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e-companion

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Electronic Companion—"The Impact of Delaying the Delay Announcements" by Gad Allon and Achal Bassamboo, *Operations Research*, http://dx.doi.org/10.1287/opre.1110.0972.

Appendix A: Proofs

Proof of Theorem 4.1

To prove the result, we shall use the Markov Decision Process theory. However, note that it is not possible to employ the concept of uniformization directly on the system as the service rate in IVR system grows without bound. To this end, we shall consider a sequence of systems index by κ. The κ^{th} system is identical to the one defined before with the modification that the IVR has κ servers and there is no waiting in the IVR, i.e., IVR is a pure loss system. We will make another transformation that will be useful for analyzing this system, we assume that the customer pays v immediately upon entering the queue for the ABS, and the holding cost of a customer in IVR is $h+\alpha v$. It is easy to see that the above is equivalent to the setting where the holding cost in ABS is h, and the customer pays v upon service completion. For this system we can use the uniformization approach to obtain the following optimality equations:

$$\begin{aligned}
V^{\kappa}(q_{IVR},q) =& \frac{1}{\alpha+\kappa\mu_{IVR}+N\mu+\lambda}\left[-hq_{IVR}-h(q-N)^{+}-\alpha vq\right.\\
&+\ \lambda V^{\kappa}(q_{IVR}+\mathbb{I}_{q_{IVR}<\kappa},q)+\mu(\min\{q,N\}V^{\kappa}_{n-1}(q_{IVR},q-1)+(N-q)^{+}V^{N}_{n-1}(q_{IVR},q))+\\
&+\ \mu_{IVR}\min\{q_{IVR},\kappa\}\max\{V^{\kappa}(q_{IVR}-1,q+1)+v,V^{\kappa}(q_{IVR}-1,q)\}+(\kappa-q_{IVR})\mu_{IVR}V^{\kappa}(q_{IVR},q)]
\end{aligned}$$

Here x^{+} is define as $\max 0,x$.

Next, we define a map $\mathcal{L}$ on real valued functions on $\mathbb{Z}^2$, and a sequence V^{N}_{n} for $n=1,2,\ldots$, where $V^{\kappa}_{0}\equiv 0$ and $V^{\kappa}_{n}=\mathcal{L}V^{\kappa}_{n-1}$ is given by

$$\begin{aligned}
V^{\kappa}_{n}(q_{IVR},q) =& \frac{1}{\alpha+\kappa\mu_{IVR}+N\mu+\lambda}\left[-hq_{IVR}-h(q-N)^{+}-\alpha vq+\lambda V^{\kappa}_{n-1}(q_{IVR}+\mathbb{I}_{q_{IVR}<\kappa},q)+\right.\\
&+\ \mu(\min\{q,N\}V^{\kappa}_{n-1}(q_{IVR},q-1)+(N-q)^{+}V^{N}_{n-1}(q_{IVR},q))+\\
&+\ \mu_{IVR}\min\{q_{IVR},\kappa\}\max\left\{V^{\kappa}_{n-1}(q_{IVR}-1,q+1)+v,V^{\kappa}_{n-1}(q_{IVR}-1,q)\right\}\\
&+\ (\kappa-q_{IVR})\mu_{IVR}V^{\kappa}_{n-1}(q_{IVR},q)].
\end{aligned}$$

It is clear that V^{κ} is a fixed point of this mapping. Using the theory of the semi-Markov decision process, we also have that $V^{\kappa}_{n}\to V^{\kappa}$ as $n\to\infty$. To show concavity of V^{κ}, we shall show that V^{κ}_{n} is concave and non-increasing in q. The proof is by induction. The result holds for $n=0$ by definition as $V^{\kappa}_{0}\equiv 0$.

