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On-line Appendix for "Cargo Capacity Management with Allotments and Spot Market Demand"

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Online Appendix

EC.1. Proofs

Proof of Proposition 1

We show the result by using induction over the time periods. We begin by showing the result for time period $\tau+1$. Rearranging the objective function of problem (9)-(10), we observe that this problem is equivalent to maximizing

$$\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} x_i\Big\{[\hat{C}^a_{ij}-C^a_{ij}-\lambda_j\,V^a_{ij}]\,z^a_{ij}-\hat{C}^a_{ij}]\Big\}+\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}}\sum_{l=1}^{n_{jk}}\Big\{[\hat{C}^s_{jkl}-C^s_{jkl}-\lambda_j\,V^s_{jkl}]\,z^s_{jkl}-\hat{C}^s_{jkl}\Big\}+\sum_{j\in\mathcal{J}}\lambda_j\,\bar{V}_j$$

subject to (3)-(4). By constraints (3), the optimal value of the decision variable z^a_{ij} is one if $\hat{C}^a_{ij}\geq C^a_{ij}+\lambda_j\,V^a_{ij}$ and zero otherwise. By constraints (4), the optimal value of the decision variable z^s_{jkl} is one if $\hat{C}^s_{jk}\geq C^s_{jk}+\lambda_j\,V^s_{jkl}$ and zero otherwise. Thus, problem (9)-(10) has the optimal objective value

$$\begin{aligned}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} x_i\Big\{[\hat{C}^a_{ij}-C^a_{ij}-\lambda_j\,V^a_{ij}]^+-\hat{C}^a_{ij}]\Big\}+\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}}\sum_{l=1}^{n_{jk}}\Big\{[\hat{C}^s_{jkl}-C^s_{jkl}-\lambda_j\,V^s_{jkl}]^+-\hat{C}^s_{jkl}\Big\}+\sum_{j\in\mathcal{J}}\lambda_j\,\bar{V}_j\\ =-\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} x_i\,\min(C^a_{ij}+\lambda_j\,V^a_{ij},\hat{C}^a_{ij})-\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}}\sum_{l=1}^{n_{jk}}\min(C^s_{jkl}+\lambda_j\,V^s_{jkl},\hat{C}^s_{jkl})+\sum_{j\in\mathcal{J}}\lambda_j\,\bar{V}_j.\end{aligned}$$

Taking expectations in the expression above, noting the definitions of $B^a_{ij}(\lambda_j)$ and $B^s_{jk}(\lambda_j)$ and the fact that $\mathbb{E}\{\min(C^s_{jkl}+\lambda_j\,V^s_{jkl},\hat{C}^s_{jkl})\}=\mathbb{E}\{\min(C^s_{jk}+\lambda_j\,V^s_{jk},\hat{C}^s_{jk})\}=B^s_{jk}(\lambda_j)$, we obtain

$$\mathbb{E}\{\tilde{\Gamma}(x,n,\lambda,U)\}=-\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} x_i\,B^a_{ij}(\lambda_j)-\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} n_{jk}\,B^s_{jk}(\lambda_j)+\sum_{j\in\mathcal{J}}\lambda_j\,\bar{V}_j$$

and the result holds for time period $\tau+1$.

Assuming that the result holds for time period $t+1$, noting that $\tilde{J}_{t+1}(x,n+e_{jk},\lambda)-\tilde{J}_{t+1}(x,n,\lambda)=-B^s_{jk}(\lambda_j)$ and plugging this expression in (11), we obtain

$$\begin{aligned}\tilde{J}_t(x,n,\lambda)=\max_{\mathcal{S}\subseteq\mathcal{J}\times\mathcal{K}}&\left\{\sum_{(j,k)\in\mathcal{S}} P_{jkt}(\mathcal{S})\big[\mathbb{E}\{R^s_{jk}\}-B^s_{jk}(\lambda_j)]\right\}\\&+\sum_{j\in\mathcal{J}}\lambda_j\,\bar{V}_j-\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} x_i\,B^a_{ij}(\lambda_j)-\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}} B^s_{jk}(\lambda_j)\,n_{jk}\\&+\sum_{t'=t+1}^{\tau}\max_{\mathcal{S}\subseteq\mathcal{J}\times\mathcal{K}}\left\{\sum_{(j,k)\in\mathcal{S}} P_{jkt'}(\mathcal{S})\,\big[\mathbb{E}\{R^s_{jk}\}-B^s_{jk}(\lambda_j)\big]\right\}\end{aligned}$$

and the result holds at time period t. This concludes the proof.

