

e - c o m p a n i o n

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—“A New Stochastic Derivative Estimator for Discontinuous Payoff Functions with Application to Financial Derivatives” by Yongqiang Wang, Michael C. Fu, and Steven I. Marcus, *Operations Research*, <http://dx.doi.org/10.1287/opre.1110.1018>.

e-companion

EC.1. Proof of Theorem 1

Proof. Assume that we are interested in the gradient estimator for $\frac{\partial \bar{J}(\theta_1, \theta_2)}{\partial \theta_2}$. To indicate that the probability measure \mathbb{P}_Y depends on the parameter θ_2 , we rewrite it as $\mathbb{P}_Y(\theta_2)$ and write the Radon-Nikodym derivative $\frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}$ as $\frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2)$ in the following part of the proof.

For simplicity, denote

$$\begin{aligned}\varphi(\theta_2) &= E_{\hat{\mathbb{Q}}} \left[L((\theta_2 - a)Y + a, Z; \theta_2) 1\{Y > 1\} \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2) \right] \\ M(Y, Z; \theta_2) &= L((\theta_2 - a)Y + a, Z; \theta_2) 1\{Y > 1\}.\end{aligned}$$

Assume δ is sufficiently small, such that $\theta_2 + \delta \in B(\epsilon, \theta_2)$. We have

$$\frac{\varphi(\theta_2 + \delta) - \varphi(\theta_2)}{\delta} = E_{\hat{\mathbb{Q}}} \left[\frac{1}{\delta} \left(M(Y, Z; \theta_2 + \delta) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) - M(Y, Z; \theta_2) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2) \right) \right].$$

By A1, A2, and the mean value theorem, there exist $\eta_1 \in [\theta_2 - \delta, \theta_2 + \delta]$ and $\eta_2 \in [\theta_2 - \delta, \theta_2 + \delta]$, such that

$$\begin{aligned}& E_{\hat{\mathbb{Q}}} \left[\frac{1}{\delta} \left(M(Y, Z; \theta_2 + \delta) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) - M(Y, z; \theta_2) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2) \right) \right] \\ &= E_{\hat{\mathbb{Q}}} \left[\frac{1}{\delta} \left(M(Y, Z; \theta_2 + \delta) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) - M(Y, Z; \theta_2) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) \right) \right. \\ &\quad \left. + \frac{1}{\delta} \left(M(Y, Z; \theta_2) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) - M(Y, Z; \theta_2) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2) \right) \right] \\ &= E_{\hat{\mathbb{Q}}} \left[\frac{\partial}{\partial \theta_2} M(Y, Z; \tilde{\theta}_2) \Big|_{\tilde{\theta}_2 = \eta_1(Y, Z)} \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) + M(Y, Z; \theta_2) \frac{\partial}{\partial \tilde{\theta}_2} \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \tilde{\theta}_2) \Big|_{\tilde{\theta}_2 = \eta_2(Y, Z)} \right] \\ &= E_{\hat{\mathbb{Q}}} \left[\frac{\partial}{\partial \theta_2} M(Y, Z; \tilde{\theta}_2) \Big|_{\tilde{\theta}_2 = \eta_1(Y, Z)} \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) \right. \\ &\quad \left. + M(Y, Z; \theta_2) \frac{\partial}{\partial \tilde{\theta}_2} \ln \left(\frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \tilde{\theta}_2) \right) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \tilde{\theta}_2) \Big|_{\tilde{\theta}_2 = \eta_2(Y, Z)} \right] \tag{EC.1}\end{aligned}$$

Therefore, by A3 and (EC.1), we have

$$\lim_{\delta \rightarrow 0} E_{\hat{\mathbb{Q}}} \left[\frac{1}{\delta} \left(M(Y, Z; \theta_2 + \delta) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2 + \delta) - M(y, z; \theta_2) \frac{d\mathbb{P}_Y}{d\hat{\mathbb{Q}}}(Y, \theta_2) \right) \right]$$