Assume that V^{κ}_{n-1} is concave and non-increasing in q we shall show that V^{κ}_{n} is concave and non-increasing in q. First, note that $-hq_{IVR} - h(q-N)^{+} - \alpha vq$ is concave, and $\lambda V^{\kappa}_{n-1}(q_{IVR} + \mathbb{I}_{q_{IVR}<\kappa}, q)$ and $(\kappa - q_{IVR})\mu_{IVR}V^{\kappa}_{n-1}(q_{IVR}, q)$ are both concave in q, as V^{κ}_{n-1} is concave in q. Second, consider $f_1(q_{IVR}, q) \equiv (\min(q,N)V^{\kappa}_{n-1}(q_{IVR}, q-1) + (N-q)^{+}V^{\kappa}_{n-1}(q_{IVR}, q))$. Using the fact that V^{κ}_{n-1} is concave and decreasing in q and appealing to Lemma 3.1 Koole (1998), we obtain that $f_1(q_{IVR}, q)$ is decreasing and concave in q. Lastly, define $f_2(q_{IVR}, q) \equiv \max\{V^{\kappa}_{n-1}(q_{IVR}-1, q+1) + v, V^{\kappa}_{n-1}(q_{IVR}-1, q)\}$. Fix q_{IVR}. Noting that $V^{\kappa}_{n-1}(q_{IVR}-1, q)$ is concave decreasing in q, there exists a q^{*} such that $V^{\kappa}_{n-1}(q_{IVR}-1, q) - V^{\kappa}_{n-1}(q^{IVR}, q+1) \leq v$ if $0 \leq q \leq q^{*}$, and $V^{\kappa}_{n-1}(q_{IVR}-1, q) - V^{\kappa}_{n-1}(q^{IVR}, q+1) > v$ if $q > q^{*}$. Thus, we have that

$$f_2(q_{IVR}, q) = \begin{cases} V^{\kappa}_{n-1}(q_{IVR}-1, q+1) + v & \text{if } q \leq q^{*} \\ V^{\kappa}_{n-1}(q_{IVR}-1, q) & \text{otherwise.} \end{cases} \tag{EC.1}$$

We need to show that $f_2(q_{IVR}, q)$ is concave and non increasing in q. Since $V^{\kappa}_{n-1}(q_{IVR}-1, q)$ is concave and non-increasing, it suffices to show the following:

$$f_2(q_{IVR}, q^{*}) \geq f_2(q_{IVR}, q^{*}+1) \tag{EC.2}$$

$$f_2(q_{IVR}, q^{*}-1) - f_2(q_{IVR}, q^{*}) \leq f_2(q_{IVR}, q^{*}) - f_2(q_{IVR}, q^{*}+1) \tag{EC.3}$$

The first follows by noting that $v \geq 0$ and the representation of f_2 in (EC.1). For the second equality note that using (EC.1) the left hand side equals $V^{\kappa}_{n-1}(q_{IVR}-1, q^{*}) - V^{\kappa}_{n-1}(q_{IVR}-1, q^{*}+1)$ and the right hand side equals v. The second equality then follows by the definition of q^{*}.

Thus, we get that $V^{\kappa}_{n}(q_{IVR}, q)$ is concave and non-increasing in q. Thus, we establish the fact that $V^{\kappa}(q_{IVR}, q)$ is concave.

Next we will show that for large κ, the system performance obtained, when modelling the IVR as $M/M/\kappa/\kappa$ system is close to the one obtained when modeling it as an $M/M/\infty$ system. We define two random variables: $Q_{M/M/\kappa/\kappa}$ which represents the number of customers in an $M/M/\kappa/\kappa$ queue in steady state with arrival rate λ and service μ_{IVR}; and $Q_{M/M/\infty}$ which represents the number of customers in an $M/M/\infty$ queue in steady state with arrival rate λ and service μ_{IVR}. Let $V(q_{IVR}, q)$ be the optimal profit for the firm when the IVR has infinite capacity starting from the state (q_{IVR}, q). Then, it is easy to see that

$$V^{N}(q_{IVR}, q) - \lambda\mathbb{P}(Q_{M/M/\kappa/\kappa} = N)h\mathbb{E}[w] \leq V(q_{IVR}, q).$$

Further, we know that $\mathbb{E}[w] = 1/\mu_{IVR}$, and $\mathbb{P}(Q_{M/M/\kappa/\kappa} = \kappa) \leq \mathbb{P}(Q_{M/M/\infty} \geq \kappa)$. Also, we can show

$$V(q_{IVR}, q) \leq V^{\kappa}(q_{IVR}, q) + \lambda v \mathbb{P}(Q_{M/M/\kappa/\kappa} = \kappa) \leq V^{\kappa}(q_{IVR}, q) + \lambda v \mathbb{P}(Q_{M/M/\infty} \geq \kappa).$$

Noting that the number in system for an $M/M/\infty$ system is Poisson distributed with mean $\lambda\mu_{IVR}$, there exists $\beta > 0$ such that for N large

$$\mathbb{P}(Q_{M/M/\infty} \geq N) \leq e^{-\beta\kappa}.$$

Thus we have for N large, there exist a finite K such that

$$\|V^{\kappa} - V\| \leq K e^{-\beta\kappa}.$$

Thus, one gets $V^{\kappa} \to V$ as $\kappa \to \infty$. Thus, concavity of V^{κ} in q results in concavity of V in q. Hence, we obtain that there exists a threshold $\eta(q_{IVR})$ such that the firm would accept the customer into ABS from IVR if and only if $q \leq \eta(q_{IVR})$. Thus we have the optimal policy is a threshold policy.

