\bar{q}_k(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) = \bar{q}_k(\mathbf{x}+\mathbf{E}_i)$,
$q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \leq q_k^*(\mathbf{x}+\mathbf{E}_i) \leq q^*$, $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1) \leq q_k^*(\mathbf{x}+\mathbf{e}_1) \leq q^*$, where

$$q_k^*(\mathbf{x}+\mathbf{E}_i)=\begin{cases} q^*-1 & \text{if } q^*\ge 1,\\ 0 & \text{otherwise.}\end{cases},\text{ and } q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)=\begin{cases} q_k^*(\mathbf{x}+\mathbf{e}_1)-1 & \text{if } q_k^*(\mathbf{x}+\mathbf{e}_1)\ge 1,\\ 0 & \text{otherwise.}\end{cases}$$

This leads to three sub-cases:

1. $q^*=0$. In this case, $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)=q_k^*(\mathbf{x}+\mathbf{E}_i)=q_k^*(\mathbf{x}+\mathbf{e}_1)=q^*=0$. Hence, $\Delta_{i-P(i),1}T_k v(\mathbf{x})=\Delta_{i\_P(i),1}v(\mathbf{x})\ge 0$ (by B1).
2. $q^*\ge 1$ and $q_k^*(\mathbf{x}+\mathbf{e}_1)=0$. In this case, $q_k^*(\mathbf{x}+\mathbf{E}_i)=q^*-1$ and $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)=q_k^*(\mathbf{x}+\mathbf{e}_1)=0$. Hence, $\Delta_{i-P(i),1}T_k v(\mathbf{x})=\Delta_{i-P(i)}v(\mathbf{x}+\mathbf{e}_1)$. But according to OA-2 $q_k^*(\mathbf{x}+\mathbf{e}_1)=0$ implies $\Delta_{i-P(i)}v(\mathbf{x}+\mathbf{e}_1)\ge 0$.
3. $q^*\ge 1$ and $q_k^*(\mathbf{x}+\mathbf{e}_1)\ge 1$. In this case, we have $q_k^*(\mathbf{x}+\mathbf{E}_i)=q^*-1$ and $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{e}_1)=q_k^*(\mathbf{x}+\mathbf{e}_1)-1$. Hence, $\Delta_{i-P(i),1}T_k v(\mathbf{x})=0$.

It follows that $T_k v(\mathbf{x})$ satisfies property B1.

<u>Property A5</u>

In order to show that $T_k v(\mathbf{x})$ satisfies A5, we need to show that $\Delta_{i-P(i),j-P(j)}T_k v(\mathbf{x})\le 0$ for all *k*, *i*, and $j\ne i$. First note that

$$\begin{aligned}\Delta_{i-P(i),j-P(j)}T_k v(\mathbf{x})&=T_k v(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)-T_k v(\mathbf{x}+\mathbf{E}_i)-T_k v(\mathbf{x}+\mathbf{E}_j)+T_k v(x)\\ &=v\left(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j+q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)\mathbf{E}_k\right)-v\left(\mathbf{x}+\mathbf{E}_i+q_k^*(\mathbf{x}+\mathbf{E}_i)\mathbf{E}_k\right)\\ &\quad -v\left(\mathbf{x}+\mathbf{E}_j+q_k^*(\mathbf{x}+\mathbf{E}_j)\mathbf{E}_k\right)+v\left(\mathbf{x}+q^*\mathbf{E}_k\right).\end{aligned}$$

There are three possible cases.

Case I. $i\ne k, j\ne k$

Using Lemma OA-2, we have $\bar{q}_k(\mathbf{x}+\mathbf{E}_i)\ge\bar{q}_k(\mathbf{x})$, $\bar{q}_k(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)\ge\bar{q}_k(\mathbf{x}+\mathbf{E}_i)$, $\bar{q}_k(\mathbf{x}+\mathbf{E}_i)\ge\bar{q}_k(\mathbf{x})$, $\bar{q}_k(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)\ge\bar{q}_k(\mathbf{x}+\mathbf{E}_j)$, $q^*\le q_k^*(\mathbf{x}+\mathbf{E}_i)\le q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)$, and $q^*\le q_k^*(\mathbf{x}+\mathbf{E}_j)\le q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)$.