$$\begin{aligned} &\leq \lim_{\delta \rightarrow 0} E_{\tilde{\mathbb{Q}}} \left[\sup_{\tilde{\theta}_2 \in B(\epsilon, \theta_2)} \left| \frac{\partial}{\partial \tilde{\theta}_2} M(Y, Z; \tilde{\theta}_2) \frac{d\mathbb{P}_Y}{d\tilde{\mathbb{Q}}}(Y, \theta_2 + \delta) \right|_{\tilde{\theta}_2 = \eta_1(Y, Z)} \right] \\ &\quad + \lim_{\delta \rightarrow 0} E_{\tilde{\mathbb{Q}}} \left[\sup_{\tilde{\theta}_2 \in B(\epsilon, \theta_2)} \left| M(Y, Z; \theta_2) \frac{\partial}{\partial \tilde{\theta}_2} \ln \frac{d\mathbb{P}_Y}{d\tilde{\mathbb{Q}}}(Y, \tilde{\theta}_2) \frac{d\mathbb{P}_Y}{d\tilde{\mathbb{Q}}}(Y, \tilde{\theta}_2) \right|_{\tilde{\theta}_2 = \eta_2(Y, Z)} \right] < \infty. \end{aligned}$$

By the dominated convergence theorem, we have

$$\frac{\partial}{\partial \theta_2} E_{\tilde{\mathbb{Q}}} \left[M(Y, Z; \theta_2) \frac{d\mathbb{P}_Y}{d\tilde{\mathbb{Q}}}(Y, \theta_2) \right] = E_{\tilde{\mathbb{Q}}} \left[\left(\frac{\partial}{\partial \theta_2} M(Y, Z; \theta_2) + M(Y, Z; \theta_2) \frac{\partial}{\partial \theta_2} \ln \frac{d\mathbb{P}_Y}{d\tilde{\mathbb{Q}}}(Y, \theta_2) \right) \frac{d\mathbb{P}_Y}{d\tilde{\mathbb{Q}}}(Y, \theta_2) \right].$$

Hence, the estimator (11) is unbiased. The theorem can be proved analogously when $\theta = \theta_{1i}$ for $i = 1, \dots, d$. \square

EC.2. Proof of Theorem 2

Proof. This theorem follows from an analogous argument as in the proof of Theorem 1. \square

EC.3. Gradient Estimators for American Option

The SLRIPA estimators for $E[J_T]$ with respect to parameters K , S_0 , σ , r , and D written in terms of original random variables are given as follows. To simplify the notation, we define $M = (\ln(S_{t_1^-} - D) - \ln \tilde{S}_0 - (r - \sigma^2/2)\tau_1)$.

1: $\theta = K$.

$$e^{-rt_1} \mathbf{1}\{S_{t_1^-} > s\} \mathbf{1}\{S_{t_1^-} > K\} - e^{-rT} \mathbf{1}\{S_{t_1^-} \leq s\} \mathbf{1}\{S_T > K\}.$$

2: $\theta = S_0$.

$$e^{-rt_1} \mathbf{1}\{S_{t_1^-} > s\} (S_{t_1^-} - K)^+ \frac{1}{\sigma^2 \tilde{S}_0 \tau_1} M + e^{-rT} \mathbf{1}\{S_{t_1^-} \leq s\} (S_T - K)^+ \frac{1}{\sigma^2 \tilde{S}_0 \tau_1} M.$$

3: $\theta = \sigma$

$$\begin{aligned} &e^{-rt_1} \mathbf{1}\{S_{t_1^-} > s\} (S_{t_1^-} - K)^+ \left[\frac{-1}{\sigma} + \frac{1}{\sigma^3 \tau_1} \left((\ln(S_{t_1^-} - D) - \ln \tilde{S}_0 - r\tau_1)^2 - \frac{\sigma^4 (\tau_1)^2}{4} \right) \right] \\ &+ e^{-rT} \mathbf{1}\{S_{t_1^-} \leq s\} \mathbf{1}\{S_T > K\} S_T (-\sigma\tau_2 + \sqrt{\tau_2} Z_2) \\ &+ e^{-rT} \mathbf{1}\{S_{t_1^-} \leq s\} (S_T - K)^+ \left[\frac{-1}{\sigma} + \frac{1}{\sigma^3 \tau_1} \left((\ln(S_{t_1^-} - D) - \ln \tilde{S}_0 - r\tau_1)^2 - \frac{\sigma^4 (\tau_1)^2}{4} \right) \right]. \end{aligned}$$