Proof of Proposition 2

The proof is immediate by the relaxation argument since $\tilde{\Gamma}(x,n,\lambda,U)$ is an upper bound for $\Gamma(x,n,U)$ as long as $\lambda \geq 0$. The optimality equations in (11) is identical to the one in (5), except that its boundary condition is an upper bound on the boundary condition of the optimality equation in (5). Therefore, the value functions computed through the optimality equation in (11) are upper bounds on those computed through the optimality equation in (5).

Proof of Proposition 3

Suppose the algorithm terminated. Consider (x^S, y^S) and λ^{S+1} at termination. Since λ^{S+1} is within $\frac{\epsilon}{3}$ of the optimum, that is

$$\tilde{J}_1(x^S, \bar{0}, \lambda^{S+1}) \leq \min_{\lambda \geq 0} \ \tilde{J}_1(x^S, \bar{0}, \lambda) + \frac{\epsilon}{3},$$

and $\tilde{J}_1(x^S, \bar{0}, \lambda^{S+1}) \geq y^S - \frac{\epsilon}{3}$ by the termination criterion, it follows that

$$y^S - \frac{2\epsilon}{3} \leq \min_{\lambda \geq 0} \ \tilde{J}_1(x^S, \bar{0}, \lambda),$$

which can be written as

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}_i} x_i^S \, B_{ij}^a(\lambda) + y^S - \frac{2\epsilon}{3} \leq \sum_{j \in \mathcal{J}} \lambda_j \, \bar{V}_j + \sum_{t=1}^{\tau} \Phi_t(\lambda) \qquad \text{for all } \lambda \geq 0.$$

That is, $(x^S, y^S - 2\epsilon/3)$ is feasible for problem (P). Since the optimal objective value of problem (R-$\{\lambda^1, \ldots, \lambda^S\}$) is within $\epsilon/3$ of the objective value provided by the solution (x^S, y^S), the optimal objective value of problem (R-$\{\lambda^1, \ldots, \lambda^S\}$) is within ϵ of the objective value provided by the solution $(x^S, y^S - 2\epsilon/3)$. Since problem (R-$\{\lambda^1, \ldots, \lambda^S\}$) is a relaxation of problem (P), this implies that $(x^S, y^S - 2\epsilon/3)$ has a value within ϵ of the optimal solution to problem (P).

To prove finite termination, suppose on the contrary that the algorithm does not terminate in a finite number of steps. We observe that $y^S = \min_{s=1,\ldots,S} \tilde{J}_1(x^S, \bar{0}, \lambda^s)$ at any iteration of the algorithm. Moreover, since there is only a finite number of feasible values for the binary x-variables, at least one of them must be visited more than once. That is $x^S = x^{s'}$ for some $s' < S$. Since the algorithm does not terminate at step S, it must be that

$$y^S > \tilde{J}_1(x^S, \bar{0}, \lambda^{S+1}) + \frac{\epsilon}{3} \geq \min_{\lambda \geq 0} \ \tilde{J}_1(x^S, \bar{0}, \lambda) + \frac{\epsilon}{3}.$$

On the other hand, since $\lambda^{s'}$ is $\epsilon/3$-optimal at $x^{s'} = x^S$, the fact that $y^S = \min_{s=1,\dots,S} \tilde{J}_1(x^S, \bar{0}, \lambda^s)$ implies

$$y^S \le \tilde{J}_1(x^{s'}, \bar{0}, \lambda^{s'}) \le \min_{\lambda \ge 0} \; \tilde{J}_1(x^{s'}, \bar{0}, \lambda) + \frac{\epsilon}{3},$$

which is a contradiction.

