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Online Supplement

Pricing Asian Options under a Hyper-Exponential Jump Diffusion Model

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More Discussion on the Numerical Algorithm

1 Stability of the Method

To study the stability of the method, we perform some numerical experiments under the BSM to show the absolute and relative errors of our double-Laplace inversion method against various choices of parameters A_1, A_2 and X. The "true" prices are obtained by using Monte Carlo simulation with a control variate being $\int_0^t e^{X(s)}ds$, because it is easy to compute $E\left[\int_0^t e^{X(s)}ds\right] = (e^{rt}-1)/r$, and $\int_0^t e^{X(s)}ds$ has a high degree of correlation with the payoff function. In addition, Richardson extrapolation is also employed to reduce the discretization bias generated when we discretize the sample path to approximate the integral. More precisely, let $M(h)$ be the Monte Carlo estimator without Richardson extrapolation when the discretization step size is set to be h. Then we use $(4M(h) - M(2h))/3$ rather than $M(h)$ as the final estimator to achieve the discretization bias reduction. For more details about the technique of control variates and Richardson extrapolation, see Glasserman [24].

Figure 1 shows how the absolute and relative errors change as A_1, A_2 and X vary in the case of low volatility $\sigma = 0.05$, illustrating that our algorithm is insensitive to the selection of parameters A_1, A_2 and X. For normal volatilities, our method becomes even more stable and associated plots can be obtained on request.

2 Discretization Error Bounds of Euler Inversion Algorithm under the BSM

The discretization error bound of the Euler inversion algorithm was first studied by Abate and Whitt [1], and was extended to a two-sided Laplace inversion case by Petrella [37]. In this