4: $\theta = r$

$$e^{-rt_1} 1\{S_{t_1^-} > s\} (T - t_1) (S_{t_1^-} - K)^+ + e^{-rt_1} 1\{S_{t_1^-} > s\} (S_{t_1^-} - K)^+ \frac{M}{\sigma^2} \\ + e^{-rT} 1\{S_{t_1^-} \leq s\} \tau_2 (S_T - K)^+ + e^{-rT} 1\{S_{t_1^-} \leq s\} (S_T - K)^+ \frac{M}{\sigma^2}.$$

5: $\theta = D$

$$e^{-rt_1} 1\{S_{t_1^-} > s\} 1\{S_{t_1^-} > K\} \left(-\frac{S_{t_1^-} - D}{s - D} + 1 \right) + e^{-rt_1} 1\{S_{t_1^-} > s\} (S_{t_1^-} - K)^+ \frac{M}{\sigma^2 \tau_1 (s - D)} \\ + e^{-rT} 1\{S_{t_1^-} \leq s\} 1\{S_T > K\} \frac{S_T}{-s + D} + e^{-rT} 1\{S_{t_1^-} \leq s\} (S_T - K)^+ \frac{M}{\sigma^2 \tau_1 (s - D)}.$$

EC.4. Derivative Estimate for Barrier Option

Here we give a detailed derivation of the derivative estimate for the barrier option considered in Section 4.2. The discounted payoff function is given by

$$J_B(S_m) = e^{-rT} (S_m - K) 1\{S_m \geq K\} \prod_{i=1}^m 1\{S_i \leq H\},$$

where S_i denotes the discretely monitored price S_{t_i} for $i = 1, \dots, m$ and $\{0 = t_0 < t_1 < \dots < t_m = T\}$.

Assume S_t follows a geometric Brownian motion, i.e., $S_t = S_0 e^{(r - \sigma^2/2)t + \sigma B_t}$, where r is the riskless rate, σ is the volatility parameter, and B_t is a standard Brownian motion. Suppose we are interested in estimating $\frac{\partial E[J_B(S_m)]}{\partial \theta}$, where θ could be S_0, r, σ , and H .

Now we derive SLRIPA derivative estimators. Note that $S_{i+1} = S_i e^{(r - \sigma^2/2)(t_{i+1} - t_i) + \sigma \sqrt{t_{i+1} - t_i} Z_{i+1}}$, where $\{Z_{i+1}\}$ are independent and identically distributed standard normal random variables. Define

$$Y = S_1/H, \quad X_{i+1} = e^{(r - \sigma^2/2)(t_{i+1} - t_i) + \sigma \sqrt{t_{i+1} - t_i} Z_{i+1}}, \quad \forall i = 1, \dots, m - 1.$$

It is easy to see that $S_1 = YH$ and $S_i = S_1 \prod_{k=2}^i X_k$ for $i > 1$; $S_i \leq H$ is equivalent to $Y \leq 1$ for $i = 1$, and $Y \prod_{k=2}^i X_k \leq 1$ for $i > 1$. Let f_Y denote the probability density function of Y , and let f_i denote the probability density function of X_i for $i = 2, \dots, m$. We have $f_Y(y) = \frac{1}{\sigma \sqrt{t_1}} n(d)$ and $f_i(x_i) = \frac{1}{x_i \sigma \sqrt{t_i - t_{i-1}}} n(d_i)$, where $d = \frac{\ln(Hy/S_0) - (r - \sigma^2/2)t_1}{\sigma \sqrt{t_1}}$, and $d_i = \frac{\ln(x_i) - (r - \sigma^2/2)(t_i - t_{i-1})}{\sigma \sqrt{t_i - t_{i-1}}}$. Hence we have

$$E[J_B(S_m)] = E \left[e^{-rT} (S_m - K) 1\{S_m \geq K\} \prod_{i=1}^m 1\{S_i \leq H\} \right]$$

$$\begin{aligned}
&= E \left[e^{-rT} (HY \prod_{k=2}^m X_k - K) 1\{HY \prod_{k=2}^m X_k \geq K\} 1\{Y \leq 1\} \left\{ \prod_{i=2}^m 1\{Y \prod_{k=2}^i X_k \leq 1\} \right\} \right] \\
&= \int e^{-rT} (Hy \prod_{k=2}^m x_k - K) 1\{Hy \prod_{k=2}^m x_k \geq K\} 1\{y \leq 1\} \left\{ \prod_{i=2}^m 1\{y \prod_{k=2}^i x_k \leq 1\} \right\} \\
&\quad \times f_Y(y) \prod_{k=2}^m f_k(x_k) dy dx_2 \dots dx_m. \tag{EC.2}
\end{aligned}$$

