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Online Supplement to "Optimal Product Acquisition, Pricing and Inventory Management for Systems with Remanufacturing"

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Part A: Derivation of Dynamic Program (1)

The additional notation needed here is summarized in the following, with the subscript representing period n.

$I_n =$ the starting inventory level of serviceable products;

$J_n =$ the starting inventory level of cores;

$i_n =$ the inventory level of serviceable products after manufacturing and remanufacturing;

$j_n =$ the inventory level of cores after remanufacturing and disposal;

$w_n =$ the remanufacturing quantity of cores.

To avoid confusion, we refer to the dynamic program (1) in the paper as (D-1) in the following derivation.

Given the starting inventory levels of serviceable product I_n and core J_n, let $v_n(I_n, J_n)$ be the optimal total discounted profit from period n to the end of the planning horizon. The problem is formulated as the following dynamic program.

$$\begin{aligned} v_n(I_n, J_n) \quad = \quad & \max_{i_n, j_n, w_n, e_n, p_n} \Big\{ p_n \mathsf{E}[D_n(p_n)] - r w_n - c(i_n - I_n - w_n) - s j_n \\ & -\mathsf{E}[G(i_n - D(p_n))] - g(e_n) + \alpha \mathsf{E}\big[v_{n+1}\big(i_n - D_n(p_n), j_n + R_n(e_n)\big)\big] \Big\}, \end{aligned} \tag{1}$$

subject to $0 \le w_n \le J_n - j_n$, $0 \le w_n \le i_n - I_n$, $0 \le j_n \le J_n$, $p_n \in [p_l, p_u]$ and $e_n \ge 0$. We assume $v_{N+1}(I_{N+1}, J_{N+1}) = 0$.

In the bracketed expression of (1), the first term is the expected revenue; the second term is the remanufacturing cost of cores; the third term is the production cost; the fourth term is the storage cost of cores; the fifth term is the inventory holding and shortage costs of serviceable products; the sixth term is the cost of the effort exerted to collect the cores; and the last term is the maximal discounted expected total profit from period $n+1$ to period N. The first constraint requires that the remanufactured quantity of cores is nonnegative and less than or equal to the difference between the inventory levels of cores before and after the decisions, as the amount of cores disposed of is equal to $J_n - j_n - w_n \ge 0$; the second constraint states that the total remanufactured quantity must be less than or equal to the increment of the serviceable inventory level, because the manufacturing quantity from raw materials is $i_n - I_n - w_n \ge 0$; the third constraints ensures that the ending inventory of cores is nonnegative and no more than the starting inventory; and the last two constraints require effort to be nonnegative and price to be within a feasible set.

Because $c \ge r$, it can be seen from (1) that the optimal remanufacturing quantity $w_n = \min\{J_n - j_n, i_n - I_n\}$. After substituting this into (1), only i_n, j_n, e_n and p_n remain as the decision variables, and problem (1) can be written as the following.

$$\begin{aligned} v_n(I_n, J_n) \quad = \quad & \max_{i_n, j_n, e_n, p_n} \Big\{ p_n d_n(p_n) + (c - r)\min\{J_n - j_n, i_n - I_n\} - c(i_n - I_n) - s j_n \\ & -\mathsf{E}[G(i_n - D_n(p_n))] - g(e_n) + \alpha \mathsf{E}\big[v_{n+1}\big(i_n - D_n(p_n), j_n + R_n(e_n)\big)\big] \Big\}, \end{aligned} \tag{2}$$

subject to $i_n \ge I_n$, $0 \le j_n \le J_n$, $p_n \in [p_l, p_u]$ and $e_n \ge 0$.

Note that, given the definition of x_n, u_n, y_n, and z_n, the following relationships hold

$$y_n = i_n, \; z_n = i_n + j_n, \; x_n = I_n \text{ and } u_n = I_n + J_n. \tag{3}$$

After applying these transformations and performing some additional algebra on (2), we can obtain (D-1).

Part B: Proofs

For ease of exposition, for a continuous and twice differentiable function $f(x,y)$, we use $(f(x,y))'_x$ and $(f(x,y))'_y$ to denote its partial derivative with respect to x and y, respectively. Meanwhile, $(f(x,y))''_{i,j}$, $i,j = x,y$ represents the second-order (mixed) partial derivative. In addition, we use $(f(\cdot,\cdot))'_1$ and $(f(\cdot,\cdot))'_2$ to denote the partial derivative with respect to the first and second argument of $f(\cdot,\cdot)$, respectively. And $(f(\cdot,\cdot))''_{i,j}$, $i,j = 1,2$ represents the second-order (mixed) partial derivative with respect to arguments i and j.

Proofs of Lemma 1, Proposition 2 and Theorem 3

In the following, we prove Lemma 1, Proposition 2 and Theorem 3 together by induction on n since they are closely related to one another. For notational convenience, we denote the demand $D_n(p)$ as D_n in this case and $\mathsf{E}[D_n] = d_n$.

As $v_{N+1}(x,u) = 0$, it is easy to show that Lemma 1 and Proposition 2 hold for $N+1$. Suppose Lemma 1 and Proposition 2 hold for $n = k+1$. We next prove them for $n = k$.

We first prove parts (a) and (b) of Lemma 1 for $n = k$. And then we verify Proposition 2 for $n = k$. After these, we prove Theorem 3. Finally, we show part (c) of Lemma 1 for $n = k$.

From the inductive assumption of Proposition 2,

$$\begin{aligned} v_k(x,u) &= \max_{y,z,e}\Big\{pd - (r-s)y - (c-r+s)z + (r-c)(u-z)^+ - \mathsf{E}[G(y-D)] - g(e) \\ &\quad +\alpha\mathsf{E}[U_{k+1}(y-D)] + \alpha\mathsf{E}\big[V_{k+1}\big(z+R(e)-D\big)\big]\Big\} + cu - r(u-x), \end{aligned}$$

subject to a set of linear constraints. It is not hard to see that every term in the first line of the bracketed expression is concave (the concavity of the fourth term is due to $c \geq r$). In the second line, the first term is concave by induction while the concavity of the second term on (z,e) is due to concavity of $f(e)$, the inductive assumption on the concavity of $V_{k+1}(\cdot)$ and Lemma 1 (c) that $v_{k+1}(x,u)$ is increasing in u. Therefore, the bracketed expression is concave and by applying Proposition B2 of Heyman and Sobel (1984), the concavity of $v_k(x,u)$ follows. The concavity of $h_{n,1}(y,z,e)$ and $h_{n,2}(y,z,e)$ are implied in the preceding arguments. Part (b) follows from $c - r + s > 0$ and the inductive assumption of part (c).