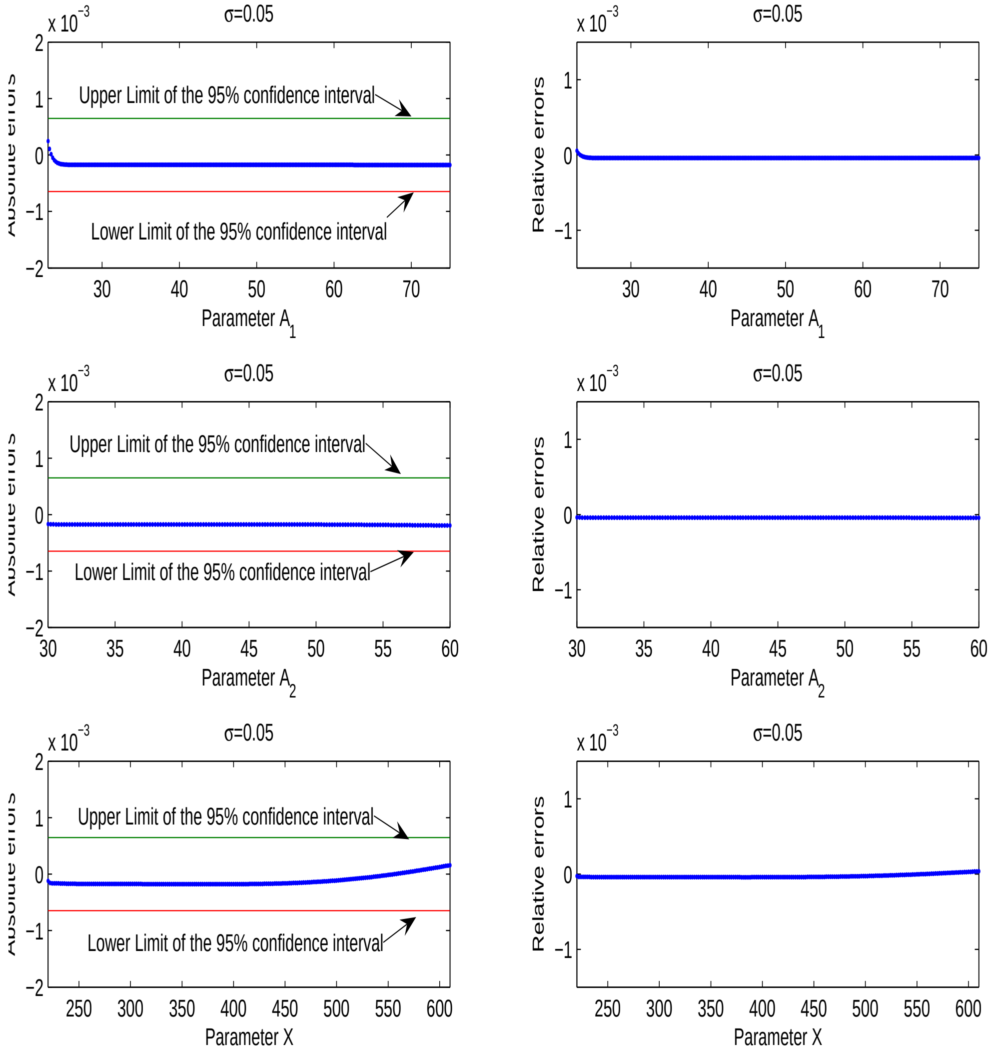

Figure 1: The stability and accuracy of the algorithm as A_1, A_2 and X vary in the case of low volatility $\sigma = 0.05$. The default choices for unvarying algorithm parameters are $A_1 = 28$, $A_2 = 40$ and X given by (27). The absolute errors and relative errors are reported on the left and right graphs, respectively. For broad regions of A_1, A_2 and X our algorithm appears to be stable and accurate, all within the 95% confidence intervals. In fact, the relative errors are all smaller than 0.02%.

subsection, by extending the results in Petrella [37], we provide discretization error bounds of the inversion algorithm for our specific case of Asian option pricing under the BSM. The discretization error bounds decay exponentially, therefore leading to a fast convergence.

Recall that what we want to invert is $\mathcal{L}(\mu,\nu)=\int_0^\infty\int_{-\infty}^\infty e^{-\mu t}e^{-\nu k}f(t,k)dkdt$, where $f(t,k)=XE(\frac{S_0}{X}A_t-e^{-k})^+$. Then we can prove the following theorem for the error bounds.

Theorem 2.1. *Suppose $t\in(0,\frac{A_1}{2(\theta_1+\theta_2)})$ and $k>\frac{A_2}{\theta_2}$, for some constant $\theta_2>0$, where $\theta_1=1+\tilde{r}+\sigma^2/2>0$ and $\tilde{r}=\max(r-\frac{\sigma^2}{2},0)$. Then the discretization error bounds e_d^+ and e_d^- satisfy*

$$e_d^+\leq\frac{C^+(\theta_1)}{1-e^{-(A_1-2\theta_1 t)}}\left\{\frac{e^{-A_2}}{1-e^{-A_2}}+e^{-(A_1-2\theta_1 t)}\right\},\tag{32}$$

$$e_d^-\leq C^-(\theta_1,\theta_2)\frac{1}{1-e^{-(A_1-2(\theta_1+\theta_2)t)}}\frac{e^{-(\theta_2k-A_2)}}{1-e^{-(\theta_2k-A_2)}},\tag{33}$$

with $C^+(\theta_1):=2S_0e^{\theta_1 t}$ and $C^-(\theta_1,\theta_2):=2S_0e^{t\sigma^2\theta_2^2/2+[(1+\tilde{r}+\sigma^2)t-1]\theta_2+\theta_1 t}$.

Before proving this theorem, we give an example to illustrate how to apply it in real situations. Consider the case where $r=0.09$, $\sigma=0.2$ and $t=1$, and we use $A_1=50$, $A_2=40$ and $\theta_2=20$. Then $\tilde{r}=0.07$, $\theta_1=1.09$, $t=1\in\left(0,\frac{A_1}{2\theta_1}\right)\equiv(0,1.19)$, and $k=4\in\left(\frac{A_2}{\theta_2},+\infty\right)\equiv(2,+\infty)$. Simple algebra yields that $C^+(\theta_1)\approx e^{6.39}$ and $C^-(\theta_1,\theta_2)\approx e^{16.59}$. Plugging them into (32) and (33), we can get discretization error bounds: $e_d^+\leq 2.53\times10^{-15}$ and $e_d^-\leq 6.80\times10^{-11}$. Hence, the discretization error for Asian option price is theoretically no more than $6.80\times10^{-11}\times e^{-rt}/t\approx 6.22\times10^{-11}$.