Proof of Proposition 4

We begin by defining the partial ordering $\succeq_{jk}$ to write $\lambda^+ \succeq_{jk} \lambda^-$ when $B^s_{jk}(\lambda^+) \ge B^s_{jk}(\lambda^-)$. To prove the proposition, we equivalently show that $\lambda^*(x, n+e_{jk}) \succeq_{jk} \lambda^*(x,n)$. For notational brevity, we let $\lambda^+ = \lambda^*(x, n+e_{jk})$ and $\lambda^0 = \lambda^*(x,n)$. We choose λ^- such that $\lambda^0 \succeq_{jk} \lambda^-$. By (12), we have

$$\begin{aligned}\tilde{J}_t(x,n,\lambda^0) - \tilde{J}_t(x,n+e_{jk},\lambda^0) &= B^s_{jk}(\lambda^0)\\ &\ge B^s_{jk}(\lambda^-) = \tilde{J}_t(x,n,\lambda^-) - \tilde{J}_t(x,n+e_{jk},\lambda^-),\end{aligned}$$

where the inequality follows from the fact that $\lambda^0 \succeq_{jk} \lambda^-$. Arranging the terms, the expression above can be written as

$$\tilde{J}_t(x,n+e_{jk},\lambda^0) \le \tilde{J}_t(x,n+e_{jk},\lambda^-) + \tilde{J}_t(x,n,\lambda^0) - \tilde{J}_t(x,n,\lambda^-). \tag{EC.1}$$

Since λ^0 is an optimal solution to problem $\min_{\lambda \ge 0} \tilde{J}_t(x,n,\lambda)$, we have $\tilde{J}_t(n,x,\lambda^0) \le \tilde{J}_t(n,x,\lambda^-)$. In this case, by (EC.1), we have

$$\tilde{J}_t(x,n+e_{jk},\lambda^0) \le \tilde{J}_t(x,n+e_{jk},\lambda^-) \tag{EC.2}$$

for all λ^- such that $\lambda^0 \succeq_{jk} \lambda^-$. This result immediately implies that $\lambda^+ \succeq_{jk} \lambda^0$. In particular, if, on the contrary, $\lambda^+ \not\succeq_{jk} \lambda^0$, then we have $B^s_{jk}(\lambda^0_j) > B^s_{jk}(\lambda^+_j)$. This implies that $\lambda^0 \succeq_{jk} \lambda^+$, in which case, we can use (EC.2) with $\lambda^- = \lambda^+$ to obtain $\tilde{J}_t(x,n+e_{jk},\lambda^0) \le \tilde{J}_t(x,n+e_{jk},\lambda^+)$. This inequality and the fact that $B^s_{jk}(\lambda^0_j) > B^s_{jk}(\lambda^+_j)$ contradict the fact that (λ^+) is an optimal solution to the problem $\min_{\lambda \ge 0} \tilde{J}_t(x,n+e_{jk},\lambda)$ that is largest according to the partial ordering $\succeq_{jk}$.

EC.2. Elements of Experimental Setup

In this section, we describe further elements of experimental setup, including the distributions used in the random sampling of data, the volume and weight distributions of allotments and spot market cargo, as well as the details of the revenue, cost, and penalty models for the cargo.

We start by describing the spot market cargo types and the model for their volumes, weights, revenues, costs and penalties. As already mentioned, the weight of each cargo type is assumed to be constant in the experiments. The value of constant weight V^s_{2k} of each type k is randomly sampled from the exponential distribution with the mean of 0.25 truncated to the interval $[0.05, 3.5]$ (all values in tons). The reference density δ_k is sampled from the normal distribution with the mean 0.167 and the standard deviation 0.04 truncated to the interval $[0.03, 0.4]$ (all values in ton/m^3). The volume V^s_{1k} of cargo type k is then a lognormal random variable with the mean V^s_{2k}/δ_k and the coefficient of variation θ^s (common for all types and varied in a controlled fashion). The revenue of type k cargo (payment by a customer to the airline) is computed as $R_k = \rho_k \max\{V^s_{2k}, V^s_{1k}/\gamma\}$, where $\gamma = 6$ m^3/ton is an industry-standard constant and ρ_k is the shipping rate randomly sampled from the interval $[1500, 3000]$ (in \$/ton). The regular shipping cost and penalty of type k are proportional to the shipping rate and are computed in the experiments as $C^s_k = \rho_k(0.04V^s_{1k} + 0.4V^s_{2k})$ and $\hat{C}^s_k = \rho_k(0.15V^s_{1k} + 1.5V^s_{2k})$ (a lower scale in volume is selected primarily to adjust for the units of measurement).