The first equality above is obtained by a change of variables. Note that in (EC.2), the integrand is continuous at $Hy \prod_{k=2}^m x_k = K$, and hence we can apply IPA to $e^{-rT}(Hy \prod_{k=2}^m x_k - K)$ and LR to $f_Y(y) \prod_{k=2}^m f_k(x_k)$ as in Section 2.2 to obtain the following SLRIPA estimator

$$\begin{aligned}
\mathcal{D}_\theta &= \frac{\partial e^{-rT} (HY \prod_{k=2}^m X_k - K)}{\partial \theta} 1\{HY \prod_{k=2}^m X_k \geq K\} 1\{Y \leq 1\} \left\{ \prod_{i=2}^m 1\{Y \prod_{k=2}^i X_k \leq 1\} \right\} \\
&\quad + e^{-rT} (HY \prod_{k=2}^m X_k - K) 1\{HY \prod_{k=2}^m X_k \geq K\} 1\{Y \leq 1\} \prod_{i=2}^m 1\{Y \prod_{k=2}^i X_k \leq 1\} \left(\frac{\partial \ln f_Y}{\partial \theta} + \sum_{i=1}^m \frac{\partial \ln f_i}{\partial \theta} \right).
\end{aligned}$$

For parameter $\theta = S_0$, the SLRIPA estimator for δ is given by

$$\begin{aligned}
\mathcal{D}_{S_0} &= e^{-rT} (HY \prod_{k=2}^m X_k - K) 1\{HY \prod_{k=2}^m X_k \geq K\} 1\{Y \leq 1\} \prod_{i=2}^m 1\{Y \prod_{k=2}^i X_k \leq 1\} \left(\frac{\partial \ln f_Y}{\partial S_0} \right) \\
&= e^{-rT} (HY \prod_{k=2}^m X_k - K) 1\{HY \prod_{k=2}^m X_k \geq K\} 1\{Y \leq 1\} \prod_{i=2}^m 1\{Y \prod_{k=2}^i X_k \leq 1\} \\
&\quad \times \left(\frac{\ln(HY) - \ln S_0 - (r - \sigma^2/2)t_1}{S_0 \sigma^2 t_1} \right).
\end{aligned}$$

Through a reverse change of variables, the SLRIPA derivative estimator for δ can be written in terms of the original random variables S_1, \dots, S_m as

$$\mathcal{D}_{S_0} = e^{-rT} (S_m - K) 1\{S_m \geq K\} \prod_{i=1}^m 1\{S_i \leq H\} \left(\frac{\ln S_1 - \ln S_0 - (r - \sigma^2/2)t_1}{S_0 \sigma^2 t_1} \right).$$

Similarly, the SLRIPA derivative estimator for $\frac{\partial E[J_B(S_m)]}{\partial H}$ is given by

$$\begin{aligned}
\mathcal{D}_H &= e^{-rT} \frac{S_m}{H} 1\{S_m \geq K\} \prod_{i=1}^m 1\{S_i \leq H\} + e^{-rT} (S_m - K) 1\{S_m \geq K\} \\
&\quad \times \prod_{i=1}^m 1\{S_i \leq H\} \left(-\frac{\ln S_1 - \ln S_0 - (r - \sigma^2/2)t_1}{H \sigma^2 t_1} \right).
\end{aligned}$$

EC.5. Proof of Theorem 3

Proof. The proof consists in finding a bound required by the dominated convergence theorem to justify the interchange of integration and differentiation. Note that $s \geq K$ and we have

$$\begin{aligned} E[S_{t_1^-}] &= E[\tilde{S}_0 e^{(r-\sigma^2/2)\tau_1 + \sigma\sqrt{\tau_1}Z_1} + D] < \infty, \\ E[S_T] &= E[\tilde{S}_0 e^{(r-\sigma^2/2)\tau_1 + \sigma\sqrt{\tau_1}Z_1} e^{(r-\sigma^2/2)(T-t_1) + \sigma\sqrt{T-t_1}Z_2}] < \infty. \end{aligned}$$

Therefore

$$E\left[e^{-rt_1} \sup_s \left\{1\{S_{t_1^-} > s\}1\{S_{t_1^-} - K > 0\} \frac{S_{t_1^-} - D}{s - D}\right\}\right] \leq E\left[e^{-rt_1} \frac{S_{t_1^-} - D}{K - D}\right] < \infty.$$