Proof of Theorem 2.1: First, since the scaling factor $X>S_0$, we have that

$$f(t,k)=XE\left(\frac{S_0}{X}A_t-e^{-k}\right)^+\leq XE\left(\frac{S_0}{X}A_t-\frac{S_0}{X}e^{-k}\right)^+=S_0E(A_t-e^{-k})^+,$$

where $k=\log(\frac{X}{Kt})$. On the other hand, we can bound A_t as follows

$$A_t=\int_0^t e^{(r-\frac{\sigma^2}{2})s+\sigma W(s)}ds\leq\int_0^t\exp\left(\tilde{r}t+\sigma\max_{\{0\leq s\leq t\}}W(s)\right)ds=t\exp\left\{\tilde{r}t+\sigma\max_{\{0\leq s\leq t\}}W(s)\right\},$$

where $\tilde{r} := \max(r - \frac{\sigma^2}{2}, 0)$. Since $\max_{\{0\leq s\leq t\}} W(s) =_d |W(t)|$, it follows that

$$\begin{aligned}
f(t,k) &\leq S_0 E\left[t\exp\{\tilde{r}t + \sigma|W(t)|\} - e^{-k}\right]^+ \\
&= S_0 E\left[\left(t\exp\{\tilde{r}t + \sigma W(t)\} - e^{-k}\right) I_{\{W(t)\geq 0,\, t\exp(\tilde{r}t+\sigma W(t))>e^{-k}\}}\right] \\
&\quad + S_0 E\left[\left(t\exp\{\tilde{r}t - \sigma W(t)\} - e^{-k}\right) I_{\{W(t)<0,\, t\exp(\tilde{r}t-\sigma W(t))>e^{-k}\}}\right] \\
&\leq S_0 E\left[\left(t\exp\{\tilde{r}t + \sigma W(t)\} - e^{-k}\right) I_{\{t\exp(\tilde{r}t+\sigma W(t))>e^{-k}\}}\right] \\
&\quad + S_0 E\left[\left(t\exp\{\tilde{r}t - \sigma W(t)\} - e^{-k}\right) I_{\{t\exp(\tilde{r}t-\sigma W(t))>e^{-k}\}}\right] \\
&= 2S_0 E\left[\left(t\exp\{\tilde{r}t + \sigma W(t)\} - e^{-k}\right) I_{\{t\exp(\tilde{r}t+\sigma W(t))>e^{-k}\}}\right],
\end{aligned}$$

via the symmetric property of standard Brownian motion.

Next, introduce a new measure $\bar{P}$ such that $\frac{d\bar{P}}{dP} = e^{Y_t - (\tilde{r}+\sigma^2/2)t}$, where $Y_t := \tilde{r}t + \sigma W(t)$. Then the change of measure leads to

$$\begin{aligned}
f(t,k) &\leq 2S_0 \bar{E}\left[t\exp\{\tilde{r}t + \sigma W_t\} I_{\{t\exp(\tilde{r}t+\sigma W(t))>e^{-k}\}} \times e^{-Y_t + (\tilde{r}+\sigma^2/2)t}\right] \\
&= 2S_0 t e^{(\tilde{r}+\sigma^2/2)t} \bar{P}\{t\exp(Y_t) > e^{-k}\} \\
&\leq 2S_0 e^{(1+\tilde{r}+\sigma^2/2)t} \bar{P}\{Y_t > -k - \log(t)\} \\
&= 2S_0 e^{\theta_1 t} \bar{P}\{Y_t > -k - \log(t)\},
\end{aligned}$$

where $\theta_1 := 1 + \tilde{r} + \sigma^2/2 > 0$ and the last inequality holds because $t < e^t$ for any $t > 0$.