Hence, we distinguish two sub-cases.

1. $q_k^*(\mathbf{x}+\mathbf{E}_i)\ge q_k^*(\mathbf{x}+\mathbf{E}_j)$. In this case, we have

$$\begin{aligned}\Delta_{i-P(i),j-P(j)}T_k v(\mathbf{x})&\le v\left(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j+q_k^*(\mathbf{x}+\mathbf{E}_i)\mathbf{E}_k\right)-v\left(\mathbf{x}+\mathbf{E}_i+q_k^*(\mathbf{x}+\mathbf{E}_i)\mathbf{E}_k\right)\\ &\quad -v\left(\mathbf{x}+\mathbf{E}_j+q_k^*(\mathbf{x}+\mathbf{E}_j)\mathbf{E}_k\right)+v\left(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{E}_j)\mathbf{E}_k\right)\\ &=\Delta_{j-P(j)}v\left(\mathbf{x}+\mathbf{E}_i+q_k^*(\mathbf{x}+\mathbf{E}_i)\mathbf{E}_k\right)-\Delta_{j-P(j)}v\left(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{E}_j)\mathbf{E}_k\right)\\ &\le\Delta_{j-P(j)}v\left(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{E}_i)\mathbf{E}_k\right)-\Delta_{j-P(j)}v\left(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{E}_j)\mathbf{E}_k\right)\le 0.\end{aligned}$$

The first inequality is due to the definition of $T_k v(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j)$ and $T_k v(\mathbf{x})$. The last inequality is due to A5 and the fact that $q_k^*(\mathbf{x}+\mathbf{E}_i)\ge q_k^*(\mathbf{x}+\mathbf{E}_j)$.

2. $q_k^*(\mathbf{x}+\mathbf{E}_i)<q_k^*(\mathbf{x}+\mathbf{E}_j)$. The proof is similar to the previous case and is omitted for brevity.

Case II. $i=k, j\ne k$

Using Lemma OA-2, we have $\bar{q}_k(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j) \geq \bar{q}_k(\mathbf{x}+\mathbf{E}_i)$, $\bar{q}_k(\mathbf{x}+\mathbf{E}_j) \geq \bar{q}_k(\mathbf{x})$, $q_k^*(\mathbf{x}+\mathbf{E}_i) \leq q^* \leq q_k^*(\mathbf{x}+\mathbf{E}_j)$, $q_k^*(\mathbf{x}+\mathbf{E}_i) \leq q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j) \leq q_k^*(\mathbf{x}+\mathbf{E}_j)$ where

$$q_k^*(\mathbf{x}+\mathbf{E}_i) = \begin{cases} q^*-1 & \text{if } q^* \geq 1, \\ 0 & \text{otherwise.} \end{cases}, \text{ and } q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j) = \begin{cases} q_k^*(\mathbf{x}+\mathbf{E}_j)-1 & \text{if } q_k^*(\mathbf{x}+\mathbf{E}_j) \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we distinguish three sub-cases:

1. $q_k^*(\mathbf{x}+\mathbf{E}_j)=0.$ In this case, we have $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j) = q_k^*(\mathbf{x}+\mathbf{E}_i) = q_k^*(\mathbf{x}+\mathbf{E}_j) = q^* = 0.$ Hence, $\Delta_{i-P(i),j-P(j)} T_k v(\mathbf{x}) = \Delta_{i-P(i),j-P(j)} v(\mathbf{x}) \leq 0.$
2. $q_k^*(\mathbf{x}+\mathbf{E}_j) \geq 1$ and $q^*=0.$ In this case, $q_k^*(\mathbf{x}+\mathbf{E}_i+\mathbf{E}_j) = q_k^*(\mathbf{x}+\mathbf{E}_j)-1$ and $q_k^*(\mathbf{x}+\mathbf{E}_i) = q^* = 0.$ Hence, $\Delta_{i-P(i),1} T_k v(\mathbf{x}) = -\Delta_{i-P(i)} v(\mathbf{x}).$ Using OA-2, $q^*=0$ implies $\Delta_{i-P(i)} v(\mathbf{x}) \geq 0.$ Hence, $\Delta_{i-P(i),1} T_k v(\mathbf{x}) \leq 0.$
3. $q_k^*(\mathbf{x}+\mathbf{e}_J) \geq 1$ and $q^* \geq 1.$ In this case, we have $q_k^*(\mathbf{x}+\mathbf{e}_I+\mathbf{e}_J) = q_k^*(\mathbf{x}+\mathbf{e}_J)-1$ and $q_k^*(\mathbf{x}+\mathbf{E}_i) = q^*-1.$ Hence, $\Delta_{i-P(i),j-P(j)} T_k v(\mathbf{x}) = 0.$