We next show Theorem 3 for period k. Note that

$$\begin{aligned} h_{k,1}(y,z,e) &= pd - (r-s)y - \mathsf{E}[G(y-D)] - g(e) - sz \\ &\quad +\alpha\mathsf{E}[U_{k+1}(y-D)] + \alpha\mathsf{E}\big[V_{k+1}\big(z+R(e)-D\big)\big], \end{aligned} \tag{4}$$

and

$$
\begin{aligned}
h_{k,2}(y,z,e) &= pd-(r-s)y-\mathsf{E}[G(y-D)]-g(e)-(c-r+s)z \\
&\quad +\alpha\mathsf{E}[U_{k+1}(y-D)]+\alpha\mathsf{E}\big[V_{k+1}\big(z+R(e)-D\big)\big]. \qquad (5)
\end{aligned}
$$

From the inductive assumption of Proposition 2 and the definitions,

$$
\begin{aligned}
\theta_k &= \arg\max_{y}\{-(r-s)y-\mathsf{E}[G(y-D)]+\alpha\mathsf{E}[U_{k+1}(y-D)]\}, \\
(\xi_k,\bar{e}_k) &= \arg\max_{y,e}\big\{-cy-\mathsf{E}[G(y-D)]-g(e) \\
&\qquad +\alpha\mathsf{E}[U_{k+1}(y-D)]+\alpha\mathsf{E}\big[V_{k+1}\big(y+R(e)-D\big)\big]\big\}, \\
(\eta_k,\widetilde{e}_k) &= \arg\max_{z,e}\big\{-sz-g(e)+\alpha\mathsf{E}\big[V_{k+1}\big(z+R(e)-D\big)\big]\big\}, \\
e_k(z) &= \arg\max_{e}\big\{-g(e)+\alpha\mathsf{E}\big[V_{k+1}\big(z+R(e)-D\big)\big]\big\}.
\end{aligned}
$$

It is obvious that $e_k(\xi_k)=\bar{e}_k$, and $e_k(\eta_k)=\widetilde{e}_k$. That $\xi_k\le\theta_k$ results from Lemma 1 (b), (c) and the concavity of $h_{k,2}(y,z,e)$. Furthermore, if $\eta_k<\theta_k$, then we redefine θ_k as $\theta_k=\eta_k$, where η_k is defined by $(\eta_k,\widetilde{e}_k)=\arg\max_{x,e}\{h_{k,1}(x,x,e)\}$. Thus, the relationship of all constants presented in Theorem 3 is satisfied.

Substitute $e_k(z)$ into (4) and (5), based on the definition of the control parameters, the optimal remanufacturing-production-disposal policy can be proved similarly as in Zhou et al. (2007) and so we skip the details here. After the optimal remanufacturing-production-disposal policy is derived, the corresponding optimal effort level follows.

With the optimal policy presented in Theorem 3, we are ready to verify Proposition 2 for period k. It is sufficient to show that $(v_k(x,u))'_x$ (resp., $(v_k(x,u))'_u$) only depends on x (resp., u). With different values of (x,u), the optimal policy divides the state space into different regions. Within the interior of each region, it is not hard to see the result is true, since $h_{k,1}(y,z,e_k(z))$ and $h_{k,2}(y,z,e_k(z))$ can be decomposed by the inductive assumption. So we only need to check the boundaries between different regions. We use one case to illustrate that, even at the boundary, $v_k(x,u)$ can still be decomposed. Other cases can be similarly proved. Consider the boundary of $u=\eta_k$. The right-sided derivative is

$$
\begin{aligned}
(v_k(x,u))'_{u=\eta_k+} &= \lim_{\Delta\to 0}\frac{v_k(x,\eta_k+\Delta)-v_k(x,\eta_k)}{\Delta} \\
&= \lim_{\Delta\to 0}\frac{h_{k,1}\big(\max\{x,\theta_k\},\eta_k,e_k(\eta_k)\big)-h_{k,1}\big(\max\{x,\theta_k\},\eta_k,e_k(\eta_k)\big)}{\Delta}+c-r \\
&= c-r,
\end{aligned}
$$

and the left-sided derivative is

$$
(v_k(x,u))'_{u=\eta_k-}
$$

$$
\begin{aligned}
&= \lim_{\Delta\to 0} \frac{v_k(x,\eta_k) - v_k(x,\eta_k-\Delta)}{\Delta} \\
&= \lim_{\Delta\to 0} \frac{h_{k,1}\big(\max\{x,\theta_k\},\eta_k,e_k(\eta_k)\big) - h_{k,1}\big(\max\{x,\theta_k\},\eta_k-\Delta,e_k(\eta_k-\Delta)\big)}{\Delta} + c - r
\end{aligned}
$$

which is independent of x because $v_{k+1}(\cdot,\cdot)$ is decomposable. Thus, both one-sided derivatives are independent of x.

Finally, we prove part(c) of Lemma 1 for $n = k$. Note that, for cases (i), (iv) and (v) presented in Theorem 3, the result is immediate as $(v_k(x,u))'_u = c - r$. So it is sufficient to show cases (ii) and (iii). For case (ii), i.e., $\xi_k \le u < \theta_k$ and $y_k^* = z_k^* = u$,

$$
\begin{aligned}
(v_k(x,u))'_u &= (c-r) + \big(h_{k,2}\big(u,u,e_k(u)\big)\big)' \\
&\le c - r,
\end{aligned}
$$

because $\xi_k \le u$ and the concavity of $h_{k,2}(y,y,e_k(y))$. Moreover,

$$
\begin{aligned}
(v_k(x,u))'_u &= \big(h_{k,1}\big(u,u,e_k(u)\big)\big)' \\
&\ge 0,
\end{aligned}
$$

because $u \le \theta_k \le \eta_k$.

For case (iii), i.e., $\theta_k \le u < \eta_k$ and $y_k^* = \max\{x,\theta_k\}$, $z_k^* = u$,

$$
\begin{aligned}
(v_k(x,u))'_u &= (c-r) + \big(h_{k,2}\big(\max\{x,\theta_k\},u,e_k(u)\big)\big)'_u \\
&\le c - r,
\end{aligned}
$$

where the inequality follows from that $h_{k,2}(y,z,e)$ is decreasing in z. Moreover,

$$
\begin{aligned}
(v_k(x,u))'_u &= (c-r) + \big(h_{k,2}\big(\max\{x,\theta_k\},u,e_k(u)\big)\big)'_u \\
&= \big(h_{k,1}\big(\max\{x,\theta_k\},u,e_k(u)\big)\big)'_u \\
&\ge 0,
\end{aligned}
$$

in which the inequality is due to the concavity of $h_{k,1}(y,z,e_k(z))$, and $u \le \eta_k$.