Therefore, when $j_1 \geq 0$ and $j_2 \geq 0$, we have

$$\begin{aligned}
f((2j_1+1)t, (2j_2+1)k) &\leq 2S_0 e^{\theta_1(2j_1+1)t} \bar{P}\{Y_t > -(2j_2+1)k - \log((2j_1+1)t)\} \\
&\leq 2S_0 e^{\theta_1 t} e^{2\theta_1 j_1 t} = C^+(\theta_1) e^{2\theta_1 j_1 t},
\end{aligned}$$

where $C^+(\theta_1) := 2S_0 e^{\theta_1 t}$. On the other hand when $j_1 \geq 0$ and $j_2 \leq -1$, we have that for any $\theta_2 > 0$,

$$\begin{aligned}
f((2j_1+1)t, (2j_2+1)k) &\leq 2S_0 e^{\theta_1(2j_1+1)t} \bar{P}\{Y_t > -(2j_2+1)k - \log((2j_1+1)t)\} \\
&\leq 2S_0 e^{\theta_1(2j_1+1)t} \bar{P}\{Y_t > -j_2 k - \log((2j_1+1)t)\} \\
&\leq 2S_0 e^{\theta_1(2j_1+1)t} \bar{E}\left(e^{\theta_2 Y_t}\right) e^{\theta_2 j_2 k + \theta_2 \log((2j_1+1)t)},
\end{aligned}$$

where the second inequality holds as $j_2 \le -1$ and the third inequality comes from Markov's inequality. Since $x+1 \le e^x$ for any $x > -1$, we obtain that $e^{\theta_2 \log((2j_1+1)t)} \le e^{\theta_2[(2j_1+1)t-1]}$ and

$$\begin{aligned}
f((2j_1+1)t,(2j_2+1)k) &\le 2S_0 e^{\theta_1(2j_1+1)t}\bar{E}\left(e^{\theta_2 Y_t}\right)e^{\theta_2 j_2 k+\theta_2[(2j_1+1)t-1]} \\
&= 2S_0 e^{(\theta_1+\theta_2)t-\theta_2}\bar{E}\left(e^{\theta_2 Y_t}\right)\cdot e^{2(\theta_1+\theta_2)j_1 t}e^{\theta_2 j_2 k} \\
&= C^-(\theta_1,\theta_2)e^{\theta_2 j_2 k+2(\theta_1+\theta_2)j_1 t},
\end{aligned}$$

where

$$C^-(\theta_1,\theta_2) := 2S_0 e^{(\theta_1+\theta_2)t-\theta_2}\cdot\bar{E}\left(e^{\theta_2 Y_t}\right) = 2S_0 e^{(\theta_1+\theta_2)t-\theta_2}\cdot E\left(e^{(\theta_2+1)Y_t-(\tilde{r}+\sigma^2/2)t}\right).$$

Recall that $Y_t = \tilde{r}t+\sigma W(t)$. Simple algebra yields

$$C^-(\theta_1,\theta_2) = 2S_0 e^{t\sigma^2\theta_2^2/2+[(1+\tilde{r}+\sigma^2)t-1]\theta_2+\theta_1 t}.$$

If we have $t \in \left(0,\frac{A_1}{2\theta_1}\right)$, according to the definition of e_d^+ and the bound of function $f((2j_1+1)t,(2j_2+1)k)$ obtained above, we can get