<u>Case III:</u> $i \neq k,\ j = k$

The proof is similar to the case $i=k,\ j \neq k.$

It follows that $T_k v(\mathbf{x})$ satisfies property A5.

<u>Property A6</u>

In order to show that $T_k v(\mathbf{x})$ satisfies A6, we need to show that $\Delta_1 T_k v(\mathbf{x}) \geq -c_1$ for all *k*. First, note that $\Delta_1 T_k v(\mathbf{x}) = T_k v(\mathbf{x}+\mathbf{e}_1) - T_k v(x).$ Using Lemma OA-2, we have $q_k^*(\mathbf{x}+\mathbf{e}_1) \leq q^*$. Hence,

$$\begin{aligned} \Delta_1 T_k v(\mathbf{x}) &= T_k v(\mathbf{x}+\mathbf{e}_1) - T_k v(\mathbf{x}) \\ &= v(\mathbf{x}+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{e}_1)\mathbf{E}_k) - v(\mathbf{x}+q^*\mathbf{E}_k) \\ &\geq v(\mathbf{x}+\mathbf{e}_1+q_k^*(\mathbf{x}+\mathbf{e}_1)\mathbf{E}_k) - v(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{e}_1)\mathbf{E}_k) \\ &= \Delta_1 v(\mathbf{x}+q_k^*(\mathbf{x}+\mathbf{e}_1)\mathbf{E}_k) \\ &\geq -c_1. \end{aligned}$$

The first inequality is due to the definition of $v(\mathbf{x}+q^*\mathbf{E}_k).$
It follows that $T_k v(\mathbf{x})$ satisfies property A6.

Operator *T*

It is straightforward to show that *h*(**x**) satisfies B1 and A5 since *h*(**x**) is separable and convex in each of the variables x_i. Also, since *V* is closed under multiplication by a scalar and addition, *Tv* satisfies properties B1 and A5. To show that *Tv* also satisfies A6, note that

$$\Delta_1 Tv(\mathbf{x}) = \Delta_1 h(\mathbf{x}) + \sum_{k=1}^{m} \mu_k \Delta_1 T_k v^*(\mathbf{x}) + \sum_{l=1}^{n} \lambda_l \sum_{t=1}^{q_l} p_t^l \Delta_1 T_t^l v^*(\mathbf{x}) \geq -c_1 \left(\sum_{k=1}^{m} \mu_k + \sum_{l=1}^{n} \lambda_l \left(\sum_{t=1}^{q_l} p_t^l \right) \right)$$
$$= -c_1 \left(\sum_{k=1}^{m} \mu_k + \sum_{l=1}^{n} \lambda_l \right)$$
$$= -c_1 (1-\alpha) \geq -c_1.$$

Hence, $Tv \in \mathcal{V}$.

To complete the proof of Lemma 1, we need to show that $v^* \in \mathcal{V}$. This can be easily done using the fact that (1) $v^* = \lim_{n\to\infty} T^{(n)} v$ for any $v \in \mathcal{V}$, where $T^{(n)}$ refers to n compositions of operator T (see Proposition 3.1.5 and 3.1.6 of Bertsekas (1995)) and (2) $T^{(n)} v \in \mathcal{V}$ since $Tv \in \mathcal{V}$ for any $v \in \mathcal{V}$.