■

Proof of Proposition 5.1 For the above delayed cheap talk game, if an informative equilibria exists, it must be the case that the firm obtains its first best. This can be argued as follows: suppose there exists an informative equilibrium where the firm does not obtain its first best. It must be the case that there are at least two signals which are used by the firm and the customer joins when she receives one signal and balks when she receives the other signal. Given this strategy for the customer, the firm would have a profitable deviation if it does not achieve its first best. Thus, we have the result. ■

Proof of Theorem 5.2 The firm clearly has no incentive to deviate from the Full Control solution. Thus, to ensure equilibrium with influential cheap talk we need to ensure that conditions 1 and 2 of the Definition 5.1 hold. Using the arguments presented above we have that condition 1 is satisfied, if and only if, the inequalities in (11) holds. For Condition 2 which requires that the overall expected utility of customers is non-negative, we need to satisfy the following condition:

$$R \sum_{q_{IVR}=0}^{\infty} \sum_{q=0}^{\eta(q_{IVR})-1} p(q_{IVR}, q) - \frac{c}{N\mu} \sum_{q_{IVR}=0}^{\infty} \sum_{q=0}^{\eta(q_{IVR})-1} (q+1-N)^{+} p(q_{IVR}, q) + R_{IVR} - \frac{c}{\mu_{IVR}} \geq 0 \tag{EC.4}$$

Using the definition of threshold $\underline{\eta}^c$, $\delta(\eta)$, and K, we can re-express this condition as follows:

$$\left(\frac{c}{\mu_{IVR}} - R_{IVR}\right) + \frac{c}{N\mu}\underline{\eta}^c \sum_{q_{IVR}=0}^{\infty} \sum_{q=N}^{\eta(q_{IVR})-1} p(q_{IVR}, q) \le R \sum_{q_{IVR}=0}^{\infty} \sum_{q=0}^{\eta(q_{IVR})-1} p(q_{IVR}, q).$$

We can then restate the above condition as follows:

$$\frac{\underline{\eta}^c + K}{(1+\delta(\eta))} \le \frac{RN\mu}{c}, \tag{EC.5}$$

where K and δ are as defined in (12). Note that the system dynamics under full control dictate that $Q_{ABS}(t) \le \sup_q \eta^*(q)$. Thus, we have $\bar{\eta}^c \le \sup_q \eta^*(q)$. Combining the three inequalities in (11) and (EC.5), we have the desired result. This completes the proof. ■

Proof of Proposition 6.1 For part (a) note that the contribution of a customer to the profit of the firm is $v - hW$, where W is the waiting time in the system. Clearly, if the contribution of the customer is negative, the firm will not admit him. Note, however, that due to the need to account for the dynamics of the model, a customer with positive contribution is not necessarily guaranteed admittance. Thus, if the number of customers in the system exceeds $v\mu/h$, the expected waiting time would be greater than v/h, thus his contribution will be negative. This completes the proof of part (a). Proof of part (b) is analogous to the above proof, and uses the observation that if a customer provides only negative contribution based on the number of customers in the ABS, the firm can disregard the number of customers waiting in the IVR. ■

Proof of Proposition 6.2

(a) For an informative equilibrium to exist it must be the case that $R/c < v/h$. Further, the firm achieves its first best profit. Also, the only pure strategy non-informative equilibrium for the system is the one where no customer balks the system. Let $\mathbb{E}[\widetilde{W}]$ be the waiting time in this system. Also, let $p0$ denote the fraction of customers who are joining in an informative equilibrium for the delayed cheap talk game. Let $\mathbb{E}[W]$ denote the expected waiting time in ABS for the system in an informative equilibrium for the delayed cheap talk game. Using the fact that the firm achieves its first best under an informative equilibria, we have:

$$\lambda(v - h\mathbb{E}[\widetilde{W}]) \le \lambda p(v - h\mathbb{E}[W]).$$

The above implies

$$\frac{v}{h} \le \frac{\mathbb{E}[\widetilde{W}] - p\mathbb{E}[W]}{1-p}.$$

Appealing to the fact that $R/c < v/h$, we have that

$$\lambda(R - c\mathbb{E}[\widetilde{W}]) \leq \lambda p(R - c\mathbb{E}[W]).$$

Thus, the overall expected utility of the customers would improve in an informative equilibrium when compared to a non-informative one. Noting that the babbling equilibrium in the base model and the delayed cheap model are identical, completes the proof.