The demand for each cargo type on the spot market is subject to consumer choice among the flights departing within the next two weeks. There are a total of 8 flights within this time span. The choice between flights is described by a multinomial logit model (EC.3) in which the preference weights v_{jkt} depend on the time of the flight j within a week as well as on the number of weeks before the week of flight departure (this number can be 0, 1 or 2). The required $3 \times 4 = 12$ preference weights are generated as $0.9e^{0.1Z}$ where Z is a standard normal random variable. The no-purchase option has a constant preference weight $v_{0kt} = 0.1$.

Finally, we sample the maximum allotted weight W_i of each bid i from the normal distribution with mean 5 and standard deviation 2 tons (the same for all flights included in the bid). Utilization of capacity by allotments is described by a hierarchical model. The utilized fraction F_{ij} of the allotted weight of bid i on flight j is distributed as normal with mean 0.9 and standard deviation 1/9.7 truncated to the interval [0,1] (the parameters are selected so that utilized weight follows a practical rule of thumb – at least 80% weight utilization at least 80% of the time). Conditionally on utilized weight $V^a_{2ij} = F_{ij}W_i$, utilized volume V^a_{1ij} is distributed as a lognormal random variable with the mean V^a_{2ij}/γ and the coefficient of variation θ^a (controlled in the experiments). Allotment i revenues from flight j are $R_{ij} = \rho_i \max\{V^a_{2ij}, V^a_{1ij}/\gamma\}$. The cost and penalty models are similar to those of spot market cargo (although the costs are potentially lower and penalties are potentially higher, both unrelated to the rates). In particular, we let $C^a_{ij} = 25V^a_{1ij} + 250V^a_{2ij}$ and $\hat{C}^a_{ij} = 150V^a_{1ij} + 1500V^a_{2ij}$.

EC.3. Uncertainty Structure, Its Implications for Independence Between Flights and Possible Generalizations

In this section, we provide a detailed discussions of the uncertainty structure in our model, and, in particular, its independence assumptions. There are three source of uncertainty in the model: (1) capacity utilizations, payments and costs associated with allotments, (2) capacity utilizations, payments and costs associated with accepted spot market bookings, and (3) spot market demand resulting from consumer choice behavior. In each case, we choose to keep specifications of distributions general as long as the same modelling and analytic approach can be used. Moreover, the general form of distributions, particularly for payment and cost structures, has the advantage of capturing the industry practices which vary across different cargo carriers. The essence of independence assumptions can be summarized by two aspects: independence of distributional specifications and statistical independence.

By independence of specifications, we mean that distributions depend only on the indices of the object they describe: allotment quotes indexed by atomic bid/flight combinations, spot market cargo bookings by flight and cargo type, and spot market choice probabilities by flight, type and time. This excludes any dependence of input distributions on the allotment decisions, the current number of spot market bookings or cargo loading decisions.

The statistical independence is used in its usual sense: allotment capacity utilizations/revenues for each atomic bid/flight combination, spot market cargo capacity utilizations/revenues for each flight/cargo type combination, and spot market demand realizations for each decision period are mutually independent. The immediate implication of statistical independence is that probability measure describing future behavior of the system does not depend on any past realizations of uncertain quantities. Moreover, since all bookings which cannot be accommodated on their scheduled flight are outsourced, this probability measure also does not depend on loading decisions made at departure times of the flights.

The most important implication of independence assumption is that the state of the system can be completely described by the allotment decisions and the current numbers of spot market bookings by flight/type. Since the state is not affected by loading decisions, and the objectives/feasible sets of cargo loading problems for each flight are independent, it is irrelevant when the departures occur. This explains why it is appropriate to assume that all departures occur at the end and all loading decisions are made at the same time. In practice, the loading problems for each flight are solved on on-going basis but without affecting the system state. This does not affect the allotment selection problem since all of the flights occur in the future. For the spot market booking control, the state components which apply to departed flights are irrelevant and can be ignored in the dynamic programming formulation. In the practical approximation of booking control (described in §6 and §EC.5), we can ignore the flights which have already departed.

