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e-companion

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e-companion: Hazard Rate Scaling of the Abandonment Distribution for the GI/M/n + GI Queue in Heavy Traffic

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EC1. Proof of Proposition 6.1

For each $i \geq 1$, let us define

$$r_i(t) = 1\{p_i \leq W(\tau_i/\lambda -)\}1\{p_i \leq t - \tau_i/\lambda\} - \int_0^{(t-\tau_i/\lambda)\wedge W(\tau_i/\lambda -)\wedge p_i} h(u)du,$$

for $t \geq 0$, where we set $h(u) = 0$ for $u < 0$. We now show the following two facts.

(a) For each $i \geq 1$, $r_i = \{r_i(t), t \geq 0\}$ is a square-integrable $\mathcal{F}$-martingale with quadratic variation process

$$\langle\langle r_i \rangle\rangle_t = \int_0^{(t-\tau_i/\lambda)\wedge W(\tau_i/\lambda -)\wedge p_i} h(u)du,$$

for $t \geq 0$.

(b) For $i \neq j$, r_i and r_j are orthogonal.

These two facts together will imply the result since $E[N_A(\lambda t)] < \infty$.

We now prove (a). The square integrability of r_i is clear since $||h||_\infty < \infty$. Now let $s < t$, we next show that $E[r_i(t)|\mathcal{F}_s] = r_i(s)$. First note that we have

$$\begin{aligned} &E[r_i(t)|\mathcal{F}_s] \\ &= E\left[1\{p_i \leq W(\tau_i/\lambda -)\}1\{p_i \leq t - \tau_i/\lambda\} - \int_0^{(t-\tau_i/\lambda)\wedge W(\tau_i/\lambda -)\wedge p_i} h(u)du \;\middle|\; \mathcal{F}_s\right] \\ &= r_i(s) + E\Bigg[1\{p_i \leq W(\tau_i/\lambda -)\}1\{s - \tau_i/\lambda < p_i \leq t - \tau_i/\lambda\} \end{aligned}$$

$$-\int_{(s-\tau_i/\lambda)\wedge W(\tau_i/\lambda-)\wedge p_i}^{(t-\tau_i/\lambda)\wedge W(\tau_i/\lambda-)\wedge p_i} h(u)du \;\Big|\; \mathcal{F}_s\Big], \tag{1}$$

where the final equality follows because by (24), $r_i(s)$ is $\mathcal{F}_s$-measurable since

$$\begin{aligned}(s-\tau_i/\lambda)\wedge W(\tau_i/\lambda-)\wedge p_i = &\\ &+(s-\tau_i/\lambda)(1-1\{p_i\le W(\tau_i/\lambda-)\}1\{p_i\le s-\tau_i/\lambda\})1\{W(\tau_i/\lambda-)>s-\tau_i/\lambda\}\\ &+W(\tau_i/\lambda-)(1-1\{p_i\le W(\tau_i/\lambda-)\}1\{p_i\le s-\tau_i/\lambda\})1\{W(\tau_i/\lambda-)\le s-\tau_i/\lambda\}\\ &+p_i 1\{p_i\le W(\tau_i/\lambda-)\}1\{p_i\le s-\tau_i/\lambda\},\end{aligned}$$

and both $1\{W(\tau_i/\lambda-)\le s-\tau_i/\lambda\}$ and $W(\tau_i/\lambda-)1\{W(\tau_i/\lambda-)\le s-\tau_i/\lambda\}$ are $\mathcal{F}_s$-measurable by (4). It is also clear that $p_i 1\{p_i\le W(\tau_i/\lambda-)\}1\{p_i\le t-\tau_i/\lambda\}$ is $\mathcal{F}_t$-measurable by (24). If we now define the $\sigma-$algebra

$$\begin{aligned}\mathcal{G}_t^i =\ &\sigma\{N_A(\lambda u), u\ge 0\}\vee\sigma\{N_D(u), u\ge 0\}\vee\sigma\{p_k, k\ge 1, k\ne i\}\\ &\vee\ \sigma\{\{1\{p_i\le W(\tau_i/\lambda-)\}1\{p_i\le s-\tau_i/\lambda\}, 0\le s\le t\}\}\vee\ \sigma\{W(\tau_i/\lambda-)\},\end{aligned}$$

it then follows by the independence of p_i from $N_A, N_D, \{p_j, j\ge 1, j\ne i\}$ and $W(\tau_i-)$ that

$$E\left[1\{p_i\le W(\tau_i/\lambda-)\}1\{s-\tau_i/\lambda<p_i\le t-\tau_i/\lambda\}-\int_{(s-\tau_i/\lambda)\wedge W(\tau_i/\lambda-)\wedge p_i}^{(t-\tau_i/\lambda)\wedge W(\tau_i/\lambda-)\wedge p_i} h(u)du \;\Big|\; \mathcal{G}_s^i\right]=0.$$

Hence, since $\mathcal{F}_t\subseteq\mathcal{G}_t^i$, we have by the tower property and (1) that $E[r_i(t)|\mathcal{F}_s]=r_i(s)$ and so r_i is an $\mathcal{F}$-martingale. The fact that the quadratic variation of r_i is as in (1) is now also immediate.

We now show (b). In order to show that r_i and r_j are orthogonal for $i\ne j$, note first by the $\mathcal{F}_s$-measurability of $r_i(s)$ and $r_j(s)$ and results in (a) that

$$\begin{aligned}E[r_i(t)r_j(t)|\mathcal{F}_s] &= E[(r_i(s)+(r_i(t)-r_i(s)))(r_j(s)+(r_j(t)-r_j(s)))|\mathcal{F}_s]\\ &= E[r_i(s)r_j(s)|\mathcal{F}_s]+E[r_i(s)(r_j(t)-r_j(s))|\mathcal{F}_s]\\ &\quad +E[r_j(s)(r_i(t)-r_i(s))|\mathcal{F}_s]+E[(r_i(t)-r_i(s))(r_j(t)-r_j(s))|\mathcal{F}_s]\\ &= r_i(s)r_j(s)+r_i(s)E[(r_j(t)-r_j(s))|\mathcal{F}_s]\\ &\quad +r_j(s)E[(r_i(t)-r_i(s))|\mathcal{F}_s]+E[(r_i(t)-r_i(s))(r_j(t)-r_j(s))|\mathcal{F}_s]\\ &= r_i(s)r_j(s)+E[(r_i(t)-r_i(s))(r_j(t)-r_j(s))|\mathcal{F}_s].\end{aligned} \tag{2}$$

Defining now the σ-algebra $\mathcal{H}_t^{j,i}=\mathcal{G}_t^j\vee\sigma\{W(\tau_i/\lambda-)\}$, it follows that $(r_i(t)-r_i(s))$ is measurable with respect to $\mathcal{H}_t^{j,i}$ and hence by the independence of p_j from $W(\tau_i-)$ for $j>i$, we obtain that

$$E[(r_i(t)-r_i(s))(r_j(t)-r_j(s))|\mathcal{H}_s^{j,i}]=(r_i(t)-r_i(s))E[(r_j(t)-r_j(s))|\mathcal{H}_s^{j,i}]\ =\ 0.$$

By the fact that $\mathcal{F}_t\subseteq\mathcal{H}_t^{j,i}$, the tower property and (2), we therefore have that r_i and r_j are orthogonal to each other and hence the proof is complete. □

EC2. Proofs of the results in Section 7

Proof of Proposition 7.1. By Theorem 3.9 of (1) it suffices to show that each term converges to zero on its own. We begin by noting that by (7) and the Functional Weak Law of Large Numbers (7), $\bar{N}_A^n \Rightarrow 0$ as $n \to \infty$, and so by the Random Time Change Theorem (1), since $n^{-1}\lambda^n e \to \mu e$, we have that $\bar{N}_A^n(n^{-1}\lambda^n e) \Rightarrow 0$ as $n \to \infty$.