Similarly, we also have

$$\begin{aligned} &E\left[e^{-rt_1} \sup_s \left\{1\{S_{t_1^-} > s\}(S_{t_1^-} - K)^+ \frac{1}{\sigma^2\tau_1(s - D)} \left(\ln\left(\frac{S_{t_1^-} - D}{\tilde{S}_0}\right) - (r - \sigma^2/2)\tau_1\right)\right\}\right] \\ &\leq E\left[e^{-rt_1} (S_{t_1^-} - K)^+ \frac{1}{\sigma^2\tau_1(K - D)} \left(\ln\left(\frac{S_{t_1^-}}{\tilde{S}_0}\right) - (r - \sigma^2/2)\tau_1\right)\right] < \infty, \end{aligned}$$

and

$$E\left[e^{-rT} \sup_s \left\{1\{S_{t_1^-} \leq s\} \frac{S_T}{s - D}\right\}\right] \leq E\left[e^{-rT} \frac{S_T}{K - D}\right] < \infty.$$

We also have

$$\begin{aligned} &E\left[e^{-rT} \sup_s \left\{1\{S_{t_1^-} \leq s\}(S_T - K)^+ \frac{1}{\sigma^2\tau_1(s - D)} \left(\ln\left(\frac{S_{t_1^-} - D}{\tilde{S}_0}\right) - (r - \sigma^2/2)\tau_1\right)\right\}\right] \\ &\leq E\left[e^{-rT} (S_T - K)^+ \frac{1}{\sigma^2\tau_1(K - D)} \left(\ln\left(\frac{S_{t_1^-}}{\tilde{S}_0}\right) - (r - \sigma^2/2)\tau_1\right)\right] < \infty. \end{aligned}$$

Therefore condition A3(b) in Section 3 holds. Conditions A1 and A2 hold by the property of the payoff function J_T . The proof is completed by Theorem 1. \square

EC.6. The American Call Option with a Different Stock Price Model

Every m days, the continuous part of the stock price is given as: $\tilde{S}_t = \tilde{S}_0 e^{rt} X$, where $X \sim U(1 - \sigma\sqrt{t}, 1 + \sigma\sqrt{t})$ and $\sigma\sqrt{t} < 1$. Let $h_i(Z_i; S, t, r, \sigma) = S e^{rt} (1 + \sigma\sqrt{t}Z_i)$, where $\{Z_i\}$ are iid random variables distributed as $U(-1, 1)$. Define h_i as

$$h_i(S; t, r, \sigma) = S e^{rt} \left[\prod_{j=1}^i (1 + \sigma\sqrt{m}Z_j) \right] (1 + \sigma\sqrt{t - im}Z_{i+1})$$

for $t \in [im, (i+1)m]$. For simplicity, we will assume $\tau_1 = m$, $\tau_2 < m$, $T = \tau_1 + \tau_2$. Then

$$\begin{aligned} h_1^{-1}(y; \tilde{S}_0, \tau_1, r, \sigma) &= \frac{1}{\sigma\sqrt{\tau_1}} \left(\frac{y}{\tilde{S}_0 e^{r\tau_1}} - 1 \right); \\ \tilde{S}_0 &= S_0 - D e^{-r\tau_1}; \\ \tilde{S}_{t_1^-} &= h_0(\tilde{S}_0; \tau_1, r, \sigma) = \tilde{S}_0 e^{r\tau_1} (1 + \sigma\sqrt{\tau_1} Z_1); \\ S_{t_1^-} &= \tilde{S}_{t_1^-} + D = \tilde{S}_0 e^{r\tau_1} (1 + \sigma\sqrt{\tau_1} Z_1) + D; \\ \tilde{S}_T &= (S_{t_1^-} - D) e^{r\tau_2} (1 + \sigma\sqrt{T - \tau_1} Z_2). \end{aligned}$$