So the proofs of Lemma 1, Proposition 2 and Theorem 3 are thus completed. $\square$

Proof of Proposition 4

We first show that $e_n(x)$ is decreasing in x. To prove this, it is sufficient to show that $-g(e) + \alpha\mathsf{E}[V_{n+1}(x + R(e) - D)]$ is submodular in x and e from Topkis (1998). Note that

$$
(-g(e) + \alpha\mathsf{E}[V_{n+1}(x + R(e) - D)])''_{e,x}
$$

$$
\begin{aligned}
&= \alpha \mathsf{E}[(V_{n+1}(x+R(e)-D))'' f'(e)] \\
&\leq 0
\end{aligned}
$$

because $V_{n+1}(\cdot)$ is concave and $f(e)$ is increasing in e. So, the submodularity follows. Moreover, by the definition of e_n^* in Theorem 3, e_n^* is decreasing in z^*. $\square$

Proof of Proposition 5

That θ_n is independent of e follows from its definition and that $v_n(x,u)$ is decomposable from Proposition 2. The proof in the above proposition also shows that $h_{n,1}(y,z,e)$ is submodular in (z,e), which implies that η_n is decreasing in e. An analogous argument can show that ξ_n is decreasing in e. $\square$

To prove Lemmas 6, 7, Theorem 8, and Proposition 9, we first define a class of functions $\mathfrak{V}$, which includes continuous and twice differentiable functions $\varphi(x,y) \in \mathfrak{V}$: $\Re^2 \to \Re$ such that

(P-a) $\varphi(x,y)$ is concave and $0 \leq (\varphi(x,y))'_y \leq c-r$;

(P-b) $(\varphi(x,y))''_{x,x} + (\varphi(x,y))''_{x,y} \leq 0$;

(P-c) $(\varphi(x,y))''_{y,y} + (\varphi(x,y))''_{x,y} \leq 0$; and

(P-d) $\varphi(x,y)$ is supermodular, i.e., $(\varphi(x,y))''_{x,y} \geq 0$.

To proceed, we need to have the following result.

Lemma 11 *For $n = 1, ..., N$, if $v_{n+1}(x,u) \in \mathfrak{V}$, then there exists optimal $p_n(x,u)$ and $e_n(x,u)$ such that $u + bp_n(x,u)$ is increasing in u and $u + f(e_n(x,u))$ is increasing in u.*

Proof. Let $\widehat{p} := z + bp$. As for a given z, $\widehat{p}$ and p are one-to-one correspondent, we can analyze $(y,z,\widehat{p},e)$ instead of (y,z,p,e). Define

$$
\begin{aligned}
\tilde{h}_{n,1}(y,z,\widehat{p},e) &= h_{n,1}(y,z,z+bp,e) \\
&= (z+bp-z)\frac{1}{b}(a-(z+bp-z)+\mathsf{E}[\epsilon^d]) - (r-s)y - \mathsf{E}[G(y+z+bp-z-a-\epsilon^d)] \\
&\quad -g(e) - sz + \alpha \mathsf{E}\big[v_{n+1}\big(y+z+bp-z-a-\epsilon^d, z+bp+R(e)-a-\epsilon^d\big)\big] \\
&= \frac{1}{b}(\widehat{p}-z)(a-(\widehat{p}-z)+\mathsf{E}[\epsilon^d]) - (r-s)y - \mathsf{E}[G(\widehat{p}+y-z-a-\epsilon^d)] - g(e) - sz \\
&\quad +\alpha \mathsf{E}\big[v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)\big].
\end{aligned}
$$

We first show that the optimal $e(\widehat{p}, y, z)$ is decreasing in $\widehat{p}$, which follows from the concavity of $\tilde{h}_{n,1}(y, z, \widehat{p}, e)$ and submodularity on $(e, \widehat{p})$ as

$$\begin{aligned}&\Big(\mathsf{E}\big[v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)''_{2,1}\big]\\&+\mathsf{E}\big[v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)''_{2,2}\big]\Big)\, f'(e) \leq 0.\end{aligned}$$

It is not hard to see that $\tilde{h}_{n,1}(y, z, \widehat{p}, e(\widehat{p}, y, z))$ is supermodular in z and $\widehat{p}$ as

$$\begin{aligned}&\frac{2}{b}+\mathsf{E}[G(\widehat{p}+y-z-a-\epsilon^d)'']-\mathsf{E}\big[v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)\big]''_{1,1}\\&-\mathsf{E}\big[v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)''_{1,2}\big]f'(e)((e(\widehat{p}, y, z))'_{\widehat{p}}+1)\\=\;&\frac{2}{b}+\mathsf{E}[G(\widehat{p}+y-z-a-\epsilon^d)'']-\mathsf{E}\big[v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)''_{1,1}\\&+v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)''_{1,2}\big]\\&-\mathsf{E}\big[v_{n+1}\big(\widehat{p}+y-z-a-\epsilon^d, \widehat{p}+R(e)-a-\epsilon^d\big)''_{1,2}(f'(e)((e(\widehat{p}, y, z))'_{\widehat{p}})\\\geq\;& 0\end{aligned}$$

from the convexity of $G(\cdot)$, Property (P-b) and (P-d) for $v_{n+1}(x, u)$, and $f'(e) \geq 0$. Therefore, at the optimum, $\widehat{p}(y, z) = z + bp(y, z)$ is increasing in z.

To show $z + f(e(y, z))$ is increasing in z, for simplicity, let $F := f(e)$ and $\widehat{e} := z + f(e) = z + F$. As $f(e)$ is strictly increasing, $e = f^{-1}(F)$. We define

$$\begin{aligned}\tilde{h}_{n,1}(y, z, p, \widehat{e}) &= h_{n,1}(y, z, p, z+F)\\&= pd(p)-(r-s)y-\mathsf{E}[G(y-D(p))]-g(f^{-1}(z+F-z))-sz\\&\quad+\alpha\mathsf{E}\big[v_{n+1}\big(y-D(p), z+F-D(p)+\epsilon^r\big)\big]\\&= pd(p)-(r-s)y-\mathsf{E}[G(y-D(p))]-g(f^{-1}(\widehat{e}-z))-sz\\&\quad+\alpha\mathsf{E}\big[v_{n+1}\big(y-D(p), \widehat{e}-D(p)+\epsilon^r\big)\big].\end{aligned}$$

