Supplementary Material

This e-companion contains material that has been omitted from the main paper for the sake of readability and/or brevity. The e-companion is structured as follows. The first section shows that the optimal solutions of the three optimization problems in (2) can differ in general. Section EC.2 provides a synopsis of the various $\mathcal{RVRP}$ formulations, and Section EC.3 relates the robust CVRP to a distributionally robust variant of the chance-constrained CVRP. Section EC.4 provides further computational results that illustrate the performance of Formulation (2VF). We close with the proofs of all propositions and corollaries from the main text.

EC.1. An Example where $\overline{\mathcal{CVRP}} \neq \mathcal{RVRP} \neq \overline{\mathcal{RVRP}}$

Consider a robust CVRP instance with $m = 3$ vehicles of capacity $Q = 10$ and $n = 4$ customers with demand support $\mathcal{Q} = \left\{ q \in \mathbb{R}^4_+ : q_1, q_2 \in [0, 10], \, q_1 + q_2 \leq 10, \, q_3, q_4 \in [0, 1] \right\}$. The graph $G = (V, A)$ and the symmetric transportation costs c_{ij} are defined in Figure EC.1. Figure EC.2 illustrates optimal solutions to $\overline{\mathcal{CVRP}}$, $\mathcal{RVRP}$ and $\overline{\mathcal{RVRP}}$. Note that the optimal solution to $\overline{\mathcal{CVRP}}$ depends on the demand realization $q \in \mathcal{Q}$: for $q_1 \leq 9$, an optimal solution is illustrated in Figure EC.2, whereas for $q_1 > 9$, an optimal solution is given by $\mathbf{R} = (R_1, R_2, R_3)$ with $R_1 = (1)$, $R_2 = (2, 3)$ and $R_3 = (4)$. The optimal objective values are 61, 65 and 80 for $\overline{\mathcal{CVRP}}$, $\mathcal{RVRP}$ and $\overline{\mathcal{RVRP}}$, respectively.

Figure EC.1 Example CVRP instance. The weight of edge $\{i, j\}$ denotes the symmetric transportation costs $c_{ij} = c_{ji}$.

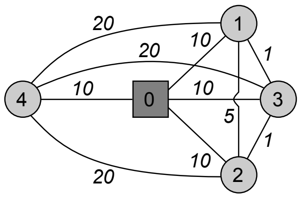
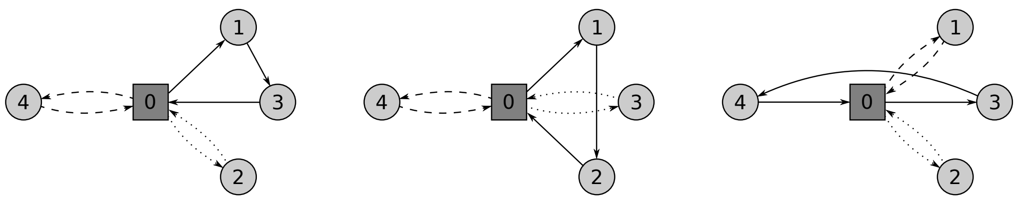


Figure EC.2 Optimal solutions to the CVRP instance from Figure EC.1. From left to right, the graphs illustrate optimal solutions to $\overline{\mathcal{CVRP}}$ (for $q_1 \leq 9$), $\mathcal{RVRP}$ and $\overline{\mathcal{RVRP}}$, respectively.



EC.2. Synopsis of the Robust CVRP Formulations

Table EC.1 summarizes the numbers of variables and constraints required by each of the proposed $\mathcal{RVRP}$ formulations. Note that the transportation costs are often symmetric, that is, $c_{ij} = c_{ji}$ for all $(i, j) \in A$. In this case, one can use the technique presented in Remark 5 of the main text to reduce the number of binary variables x_{ij} in Formulations (2VF) and (VA1). This observation is already reflected in the table, and it is these variants of (2VF) and (VA1) that we used in our computational results. Also note that (2VF) and (VA1) contain an exponential number of robust RCI constraints (7). This renders their direct solution computationally prohibitive for all but the smallest instances. In practice, mixed-integer linear problems with exponentially many constraints are solved using branch-and-cut algorithms that iteratively add only those constraints that are violated. Section 5 of the main text elaborates upon this point.