(b) Combining Propositions 4.5 and 5.1 from Allon et al. (2007), we obtain that in the base model the customers and the firm prefer an informative equilibrium over a babbling one. Further, note that the non-informative equilibrium in the base model and the delayed cheap model are identical. This completes the proof.

(c) The proof follows in an analogous manner to part (a) with the exception that there will be a probability of joining for both the base model and delay announcement model. Further, since $R/c < v/h$ for the neutral region, we obtain the result.

■

Proof of Proposition B.1 Note that under the proposed equilibrium the system operates as an $M/M/N/k$ queueing system. We next show that the customers who wait in this queuing system experience waiting time with increasing hazard rate.

In an M/M/N/k system using the steady state analysis and the PASTA property, the number of customers waiting in the queue when a customer arrives and joins the system (and waits) has a geometric distribution conditioned on it being less than k. Thus, we have that the waiting time in the system for a customer has the same distribution as $\sum_{j=1}^{Q+1} X_j$, where X_j are exponential with mean $1/(N\mu)$ and Q has the same distribution as the number in system of $M/M/N/k$ conditioned on the event that the number of queue is less than N and the arriving customer waits. We also know that the $Q+1$ has an increasing hazard rate. Using Theorem 7.1 in Ross et al. (2005), we have that the waiting time will also have an increasing hazard rate.

Using the arguments similar to the proof of Proposition 2(i) in Mandelbaum and Shimkin (2000), we obtain that the customers will never abandon the system if they join the system. This shows that the equilibria described earlier are abandonment-proof from the customer's perspective. Given that the customers do not abandon and the firm had no profitable deviation from its signaling rule, it follows that the firm will not have any profitable deviation when the customers are allowed to deviate. This completes the proof. ■

Proof of Proposition B.2 The proof follows by appealing to Theorem 7.1 in Ross et al. (2005) and Proposition 2(i) in Mandelbaum and Shimkin (2000), and is along the same lines as Proposition B.1. ■

Proof of the Proposition B.4 Consider the two full control problem on the same probability space. All the service times and arrival times for the two systems are identical. Consider the optimal policy of the full control problem under immediate announcement. It is easy to see that if we employ a policy for the delayed announcement that rejects exactly those customers rejected in the immediate announcement system then the ABS part of the system behaves in an identical manner for the two systems. Moreover, since we have $v_{IVR} > h/\mu_{IVR}$, the delayed announcement system generates more profit. Thus, we have the desired result.

■

Appendix B: Extensions

In this section, we extend our findings in two directions. First, we allow the customers to abandon the system, i.e., the customers, once they join, are not required to get served. Second, we study a related model where the IVR is essential to the system and the customers may be given the delay announcement before or after this essential IVR.

B.1. Abandonments

In many service settings, customers can make a decision not only regarding joining versus balking, but also about leaving the system after joining without receiving the service. So far, we have focused on the first two decisions, while disallowing customer abandonment. In this section we first show that in the base model, where the information is provided immediately upon arrival, if the customers are allowed to update their beliefs about the system and renege the queue, the equilibria characterized remain unchanged. We then provide conditions under which a similar property is exhibited by the delayed cheap talk model.

PROPOSITION B.1. In the equilibria identified for the base model in Proposition 3.1 a rational customer will not abandon even if allowed to, and the firm will not deviate from its signaling rule. In this sense, these equilibria are abandonment-proof.

The proposition states that a rational customer who updates his belief on the state of the system *after* joining the system, would not abandon. While the customer might realize the fact that he was lured to join in a state he otherwise would not join, he is in a better position compared to the one in which he decided to join. This is somewhat equivalent to treating the elapsed time as "sunk cost," and explains the reason why the customers do not have any profitable deviation. On the other hand, one can easily see that given that the customers who join the system are not interested in leaving the system without service, the firm's full control solution remains unchanged. Thus, the firm will not have any profitable deviation either.

Hence, in this setting abandonments will not arise endogenously. Other more complex settings, such as the one in which the valuation varies over time (see Haviv and Ritov 2001) or one in which the customers feel that they have been left out of the system without being informed (see Mandelbaum and Shimkin 2000), can lead to rational abandonments.