Take the second-order partial derivative of $h_{n,1}(y, z, p, \widehat{e})$ with respect to z and $\widehat{e}$,

$$\begin{aligned}&\left(g'(f^{-1}(\widehat{e}-z))\frac{1}{f'(f^{-1}(\widehat{e}-z))}\right)'_{\widehat{e}}\\=\;& g''(f^{-1}(\widehat{e}-z))\left(\frac{1}{f'(f^{-1}(\widehat{e}-z))}\right)^2+g'(f^{-1}(\widehat{e}-y))\frac{1}{(f'(f^{-1}(\widehat{e}-z)))^2}\frac{-f''(f^{-1}(\widehat{e}-z))}{f'((f^{-1}(\widehat{e}-z))}\\\geq\;& 0,\end{aligned}$$

since $g(\cdot)$ is convex increasing and $f(e)$ is concave increasing. Hence, $\widehat{e}(y, z) = z + f(e(y, z))$ is increasing in z. $\square$

The road map of the following proofs is as follows. We first assume that $v_{n+1}(x, u) \in \mathfrak{V}$ for some $n + 1$. As a result, we can easily show part (b) of Lemma 6 for n as the proof of Lemma

1 since $D_n(p)$ is linear in p. We thus skip its detailed proof here. Then we prove Lemma 7 and Theorem 8 for period n assuming Proposition 9 is valid for n. After that, we prove Proposition 9 for n. Finally, we show that $v_n(x,u) \in \mathfrak{V}$ in Lemma 12, which simultaneously proves parts (a) and (c) of Lemma 6.

Proof of Lemma 7

For (a), from the concavity of $h_{n,1}(y,z,e,p)$ and $h_{n,2}(y,z,e,p)$ (Lemma 6 (b)), by definition, $\delta_n(y)$ is the solution of $\big(h_{n,1}\big(y,z,e_n(y,z),p_n(y,z)\big)\big)'_z = 0$, or

$$\big(h_{n,1}\big(y,\delta_n(y),e_n(y,\delta_n(y)),p_n(y,\delta_n(y))\big)\big)'_z = 0,$$

and $e_n(y,z)$, $p_n(y,z)$ are the solutions to the following system of equations:

$$\begin{aligned}(h_{n,1}(y,z,e,p))'_e &= -g'(e) + \alpha \mathsf{E}\Big[\big(v_{n+1}(y-D(p), z+R(e)-D(p))\big)'_2 f'(e)\Big] = 0, & (6)\\ (h_{n,1}(y,z,e,p))'_p &= d(p) - pb - \mathsf{E}[G'(y-D(p))b] + \alpha \mathsf{E}\Big[\big(v_{n+1}(y-D(p), z+R(e)-D(p))\big)'_1 b\Big] \\ &\quad + \alpha \mathsf{E}\Big[\big(v_{n+1}(y-D(p), z+R(e)-D(p))\big)'_2 b\Big] = 0. & (7)\end{aligned}$$

For brevity, in what follows, we suppress the subscript n unless confusion may arise and let p'_y (resp., p'_z) and e'_y (resp., e'_z) denote $(p(y,z))'_y$ (resp., $(p(y,z))'_z$) and $(e(y,z))'_y$ (resp., $(e(y,z))'_z$), respectively.

Note that, from Lemma 6 (b), $h_{n,1}(y,z,e(y,z),p(y,z))$ is concave and

$$\big(h_{n,1}\big(y,z,e(y,z),p(y,z)\big)\big)''_{zz} \le 0.$$

Based on the Implicit Function Theorem,

$$(\delta_n(y))' = -\frac{\big(h_{n,1}\big(y,\delta_n(y),e(y,\delta_n(y)),p(y,\delta_n(y))\big)\big)''_{y,z}}{\big(h_{n,1}\big(y,\delta_n(y),e(y,\delta_n(y)),p(y,\delta_n(y))\big)\big)''_{z,z}},$$

in which

$$\begin{aligned}&\big(h_{n,1}\big(y,z,e(y,z),p(y,z)\big)\big)''_{y,z}\\ =\ &\Big\{(p'_y d(p(y,z)) - pbp'_y - \mathsf{E}[G'(y-D(p(y,z)))(1+bp'_y)] - g'(e(y,z))e'_y - (r-c)\\ &+\alpha \mathsf{E}\big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)'_1 (1+bp'_y)\big]\\ &+\alpha \mathsf{E}\big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)'_2 (f'(e(y,z))e'_y + bp'_y)\big]\Big\}'_z\\ =\ &\Big\{-\mathsf{E}[G'(y-D(p(y,z)))] + \alpha \mathsf{E}\big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)\big]'_1\Big\}'_z\end{aligned}$$

$$\begin{aligned}
&= -\mathsf{E}[G''(y-D(p(y,z)))bp'_z] + \alpha\mathsf{E}\big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{1,1}bp'_z\big] \\
&\quad +\alpha\mathsf{E}\big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{1,2}(1+f'(e(y,z))e'_z+bp'_z)\big] \\
&\geq 0
\end{aligned}$$

where the second equality follows from the optimality Eqns. (6) and (7) of $p(y,z)$ and $e(y,z)$, and the inequality follows from Lemma 11, Properties (P-b) (P-d) of $v_{n+1}(x,u)$, and Proposition 9. This also implies that $h_{n,1}(y,z,e(y,z),p(y,z))$ is supermodular. Therefore, $(\delta_n(y))' \geq 0$.

Moreover,

$$\begin{aligned}
&(h_{n,1}(y,z,e(y,z),p(y,z)))''_{z,z} \\
&= \alpha\mathsf{E}\big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{2,1}bp'_z\big] \\
&\quad +\alpha\mathsf{E}\big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{2,2}(1+f'(e(y,z))e'_z+bp'_z)\big].
\end{aligned}$$

At $p(y,z)$ and $e(y,z)$, take the partial derivative of (7) with respect to z:

$$\begin{aligned}
&\Big[(h_{n,1}(y,z,e(y,z),p(y,z)))'_p\Big]'_z \\
&= b\Bigg(p'_z\Big(-2-\mathsf{E}[G''(y-D(p(y,z)))b] + \alpha\mathsf{E}\Big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{1,1}b\Big]\Big) \\
&\quad +\alpha\mathsf{E}\Big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{1,2}(1+f'(e(y,z))e'_z+bp'_z)\Big] \\
&\quad +\alpha\mathsf{E}\Big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{2,1}bp'_z\Big] \\
&\quad +\alpha\mathsf{E}\Big[\big(v_{n+1}(y-D(p(y,z)), z+R(e(y,z))-D(p(y,z)))\big)''_{2,2}(1+f'(e(y,z))e'_z+bp'_z)\Big]\Bigg) \\
&= 0. \qquad (8)
\end{aligned}$$

Therefore, apply (8),

$$\begin{aligned}
&(h_{n,1}(y,z,e(y,z),p(y,z)))''_{y,z} + (h_{n,1}(y,z,e(y,z),p(y,z)))''_{z,z} \\
&= \frac{1}{b}\left(((h_{n,1}(y,z,e(y,z),p(y,z)))'_p\right)'_z + 2p'_z = 2p'_z \leq 0,
\end{aligned}$$

because of Proposition 9. And so $0 \leq (\delta_n(y))' \leq 1$.