Since the ability to treat flight departures in independent fashion is clearly very attractive from an analytic point of view and important in our approach, it is also interesting how independence assumptions can be relaxed without affecting the independence of the flights. One concern is that allotment utilization/revenues and spot market demand usually depend on macroeconomic environment characterized by such factors as interest rates, consumer confidence, and others. It is possible to extend our approach by introducing an uncertain but observable discrete *states of the environment* with Markovian dynamics. (The notion of a "fluctuating demand environment" of this form is discussed, for example, by Song and Zipkin (1993).) All steps in the analysis and computational approach go through with minimal changes. Another concern is a likely dependence of spot market demand on allotment decisions. In terms of our model, this means dependence of the booking probabilities $P_{jkt}(\cdot)$ on allotment decisions. While handling general dependence is difficult, we can extend our approach to the case where the dependence of $P_{jkt}(\cdot)$ on x_i's is *linear*. This is reasonable, for example, if each accepted spot market booking is estimated to cause a given reduction in booking probabilities:

$$P_{jkt}(x, \mathcal{S}) = P^0_{jkt}(\mathcal{S}) - \sum_{i \in \mathcal{I}} x_i \, \pi^i_{jkt}(\mathcal{S}),$$

where $P^0_{jkt}(\mathcal{S})$ is a booking probability when no bids are granted, and $\pi^i_{jkt}(\mathcal{S})$ is a reduction in the booking probability if atomic bid $i \in \mathcal{I}$ is granted. With the booking probabilities of this form, our approach is still applicable, because the Lagrangian bound remains convex in λ for fixed x and linear in x for fixed λ.

EC.4. Perfect Hindsight Upper Bound for the Case of Related Departures

In this section, we relax the assumption of independent loading decisions for different flights and allow the possibility to rebook a shipment, at a cost, for a later flight. The cost model for the shipments is generalized as follows. Let the shipping cost of a type k package originally booked for flight j but shipped on flight j' be described by a random variable $\tilde{C}^s_{jj'k}$. If it is not possible to shift cargo from flight j to j', then we can assume that $\tilde{C}^s_{jj'k}$ takes a prohibitively large value. Similarly, let the shipping cost of allotment i allocated to flight j but shipped on flight j' be $\tilde{C}^a_{ijj'}$. Finally, let flight ϕ be a dummy flight such that shipping on this flight represents outsourcing and outsourcing costs are represented by $\tilde{C}^s_{j\phi k}$ and $\tilde{C}^a_{ij\phi}$. The main difficulty in the exact handling of the loading decisions in this case is that if there are booking requests in between an earlier and a later flight, then the load on the later flight is not known at the time of departure of the early flight. Thus, loading problems of the earlier flight are subject to significant additional uncertainty. However, one can construct an upper bound to the combined loading problem of all flights by finding the loading decisions in hindsight, that is, under the assumption that all loading decisions are made simultaneously and with perfect knowledge of the number of shippings and their characteristics for all flights. Thus, the combined loading problem becomes

$$\begin{aligned}
\max \quad & -\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}}\sum_{l=1}^{n_{jk}}\sum_{j'\in\mathcal{J}\cup\{\phi\}} \tilde{C}^s_{jj'kl}\, z^s_{jj'kl} - \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i}\sum_{j'\in\mathcal{J}\cup\{\phi\}} x_i\, \tilde{C}^a_{ijj'}\, z^a_{ijj'} \\
\text{subject to} \quad & \sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{K}}\sum_{l=1}^{n_{jk}} V^s_{jkl}\, z^s_{jj'kl} + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} x_i\, V^a_{ij}\, z^a_{ijj'} \le \bar{V}_{j'} \qquad \text{for all } j'\in\mathcal{J} \\
& \sum_{j'\in\mathcal{J}\cup\{\phi\}} z^s_{jj'kl} = 1 \qquad \text{for all } l=1,\ldots,n_{jk},\ k\in\mathcal{K},\ j\in\mathcal{J}, \\
& \sum_{j'\in\mathcal{J}\cup\{\phi\}} z^a_{ijj'} = 1 \qquad \text{for all } i\in\mathcal{I},\ j\in\mathcal{J}_i, \\
& z^s_{jj'kl}\in\{0,1\} \qquad \text{for all } l=1,\ldots,n_{jk},\ k\in\mathcal{K},\ j\in\mathcal{J},\ j'\in\mathcal{J}\cup\{\phi\} \\
& 0\le z^a_{ijj'}\le 1 \qquad \text{for all } i\in\mathcal{I},\ j\in\mathcal{J}_i,\ j'\in\mathcal{J}\cup\{\phi\}.
\end{aligned}$$