Next, note that for each $t \geq 0$, $Q^n(t) \leq Q^n(0) + N_A(\lambda^n t)$, and hence for each $T \geq 0$, we have that

$$\sup_{0 \leq t \leq T} \left| \bar{N}_D^n \left(\mu \int_0^t (\bar{Q}^n(s) \wedge 1) ds \right) \right| \leq \sup_{0 \leq t \leq T} \left| \bar{N}_D^n \left(\mu \int_0^t (\bar{Q}^n(0) + \bar{N}_A^n(n^{-1}\lambda^n s) + n^{-1}\lambda^n s) ds \right) \right|. \quad (3)$$

However, since $\bar{Q}^n(0) + \bar{N}_A^n(n^{-1}\lambda^n e) + n^{-1}\lambda^n e \Rightarrow \bar{Q}(0) + \mu e$ by the assumption of the proposition and $\bar{Q}^n(0) + \bar{N}_A^n(n^{-1}\lambda^n e) + \lambda^n e$ is also an increasing function, it follows that

$$\mu \int_0^e (\bar{Q}^n(0) + \bar{N}_A^n(n^{-1}\lambda^n s) + n^{-1}\lambda^n s) ds \Rightarrow \mu \int_0^e (\bar{Q}(0) + \mu s) ds,$$

as $n \to \infty$. Hence, since by the Functional Weak Law of Large Numbers (1), $\bar{N}_D^n \Rightarrow 0$ as $n \to \infty$, it follows by the Random Time Change Theorem (1) that $\bar{N}_D^n \left(\bar{Q}^n(0) + \bar{N}_A^n(n^{-1}\lambda^n e) + \lambda^n e \right) \Rightarrow 0$ as $n \to \infty$. Thus, by the Continuous Mapping Theorem (7),

$$\sup_{0 \leq t \leq T} \left| \bar{N}_D^n \left(\bar{Q}^n(0) + \bar{N}_A^n(n^{-1}\lambda^n e) + \lambda^n e \right) \right| \Rightarrow 0,$$

as $n \to \infty$ and so, as a result of (3) it follows that

$$\bar{N}_D^n \left(\mu \int_0^e (\bar{Q}^n(s) \wedge 1) ds \right) \Rightarrow 0,$$

as $n \to \infty$.

It remains to show that $\bar{R}^n \Rightarrow 0$ as $n \to \infty$. By Proposition (6.1) we have that $\bar{R}^n$ is a square-integrable $\mathcal{F}$-martingale with quadratic variation given by

$$\begin{aligned} \langle\langle \bar{R}^n \rangle\rangle_t &= \frac{1}{n^2} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{(t - \tau_i/\lambda^n) \wedge W^n((\tau_i/\lambda^n)-) \wedge p_i^n} h^n(u) du \\ &\leq \frac{t}{n} ||h||_\infty \frac{N_A(\lambda^n t)}{n}. \end{aligned}$$

Thus, for each $T \geq 0$, we have that

$$\sup_{0 \leq t \leq T} \langle\langle \bar{R}^n \rangle\rangle_t \leq \frac{T}{n} ||h||_\infty \frac{N_A(\lambda^n T)}{n}. \quad (4)$$

However, since by the Functional Strong Law of Large Numbers (1), the assumption of the proposition and the Random Time Change Theorem we have that $n^{-1}N_A(\lambda^n T) = n^{-1}N_A(n(n^{-1}\lambda^n)T) \Rightarrow \mu T$ as $n \to \infty$, it follows by (4) that $\langle\langle \bar{R}^n \rangle\rangle \Rightarrow 0$ as $n \to \infty$ and hence by the Martingale Invariance Principle, $\bar{R}^n \Rightarrow 0$ as $n \to \infty$. □

Proof of Theorem 7.2. Let us first note that

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{N_A(\lambda^n t)}\int_0^{(t-\tau_i/\lambda^n)\wedge W^n((\tau_i/\lambda^n)-)\wedge p_i^n} h^n(s)ds \\
&\le \frac{||h||_\infty}{n}\sum_{i=1}^{N_A(\lambda^n t)}\int_0^{(t-\tau_i/\lambda^n)\wedge W^n((\tau_i/\lambda^n)-)\wedge p_i^n} ds \\
&= \frac{||h||_\infty}{n}\sum_{i=1}^{N_A(\lambda^n t)}\int_0^t 1\{0\le s-\tau_i/\lambda^n \le W^n((\tau_i/\lambda^n)-)\wedge p_i^n\}ds \\
&= \frac{||h||_\infty}{n}\int_0^t \sum_{i=1}^{N_A(\lambda^n t)} 1\{0\le s-\tau_i/\lambda^n \le W^n((\tau_i/\lambda^n)-)\wedge p_i^n\}ds \\
&= ||h||_\infty \int_0^t (\bar{Q}^n(s)-1)^+ ds.
\end{aligned}
$$

Thus, by (23) and the triangle inequality, we have that

$$
\begin{aligned}
|\bar{Q}^n(t)-1| \le\ & |\bar{Q}^n(0)-1| + |\bar{N}_A^n(n^{-1}\lambda^n t)| + \left|\bar{N}_D^n\left(\mu\int_0^t (\bar{Q}^n(s)\wedge 1)ds\right)\right| + |\bar{R}^n(t)| + |n^{-1}\lambda^n-\mu|t \\
&+\mu\int_0^t |\bar{Q}^n(s)-1|ds + ||h||_\infty \int_0^t |\bar{Q}^n(s)-1|ds.
\end{aligned}
$$

Moreover, for each $T\ge r\ge 0$,

$$
\begin{aligned}
&\sup_{0\le t\le r}|\bar{Q}^n(t)-1| \\
&\le \sup_{0\le t\le r}\left||\bar{Q}^n(0)-1| + |\bar{N}_A^n(n^{-1}\lambda^n t)| + \left|\bar{N}_D^n\left(\mu\int_0^t (\bar{Q}^n(s)\wedge 1)ds\right)\right| + |\bar{R}^n(t)| + |n^{-1}\lambda^n-\mu|t\right| \\
&\quad +(\mu+||h||_\infty)\int_0^r \sup_{0\le t\le s}|\bar{Q}^n(t)-1|ds.
\end{aligned}
$$