Proof of Theorem 1

The structure of the optimal policy and properties P.1-P.6 are consequences of properties A1-A6 and the definitions of the threshold parameters $s_k^*(\mathbf{x}_{-k})$ and $r_l^*(\mathbf{x}_{-1})$. In particular let

$$s_k^*(\mathbf{x}_{-k}) = \min\left\{ x_k \geq 0 \mid \Delta_{k-P(k)} v^*(\mathbf{x}) \geq 0 \right\},$$

and

$$r_l^*(\mathbf{x}_{-1}) = \min\left\{ x_1 \geq 0 \mid \Delta_1 v^*(\mathbf{x}) \geq -c_l \right\}.$$

Then, property A1 applied to the case $i = k$ and $j = k$, (i.e., $\Delta_{k-P(k),k} v^*(\mathbf{x}) \geq 0$), combined with the above definitions, implies

$$\begin{cases} v^*(\mathbf{x}+\mathbf{E}_k) \geq v^*(\mathbf{x}) & \text{if and only if } x_k \geq s_k^*(\mathbf{x}_{-k}), \text{ and} \\ v^*(\mathbf{x}+\mathbf{E}_k) < v^*(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Hence, the optimal production policy is a base-stock policy with the optimal base-stock level $s_k^*(\mathbf{x}_{-k})$, such that it is optimal to produce units of item k as long as the resulting inventory level does not exceed $s_k^*(\mathbf{x}_{-k})$. Property A4 applied to the case $i = 1$, $j = 1$ (i.e., $\Delta_{1,1} v^*(\mathbf{x}) \geq 0$) implies that

$$\begin{cases} v^*(\mathbf{x}) + c_l \geq v^*(\mathbf{x}-\mathbf{e}_1) & \text{if and only if } x_1 > r_l(\mathbf{x}_{-1}), \text{ and} \\ v^*(\mathbf{x}) + c_l < v^*(\mathbf{x}-\mathbf{e}_1) & \text{otherwise.} \end{cases}$$

Hence, the optimal order fulfillment policy is a rationing policy, where it is optimal to fulfill demand from class l as long as the inventory level of the end item does not drop below $r_l^*(\mathbf{x}_{-1})$. Moreover, since $c_1 > c_2 > \cdots > c_n$, we have $r_l^*(\mathbf{x}_{-1}) \geq \cdots \geq r_1^*(\mathbf{x}_{-1}) = 0$.

Property **P.1** follows from property A2, which implies $s_k^*(\mathbf{x}_{-k})$ is non-decreasing in x_j for $j \notin S(i)$, and from property A1, which implies $s_k^*(\mathbf{x}_{-k})$ is non-increasing in x_j for $j \in S(k)$ and $j \neq k$. Property **P.2** follows from property A5, which implies that $s_k^*(\mathbf{x}_{-k}+\mathbf{E}_j) \geq s_k^*(\mathbf{x}_{-k})$ for $j \neq k$. Property **P.3** follows from property A3, which implies $s_k^*(\mathbf{x}_{-k}+\mathbf{e}_j-\mathbf{e}_l) \leq s_k^*(\mathbf{x}_{-k})$ or equivalently $s_k^*(\mathbf{x}_{-k}+\mathbf{e}_j) \leq s_k^*(\mathbf{x}_{-k}+\mathbf{e}_l)$. In order to show property **P.4**, it suffices to notice that decisions are made only at times when the system state changes (an order arrival or an item completion). If it is optimal to produce an item k in state $\mathbf{x}=(x_1,\cdots,x_m)$, then it continues to be optimal to produce it if an order of size t arrives which either leaves the state unchanged (in case of unavailability the end item) or moves it to state $\mathbf{x}-u\mathbf{e}_1=(x_1-u,x_2,...,x_m)$, where $0\le u\le\min(x_1,t)$. This is because the optimal policy is a base-stock policy and the base-stock level for item k is non-increasing in x_1. Similarly, if it is optimal to produce item k in state $\mathbf{x}$, then it continues to be optimal to produce it if item j ($j\ne k$) completes assembly, by virtue of **P.3**. Property **P.5** follows from property A4 applied to the case $j=1$ (i.e., $\Delta_{i,1}v(\mathbf{x})\ge 0$), which implies that $r_l^*(\mathbf{x}_{-1})$ is non-increasing in each x_i for $i\ne 1$ and from property A6, which implies Property **P.6.**