In the problem above, the decision variable $z^s_{jj'kl}$ takes value one if the l-th booked spot market request for cargo type k on flight j is served through flight j'. Similarly, $z^a_{ijj'}$ corresponds to the portion of the cargo generated by atomic bid i on flight j that is served through flight j'. Using the problem in the boundary condition of the optimality equation in (5), all of our development goes through without any modifications to the methodology. In the resulting expressions (such as equations (12) and (13) of Proposition 1), the functions $B^a_{ij}(\lambda_j)$ and $B^s_{jk}(\lambda_j)$ are replaced by

$$\begin{aligned}
\tilde{B}^a_{ij}(\lambda) &= \mathbb{E}\Big[\min_{j\in\mathcal{J}\cup\{\phi\}}(\tilde{C}^a_{ijj'} + \lambda_j V^a_{ij})\Big], \text{ and} \\
\tilde{B}^s_{jk}(\lambda) &= \mathbb{E}\Big[\min_{j\in\mathcal{J}\cup\{\phi\}}(\tilde{C}^s_{jj'kl} + \lambda_j V^s_{jkl})\Big],
\end{aligned}$$

respectively (with a convention that $\lambda_\phi \equiv 0$).

EC.5. Consumer Choice Model for Efficient Implementation

Whether we compute the Lagrange multipliers once at the beginning of the planning horizon by solving the problem $\min_{\lambda\ge 0}\tilde{J}_1(x^*,\bar{0},\lambda)$ or recompute them at every time period by solving the problem $\min_{\lambda\ge 0}\tilde{J}_t(x^*,n,\lambda)$, the spot market booking control decisions require solving the optimization problem in (21). This is a challenging combinatorial optimization problem as it involves choosing a subset of $\mathcal{J}\times\mathcal{K}$ and there are potentially $2^{|\mathcal{J}|\times|\mathcal{K}|}$ such subsets. Enumerating over all possible subsets is clearly not an option. However, it turns out that there is an important class of consumer choice models that make the optimization problem in (21) tractable. In this section, we briefly review this class of consumer choice models.

We recall that $P_{jkt}(\mathcal{S})$ is the probability that there is a spot market booking for flight j and cargo type k at time period t given that the set of open flight and cargo type combinations at time period t is $\mathcal{S}$. To construct a tractable model for $P_{jkt}(\mathcal{S})$, we assume that a customer that is interested in making a spot market booking for cargo type k arrives into the system at time period t with probability π_{kt}. We naturally have $\sum_{k\in\mathcal{K}} \pi_{kt} \leq 1$ and the difference between the two sides of the inequality gives the probability that there is no customer arrival at time period t. Once a customer that is interested in cargo type k arrives into the system, this customer makes a choice among the flights according to the multinomial logit choice model. The multinomial logit choice model stipulates that if a customer that arrives into the system is interested in cargo type k, then this customer associates the preference weights $\{v_{jkt} : j \in \mathcal{J}\}$ with the different flights. These preference weights characterize the attractiveness of different flights to the customer. The customer also associates the preference weight v_{0kt} with the no booking option. In this case, if the set of open flight and cargo type combinations at time period t is given by $\mathcal{S}$, then the customer chooses flight j with probability $v_{jkt}/[\sum_{j':(j',k)\in\mathcal{S}} v_{j'kt} + v_{0kt}]$, which is the preference weight of flight j relative to the preference weights of all available options. Therefore, the probability that there is a spot market booking for flight j and cargo type k at time period t is given by

$$P_{jkt}(\mathcal{S}) = \pi_{kt} \frac{v_{jkt}}{\sum_{j':(j',k)\in\mathcal{S}} v_{j'kt} + v_{0kt}}. \tag{EC.3}$$

Multinomial logit choice model is a widely used tool in marketing and economics [see Anderson et al. (1992)] as well as, more recently, in revenue management [see Talluri and van Ryzin (2004) and van Ryzin and Liu (2008)].