It therefore follows by Gronwall's inequality (5) that for each $T\ge 0$,

$$
\begin{aligned}
&e^{-(\mu+||h||_\infty)T}\sup_{0\le t\le T}|\bar{Q}^n(t)-1| \qquad (5)\\
&\le \sup_{0\le t\le T}\left||\bar{Q}^n(0)-1| + |\bar{N}_A^n(n^{-1}\lambda^n t)| + \left|\bar{N}_D^n\left(\mu\int_0^t (\bar{Q}^n(s)\wedge 1)ds\right)\right| + |\bar{R}^n(t)| + |n^{-1}\lambda^n-\mu|t\right|.
\end{aligned}
$$

However since by the assumption of the proposition, Proposition 7.1 and the Continuous Mapping Theorem (7) we have that

$$
\sup_{0\le t\le T}\left||\bar{Q}^n(0)-1| + |\bar{N}_A^n(\lambda^n t)| + \left|\bar{N}_D^n\left(\mu\int_0^t (\bar{Q}^n(s)\wedge 1)ds\right)\right| + |\bar{R}^n(t)| + |n^{-1}\lambda^n-\mu|t\right| \Rightarrow 0,
$$

as $n\to\infty$, it follows by (5) that $\bar{Q}^n \Rightarrow 1$ as $n\to\infty$, which completes the proof. □

EC3. Proofs of the results in Section 8

Proof of Proposition 8.1. By the Functional Central Limit Theorem for renewal process (1) and the assumed independence of $Q^n(0), N_A$ and N_D, it follows that $(\tilde{Q}^n(0), \tilde{N}_A^n, \tilde{N}_D^n) \Rightarrow (\tilde{Q}(0), \sigma_A B_1, B_2)$ as $n \to \infty$. By assumption, we have that $n^{-1}\lambda^n \Rightarrow \mu$ as $n \to \infty$. Also, by the assumption of the proposition and Theorem 7.2 we have that $\bar{Q}^n \Rightarrow 1$ as $n \to \infty$. Hence, by the Bounded Convergence Theorem (4),

$$\mu \int_0^e (\bar{Q}^n(s) \wedge 1) ds \Rightarrow \mu e,$$

as $n \to \infty$. It therefore follows by the Random Time Change Theorem (1) that

$$\left(\tilde{Q}^n(0), \tilde{N}_A^n(n^{-1}\lambda^n e), \tilde{N}_D^n\left(\mu \int_0^e (\bar{Q}^n(s) \wedge 1) ds\right)\right) \Rightarrow (\tilde{Q}(0), \sigma_A B_1(\mu e), B_2(\mu e)),$$

as $n \to \infty$.

By Theorem 3.9 of (1), it now remains to show that $\tilde{R}^n \Rightarrow 0$ as $n \to \infty$. By Proposition 6.1 of Section 6.2, $\tilde{R}^n$ is a martingale with quadratic variation

$$\langle\langle \tilde{R}^n \rangle\rangle_t = \frac{1}{n} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{(t-\tau_i/\lambda^n) \wedge W^n((\tau_i/\lambda^n)-) \wedge p_i^n} h(u) du.$$

However, as was already demonstrated in the proof of Theorem 7.2,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{(t-\tau_i/\lambda^n) \wedge W^n(\tau_i/\lambda^n -) \wedge p_i} h(s) ds \; &\le \; \|h\|_\infty \int_0^t (\bar{Q}^n(s) - 1)^+ ds \\ &\Rightarrow 0, \end{aligned}$$

where the final convergence follows from the assumptions of the proposition and Theorem 7.2. It therefore follows from the Martingale Invariance Principle (3) that $\tilde{R}^n \Rightarrow 0$ as $n \to \infty$, which completes the proof. □

Proof of Proposition 8.2. In order to show that $\{\tilde{Q}^n, n \ge 1\}$ is tight we will verify that conditions (i) and (ii) of Theorem 13.2 of (1) are satisfied. We begin with condition (i). We must show that for each $T > 0$ and $\varepsilon > 0$ there exists a $K_\varepsilon^T > 0$ such that $\mathbb{P}\{\sup_{0 \le t \le T} |\tilde{Q}^n(t)| > K_\varepsilon^T\} < \varepsilon$ for $n \ge 1$. First note that since, as in the proof of Theorem 7.2 we have that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{(t-\tau_i/\lambda^n) \wedge W^n((\tau_i/\lambda^n)-) \wedge p_i^n} h(s) ds \le \|h\|_\infty \int_0^t \tilde{Q}^{n,+}(s) ds,$$

it follows from (26) that

$$\left|\tilde{Q}^n(t)\right| \le \left|\tilde{Q}^n(0) + \tilde{N}_A(n^{-1}\lambda^n t) - \tilde{N}_D\left(\mu \int_0^t (\bar{Q}^n(s) \wedge 1) ds\right)\right. \tag{6}$$

$$
\left. - \tilde{R}^n(t) + n^{1/2}(n^{-1}\lambda^n - \mu)t \right|
+\mu \int_0^t \tilde{Q}^{n,-}(s)ds + ||h||_\infty \int_0^t \tilde{Q}^{n,+}(s)ds.
$$

Taking supremums on both sides of (6), we obtain that for each $t \geq 0$,

$$
\begin{aligned}
&\sup_{0\leq s\leq t} |\tilde{Q}^n(s)| \\
&\leq \sup_{0\leq s\leq t} \left| \tilde{Q}^n(0) + \tilde{N}_A(n^{-1}\lambda^n t) - \tilde{N}_D\left(\mu \int_0^t (\bar{Q}^n(s)\wedge 1)ds\right) - \tilde{R}^n(t) + n^{1/2}(n^{-1}\lambda^n - \mu)t \right| \\
&\quad +\mu \int_0^t \sup_{0\leq u\leq s} |\tilde{Q}^{n,-}(u)|ds + ||h||_\infty \int_0^t \sup_{0\leq u\leq s} |\tilde{Q}^{n,+}(u)|ds,
\end{aligned}
$$

thus implying that for each $T \geq 0$ and $0 \leq t \leq T$,

$$
\begin{aligned}
&\sup_{0\leq s\leq t} |\tilde{Q}^n(s)| \\
&\leq \sup_{0\leq t\leq T} \left| \tilde{Q}^n(0) + \tilde{N}_A(n^{-1}\lambda^n t) - \tilde{N}_D\left(\mu \int_0^t (\bar{Q}^n(s)\wedge 1)ds\right) - \tilde{R}^n(t) + n^{1/2}(n^{-1}\lambda^n - \mu)t \right| \\
&\quad +(\mu + ||h||_\infty) \int_0^t \sup_{0\leq u\leq s} |\tilde{Q}^n(u)|ds.
\end{aligned}
$$

Hence, by Gronwall's inequality (7),

$$
\begin{aligned}
&e^{-(\mu+||h||_\infty)T} \sup_{0\leq t\leq T} |\tilde{Q}^n(t)| \qquad (7)\\
&\leq \sup_{0\leq t\leq T} \left| \tilde{Q}^n(0) + \tilde{N}_A(n^{-1}\lambda^n t) - \tilde{N}_D\left(\mu \int_0^t (\bar{Q}^n(s)\wedge 1)ds\right) - \tilde{R}^n(t) + n^{1/2}(n^{-1}\lambda^n - \mu)t \right|.
\end{aligned}
$$