Proof of Results for Echelon Inventory Reformulation

We restate the results presented in Section 4 for the echelon reformulation in the following theorem and then provide a proof.

Theorem 2: *The optimal policy under the echelon reformulation has the following properties.*

1. *The optimal production policy is a state-dependent echelon base-stock policy.*
2. *The optimal echelon base-stock level of any item is non-decreasing in the echelon inventory level of any other item.*
3. *The optimal order fulfillment policy is determined by state-dependent rationing levels which are non-increasing in the echelon inventory of any item.*
4. *It is always optimal to fulfill demand from class* 1 *as long there is positive echelon inventory for all items.*

Proof: To prove property 1, first observe that the decision to produce item i does not change the echelon inventory level of any item other than item i. Also note that the difference $\Delta_i v(\hat{\mathbf{x}})$ measures the *economic value* of producing an additional unit of item i when the system is in state $\hat{\mathbf{x}}$. Hence, the difference $\Delta_i v(\hat{\mathbf{x}})$ under the echelon inventory formulation is equivalent to the difference $\Delta_{i-P(i)}v(\mathbf{x})$ under the local inventory formulation. As a result, the difference $\Delta_{i,i}v(\hat{\mathbf{x}})$ is equivalent to the difference $\Delta_{i-P(i),i-P(i)}v(\mathbf{x})$. Therefore, using B2, we have $\Delta_{i,i}v(\hat{\mathbf{x}})\ge 0$. Define now $s_i^*(\hat{\mathbf{x}}_{-i})=\min\{\hat{x}_i\ge 0\,|\,\Delta_i v^*(\hat{\mathbf{x}})\ge 0\}$. Then, $\Delta_{i,i}v(\hat{\mathbf{x}})\ge 0$ implies

$$\begin{cases} v^*(\hat{\mathbf{x}}+\mathbf{e}_i) \ge v^*(\hat{\mathbf{x}}) & \text{if and only if } \hat{x}_i \ge s_i^*(\hat{\mathbf{x}}_{-k}), \text{ and} \\ v^*(\hat{\mathbf{x}}+\mathbf{e}_i) < v^*(\hat{\mathbf{x}}) & \text{otherwise.} \end{cases}$$

Hence, the optimal production policy is a state-dependent echelon base-stock policy.

To show property 2, we apply similar arguments. In particular, note that the difference $\Delta_{i,j}v(\hat{\mathbf{x}})$ is equivalent to the difference $\Delta_{i-P(i),j-P(j)}v(\mathbf{x})$. Therefore, using A5, we have $\Delta_{i,j}v(\hat{\mathbf{x}}) \le 0$ for $i \ne j$, which implies that the optimal echelon base-stock level of any item is non-decreasing in the echelon inventory level of any other item.

In order to show property 3, we observe that the decision to fulfill demand from on-hand inventory reduces the echelon inventory level of all items by an equal amount. Hence, the difference $\Delta_{\mathbf{e}}v(\hat{\mathbf{x}})$ under the echelon inventory formulation is equivalent to $\Delta_1 v(\mathbf{x})$ under the local inventory formulation. Also, since $\Delta_i v(\hat{\mathbf{x}})$ is equivalent to $\Delta_I v(\mathbf{x})$, $\Delta_{\mathbf{e},1}v(\hat{\mathbf{x}})$ is equivalent to $\Delta_{1,1-P(1)}v(\mathbf{x})$. Hence, using A1, it follows that $\Delta_{\mathbf{e},1}v(\hat{\mathbf{x}}) \ge 0$. Let us now define $r_l^*(\hat{\mathbf{x}}_{-1}) = \min\{\hat{x}_1 \ge 0 \mid \Delta_{\mathbf{e}}v^*(\hat{\mathbf{x}}) \ge -c_l\}$. Then, $\Delta_{\mathbf{e},1}v(\hat{\mathbf{x}}) \ge 0$ implies