Table EC.1 Synopsis of the robust CVRP formulations.

Formulation	# Variables		# Constraints†
	Binary	Continuous	
(2VF)	$(1/2)n^2 + \mathcal{O}(n)$	—	$2^n + \mathcal{O}(n)$
(MTZ)	$n^2 + \mathcal{O}(n)$	$n^2r + \mathcal{O}(n^2 + nr)$	$n^3 + \mathcal{O}(n^2)$
(PRE)	$n^2 + \mathcal{O}(n)$	$n^2 + nr$	$2n^3 + \mathcal{O}(n)$
(1CF)	$n^2 + \mathcal{O}(n)$	$n^3 + 2n^2r + \mathcal{O}(n^2 + nr)$	$2n^3 + \mathcal{O}(n^2)$
(2CF)	$(1/2)n^2 + \mathcal{O}(n)$	$n^3 + n^2r + \mathcal{O}(n^2 + nr)$	$(3/2)n^3 + \mathcal{O}(n^2)$
(VA1)	$(1/2)n^2 + \mathcal{O}(n)$	$mn + mr$	$2^n + \mathcal{O}(mn^2)$
(VA2)	$n^2 + \mathcal{O}(n)$	$mn + mr + \mathcal{O}(n)$	$(3/2)mn^2 + \mathcal{O}(n)$

† Excluding variable bounds.

EC.3. The Chance-Constrained CVRP

In the paper, we modeled the customer demands $q \in \mathbb{R}^n_+$ as a random vector that is supported on a set $\mathcal{Q}$, and we determined the minimum cost solution that remains feasible for all demand realizations $q \in \mathcal{Q}$. While such a solution provides maximum insurance against demand uncertainty, it can become conservative. In practice, a decision maker may tolerate a controlled degree of infeasibility if the objective function (*i.e.*, the total transportation cost) decreases significantly. Formally, the decision maker may be interested in the minimum cost vehicle routes $\mathbf{R}$ that satisfy

$$\mathbb{P}\left(\sum_{i \in R_k} q_i \leq Q\right) \geq 1 - \epsilon \qquad \forall\, k \in K, \tag{EC.1}$$

that is, the probability to exceed the capacity of vehicle $k \in K$ is bounded from above by a prespecified value $\epsilon \in (0, 1]$. Here, $\mathbb{P}$ denotes the probability distribution of the customer demands q. Constraint (EC.1) is called a *chance constraint* since it restricts the chance that the customer demands q violate the capacity of vehicle k. Clearly, we recover $\mathcal{RVRP}$ if we set $\epsilon = 0$.

Optimization problems with chance constraints pose several challenges. From a theoretical viewpoint, problems with chance constraints are typically difficult to solve. Except for a few special cases, chance constraints result in non-convex and disconnected feasible regions (Prékopa 1995). Moreover, the evaluation of the left-hand side of constraint (EC.1) involves multi-dimensional integration and becomes computationally intractable for more than four random variables. From a practical perspective, chance constraints put a heavy burden on the decision maker as they require knowledge of the entire probability distribution $\mathbb{P}$.

Robust optimization theory simultaneously alleviates both drawbacks of chance constraints by providing conservative approximations to constraint (EC.1) that are tractable and that require modest information about the probability distribution $\mathbb{P}$. In the following, we first derive a variant of the robust CVRP that can be used to solve a distributionally robust version of the chance-constrained CVRP. Afterwards, we provide a brief computational study of this formulation.

EC.3.1. Relationship of the Robust CVRP to the Chance-Constrained CVRP

The following result applies a popular approximation to $\mathcal{RVRP}$.