By Proposition 8.1 and the Continuous Mapping Theorem (7) we have that

$$
\begin{aligned}
&\tilde{Q}^n(0) + \tilde{N}_A(n^{-1}\lambda^n e) - \tilde{N}_D\left(\mu \int_0^e (\bar{Q}^n(s)\wedge 1)ds\right) - \tilde{R}^n(e) + n^{1/2}(n^{-1}\lambda^n - \mu)e \\
&\Rightarrow \tilde{Q}(0) + B_1(\mu e) - B_2(\mu e) - \beta\mu e,
\end{aligned}
$$

as $n \to \infty$. Thus, by the only if portion of Theorem 13.2 of (1) we have condition (i) holds for

$$
\left\{ \tilde{Q}^n(0) + \tilde{N}_A(n^{-1}\lambda^n e) - \tilde{N}_D\left(\mu \int_0^e (\bar{Q}^n(s)\wedge 1)ds\right) - \tilde{R}^n(e) + n^{1/2}(n^{-1}\lambda^n - \mu)e, n \geq 1 \right\},
$$

and so condition (i) holds for $\{\tilde{Q}^n, n \geq 1\}$ as well as a result of (7).

We now verify that condition (ii) is satisfied. The proof follows similarly to the proof of condition (i) above. By (26) it follows that for each $\delta \geq 0$,

$$
\begin{aligned}
&\tilde{Q}^n(t+\delta) - \tilde{Q}^n(t) \qquad (8)\\
&= (\tilde{N}_A(n^{-1}\lambda^n(t+\delta)) - \tilde{N}_A(n^{-1}\lambda^n t)) \\
&\quad - \left(\tilde{N}_D\left(\mu \int_0^{t+\delta} (\bar{Q}^n(s)\wedge 1)ds\right) - \tilde{N}_D\left(\mu \int_0^t (\bar{Q}^n(s)\wedge 1)ds\right)\right) - (\tilde{R}^n(t+\delta) - \tilde{R}^n(t))
\end{aligned}
$$

$$+n^{1/2}(n^{-1}\lambda^n-\mu)\delta+\mu\int_t^{t+\delta}\tilde{Q}^{n,-}(s)ds$$
$$-\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{N_A(\lambda^n(t+\delta))}\int_0^{(t+\delta-\tau_i/\lambda^n)\wedge W^n((\tau_i/\lambda^n)-)\wedge p_i^n}h(u)du\right.$$
$$\left.-\frac{1}{\sqrt{n}}\sum_{i=1}^{N_A(\lambda^n t)}\int_0^{(t-\tau_i/\lambda^n)\wedge W^n((\tau_i/\lambda^n)-)\wedge p_i^n}h(u)du\right).$$

Moreover, as in the proof of Theorem 7.2, it can be shown that

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{N_A(\lambda^n(t+\delta))}\int_0^{(t+\delta-\tau_i/\lambda^n)\wedge W^n((\tau_i/\lambda^n)-)\wedge p_i}h(u)du\right.$$
$$\left.-\frac{1}{\sqrt{n}}\sum_{i=1}^{N_A(\lambda^n t)}\int_0^{(t-\tau_i/\lambda^n)\wedge W^n((\tau_i/\lambda^n)-)\wedge p_i}h(u)du\right)$$
$$\leq ||h||_\infty\int_t^{t+\delta}\tilde{Q}^{n,+}(s)ds.$$

Next, for each $T\geq 0$,

$$||h||_\infty\int_t^{t+\delta}\tilde{Q}^{n,+}(s)ds\leq ||h||_\infty\delta\sup_{0\leq t\leq T}|\tilde{Q}^n(s)|$$

and

$$\mu\int_t^{t+\delta}\tilde{Q}^{n,-}(s)ds\leq \mu\delta\sup_{0\leq t\leq T}|\tilde{Q}^n(s)|.$$

It therefore follows combining the above with (8), we obtain that for each $T\geq 0$,

$$\sup_{0\leq t\leq t+\delta\leq T}|\tilde{Q}^n(t+\delta)-\tilde{Q}^n(t)|$$
$$\leq \sup_{0\leq t\leq t+\delta\leq T}|\tilde{N}_A(\mu(t+\delta))-\tilde{N}_A(\mu t)|$$
$$+\sup_{0\leq t\leq t+\delta\leq T}\left|\tilde{N}_D\left(\mu\int_0^{t+\delta}(\bar{Q}^n(s)\wedge 1)ds\right)-\tilde{N}_D\left(\mu\int_0^t(\bar{Q}^n(s)\wedge 1)ds\right)\right|$$
$$+\sup_{0\leq t\leq t+\delta\leq T}|\tilde{R}^n(t+\delta)-\tilde{R}^n(t)|+n^{1/2}(n^{-1}\lambda^n-\mu)\delta+\delta(||h||_\infty+\mu)\sup_{0\leq t\leq T}|\tilde{Q}^n(s)|.$$

Thus, condition (ii) of Theorem 13.2 of (1) is now seen to be satisfied by virtue of condition (i) above and Proposition 8.1 in conjunction with the only if portion of Theorem 13.2 of (1). □

Proof of Proposition 8.3. First recall by (4) that the virtual waiting time at time $t\geq 0$ is given by

$$W^n(t)$$
$$=\inf\left\{u\geq 0: n^{-1/2}\left(N_D\left(\mu\int_0^{t+u}(Q^n(s)\wedge n)ds\right)-N_D\left(\mu\int_0^t(Q^n(s)\wedge n)ds\right)\right)\right.$$

$$
+n^{-1/2}\left(\sum_{i=1}^{N_A(\lambda^n t)} 1\{p_i^n \le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n \le (t+u)-\tau_i/\lambda^n\}\right.
$$
$$
\left.\left. - \sum_{i=1}^{N_A(\lambda^n t)} 1\{p_i^n \le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n \le t-\tau_i/\lambda^n\}\right) > \tilde{Q}^{n,+}(t)\right\} \times 1\{\tilde{Q}^n(t)\ge 0\}.
$$