$$\begin{aligned}
e_d^+ &\le \sum_{j_2=1}^{\infty}\sum_{j_1=0}^{\infty}e^{-(j_1A_1+j_2A_2)}C^+(\theta_1)e^{2\theta_1 j_1 t}+\sum_{j_1=1}^{\infty}e^{-j_1A_1}C^+(\theta_1)e^{2\theta_1 j_1 t} \\
&= C^+(\theta_1)\sum_{j_2=1}^{\infty}\sum_{j_1=0}^{\infty}e^{-(A_1-2\theta_1 t)j_1-j_2A_2}+C^+(\theta_1)\sum_{j_1=1}^{\infty}e^{-(A_1-2\theta_1 t)j_1} \\
&= C^+(\theta_1)\frac{e^{-A_2}}{1-e^{-A_2}}\frac{1}{1-e^{-(A_1-2\theta_1 t)}}+C^+(\theta_1)\frac{e^{-(A_1-2\theta_1 t)}}{1-e^{-(A_1-2\theta_1 t)}} \\
&= \frac{C^+(\theta_1)}{1-e^{-(A_1-2\theta_1 t)}}\left\{\frac{e^{-A_2}}{1-e^{-A_2}}+e^{-(A_1-2\theta_1 t)}\right\},
\end{aligned}$$

which is exactly (32).

For e_d^- we have for any $t \in \left(0,\frac{A_1}{2(\theta_1+\theta_2)}\right)$ and $k > \frac{A_2}{\theta_2}$,

$$\begin{aligned}
e_d^- &\le \sum_{j_2=-\infty}^{-1}\sum_{j_1=0}^{\infty}e^{-(j_1A_1+j_2A_2)}C^-(\theta_1,\theta_2)e^{\theta_2 j_2 k+2(\theta_1+\theta_2)j_1 t} \\
&= C^-(\theta_1,\theta_2)\sum_{j_1=0}^{\infty}e^{-(A_1-2(\theta_1+\theta_2)t)j_1}\sum_{j_2=-\infty}^{-1}e^{j_2(\theta_2 k-A_2)} \\
&= C^-(\theta_1,\theta_2)\frac{1}{1-e^{-(A_1-2(\theta_1+\theta_2)t)}}\frac{e^{-(\theta_2 k-A_2)}}{1-e^{-(\theta_2 k-A_2)}},
\end{aligned}$$

from which (33) is proved. □

	Double-Laplace Inversion Method			
	2 decimal	3 decimal	4 decimal	5 decimal
σ=0.05	(n_1, n_2)=$(35, 35)^*$	(35,35)	(35,35)	(35,35)
(CPU time)	(3.5 secs)**	(3.5 secs)	(3.5 secs)	(3.5 secs)
σ=0.1	(n_1, n_2)=(15;15)	(15;35)	(15;35)	(15;35)
(CPU time)	(1.2 secs)	(2.0 secs)	(2.0 secs)	(2.0 secs)
	Fourier-Laplace Inversion Method			
	2 decimal	3 decimal	4 decimal	5 decimal
σ=0.05	(n_l, n_f)=(35,115)	(35,135)	(35,195)	(35,195)
(CPU time)	(17.5 secs)	(20.1 secs)	(28.6 secs)	(28.6 secs)
σ=0.1	(n_l, n_f)=(15,55)	(15,75)	(15,95)	(15,95)
(CPU time)	(4.6 secs)	(6.4 secs)	(7.7 secs)	(7.7 secs)

Table 7: Comparison of the efficiency between our double-Laplace inversion and Fusai's Fourier-Laplace inversion method. In this table, (n_1, n_2)=$(35, 35)^*$ means that (n_1, n_2) should be set roughly at least (35,35) to achieve 2-decimal accuracy. (3.5 secs)** below (n_1, n_2)=$(35, 35)^*$ means that the corresponding CPU time is 3.5 seconds. The CPU times associated with Fusai's method are obtained using the code implemented by Matlab 7.1. All computations in Table 7 are conducted on an IBM laptop with a Pentium M 1.86GHz processor. We can see that to achieve the same accuracy, our method is more efficient than Fusai's.