PROPOSITION EC.1. *Assume that the probability distribution $\mathbb{P}$ has mean $\mu \in \mathbb{R}^n_+$ and a positive definite covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$, and that the set defined in (C3) contains the support of q, that is, that $\mathbb{P}(Wq \le h) = 1$. Then, every feasible solution to any of the $\mathcal{RVRP}$ formulations from Section 4 induces a set of routes $\mathbf{R}$ that satisfies the chance constraint* (EC.1) *if we replace the set $\mathcal{Q}$ in the $\mathcal{RVRP}$ formulation with*

$$\mathcal{Q}(\epsilon) = \left\{ q \in \mathbb{R}^n_+ : \max\left\{ \frac{1}{\sqrt{n}} \left\| \Sigma^{-1/2}(q-\mu) \right\|_1, \left\| \Sigma^{-1/2}(q-\mu) \right\|_\infty \right\} \le \sqrt{\frac{1-\epsilon}{\epsilon}},\ Wq \le h \right\}.$$

The set $\mathcal{Q}(\epsilon)$ from Proposition EC.1 does not satisfy the requirements of (C3) since it involves nonlinear constraints. Nevertheless, the following result shows that $\mathcal{Q}(\epsilon)$ can be readily used in any of the $\mathcal{RVRP}$ formulations from Section 4.

COROLLARY EC.1. *For decision variables $\zeta \in \mathcal{Z}$ and functions $f : \mathcal{Z} \mapsto \mathbb{R}^n$, $b : \mathcal{Z} \mapsto \mathbb{R}$, the constraint*

$$q^\top f(\zeta) \le b(\zeta) \qquad \forall q \in \mathcal{Q}(\epsilon) \tag{EC.2}$$

is satisfied if and only if there exist $\psi^+, \psi^-, \omega^+, \omega^- \in \mathbb{R}^n_+$, $\vartheta \in \mathbb{R}_+$ and $\tau \in \mathbb{R}^r_+$ such that

$$\begin{aligned}
&\left(\Sigma^{-1/2}\mu\right)^\top \left(\psi^+ - \psi^- + \omega^+ - \omega^-\right) + \sqrt{\frac{1-\epsilon}{\epsilon}} \left(\mathrm{e}^\top \psi^+ + \mathrm{e}^\top \psi^- + \sqrt{n}\vartheta\right) + h^\top \tau \le b(\zeta) \\
&\Sigma^{-1/2}\left(\psi^+ - \psi^- + \omega^+ - \omega^-\right) + W^\top \tau \ge f(\zeta) \\
&\vartheta\, \mathrm{e} \ge \omega^+ + \omega^-.
\end{aligned} \tag{EC.3}$$

In the context of Formulation (VA), for example, we replace the constraints involving the variables ψ_{ks} with m constraints of type (EC.2) in which $\zeta = (z_{1k}, \ldots, z_{nk})^\top$, $f(\zeta) = \zeta$ and $b(\zeta) = Q$, for all $k \in K$.

The chance constraint (EC.1) ensures that for each vehicle $k \in K$, the probability to exceed the vehicle capacity is bounded from above by ϵ. Since there is one such constraint for each vehicle $k \in K$, constraint (EC.1) is called an *individual chance constraint.* In the context of our vehicle routing problem, individual chance constraints guarantee that for each customer $i \in V_C$, the probability of successful delivery is at least $1 - \epsilon$. Alternatively, a decision maker may be interested in the probability that all customer demands are met successfully. This probability can be controlled by the *joint chance constraint*

$$\mathbb{P}\left(\sum_{i \in R_k} q_i \leq Q \quad \forall k \in K\right) \geq 1 - \epsilon' \tag{EC.4}$$

that bounds the probability to exceed the capacity of any of the vehicles $k \in K$. Following the discussion in Nemirovski and Shapiro (2006), we can reformulate constraint (EC.4) using Bonferroni's inequality to obtain

$$\begin{aligned}\mathbb{P}\left(\sum_{i \in R_k} q_i \leq Q \quad \forall k \in K\right) &= 1 - \mathbb{P}\left(\sum_{i \in R_k} q_i > Q \quad \text{for some } k \in K\right) \;\geq\; 1 - \sum_{k \in K} \mathbb{P}\left(\sum_{i \in R_k} q_i > Q\right) \\ &= 1 - \sum_{k \in K}\left[1 - \mathbb{P}\left(\sum_{i \in R_k} q_i \leq Q\right)\right] \;=\; (1-m) + \sum_{k \in K} \mathbb{P}\left(\sum_{i \in R_k} q_i \leq Q\right).\end{aligned}$$