This may be rewritten as

$$
\begin{aligned}
&\tilde{W}^n(t)\\
&= \inf\left\{u\ge 0 : n^{-1/2}\left(N_D\left(\mu\int_0^{t+n^{-1/2}u}(Q^n(s)\wedge n)ds\right) - N_D\left(\mu\int_0^t (Q^n(s)\wedge n)ds\right)\right)\right.\\
&\quad +n^{-1/2}\left(\sum_{i=1}^{N_A(\lambda^n t)} 1\{p_i^n \le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n \le (t+n^{-1/2}u)-\tau_i/\lambda^n\}\right.\\
&\quad \left.\left. - \sum_{i=1}^{N_A(\lambda^n t)} 1\{p_i^n \le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n \le t-\tau_i/\lambda^n\}\right) > \tilde{Q}^{n,+}(t)\right\} \times 1\{\tilde{Q}^n(t)\ge 0\}\\
&= \inf\left\{u\ge 0 : \mu u + \left(n^{-1/2}\left(N_D\left(\mu\int_0^{t+n^{-1/2}u}(Q^n(s)\wedge n)ds\right) - N_D\left(\mu\int_0^t (Q^n(s)\wedge n)ds\right)\right) - \mu u\right)\right.\\
&\quad +n^{-1/2}\left(\sum_{i=1}^{N_A(\lambda^n t)} 1\{p_i^n \le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n \le (t+n^{-1/2}u)-\tau_i/\lambda^n\}\right.\\
&\quad \left.\left. - \sum_{i=1}^{N_A(\lambda^n t)} 1\{p_i^n \le W^n(\tau_i/\lambda^n-)\}1\{p_i^n \le t-\tau_i/\lambda^n\}\right) > \tilde{Q}^{n,+}(t)\right\} \times 1\{\tilde{Q}^n(t)\ge 0\}.
\end{aligned}
\tag{9}
$$

We now make the claim that for each $U\ge 0$ and $T\ge 0$,

$$
\sup_{0\le u\le U}\sup_{0\le t\le T}\left|n^{-1/2}\left(N_D\left(\mu\int_0^{t+n^{-1/2}u}(Q^n(s)\wedge n)ds\right) - N_D\left(\mu\int_0^t (Q^n(s)\wedge n)ds\right)\right) - \mu u\right| \Rightarrow 0,
$$

as $n\to\infty$. In order to see this, first note that for each $t\ge 0$ and $u\ge 0$,

$$
\begin{aligned}
&\left(n^{-1/2}\left(N_D\left(\mu\int_0^{t+n^{-1/2}u}(Q^n(s)\wedge n)ds\right) - N_D\left(\mu\int_0^t (Q^n(s)\wedge n)ds\right)\right) - \mu u\right)\\
&= n^{-1/2}\left(N_D\left(n\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right) - n\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right)\\
&\quad -n^{-1/2}\left(N_D\left(n\mu\int_0^t(\bar{Q}^n(s)\wedge 1)ds\right) - n\mu\int_0^t(\bar{Q}^n(s)\wedge 1)ds\right)\\
&\quad +\left(n^{1/2}\mu\int_t^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds - \mu u\right).
\end{aligned}
$$

By the assumptions of the proposition and Theorem 7.2 it is clear that

$$
\sup_{0\le u\le U}\sup_{0\le t\le T}\left|n^{1/2}\mu\int_t^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds - \mu u\right| \Rightarrow 0.
$$

Next, note that

$$
\begin{aligned}
& n^{-1/2}\left(N_D\left(n\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right)-n\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right) \\
& -n^{-1/2}\left(N_D\left(n\mu\int_0^{t}(\bar{Q}^n(s)\wedge 1)ds\right)-n\mu\int_0^{t}(\bar{Q}^n(s)\wedge 1)ds\right) \\
= {} & \tilde{N}_D^n\left(\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right)-\tilde{N}_D^n\left(\mu\int_0^{t}(\bar{Q}^n(s)\wedge 1)ds\right).
\end{aligned}
$$

It therefore follows that

$$
\begin{aligned}
& \sup_{0\le u\le U}\sup_{0\le t\le T}\left|n^{-1/2}\left(N_D\left(n\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right)-n\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right)\right. \\
& \left. -n^{-1/2}\left(N_D\left(n\mu\int_0^{t}(\bar{Q}^n(s)\wedge 1)ds\right)-n\mu\int_0^{t}(\bar{Q}^n(s)\wedge 1)ds\right)\right| \\
= {} & \sup_{0\le u\le U}\sup_{0\le t\le T}\left|\tilde{N}_D^n\left(\mu\int_0^{t+n^{-1/2}u}(\bar{Q}^n(s)\wedge 1)ds\right)-\tilde{N}_D^n\left(\mu\int_0^{t}(\bar{Q}^n(s)\wedge 1)ds\right)\right| \\
\le {} & \sup_{0\le u\le U}\sup_{0\le t\le \mu T}\left|\tilde{N}_D^n\left(t+n^{-1/2}u\right)-\tilde{N}_D^n\left(t\right)\right| \\
\Rightarrow {} & 0,
\end{aligned}
$$

as $n\to\infty$, where the final convergence follows since by the Functional Central Limit Theorem for renewal processes (1), $\tilde{N}_D^n \Rightarrow B_2$ as $n\to\infty$, where B_2 is a standard Brownian motion with $\mathbb{P}$-a.s. continuous sample paths. Thus, the claim is proven.

Next, we claim that

$$
\begin{aligned}
& \sup_{0\le u\le U}\sup_{0\le t\le T}\left|n^{-1/2}\left(\sum_{i=1}^{N_A(\lambda^n t)}1\{p_i^n\le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n\le (t+n^{-1/2}u)-\tau_i/\lambda^n\}\right.\right. \\
& \qquad \left.\left.-\sum_{i=1}^{N_A(\lambda^n t)}1\{p_i^n\le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n\le t-\tau_i/\lambda^n\}\right)\right| \\
& \Rightarrow 0,
\end{aligned}
$$

as $n\to\infty$. First note that

$$
\begin{aligned}
& n^{-1/2}\left(\sum_{i=1}^{N_A(\lambda^n t)}1\{p_i^n\le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n\le (t+n^{-1/2}u)-\tau_i/\lambda^n\}\right. \\
& \qquad \left.-\sum_{i=1}^{N_A(\lambda^n t)}1\{p_i^n\le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n\le t-\tau_i/\lambda^n\}\right) \\
\le {} & n^{-1/2}\left(\sum_{i=1}^{N_A(\lambda^n (t+n^{-1/2}u))}1\{p_i^n\le W^n(\tau_i/\lambda^n-)\}1\{p_i^n\le (t+n^{-1/2}u)-\tau_i/\lambda^n\}\right. \\
& \qquad \left.-\sum_{i=1}^{N_A(\lambda^n t)}1\{p_i^n\le W^n((\tau_i/\lambda^n)-)\}1\{p_i^n\le t-\tau_i/\lambda^n\}\right)
\end{aligned}
$$

$$= \tilde{R}^n(t+n^{-1/2}u) - \tilde{R}^n(t) + n^{-1/2}\left(\sum_{i=1}^{N_A(\lambda^n(t+n^{-1/2}u))}\int_0^{(t+n^{-1/2}u-\tau_i/\lambda^n)\wedge W((\tau_i/\lambda^n)-)\wedge p_i^n} h(u)du \right.$$
$$\left. - \sum_{i=1}^{N_A(\lambda^n t)}\int_0^{(t-\tau_i/\lambda^n)\wedge W((\tau_i/\lambda^n)-)\wedge p_i^n} h(u)du\right).$$

By the assumptions of the proposition and Proposition 8.1, $\tilde{R}^n \Rightarrow 0$ as $n \to \infty$ and hence

$$\sup_{0\le u\le U}\sup_{0\le t\le T}|\tilde{R}^n(t+n^{-1/2}u)+\tilde{R}^n(t)| \Rightarrow 0,$$

as $n\to\infty$.