Plugging the expression above for $P_{jkt}(\mathcal{S})$ into the optimization problem in (21) and dropping the term $\tilde{J}_{t+1}(x^*, n, \lambda^{**})$ that does not affect the spot market booking control decisions, we need to solve the problem

$$\max_{\mathcal{S}\subseteq\mathcal{J}\times\mathcal{K}}\left\{\sum_{(j,k)\in\mathcal{S}}\frac{\pi_{kt}\,v_{jkt}\left[\mathbb{E}\{R^s_{jk}\}-B^s_{jk}(\lambda^{**}_j)\right]}{\sum_{j':(j',k)\in\mathcal{S}}v_{j'kt}+v_{0kt}}\right\}=\max_{\mathcal{S}\subseteq\mathcal{J}\times\mathcal{K}}\left\{\sum_{k\in\mathcal{K}}\left[\frac{\sum_{j:(j,k)\in\mathcal{S}}\pi_{kt}\,v_{jkt}\left[\mathbb{E}\{R^s_{jk}\}-B^s_{jk}(\lambda^{**}_j)\right]}{\sum_{j':(j',k)\in\mathcal{S}}v_{j'kt}+v_{0kt}}\right]\right\}$$

to make the booking control decisions at time period t. In the expression above, the equality follows from the fact that a sum over all $(j,k)\in\mathcal{S}$ can be written as one sum over all $k\in\mathcal{K}$ and another sum over all $j\in\mathcal{J}$ satisfying $(j,k)\in\mathcal{S}$. The expression in the square brackets corresponds to the contribution from cargo type k and the contributions from different cargo types do not interact with each other. This implies that we can solve the problem on the right side above by maximizing the contribution from each cargo type individually. In particular, if we let $\mathcal{J}_k$ be the set of flights that are open for spot market bookings for cargo type k, then the optimal objective value of the problem on the right side above is given by $\sum_{k\in\mathcal{K}}\psi_{kt}(\lambda^{**})$, where we have

$$\psi_{kt}(\lambda^{**})=\max_{\mathcal{J}_k\subseteq\mathcal{J}}\left\{\frac{\sum_{j\in\mathcal{J}_k}\pi_{kt}\,v_{jkt}\left[\mathbb{E}\{R^s_{jk}\}-B^s_{jk}(\lambda^{**}_j)\right]}{\sum_{j'\in\mathcal{J}_k}v_{j'kt}+v_{0kt}}\right\}. \tag{EC.4}$$

Problem (EC.4) is slightly more tractable than problem (21) as this problem involves the subsets of $\mathcal{J}$, which number on the order of $2^{|\mathcal{J}|}$, whereas problem (21) involves the subsets of $\mathcal{J}\times\mathcal{K}$, which number on the order of $2^{|\mathcal{J}|\times|\mathcal{K}|}$. However, for practical applications, $\mathcal{J}$ may still have too many subsets to enumerate explicitly.

It turns out that problem (EC.4) has a very special structure that makes it solvable in a tractable fashion. To illustrate this property, we assume without loss of generality that the set of flights is indexed by $\mathcal{J}=\{1,\ldots,|\mathcal{J}|\}$ and the flights are ordered such that

$$\mathbb{E}\{R^s_{1k}\}-B^s_{1k}(\lambda^{**}_1)\geq\mathbb{E}\{R^s_{2k}\}-B^s_{2k}(\lambda^{**}_2)\geq\ldots\geq\mathbb{E}\{R^s_{|\mathcal{J}|,k}\}-B^s_{|\mathcal{J}|,k}(\lambda^{**}_{|\mathcal{J}|}).$$

In this case, Talluri and van Ryzin (2004) show that there is an optimal solution to problem (EC.4) of the form $\{1,2,\ldots,j\}\subseteq\mathcal{J}$ for some $j\in\mathcal{J}$. Therefore, the possible candidates for an optimal solution to problem (EC.4) are $\varnothing,\{1\},\{1,2\},\ldots,\{1,2,\ldots,|\mathcal{J}|\}$. Since there are only $|\mathcal{J}|+1$ possible candidates for an optimal solution to (EC.4), we can check the objective function value provided by each one of these candidates and choose the best one. This result eliminates the need to enumerate over all subsets of $\mathcal{J}$ to solve problem (EC.4).

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