$$\begin{cases} v^*(\hat{\mathbf{x}}) + c_l \ge v^*(\hat{\mathbf{x}}-\mathbf{e}) & \text{if and only if } \hat{x}_1 > r_l(\hat{\mathbf{x}}_{-1}), \text{ and} \\ v^*(\hat{\mathbf{x}}) + c_l < v^*(\hat{\mathbf{x}}-\mathbf{e}) & \text{otherwise.} \end{cases}$$

Hence, the optimal inventory allocation policy is a state-dependent rationing policy. Furthermore, using similar arguments, we can show that $\Delta_{\mathbf{e},j}v(\hat{\mathbf{x}}) \ge 0$, which implies that the optimal rationing levels are non-increasing in the echelon inventory of any item.

Finally, to show property 4, note that $\Delta_{\mathbf{e}}v(\hat{\mathbf{x}})$ under the echelon inventory formulation is equivalent to $\Delta_1 v(\mathbf{x})$ under the local inventory formulation. Therefore, $\Delta_{\mathbf{e}}v(\hat{\mathbf{x}}) \ge -c_1$, which implies that it is always optimal to fulfill demand from class 1 when on-hand inventory is available.

Deriving Optimality Equation (1)

In order to transform our continuous time Markov chain problem into a discrete time Markov chain problem, we first note that decision epochs can be restricted to points in time where a change in the state of the system occurs. These changes coincide with either the arrival of an order or the completion of the production of an item. At any decision epoch, the time until the next arrival of an order from customer class *l* is exponentially distributed with rate λ_l. However, the time until the next production completion of a batch of items of type *k* depends on whether or not we decide to produce item *k*. Hence, the time until the next decision epoch (which corresponds to a change in the state of the system due to either an arrival or a production completion) depends on the production decisions we make in the current decision epoch.

This creates difficulties in transforming our problem into a discrete time problem. For example, if we decide not to produce any items, then the time until the next transition is exponentially distributed with rate $\sum_{l=1}^{n} \lambda_l$. In contrast, if we decide to produce one item, say item *k*, then the transition rate is $\sum_{l=1}^{n} \lambda_l + \mu_k$; and if we decide to produce all items, the transition rate is $\sum_{l=1}^{n} \lambda_l + \sum_{k=1}^{m} \mu_k$. To get around this difficulty, we can adopt a uniform transition rate (corresponding to the maximum feasible rate $\beta = \sum_{l=1}^{n} \lambda_l + \sum_{k=1}^{m} \mu_k$) so that the time until the next transition is always exponentially distributed with rate β. To compensate for the fact that the actual rate might be smaller (when we decide not to produce all items), we allow for the possibility of fictitious transitions from the current state back to itself. For example, if we decide not to produce item *k* when the system is in state **x**, then the state of the system at the next decision epoch remains **x** with probability μ_k / β.

Having adopted this uniform transition rate, we now have a standard discrete time control problem where total cost is the sum of the discounted costs incurred over each period. Optimal cost in each period can be expressed as the sum of the current period cost and the expected discounted cost to go given that actions in future periods will be taken optimally. In particular, we can show that the optimality equation can be written as follows (see for example Bertsekas (1995), pages 242-249 for details):

$$v^*(\mathbf{x}) = \frac{h(\mathbf{x})}{\alpha + \beta} + \sum_{k=1}^{m} \frac{\mu_k}{\alpha + \beta} T_k v^*(\mathbf{x}) + \sum_{l=1}^{n} \sum_{i=1}^{q_l} p_i^l \frac{\lambda_l}{\alpha + \beta} T_i^l v^*(\mathbf{x}).$$

Without loss of generality, we can rescale time so that $\alpha + \beta = 1$, which then immediately leads to equation (1).