For (b), analogous to the proof of part (a), we can show that $0 \leq (\theta_n(z))' \leq 1$. □

Proof of Theorem 8

Here we show if $v_{n+1}(x,u) \in \mathfrak{V}$, the policy in Theorem 8 is optimal for n. We solve the optimization problem of $v_{n,\ell}(x,u), \ell = 1,2$ after replacing e and p by $e_n(y,z)$ and $p_n(y,z)$, and then compare their values to find $v_n(x,u)$ as well as the corresponding optimal solutions.

Let's first verify the relationship that $\xi_n \le S_{n,0} \le \eta_{n,1} \le S_{n,1}$ and $\eta_{n,0} \le \eta_{n,1}$. Note that $S_{n,1} \ge \eta_{n,1} \ge S_{n,0}$ follows directly from their definitions. Note that $\eta_{n,1} \ge \eta_{n,0}$ also by definition. In addition, note that ξ_n and $\theta_n(u)$ are the solutions of, respectively,

$$(h_{n,2}\big(\xi_n, \xi_n, e_n(\xi_n, \xi_n), p_n(\xi_n, \xi_n)\big))'_y + (h_{n,2}\big(\xi_n, \xi_n, e_n(\xi_n, \xi_n), p_n(\xi_n, \xi_n)\big))'_z = 0,$$

and

$$(h_{n,2}\big(\theta_n(u), u, e_n(\theta_n(u), u), p_n(\theta_n(u), u)\big))'_y = 0.$$

From Lemma 6 (b),

$$(h_{n,2}\big(\xi_n, \xi_n, e_n(\xi_n, \xi_n), p_n(\xi_n, \xi_n)\big))'_z \le 0.$$

Hence

$$(h_{n,2}\big(\xi_n, \xi_n, e_n(\xi_n, \xi_n), p_n(\xi_n, \xi_n)\big))'_y \ge 0.$$

By the concavity of $h_{n,2}(y, z, e, p)$, $\xi_n \le \theta_n(\xi_n)$ and $\xi_n \le \eta_{n,0}$. Thus, from the definition of $S_{n,0}$, $\xi_n \le S_{n,0}$ and the relationship of the parameters holds, i.e., $\xi_n \le S_{n,0} \le \eta_{n,1} \le S_{n,1}$ and $\eta_{n,0} \le \eta_{n,1}$.

We now consider the optimization problem $v_{n,1}(x, u)$. When $u < \eta_{n,1}$, note that the maximizer $(\eta_{n,0}, \eta_{n,1})$ of $h_{n,1}(y, z, e_n(y, z), p_n(y, z))$ with the constraints ignored is not attainable. We have the following two cases.

1. If $x < \eta_{n,0}$, because $h_{n,1}$ is concave, then the optimum must be achieved on the boundary $z^* = u$. For y^*, it should be the closest point to $\theta_n(u)$ depending on u and x.

2. If $x \ge \eta_{n,0}$, then at the optimum, $y = x$ or $z = u$. Suppose $y = x$. Since $\delta_n(x)$ is increasing and $x \ge \eta_{n,0}$, $\delta_n(x) \ge \delta_n(\eta_{n,0}) = \eta_{n,1} > u$. Therefore, by the definition of $\delta_n(x)$, $h_{n,1}(x, z, e_n(x, z), p_n(x, z))$ is increasing for $z \le \delta_n(x)$. Hence at the optimum, $z^* = u$. Moreover, by the definition of $\theta_n(u)$, $\theta_n(u) < \eta_{n,0} < x$. Thus, $h_{n,1}(x, u, e_n(x, u), p_n(x, u))$ is decreasing for $x \ge \theta_n(u)$. Therefore, the optimal solution is $y^* = x$, $z^* = u$.

When $\eta_{n,1} \le u$, there are also two cases.

1. If $x < \eta_{n,0} \le \eta_{n,1} \le u$, then $(\eta_{n,0}, \eta_{n,1})$ is attainable. Therefore, $y^* = \eta_{n,0}$, $z^* = \eta_{n,1}$.

2. If $x \ge \eta_{n,0}$, then $(\eta_{n,0}, \eta_{n,1})$ is not attainable and the optimal solution must be on the boundary of the feasible set, i.e., $y^* = x$. There are two subcases for z^*.

   - When $x < S_{n,1}$, $z^* = \min\{u, \delta_n(x)\}$ because $x < \delta_n(x)$, $e^* = e_n(x, \min\{u, \delta_n(x)\})$, and $p^* = p_n(x, \min\{u, \delta_n(x)\})$.
   - When $S_{n,1} \le x$, $z^* = x$ because $x > \delta_n(x)$, $e^* = e_n(x, x)$, and $p^* = p_n(x, x)$.

We next analyze the optimization problem $v_{n,2}(x,u)$. At first, if $u < S_{n,0}$, then $u < \theta_n(u)$; otherwise, $u \geq \theta_n(u)$, which is due to $0 \leq (\theta_n(x))' \leq 1$ and $\xi_n \leq \theta_n(\xi_n)$.

Note that $h_{n,2}\big(y, z, e_n(y,z), p_n(y,z)\big)$ is jointly concave in (y,z) and decreasing in z (Lemma 6(b)). By definition, if $\big(\theta_n(u), u\big)$ is attainable without violating the constraints, it must be the maximizer since disposal is not allowed in this case; otherwise, the optimal solution must be attained at some boundary of the region defined by the constraints. The optimal solution of $v_{n,2}(x,u)$ is specified under the following three different cases.

1. If $u < \xi_n$, then $x \leq u < \xi_n \leq S_{n,0}$ and $u \leq \theta_n(u)$ from the analysis above. It is clear that $\big(\theta_n(u), u\big)$ is not attainable. By the definition of $\theta_n(u)$, $h_{n,2}\big(y, z, e_n(y,z), p_n(y,z)\big)$ is increasing for $y \leq \theta_n(z)$. Hence $y = z$ always holds at the optimum. By the definition of ξ_n, $h_{n,2}\big(y, y, e_n(y,y), p_n(y,y)\big)$ can achieve its optimum at (ξ_n, ξ_n) without violating any constraints. Thus $y^* = z^* = \xi_n$. Moreover, $e^* = e_n(\xi_n, \xi_n) = \bar{e}_n$ and $p^* = p_n(\xi_n, \xi_n) = \bar{p}_n$.