Next note that for each $t \ge 0$,

$$\left|\sum_{i=1}^{N_A(\lambda^n(t+n^{-1/2}u))}\int_0^{(t+n^{-1/2}u-\tau_i/\lambda^n)\wedge W((\tau_i/\lambda^n)-)\wedge p_i^n} h(u)du\right.$$
$$\left. - \sum_{i=1}^{N_A(\lambda^n t)}\int_0^{(t-\tau_i/\lambda^n)\wedge W((\tau_i/\lambda^n)-)\wedge p_i^n} h(u)du\right|$$
$$= \left|\sum_{i=1}^{N_A(\lambda^n(t+n^{-1/2}u))}\int_0^{t+n^{-1/2}u} 1\{0\le s-\tau_i/\lambda^n \le W^n((\tau_i^n/\lambda^n)-)\wedge p_i^n\}h(s-\tau_i/\lambda^n)ds\right.$$
$$\left. - \sum_{i=1}^{N_A(\lambda^n t)}\int_0^{t} 1\{0\le s-\tau_i/\lambda^n \le W^n((\tau_i/\lambda^n)-)\wedge p_i^n\}h(s-\tau_i/\lambda^n)ds\right|$$
$$= \left|\int_t^{t+n^{-1/2}u}\sum_{i=1}^{N_A(\lambda^n(t+n^{-1/2}u))} 1\{0\le s-\tau_i/\lambda^n \le W^n((\tau_i/\lambda^n)-)\wedge p_i^n\}h(s-\tau_i/\lambda^n)ds\right|$$
$$\le ||h||_\infty\left|\int_t^{t+n^{-1/2}u}\sum_{i=1}^{N_A(\lambda^n(t+n^{-1/2}u))} 1\{0\le s-\tau_i/\lambda^n \le W^n((\tau_i/\lambda^n)-)\wedge p_i^n\}ds\right|$$
$$= ||h||_\infty\int_t^{t+n^{-1/2}u}(Q^n(s)-n)^+ds.$$

Hence, by the assumptions of the proposition and Theorem 7.2

$$\sup_{0\le u\le U}\sup_{0\le t\le T} n^{-1/2}\left|\sum_{i=1}^{N_A(\lambda^n(t+n^{-1/2}u))}\int_0^{(t+n^{-1/2}u-\tau_i/\lambda^n)\wedge W((\tau_i/\lambda^n)-)\wedge p_i^n} h(u)du\right.$$
$$\left. - \sum_{i=1}^{N_A(\lambda^n t)}\int_0^{(t-\tau_i/\lambda^n)\wedge W((\tau_i/\lambda^n)-)\wedge p_i^n} h(u)du\right|$$
$$\le \sup_{0\le u\le U}\sup_{0\le t\le T} n^{-1/2}||h||_\infty\int_t^{t+n^{-1/2}u}(Q^n(s)-n)^+ds$$
$$= \sup_{0\le u\le U}\sup_{0\le t\le T}||h||_\infty\int_t^{t+n^{-1/2}u}\tilde{Q}^{n,+}(s)ds$$
$$\le U||h||_\infty\sup_{0\le t\le T+n^{-1/2}U}(\bar{Q}^n(u)-1)^+$$

$\Rightarrow 0,$

as $n \to \infty$. Thus, by the triangle inequality, the claim is proven.

Now note that by Proposition 8.2, $\{\tilde{Q}^n, n \geq 1\}$ is stochastically bounded. Thus, for each $T > 0$ and $\varepsilon > 0$ there exists a $K_\varepsilon^T > 0$ such that $\mathbb{P}\{\sup_{0\leq t\leq T} |\tilde{Q}^n(t)| > K_\varepsilon^T\} < \varepsilon$ for $n \geq 1$. Thus, by the representation (9) for $\tilde{W}^n$ and the previous two results it is clear that for each $T \geq 0$, $\sup_{0\leq t\leq T} |\tilde{W}^n(t) - \mu^{-1}\tilde{Q}^{n,+}(t)| \Rightarrow 0$ as $n \to \infty$ and hence the claim of the proposition is proven. □

EC4. Proofs of the results in Section 9

Proof of Proposition 9.1. We first show that $\tilde{\varepsilon}^n \Rightarrow 0$ as $n \to \infty$. By (3), for each $t \geq 0$,

$$\begin{aligned}
&\tilde{\varepsilon}^n(t) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} (1\{p_i^n \leq W^n((\tau_i/\lambda^n)-)\}1\{p_i^n \leq t - \tau_i/\lambda^n\} - 1\{p_i^n \leq W^n((\tau_i/\lambda^n)-)\}) \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} 1\{p_i^n \leq W^n((\tau_i/\lambda^n)-)\}1\{p_i^n > t - \tau_i/\lambda^n\} \\
&= -\frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} 1\{t - \tau_i/\lambda^n < p_i^n \leq W^n((\tau_i/\lambda^n)-)\} \\
&= -\frac{1}{\sqrt{n}} \sum_{i=N_A(\lambda^n(t-\sup_{0\leq s\leq t} W^n(s))^+)}^{N_A(\lambda^n t)} 1\{t - \tau_i/\lambda^n < p_i^n \leq W^n((\tau_i/\lambda^n)-)\} \\
&\geq -\frac{1}{\sqrt{n}} \sum_{i=N_A(\lambda^n(t-\sup_{0\leq s\leq t} W^n(s))^+)}^{N_A(\lambda^n t)} 1\left\{p_i^n \leq \sup_{0\leq s\leq t} W^n(s)\right\}. \qquad (10)
\end{aligned}$$

Note that the fourth equality above follows since if $i < N_A(\lambda^n(t - \sup_{0\leq s\leq t} W^n(s))^+$, then $t - \tau_i/\lambda^n > \sup_{0\leq s\leq t} W^n(s) \geq W^n((\tau_i/\lambda^n)-)$ and so $1\{t - \tau_i/\lambda^n < p_i^n \leq W^n((\tau_i/\lambda^n)-)\} = 0$. Now let $\varepsilon > 0$. By Propositions 8.2 and 8.3, for each $T \geq 0$ there exists a C_ε^T such that $\mathbb{P}\{\sup_{0\leq s\leq t} \tilde{W}^n(s) \geq C_\varepsilon^T\} < \varepsilon$ for each $n \geq 1$. Hence, by (10), with probability at least $1 - \varepsilon$,

$$\begin{aligned}
|\tilde{\varepsilon}^n(t)| &\leq \frac{1}{\sqrt{n}} \sum_{N_A(\lambda^n(t-C_\varepsilon^T/\sqrt{n})^+)}^{N_A(\lambda^n t)} 1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\} \\
&\leq \frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} 1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\} - \frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n(t-C_\varepsilon^T/\sqrt{n})^+)} 1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\} \\
&= \frac{1}{n} \sum_{i=1}^{n(n^{-1}N_A(\lambda^n t))} \sqrt{n}1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\} - \frac{1}{n} \sum_{i=1}^{n(n^{-1}N_A(\lambda^n(t-C_\varepsilon^T/\sqrt{n})^+))} \sqrt{n}1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\} \\
&= \frac{1}{n} \sum_{i=1}^{n(n^{-1}N_A(\lambda^n t))} (\sqrt{n}1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\} - E[\sqrt{n}1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\}]) \qquad (11) \\
&\quad -\frac{1}{n} \sum_{i=1}^{n(n^{-1}N_A(\lambda^n(t-C_\varepsilon^T/\sqrt{n})^+))} (\sqrt{n}1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\} - E[\sqrt{n}1\left\{p_i^n \leq C_\varepsilon^T/\sqrt{n}\right\}])
\end{aligned}$$

$$+\frac{1}{n}(N_A(\lambda^n t) - N_A(\lambda^n (t - C_\varepsilon^T/\sqrt{n})^+))E[\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}].$$