Next, we turn our attention to the delayed cheap talk game. Let Q denote the number of customers in the queue in ABS at the departure instance from the IVR, conditioned on the fact that he joins the ABS and waits for his service. It is clear that in the delayed model, using the above logic that a customer will not abandon during his stay at the IVR. It is worth noting that this abandonment-proofness for IVR stems from the fact that the delay in the IVR is independent of the congestion in the system. Thus, the customer is not able to obtain any additional information about the system as he gets "served/delayed" at the IVR. The question of whether one will abandon while waiting for the ABS can be answered along the same lines as the base system. The following result provides sufficient condition for the abandonment-proofness of the equilibria defined for the delayed cheap talk game.

PROPOSITION B.2. If Q has an increasing hazard rate under the equilibria defined by Proposition 5.2 then a rational customer will not abandon even if allowed to in the IVR as well as in the ABS and the firm will not deviate from its signaling rule. In this sense, these equilibria are abandonment-proof.

B.2. The case of an essential IVR

In this section, we consider a system where the IVR provides an essential part of the service. Specifically, we study the following two setups that include the IVR: a) a system where the firm provides information immediately on arrival (before the IVR), and b) a system where the firm provides delayed delay announcement (after the IVR). Given that the IVR is essential, we would assume that both the customer and the firm derive positive utility from the experience at the IVR, i.e., $v_{IVR} > h_{IVR}/\mu_{IVR}$ and $R_{IVR} > c_{IVR}/\mu_{IVR}$.

Using an arguments similar to Proposition 4.1, if the firm has full control in the system with immediate delay announcement (i.e. announcing the delay before the IVR) the optimal policy has the following property.

PROPOSITION B.3. For the full control problem in the system with immediate delay announcements, there exists a threshold function $\tilde{\eta}^*(\cdot)$, such that it is optimal for the firm to accept a customer that completes service at the IVR, if and only if $Q < \tilde{\eta}^*(Q_{IVR}(t))$ and "turn him away" otherwise.

Proof: To show that $V_{IA}^{\kappa}(q_{IVR}, q)$ is concave in q for a given q_{IVR} we follow the same lines as the in proof of Proposition 4.1. To show that this holds for $V(q_{IVR}, q)$, we have to use a slightly different argument. In particular, it is easy to see that

$$V_{IA}^{\kappa}(q_{IVR}, q) < V_{IA}(q_{IVR}, q) < V_{IA}^{\kappa}(q_{IVR}, q) + (v_{IVR} + v)\mathbb{P}(Q_{M/M/\kappa/\kappa} = \kappa).$$

Then, taking limits as $\kappa \to \infty$ we get the desired result. ■

Further, under full control we can show that for any initial state, the system with immediate announcement performs worse than the system with delayed announcement. To formally state the result, let $V_{IA}(q_{IVR}, q)$ and $V_{DA}(q^{IVR}, q)$ be the optimal discounted infinite horizon profit in the immediate announcement (IA) system and the delayed announcement (DA) system starting with the state (q_{IVR}, q), respectively. Then we have the following result.

PROPOSITION B.4. We have

$$V_{IA}(q_{IVR}, q) \leq V_{DA}(q^{IVR}, q).$$

We can define the notion of pure strategy MPBNE for the cheap talk games for both the immediate announcement as we have done for the delayed announcement system. Recall that an equilibrium with influential cheap talk exists in the delayed announcement system iff the conditions specified in Theorem 5.2 hold. Similarly, one can observe that the influential influential cheap talk exists in the immediate announcement system iff the following conditions hold:

$$\sum_{q_{IVR}=0}^{\infty} \sum_{q=N}^{\eta(q_{IVR})-1} \mathcal{U}(q_{IVR}, q) p(q_{IVR}, q) \geq 0, \quad \sum_{q_{IVR}=0}^{\infty} \sum_{q=\eta(q)}^{\infty} \mathcal{U}(q_{IVR}, q) p(q_{IVR}, q) \leq 0, \tag{EC.6}$$

where $\mathcal{U}(q_{IVR}, q)$ is the expected utility obtained by a customer from the system (including IVR) joining the immediate announcement system where the firm has full control and the state of the system is (q_{IVR}, q). Using Proposition B.4 we obtain the following result.

COROLLARY B.1. If there exists a pure strategy MPBNE for both the immediate announcement cgeap talk game as well the delayed announcement cheap talk game, then the firm would earn higher profits in the latter.

The implication of this theorem is that if credibility can be obtained via delaying the delay announcement or sustained in both systems regardless of the timing of the announcement, the firm prefers the setting in which the information provision is delayed. However, it is important to note that, just as in the comparison discussed in Section 6, while the firm might expand the set of equilibria by delaying the information provision, the firm might also diminsh the set.