2. If $\xi_n \leq u < S_{n,0}$, then again $u \leq \theta_n(u)$. At the optimum, it must be true that $y = z$ or $z = u$. Suppose $y = z$. Because $h_{n,2}\big(y, y, e_n(y,y), p_n(y,y)\big)$ is decreasing in y for $y \geq \xi_n$ and so $z^* = u$ must hold at the optimum. Therefore, by the definition of $\theta_n(u)$, $h_{n,2}\big(y, u, e_n(y,u), p_n(y,u)\big)$ is increasing in y for $y \leq \theta_n(u)$. Thus $y^* = u$. The optimal solution is $y^* = z^* = u$, $e^* = e_n(u,u)$ and $p^* = p_n(u,u)$.

3. If $u \geq S_{n,0}$, then $u \geq \theta_n(u)$. Recall that if $\big(\theta_n(u), u\big)$ can be attained, it must be the optimal solution. Therefore, when $x < \theta_n(u)$, the optimal solution is $y^* = \theta_n(u)$ and $z^* = u$. When $\theta_n(u) \leq x$, at the optimum, $y^* = x$ and $z^* = u$. Combine these two, $y^* = \max\{x, \theta_n(u)\}$, and $z^* = u$. Moreover, $e^* = e_n\big(\max\{x, \theta_n\}(u), u\big)$, and $p^* = p_n\big(\max\{x, \theta_n(u)\}, u\big)$.

Finally, based on the preceding analysis of $v_{n,\ell}$, $\ell = 1, 2$, we are ready to prove the strategy described in Theorem 8 is optimal for period n by comparing $v_{n,1}(x,u)$ and $v_{n,2}(x,u)$ in different regions of (x,u).

i. If $u < \xi_n$, then the optimal solution of $h_{n,1}$ is a feasible solution of $h_{n,2}$. We can therefore only consider $h_{n,2}$, whose optimal solution is $y^* = z^* = \xi_n$, $e^* = \bar{e}_n$ and $p^* = \bar{p}_n$.

ii. If $\xi_n \leq u < S_{n,0}$, then $v_{n,2}$ still dominates $v_{n,1}$ and the optimal policy is $y^* = z^* = u$ and $e^* = e_n(u,u)$, $p^* = p_n(u,u)$.

iii. If $S_{n,0} \leq u < \eta_{n,1}$, again we have $v_n(x,u) = v_{n,2}(x,u)$, and the optimal policy is $y^* = \max\{x, \theta_n(u)\}$, $z^* = u$, $e^* = e_n\big(\max\{x, \theta_n(u)\}, u\big)$ and $p^* = p_n\big(\max\{x, \theta_n(u)\}, u\big)$.

iv. If $\eta_{n,1} \leq u$ and $x < \eta_{n,0}$, then the optimal solution of $h_{n,2}$ is just a feasible solution of $h_{n,1}$. So from the result of optimizing $h_{n,1}$, $y_n^* = \eta_{n,0}$, $z_n^* = \eta_{n,1}$ and $e^* = \widetilde{e}_n$ $p^* = \widetilde{p}_n$.

v. If $\eta_{n,1} \le u$ and $x \ge \eta_{n,0}$, there are two scenarios.

- If $x < S_{n,1}$, then the definition of $S_{n,1}$ implies $x \le \delta_n(x)$. Hence $v_n(x,u) = v_{n,1}(x,u)$, and the optimal policy is $y^* = x$, $z^* = \min\{u, \delta_n(x)\}$, $e^* = e_n\big(x, \min\{u, \delta_n(x)\}\big)$ and $p^* = p_n\big(x, \min\{u, \delta_n(x)\}\big)$.
- When $x \ge S_{n,1}$, then $x \ge \delta_n(x)$ and the optimal solution of $h_{n,2}$ is a feasible solution of $h_{n,1}$. Hence $v_n(x,u) = v_{n,1}(x,u)$, and the optimal policy is $y^* = z^* = x$.

So Theorem 8 is true for period n.

Proof of Proposition 9

We now verify Proposition 9 for n. We need to show that $e_n(y,z)$ increases with y while decreases with z and $p_n(y,z)$ decreases with both y and z. To this end, we need to first show that $h_{n,1}(y,z,e,p)$ and $h_{n,2}(y,z,e,p)$ are submodular on (y,p), (p,e), and (z,p). We only focus on $h_{n,1}(y,z,e,p)$ as $h_{n,2}(y,z,e,p)$ just differs from it by a linear term. For notational brevity, we use $\mathsf{E}[(v_{n+1})''_{ij}]$ to denote $\mathsf{E}\Big[\Big(v_{n+1}\big(y - D(p), z + R(e) - D(p)\big)\Big)''_{i,j}\Big]$ for $i,j = 1,2$.

Note that

$$
\begin{aligned}
& (h_{n,1}(y,z,e,p))''_{y,p} \\
= \;& -b\mathsf{E}[G(y - D(p))]'' + \alpha \mathsf{E}\Big[(v_{n+1})''_{1,1}\Big] b + \alpha \mathsf{E}\Big[(v_{n+1})''_{1,2}\Big] b \\
\le \;& 0,
\end{aligned}
$$

because of the convexity of $G(x)$ and (P-b) of $v_{n+1}(x,u)$. Hence $h_{n,1}(y,z,e,p)$ is submodular on (y,p). Similarly, we can show that $h_{n,1}(y,z,e,p)$ is submodular in (p,z) and (p,e). From Topkis (1998), $p(e,y,z) = \arg\max_p\{h_{n,1}(y,z,e,p)\}$ is decreasing in (e,y,z). Before we show $p(y,z) = p(e(y,z),y,z)$ is decreasing in both y and z, we now show $e(y,z)$ is increasing in y but decreasing in z. From (6),

$$
\begin{aligned}
& (h_{n,1}(y,z,e,p(e,y,z)))''_{e,y} \\
= \;& \alpha f'(e)\mathsf{E}\Big[(v_{n+1})''_{2,1}(1 + b(p(e,y,z))'_y)\Big] + \alpha f'(e)\mathsf{E}\Big[(v_{n+1})''_{2,2} b(p(e,y,z))'_y\Big] \\
= \;& \alpha f'(e)\left(\mathsf{E}\Big[(v_{n+1})''_{2,1}\Big] + \mathsf{E}\Big[((v_{n+1})''_{2,1} + (v_{n+1})''_{2,2}) b(p(e,y,z))'_y\Big]\right) \\
\ge \;& 0,
\end{aligned}
$$

because $f'(e) \ge 0$, $p'_y(e,y,z) \le 0$, (P-c) and (P-d) of v_{n+1}. So $e(y,z)$ is increasing in y.

From the property of $e(y,z)$ and $p(e,y,z)$ that we have shown, it is clear that $p(y,z)$ is decreasing in y.