Let $\theta_1 = (r, \sigma, S_0, D, K)$ and $\theta_2 = s$. Note that $S_{t_1^-} \in [a, b]$, where $a = \tilde{S}_0 e^{r\tau_1} (1 - \sigma\sqrt{\tau_1}) + D$ and $b = \tilde{S}_0 e^{r\tau_1} (1 + \sigma\sqrt{\tau_1}) + D$, hence we can apply the SLRIPA estimator in Section 2.2.2. Define a new random variable

$$Y = \Gamma(S_{t_1^-}; \theta_1, \theta_2) = \frac{(S_{t_1^-} - a)(b - s)}{(b - S_{t_1^-})(s - a)}, \quad (\text{EC.3})$$

then

$$S_{t_1^-} = \Gamma^{-1}(Y; \theta_1, \theta_2) = \frac{bY(s - a) + a(b - s)}{Y(s - a) + (b - s)}.$$

Hence

$$\begin{aligned} P(Y \leq y) &= P\left(\frac{(S_{t_1^-} - a)(b - s)}{(b - S_{t_1^-})(s - a)} \leq y\right) = P\left(S_{t_1^-} \leq \Gamma^{-1}(y; \theta_1, \theta_2)\right) \\ &= P\left(Z_1 \leq h_1^{-1}(\Gamma^{-1}(y; \theta_1, \theta_2) - D; \tilde{S}_0, \tau_1, r, \sigma)\right) \\ &= \int_{-1}^{h_1^{-1}(\Gamma^{-1}(y; \theta_1, \theta_2) - D; \tilde{S}_0, \tau_1, r, \sigma)} f_1(z_1) dz_1, \end{aligned}$$

where $f_1(z_1) = 1/2$ for $z_1 \in (-1, 1)$, so the probability density function of Y is

$$\begin{aligned} f_Y(y) &= \frac{\partial P(Y \leq y)}{\partial y} = \frac{1}{2} \frac{\partial h_1^{-1}(\Gamma^{-1}(y; \theta_1, \theta_2) - D; \tilde{S}_0, \tau_1, r, \sigma)}{\partial y} \\ &= \frac{1}{2\sigma\sqrt{\tau_1}\tilde{S}_0 e^{r\tau_1}} \frac{(b - a)(b - s)(s - a)}{(y(s - a) + (b - s))^2}. \end{aligned}$$

The payoff function for $E[\hat{J}_T^1]$ is $L(S_{t_1^-}; \theta_1, \theta_2) = (S_{t_1^-} - K)^+$, and by the analysis in Section 2.2, the SLRIPA estimator (9) for $\frac{\partial E[\hat{J}_T^1]}{\partial \theta}$ is given by:

$$1\{Y > 1\}1\{\Gamma^{-1}(Y; \theta_1, \theta_2) > K\} \frac{\partial(\Gamma^{-1}(Y; \theta_1, \theta_2) - K)}{\partial \theta} e^{r(T - \tau_1)}$$

$$+ 1\{Y > 1\}e^{r(T-\tau_1)}(\Gamma^{-1}(Y; \theta_1, \theta_2) - K)^+ \frac{\partial \ln f_Y(Y)}{\partial \theta}.$$

Consider estimating the derivative of $E[\hat{J}_T^2]$ with respect to θ . Note that $S_{t_1^-} \leq s$ is equivalent to $\psi(Y) = -Y + 1 \geq 0$, and the payoff function is $L(S_{t_1^-}, Z_2; \theta_1, \theta_2) = [h(Z_2; S_{t_1^-} - D, \tau_2, r, \sigma) - K]^+$. By the analysis in Section 2.2, then the SLRIPA estimator (9) for $\frac{\partial E[\hat{J}_T^2]}{\partial \theta}$ is given by

$$1\{Y \leq 1\} \frac{\partial(h_2(Z_2; \Gamma^{-1}(Y; \theta_1, \theta_2) - D, \tau_2, r, \sigma) - K)^+}{\partial \theta} \\ + 1\{Y \leq 1\}(h_2(Z_2; \Gamma^{-1}(Y; \theta_1, \theta_2) - D, \tau_2, r, \sigma) - K)^+ \frac{\partial \ln f_Y(Y)}{\partial \theta}.$$