However, since

$$E[\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}] \to \int_0^{C_\varepsilon^T} h(u)du,$$

as $n \to \infty$ and both $n^{-1}N_A(\lambda^n e) \Rightarrow \mu e$ and $n^{-1}N_A(\lambda^n(e - C_\varepsilon^T/\sqrt{n})^+) \Rightarrow \mu e$ as $n \to \infty$, it is immediate that

$$\frac{1}{n}(N_A(\lambda^n e) - N_A(\lambda^n (e - C_\varepsilon^T/\sqrt{n})^+))E[\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}] \Rightarrow 0,$$

as $n \to \infty$. In order show that $\tilde{\varepsilon}^n \Rightarrow 0$ as $n \to \infty$, it remains to show that the first two terms in (11) converge to zero as n tends to ∞

For each $n \ge 1$, consider the process

$$\bar{C}^n = \frac{1}{n}\sum_{i=1}^{ne}(\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\} - E[\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}]).$$

It is a martingale with quadratic variation

$$\begin{aligned}\langle\langle \bar{C}^n \rangle\rangle &= \frac{1}{n^2}\sum_{i=1}^{\lfloor ne \rfloor}(\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\} - E[\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}])^2 \\ &\le \frac{1}{n}\sum_{i=1}^{\lfloor ne \rfloor}(1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\} - E[1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}])^2.\end{aligned}$$

Taking expectations, we obtain that for each $T \ge 0$,

$$\begin{aligned}E[\langle\langle \bar{C}^n \rangle\rangle_T] &\le TE[(1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\} - E[1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}])^2 \\ &= TE[1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}](1 - E[1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}]) \\ &\Rightarrow 0,\end{aligned}$$

as $n \to \infty$, since $E[1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}] \to 0$ as $n \to \infty$. However, since $\langle\langle \bar{C}^n \rangle\rangle$ is a non-decreasing process, this then implies that $\langle\langle \bar{C}^n \rangle\rangle \Rightarrow 0$ as $n \to \infty$. Hence, by the martingale invariance principle (3), $\bar{C}^n \Rightarrow 0$ as $n \to \infty$. However, since $n^{-1}N_A(\lambda^n e) \Rightarrow \mu e$ and $n^{-1}N_A(\lambda^n(e - C_\varepsilon^T/\sqrt{n})^+) \Rightarrow \mu e$ as $n \to \infty$, this then implies by the random time change theorem (1), that

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n(n^{-1}N_A(\lambda^n t))}(\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\} - E[\sqrt{n}1\left\{p_i^n \le C_\varepsilon^T/\sqrt{n}\right\}]) \\ &= \bar{C}^n(n^{-1}N_A(\lambda^n t)) \\ &\Rightarrow \bar{C}^n(\mu e) = 0,\end{aligned}$$

as $n \to \infty$, and, similarly,

$$\frac{1}{n}\sum_{i=1}^{n(n^{-1}N_A(\lambda^n(t-C_\varepsilon^T/\sqrt{n})^+))}(\sqrt{n}1\{p_i^n \le C_\varepsilon^T/\sqrt{n}\} - E[\sqrt{n}1\{p_i^n \le C_\varepsilon^T/\sqrt{n}\}]) \Rightarrow 0,$$

as $n \to \infty$, which completes the proof that $\tilde{\varepsilon}^n \Rightarrow 0$ as $n \to \infty$.

We next show that $\tilde{M}^n \Rightarrow 0$ as $n \to \infty$. First note that as in (6), $\tilde{M}^n$ is a martingale with quadratic variation

$$\langle\langle \tilde{M}^n \rangle\rangle_t = \frac{1}{n}\sum_{i=1}^{N_A(\lambda^n t)} F^n(W^n((\tau_i/\lambda^n)-)).$$

Next, for each $\delta, \varepsilon \ge 0$,

$$\begin{aligned}\mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{N_A(\lambda^n T)} F^n(W^n((\tau_i/\lambda^n)-)) \ge \delta\right\} &\le \mathbb{P}\left\{\sup_{0\le t\le T}|\tilde{W}^n(t)| > C_\varepsilon^T\right\} \\ &\quad + \mathbb{P}\left\{\frac{1}{n}N_A(\lambda^n T)F^n(n^{-1/2}C_\varepsilon^T) \ge \delta\right\} \\ &\le \varepsilon + \mathbb{P}\left\{\frac{1}{n}N_A(\lambda^n T)F^n(n^{-1/2}C_\varepsilon^T) \ge \delta\right\}.\end{aligned}$$

However, since by the Functional Strong Law of Large Numbers (1), and (7) and the Random Time Change Theorem (1) we have that $n^{-1}N(\lambda^n T) = n^{-1}N(n(n^{-1}\lambda^n)T) \Rightarrow \mu T$ and also

$$\begin{aligned}F^n(n^{-1/2}C_\varepsilon^T) &= 1 - \exp\left(\int_0^{n^{-1/2}C_\varepsilon^T} h^n(u)du\right) \\ &= 1 - \exp\left(-\frac{1}{\sqrt{n}}\int_0^{C_\varepsilon^T} h(u)du\right) \\ &\le 1 - \exp\left(-\frac{1}{\sqrt{n}}||h||_\infty C_\varepsilon^T\right) \\ &\to 0,\end{aligned}$$

as $n \to \infty$, it follows that

$$\limsup_{n\to\infty}\mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^{N_A(\lambda^n T)} F^n(W^n((\tau_i/\lambda^n)-)) \ge \delta\right\} \le \varepsilon.$$

Thus, since δ and ε were arbitrary and $\langle\langle \tilde{M}^n \rangle\rangle$ is an increasing process, it follows that $\langle\langle \tilde{M}^n \rangle\rangle \Rightarrow 0$ as $n \to \infty$. The result now follows by the Martingale Invariance Principle (3). □

Proof of Proposition 9.2. First note that

$$\tilde{\delta}^n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{N_A(\lambda^n t)} F^n(W^n((\tau_i/\lambda^n)-)) - \int_0^t\left(\int_0^{(1/\mu)\tilde{Q}^{n,+}(s)} h(u)du\right)ds$$

$$= \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} F^n(W^n((\tau_i/\lambda^n)-)) - \frac{1}{n} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{\tilde{W}^n((\tau_i/\lambda^n)-)} h(u)du \right) \tag{12}$$

$$+ \left(\frac{1}{n} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{\tilde{W}^n((\tau_i/\lambda^n)-)} h(u)du - \mu \int_0^t \int_0^{\tilde{W}^n(s)} h(u)duds \right) \tag{13}$$

$$+\mu \left(\int_0^t \int_0^{\tilde{W}^n(s)} h(u)duds - \int_0^t \int_0^{(1/\mu)\tilde{Q}^{n,+}(s)} h(u)duds \right). \tag{14}$$

We now claim that (12)-(14) each converges weakly to 0 as $n \to \infty$.