We remain to show that $e(y,z)$ and $p(y,z)$ are decreasing in z. We show the result for $p(y,z)$ first. From (8),

$$
\begin{aligned}
& (p(y,z))'_z \\
= & \frac{-\alpha(1+f'(e(y,z))e'_z)\Big(\mathsf{E}\Big[(v_{n+1})''_{1,2}\Big]+\mathsf{E}\Big[(v_{n+1})''_{2,2}\Big]\Big)}{\Big(-2-\mathsf{E}[G''(y-D(p(y,z)))b]+b\alpha\Big(\mathsf{E}\Big[(v_{n+1})''_{1,1}\Big]+\mathsf{E}\Big[(v_{n+1})''_{1,2}\Big]+\mathsf{E}\Big[(v_{n+1})''_{2,1}\Big]+\mathsf{E}\Big[(v_{n+1})''_{2,2}\Big]\Big)} \\
\le & \; 0,
\end{aligned}
$$

because of Lemma 11, properties (P-b) and (P-c) of $v_{n+1}(x,u)$. Similarly, for $e(y,z)$,

$$
\begin{aligned}
& (e(y,z))'_z \\
= & \frac{\alpha f'(e)\Big((1+bp'_z)\Big(\mathsf{E}\Big[(v_{n+1})''_{2,2}\Big]+\mathsf{E}\Big[(v_{n+1})''_{2,1}bp'_z\Big]\Big)}{g''(e)-\alpha\Big(\mathsf{E}\Big[(v_{n+1})'_{2}\Big]f''(e(y,z))+(f'(e(y,z)))^2\mathsf{E}\big[(v_{n+1})''_{2,2}\big]\Big)} \\
\le & \; 0,
\end{aligned}
$$

where the numerator is negative because of $f'(e)>0$, Lemma 11, $p'_z \le 0$, and properties (P-a) and (P-d) of $v_{n+1}(x,u)$ while the denominator is positive because of convexity of $g(e)$, concavity of $f(e)$ and (P-a) of $v_{n+1}(x,u)$.

Finally, based on Theorem 8, it is not hard to see that e_n^* is increasing in y^* but decreasing in z^* and p_n^* is decreasing in y^* and z^*. $\square$

Lemma 12 *For $n=1,\ldots,N$, $v_n(x,u)\in\mathfrak{V}$.*

Proof. We prove this result by induction. It can be shown for period $n=N$ easily as $v_{N+1}(x,u)=0$. Suppose it is true for $n=k+1$, i.e., $v_{k+1}(x,u)\in\mathfrak{V}$.

Now we verify (P-a) for $n=k$. The concavity of $v_k(x,u)$ can be easily shown as the proof of Lemma 1 because $D_n(p)$ is linear and the resulting $h_{k,\ell}(y,z,e,p)$, $\ell=1,2$, is concave in (y,z,e,p). For the second part of (P-a), we only check one case to illustrate the idea, i.e., $S_{k,0}\le u<\eta_{k,1}$. In such case, $z^*=u$ and $y^*=\max(\theta_k(u),x)$.

If $x\le\theta_k(u)$, then $y^*=\theta_k(u)$.

$$
\begin{aligned}
(v_k(x,u))'_u &= (c-r)+\Big(h_{k,2}\big(\theta_k(u),u,e_k(\theta_k(u),u),p_k(\theta_k(u),u)\big)\Big)' \\
&= (c-r)-(c-r+s)+\alpha\mathsf{E}\big[v_{k+1}\big(\theta_k(u)-D(p_k(\theta_k(u),u)), \\
&\qquad u+R(e_k(\theta_k(u),u))-D(p_k(\theta_k(u),u))\big)\big]'_2 \\
&\le c-r,
\end{aligned}
$$

in which the second equality follows from the optimality of $\theta_k(u)$, $p_k(\theta_k(u), u)$, and $e_k(\theta_k(u), u)$ while the inequality is due to the inductive assumption that $[v_{k+1}(x, u)]'_u \le c - r$.
Meanwhile,

$$\begin{aligned}(v_k(x,u))'_u &= (c-r) + \Big(h_{k,2}\big(\theta_k(u), u, e_k(\theta_k(u), u), p_k(\theta_k(u), u)\big)\Big)' \\ &= \big(h_{k,1}\big(\theta_k(u), u, e_k(\theta_k(u), u), p_k(\theta_k(u), u)\big)\big)' \\ &\ge 0,\end{aligned}$$

where the inequality is due to the concavity of $h_{k,1}\big(\theta_k(u), u, e_k(\theta_k(u), u), p_k(\theta_k(u), u)\big)$ and $u < \eta_{k,1}$.

If $x > \theta_k(u)$, then $y^* = x$. From the optimality of $e_k(x, u)$ and $p_k(x, u)$,

$$\begin{aligned}(v_k(x,u))'_u &= (c-r) + \Big(h_{k,2}\big(x, u, e_k(x,u), p_k(x,u)\big)\Big)'_u \\ &= (c-r) - (c-r+s) + \alpha\mathsf{E}\big[v_{k+1}\big(x - D(p_k(x,u)), \\ &\quad\; u + R(e_k(x,u)) - D(p_k(x,u))\big)\big]'_2 \\ &\le c - r.\end{aligned}$$

Meanwhile, as $u \le \eta_{k,1} = \delta_k(\eta_{k,0})$ and $x \ge \theta_k(u) \ge \theta_k(\eta_{k,1}) = \eta_{k,0}$, $u \le \delta_k(x)$ from the fact that $\delta_k(x)$ is increasing. Therefore,

$$\begin{aligned}(v_k(x,u))'_u &= (c-r) + \big(h_{k,2}\big(x, u, e_k(x,u), p_k(x,u)\big)\big)'_u \\ &= \big(h_{k,1}\big(x, u, e_k(x,u), p_k(x,u)\big)\big)'_u \\ &\ge 0,\end{aligned}$$

where the inequality is due to the concavity of $h_{k,1}\big(y, z, e_k(y,z), p_k(y,z)\big)$ in z. So (P-a) is valid for k. Therefore, parts (a) and (c) of Lemma 6 is valid.

For (P-b)-(P-d) of $v_k(x, u)$, to avoid a tedious proof, we also only illustrate the idea of proof by using the case $S_{k,0} \le u < \eta_{k,1}$.