So the full SLRIPA estimator for $E[J_T]$ with respect to θ is given by

$$e^{-r\tau_1} 1\{Y > 1\} 1\{\Gamma^{-1}(Y; \theta_1, \theta_2) > K\} \frac{\partial(\Gamma^{-1}(Y; \theta_1, \theta_2) - K)}{\partial \theta} \\ + e^{-r\tau_1} 1\{Y > 1\} (\Gamma^{-1}(Y; \theta_1, \theta_2) - K)^+ \frac{\partial \ln f_Y(Y)}{\partial \theta} \\ + e^{-rT} 1\{Y \leq 1\} \frac{\partial(h_2(Z_2; \Gamma^{-1}(Y; \theta_1, \theta_2) - D, \tau_2, r, \sigma) - K)^+}{\partial \theta} \\ + e^{-rT} 1\{Y \leq 1\} (h_2(Z_2; \Gamma^{-1}(Y; \theta_1, \theta_2) - D, \tau_2, r, \sigma) - K)^+ \frac{\partial \ln f_Y(Y)}{\partial \theta} \\ - [e^{-r\tau_1} 1\{Y > 1\} (\Gamma^{-1}(Y; \theta_1, \theta_2) - K)^+ \\ + e^{-rT} 1\{Y \leq 1\} (h_2(Z_2; \Gamma^{-1}(Y; \theta_1, \theta_2) - D, \tau_2, r, \sigma) - K)^+] \frac{\partial(rT)}{\partial \theta}. \quad (\text{EC.4})$$

EC.7. The American Call Option with Multiple Ex-Dividend Dates

We now extend the results in Section 4.3 to the cases where there are multiple ex-dividend dates.

To simplify the problem, we denote $d_j = \sum_{i=j}^{\eta(T)} D_i \exp\left(-r \sum_{k=j+1}^i \tau_k\right)$, $j = 1, \dots, \eta(T)$.

Since the support of the distribution of the random variables $S_{t_i^-}$, $i = 1, \dots, \eta(T)$ depends on the parameters of interest and the indicator function also contains parameters, we do the following change of random variables to move the parameters out of the indicator functions and make the support independent of the parameters of interests.

$$Y_1 = g_1(S_{t_1^-}) = \frac{S_{t_1^-} - d_1}{s_1 - d_1}; \\ \vdots \\ Y_{\eta(T)} = g_{\eta(T)}(S_{t_{\eta(T)}^-}) = \frac{S_{t_{\eta(T)}^-} - d_{\eta(T)}}{s_{\eta(T)} - d_{\eta(T)}};$$

$$Y_{\eta(T)+1} = g_{\eta(T)+1}(S_T) = S_T.$$

Therefore we have

$$\begin{aligned} \frac{\partial E[\hat{J}_T]}{\partial \theta} &= \frac{\partial}{\partial \theta} E \left[\sum_{i=1}^{\eta(T)} \left(\prod_{j=1}^{i-1} 1\{S_{t_j^-} \leq s_j\} 1\{S_{t_i^-} > s_i\} (S_{t_i^-} - K)^+ e^{r(T-t_i)} \right) \right. \\ &\quad \left. + \prod_{j=1}^{\eta(T)} 1\{S_{t_j^-} \leq s_j\} (S_T - K)^+ \right] \\ &= \frac{\partial}{\partial \theta} \left\{ \sum_{i=1}^{\eta(T)} \left(\int_0^\infty \prod_{j=1}^{i-1} 1\{y_j \leq 1\} 1\{y_i > 1\} (y_i(s_i - d_i) + d_i)^+ e^{r(T-t_i)} \right. \right. \\ &\quad \times f_1(y_1) f_{2;1}(y_2; y_1) \cdots f_{i;i-1}(y_i; y_{i-1}) dy_1 \cdots dy_i \Big) \\ &\quad \left. + \int_0^\infty \prod_{j=1}^{\eta(T)} 1\{y_j \leq 1\} (y_{\eta(T)+1} - K)^+ f_1(y_1) f_{2;1}(y_2; y_1) \cdots f_{\eta(T)+1;\eta(T)}(y_{\eta(T)+1}; y_{\eta(T)}) \right. \\ &\quad \left. \times dy_1 \cdots dy_{\eta(T)+1} \right\}, \end{aligned}$$

where f_1 is the probability density function of the random variable Y_1 , and $f_{i;i-1}$ is the probability density function of the random variable Y_i given Y_{i-1} , which can be easily obtained. Assuming interchangeability of integral and derivative, we have $\frac{\partial E[\hat{J}_T]}{\partial \theta} = \frac{\partial E[\hat{J}_T^1]}{\partial \theta} + \frac{\partial E[\hat{J}_T^2]}{\partial \theta}$, where