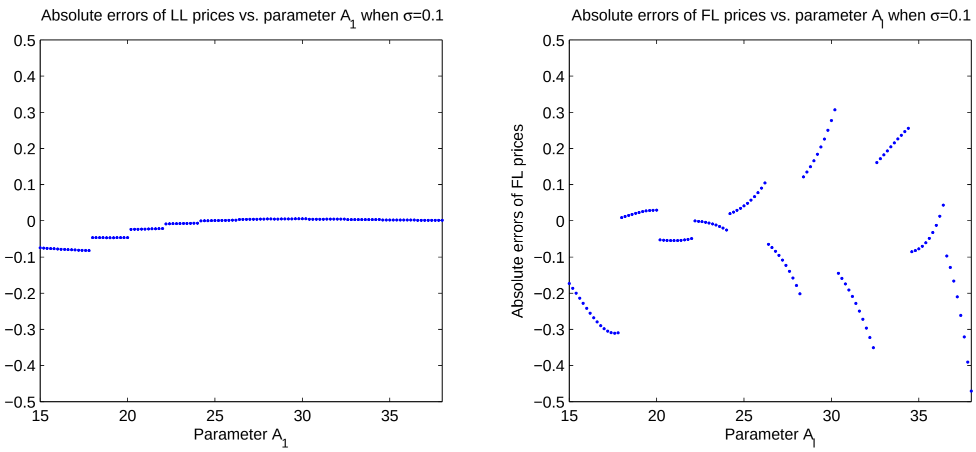

Figure 2: Comparison of the stability and accuracy between the double-Laplace inversion and the Fourier-Laplace inversion method in the case of low volatility $\sigma = 0.1$ under Kou's model, where other parameters are $K = 100$, $S_0 = 100$, $t = 1$, $\lambda = 3$, $r = 0.09$, $p_1 = 0.6$, $q_1 = 0.4$, and $\eta_1 = \theta_1 = 25$;. The absolute errors of LL prices (obtained by the double-Laplace inversion method) and FL prices (obtained by the Fourier-Laplace inversion method) are reported on the left and right graphs, respectively. Other parameters for the left graph are $n_1 = 35$, $n_2 = 55$, $A_2 = 40$ and $X = 5460$; while other parameters for the right graph are $n_l = 35$, $n_f = 135$ and $A_f = 40$. We can see that LL prices are quite accurate and stable when A_1 varies between [22.2,38], but FL prices are so desultory that we cannot decide which A_l we should choose.

3 Comparison with the Fourier and Laplace Inversion Algorithm

Under the BSM, Fusai [22] gave a closed-form for the Fourier-Laplace transform of Asian option price w.r.t. $k = \ln(\sigma^2 Kt/(4S_0))$ and $h = \sigma^2 t/4$, respectively. Despite some similarities, there are some key differences between our method and Fusai's method. (1) Our method performs better for low volatility, e.g., $\sigma = 0.05$ or 0.1. Specifically, for Fusai's method, a large number of terms are needed to do the Euler inversion to achieve a desired accuracy. In comparison, our algorithm in general requires far fewer terms in computation, especially for low volatility (See Table 7). This is mainly because we use the latest inversion method with a scaling factor in Petrella [37]. (2) Our method performs better in jump diffusion models. Specifically, the Fourier-Laplace inversion method seems unstable in the case of low volatilities under Kou's model because it is sensitive to parameters. To illustrate the sensitivity, we fix $A_f = 40$ and let A_l change from 15 to 38. The right panel of Figure 2 illustrates how the difference between the numerical result and the true value changes as A_l varies in the case of $\sigma = 0.1$ and $K = 100$. In comparison with the double-Laplace inversion on the left panel of Figure 2, FL prices seem unstable. Figure 2 seems to indicate that, with jumps, our double-Laplace inversion method works in a more stable manner than the Fourier-Laplace method, especially in the case of low volatility. (3) Under the BSM, we can derive a theoretical discretization error bound for the double-Laplace inversion. See Section 2 in the online supplement. (4) In terms of the main theoretical difference, note that the recursion used in Fusai's paper, namely (8), has no unique but infinitely many solutions. We spend considerable efforts to overcome this difficulty; see Section 3.