If $x \le \theta_k(u)$, then $y^* = \theta_k(u)$. In this proof, $\mathsf{E}[(v_{k+1})''_{ij}] := \mathsf{E}\Big[\Big(v_{k+1}\big(y - D(p_k(y,z), z + R(e_k(y,z)) - D(p_k(y,z))\big)\Big)''_{i,j}\Big]$ for $i, j = 1, 2$ and $p_k := p_k(x, u)$, $e_k := e_k(x, u)$. (P-b) of $v_k(x, u)$ follows since

$$(v_k(x,u))''_{x,x} + (v_k(x,u))''_{x,u} = 0.$$

(P-b) is true because

$$(v_k(x,u))''_{u,u} + (v_k(x,u))''_{x,u} = (v_k(x,u))''_{u,u} \le 0,$$

while (P-d) is also valid as $(v_k(x,u))''_{x,u} = 0$.

If $x > \theta_k(u)$, then $y^* = x$ and (P-b) of $v_k(x,u)$ is true as

$$\begin{aligned}
& (v_k(x,u))''_{x,x} + (v_k(x,u))''_{x,u} \\
= \; & -\mathsf{E}[G''(x - D(p_k))(1 + bp'_x)] + \alpha \mathsf{E}\big[(v_{k+1})''_{1,1}(1 + bp'_x)\big] + \alpha \mathsf{E}\big[(v_{k+1})''_{1,2}(f'(e_k)e'_x + bp'_x)\big] \\
& + \alpha \mathsf{E}\big[(v_{k+1})''_{2,1}(1 + bp'_x)\big] + \alpha \mathsf{E}\big[(v_{k+1})''_{2,2}(f'(e_k)e'_x + bp'_x)\big] \\
= \; & \frac{1}{b}\Big[(h_{k,1}(x,u,p_k,e_k))'_p\Big]'_x + 2p'_x \\
= \; & 2p'_x \\
\leq \; & 0
\end{aligned}$$

where the last equality follows from that $\Big[(h_{k,1}(x,u,p_k,e_k))'_p\Big]'_x = 0$ and the inequality is due to Proposition 9.

Similarly, (P-b) follows from that

$$\begin{aligned}
& (v_k(x,u))''_{u,u} + (v_k(x,u))''_{x,u} \\
= \; & \alpha \mathsf{E}\big[(v_{k+1})''_{2,1} bp'_u\big] + \alpha \mathsf{E}\big[(v_{k+1})''_{2,2}(1 + f'(e_k)e'_u + bp'_u)\big] - \mathsf{E}[G''(x - D(p_k))bp'_u] \\
& + \alpha \mathsf{E}\big[(v_{k+1})''_{1,1} bp'_u\big] + \alpha \mathsf{E}\big[(v_{k+1})''_{1,2}(1 + f'(e_k)e'_u + bp'_u)\big] \\
= \; & \big(h_{k,1}(x,u,p_k,e_k)'_p\big)'_u + 2p'_u \\
\leq \; & 0
\end{aligned}$$

which follows from $\big(h_{k,1}(x,u,p_k,e_k)'_p\big)'_u = 0$ and $p'_u \leq 0$. And (P-d) follows from

$$\begin{aligned}
& (v_k(x,u))''_{x,u} \\
= \; & -\mathsf{E}[G''(x - D(p_k))bp'_u] + \alpha \mathsf{E}\big[(v_{k+1})''_{1,1} bp'_u\big] + \alpha \mathsf{E}\big[(v_{k+1})''_{1,2}(1 + f'(e_k)e'_u + bp'_u)\big] \\
= \; & -\mathsf{E}[G''(x - D(p_k))bp'_u] + \alpha \mathsf{E}\big[((v_{k+1})''_{1,1} + (v_{k+1})''_{1,2})bp'_u\big] + \alpha \mathsf{E}\big[(v_{k+1})''_{1,2}(1 + f'(e_k)e'_u)\big] \\
\geq \; & 0
\end{aligned}$$

in which the inequality is due to Lemma 11, (P-b), (P-d) of $v_{k+1}(x,u)$ and Proposition 9. This concludes the proof. $\square$

Proof of Proposition 10

That ξ_n decreases with e and p can be proved by showing that $h_{n,2}(y,y,e,p)$ is submodular in (y,e) and (y,p). We verify the submodularity on (y,e).

$$(h_{n,2}(y,y,e,p))''_{y,e} = f'(e)\alpha(\mathsf{E}\big[(v_{n+1})''_{1,2}\big] + \mathsf{E}\big[(v_{n+1})''_{2,2}\big]) \leq 0,$$

from (P-d) of v_{n+1} and $f'(e) \geq 0$. Similarly, one can show the submodularity on (y,p).

That $\delta_n(y)$ decreases with e and p follows from the submodularity of $h_{n,1}(y,z,e,p)$ on (z,e) and (z,p), which can be easily shown. The monotonicity of $\theta_n(z)$ in e and p follows from the supermodularity of $h_{n,2}(y,z,e,p)$ on (y,e) and the submodularity on (y,p), which have been verified in the proof of Proposition 9. Finally, to show the monotonicity of $\eta_{n,0}$ in e and p, it is sufficient to show $h_{n,1}(y,z(y,e,p),e,p)$ is supermodular in (y,e) while submodular in (y,p) with $z(y,e,p) = \arg\max_z\{h_{n,1}(y,z,e,p)\}$. Note that

$$\begin{aligned}(h_{n,1}(y,z(y,e,p),e,p))''_{e,y} &= \alpha f'(e)\mathsf{E}\big[\big(v_{n+1}(y-D(p),z(y,e,p)+R(e)-D(p))\big)''_{2,1} \\ &\quad +\big(v_{n+1}(x-D(p),z(y,e,p)+R(e)-D(p))\big)''_{2,2}(z(y,e,p))'_y\big] \geq 0,\end{aligned}$$

where the inequality follows from (P-b) of $v_{n+1}(x,u)$, the concavity of $v_{n+1}(x,u)$, and $0 \leq (z(y,e,p))'_y \leq 1$ which is implied by (P-b) of $v_{n+1}(x,u)$ and the Implicit Function Theorem. So the supermodularity follows. And

$$\begin{aligned}&(h_{n,1}(y,z(y,e,p),e,p))''_{y,p} \\ =\ & -b\mathsf{E}[G''(y-D(p))] + \alpha\mathsf{E}\big[\big(v_{n+1}(y-D(p),z(y,e,p)+R(e)-D(p))\big)''_{1,1}b\big] \\ & +\alpha\mathsf{E}\big[\big(v_{n+1}(y-D(p),z(y,e,p)+R(e)-D(p))\big)''_{1,2}(b+(z(y,e,p))'_p)\big] \leq 0,\end{aligned}$$

which follows from (P-d), (P-d) of $v_{n+1}(x,u)$, and $(z(y,e,p))'_p \leq 0$ as $h_{n,1}(y,z,e,p)$ is submodular on (z,p). So the submodularity of $h_{n,1}(y,z(y,e,p),e,p)$ on (y,p) is valid. The property of $\eta_{n,1}$ can be similarly proved. $\square$

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