$$\begin{aligned} \frac{\partial E[\hat{J}_T^1]}{\partial \theta} &= \sum_{i=1}^{\eta(T)} \left(\int_0^\infty \prod_{j=1}^{i-1} 1\{y_j \leq 1\} 1\{y_i > 1\} \frac{\partial (y_i(s_i - d_i) + d_i)^+ e^{r(T-t_i)}}{\partial \theta} \right. \\ &\quad \times f_1(y_1) \cdots f_{i;i-1}(y_i; y_{i-1}) dy_1 \cdots dy_i \Big) \\ &\quad + \sum_{i=1}^{\eta(T)} \left(\int_0^\infty \prod_{j=1}^{i-1} 1\{y_j \leq 1\} 1\{y_i > 1\} (y_i(s_i - d_i) + d_i)^+ e^{r(T-t_i)} \right. \\ &\quad \times \left(\sum_{k=1}^i \frac{\partial \ln f_{i;i-1}(y_i; y_{i-1})}{\partial \theta} \right) f_1(y_1) f_{2;1}(y_2; y_1) \cdots f_{i;i-1}(y_i; y_{i-1}) dy_1 \cdots dy_i \Big), \\ \frac{\partial E[\hat{J}_T^2]}{\partial \theta} &= \int_0^\infty \prod_{j=1}^{\eta(T)} 1\{y_j \leq 1\} \frac{\partial (y_{\eta(T)+1} - K)^+}{\partial \theta} \\ &\quad \times f_1(y_1) f_{2;1}(y_2; y_1) \cdots f_{\eta(T)+1;\eta(T)}(y_{\eta(T)+1}; y_{\eta(T)}) dy_1 \cdots dy_{\eta(T)+1} \\ &\quad + \int_0^\infty \prod_{j=1}^{\eta(T)} 1\{y_j \leq 1\} (y_{\eta(T)+1} - K)^+ \left(\sum_{i=1}^{\eta(T)+1} \frac{\partial \ln f_{i;i-1}(y_i; y_{i-1})}{\partial \theta} \right) \\ &\quad \times f_1(y_1) f_{2;1}(y_2; y_1) \cdots f_{\eta(T)+1;\eta(T)}(y_{\eta(T)+1}; y_{\eta(T)}) dy_1 \cdots dy_{\eta(T)+1}. \end{aligned}$$

The SLRIPA estimator for $\frac{\partial E[J_T]}{\partial \theta}$ is given by

$$\sum_{i=1}^{\eta(T)} \left(e^{-rt_i} \prod_{j=1}^{i-1} 1\{Y_j \leq 1\} 1\{Y_i > 1\} \frac{\partial (Y_i(s_i - d_i) + d_i)^+}{\partial \theta} \right)$$

$$\begin{aligned}
& + \sum_{i=1}^{\eta(T)} \left(e^{-rt_i} \prod_{j=1}^{i-1} 1\{Y_j \leq 1\} 1\{Y_i > 1\} (Y_i(s_i - d_i) + d_i) + \left(\sum_{k=1}^i \frac{\partial \ln f_{i;i-1}(Y_i; Y_{i-1})}{\partial \theta} \right) \right) \\
& + e^{-rT} \prod_{j=1}^{\eta(T)} 1\{Y_j \leq 1\} \frac{\partial (Y_{\eta(T)+1} - K)^+}{\partial \theta} \\
& + e^{-rT} \prod_{j=1}^{\eta(T)} 1\{Y_j \leq 1\} (Y_{\eta(T)+1} - K)^+ \left(\sum_{i=1}^{\eta(T)+1} \frac{\partial \ln f_{i;i-1}(Y_i; Y_{i-1})}{\partial \theta} \right) \\
& - \left(\sum_{i=1}^{\eta(T)} e^{-rt_i} \prod_{j=1}^{i-1} 1\{Y_j \leq 1\} 1\{Y_i > 1\} (Y_i(s_i - d_i) - K)^+ + e^{-rT} \prod_{j=1}^{\eta(T)} 1\{Y_j \leq 1\} (Y_{\eta(T)+1} - K)^+ \right) \frac{\partial (rT)}{\partial \theta}.
\end{aligned} \tag{EC.5}$$

REMARK EC.1. Although the estimator (EC.5) is written in terms of $\{Y_i\}$, it can be rewritten in terms of $\{S_{t_i^-}\}$ by applying $Y_i = g_i(S_{t_i^-})$.