We begin with (12). Note that

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} F^n(W^n((\tau_i/\lambda^n)-)) - \frac{1}{n} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{\tilde{W}^n((\tau_i/\lambda^n)-)} h(u)du \qquad (15) \\ &= \frac{1}{n} \sum_{i=1}^{N_A(\lambda^n t)} \left(\sqrt{n} F^n(W^n((\tau_i/\lambda^n)-)) - \int_0^{\tilde{W}^n((\tau_i/\lambda^n)-)} h(u)du \right) \\ &\leq \frac{1}{n} N_A(\lambda^n T) \sup_{0 \leq t \leq T} \left(\sqrt{n} F^n(W^n(t)) - \int_0^{\tilde{W}^n(t)} h(u)du \right). \end{aligned}$$

Now note that for each $t \geq 0$,

$$1 - e^{-x} = x - \int_0^x e^{-t}(x-t)dt,$$

and hence

$$\begin{aligned} \sqrt{n} F^n(W^n(t)) &= \sqrt{n} \left(1 - \exp\left(- \int_0^{W^n(t)} h^n(u)du \right) \right) \\ &= \sqrt{n} \left(1 - \exp\left(- \frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du \right) \right) \\ &= \int_0^{\tilde{W}^n(t)} h(u)du - \sqrt{n} \int_0^{\frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du} e^{-t} \left(\frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du - t \right) dt, \end{aligned}$$

and hence by (15), for each $T \geq 0$,

$$\begin{aligned} &\sup_{0 \leq t \leq T} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{N_A(\lambda^n t)} F^n(W^n((\tau_i/\lambda^n)-)) - \frac{1}{n} \sum_{i=1}^{N_A(\lambda^n t)} \int_0^{\tilde{W}^n((\tau_i/\lambda^n)-)} h(u)du \right| \\ &\leq \frac{1}{n} N_A(\lambda^n T) \sup_{0 \leq t \leq T} \left(\sqrt{n} \int_0^{\frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du} e^{-t} \left| \frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du - t \right| dt \right) \\ &\leq \frac{1}{n} N_A(\lambda^n T) \sup_{0 \leq t \leq T} \left(\sqrt{n} \int_0^{\frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du} e^{-t} \left| \frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du + t \right| dt \right) \\ &\leq \frac{2}{n} N_A(\lambda^n T) \sup_{0 \leq t \leq T} \sqrt{n} \left(\frac{1}{\sqrt{n}} \int_0^{\tilde{W}^n(t)} h(u)du \right)^2 \end{aligned}$$

$$\leq \frac{2||h||_\infty}{\sqrt{n}} \left(\frac{1}{n} N_A(\lambda^n T)\right) \left(\sup_{0\leq t\leq T} (\tilde{W}^n(t))^2\right).$$

However, since as in the proof of Proposition 9.1, $n^{-1}N_A(\lambda^n T) \Rightarrow \mu T$ as $n \to \infty$ and since by Proposition 8.2 we have that $\{\sup_{0\leq t\leq T}(\tilde{W}^n(t))^2, n \geq 1\}$ is stochastically bounded, it follows by Slutsky's Theorem (3) that

$$\frac{2||h||_\infty}{\sqrt{n}} \left(\frac{1}{n} N_A(\lambda^n T)\right) \left(\sup_{0\leq t\leq T} (\tilde{W}^n(t))^2\right) \Rightarrow 0, \tag{16}$$

as $n \to \infty$ and hence the first part is proven.

We next proceed to (13). Note that by Propositions 8.2 and 8.3 we have that $\{\tilde{W}^n, n \geq 1\}$ is tight and hence by Theorem 5.1 of (1) it is relatively compact. Let $\{n_k\}$ be a subsequence along which $\{\tilde{W}^n\}$ converges to some limit $\tilde{W}$. It then follows since as in the proof of Proposition 9.1 that $n^{-1}N_A(\lambda^n e) \Rightarrow \mu e$ as $n \to \infty$, we have by Theorem 3.9 of (1) that $(\tilde{W}^{n_k}, {n_k}^{-1}N_A(\lambda^{n_k} e)) \Rightarrow (\tilde{W}, \mu e)$ as $n \to \infty$. By the Skorohod Representation Theorem (1), there exists an alternate probability space on which we may assume that $(\tilde{W}^{n_k}, {n_k}^{-1}N_A(\lambda^{n_k} e)) \to (\tilde{W}, \mu e)$ $\mathbb{P}$-a.s. It then follows by Lemma 8.3 of (2) and the fact that for each $T \geq 0$ we have that $\{\sup_{0\leq t\leq T}|\tilde{W}^{n_k}(t)|, n_k \geq 1\}$ is $\mathbb{P}$-a.s. bounded that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left|\frac{1}{n_k}\sum_{i=1}^{N_A(\lambda^{n_k}t)}\int_0^{\tilde{W}^{n_k}((\tau_i/\lambda^{n_k})-)} h(u)du - \mu\int_0^t\int_0^{\tilde{W}^{n_k}(s)} h(u)duds\right| \\
&= \sup_{0\leq t\leq T}\left|\int_0^t\int_0^{\tilde{W}^{n_k}(s-)} h(u)dud\left(\frac{1}{n_k}N_A(\lambda^{n_k}s)\right) - \mu\int_0^t\int_0^{\tilde{W}^{n_k}(s)} h(u)duds\right| \\
&\to 0,
\end{aligned}$$

as $k \to \infty$. This, then implies that

$$\frac{1}{n_k}\sum_{i=1}^{N_A(\lambda^{n_k}e)}\int_0^{\tilde{W}^{n_k}((\tau_i/\lambda^{n_k})-)} h(u)du - \mu\int_0^e\int_0^{\tilde{W}^{n_k}(s)} h(u)duds \Rightarrow 0,$$

as $k \to \infty$. However, since the choice of $\{n_k\}$ was arbitrary, the result follows.

Finally, for (14) note that by Proposition 8.3 we have that for each $T \geq 0$,

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left|\mu\left(\int_0^t\int_0^{\tilde{W}^n(s)} h(u)duds - \int_0^t\int_0^{(1/\mu)\tilde{Q}^{n,+}(s)} h(u)duds\right)\right| \\
&\leq \mu||h||_\infty T \sup_{0\leq t\leq T}|\tilde{W}^n(s) - (1/\mu)\tilde{Q}^{n,+}(s)| \\
&\Rightarrow 0,
\end{aligned}$$

as $n \to \infty$ and hence the proof is complete. □

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