We thus conclude that for any $\epsilon' \in (0,1]$, the joint chance constraint (EC.4) is satisfied whenever the individual chance constraint (EC.1) is satisfied for $\epsilon = \epsilon'/m$. More elaborate approaches to solve problems with joint chance constraints are discussed in Chen et al. (2010), and Nemirovski and Shapiro (2006).

EC.3.2. Computational Study of the Chance-Constrained CVRP

We assume that the uncertain customer demands are supported on the hyper-rectangle $[(1-\alpha)q^0, (1+\alpha)q^0]$, where q^0 denotes the nominal demands from the benchmark instances and $\alpha = 0.2$. We further assume that the demands are independent with expected values $\mu_i = q_i^0$ and standard deviations $\sigma_i = (1/25)q_i^0$. We remark that independency of demands is only assumed here for ease of exposition and that the results in Section EC.3.1 equally apply to the generic case where the customer demands are dependent.

In principle, we can solve the chance-constrained CVRP with any of the $\mathcal{RVRP}$ formulations presented in Section 4. However, since the set $\mathcal{Q}(\epsilon)$ associated with the chance-constrained CVRP cannot be expressed as a budget support or a factor model support, the separation of robust RCI cuts requires the repeated solution of linear programs and becomes computationally expensive.

Table EC.2 Transportation costs for the chance-constrained CVRP. Different values of ϵ and ϵ' correspond to different tolerances for the individual and joint chance constraints, respectively.

Problem	Robust solution	ϵ				ϵ'			
		1%	5%	10%	20%	1%	5%	10%	20%
E-n22-k4	375	375	373	373	373	375	375	375	373
E-n23-k3	569	569	564	564	544	569	569	569	564
P-n16-k8	450	450	450	439	439	450	450	450	450
P-n19-k2	212	212	195	195	195	212	195	195	195
P-n20-k2	216	216	209	208	208	216	209	209	208
P-n21-k2	211	211	211	208	208	211	211	211	208
P-n22-k2	216	216	215	213	213	216	216	215	213
Savings†	0.0%	0.0%	1.6%	3.6%	5.2%	0.0%	0.4%	0.4%	1.1%

†Average cost savings relative to the robust solutions across the 20 benchmark instances from Table 1.

Hence, we did not use formulations (2VF) and (VA1), which require the separation of robust RCI cuts to guarantee robust feasibility, and instead chose Formulation (VA2), which performed best among the remaining formulations in Table 1.

Table EC.2 reports the results for individual and joint chance constraints over a set of benchmark problems. As expected, the transportation costs decrease with growing tolerance levels. In the extreme case where the tolerance level approaches zero, the chance-constrained CVRP reduces to the robust CVRP with support $[(1-\alpha)q^0,(1+\alpha)q^0]$. The table shows that for the same tolerance level, joint chance constraints result in higher transportation costs than individual chance constraints. This is due to the fact that individual chance constraints only provide a probabilistic guarantee that any individual route $k \in K$ is feasible, whereas joint chance constraints provide a probabilistic guarantee that all routes are feasible simultaneously.

EC.4. Detailed Performance of Formulation (2VF)

Table EC.3 presents detailed results that illustrate the performance of Formulation (2VF) over the full suite of 90 benchmark instances considered in the paper.

Table EC.3 Performance of Formulation (2VF) with robust RCI cuts and $(\alpha, \beta) = (0.1, 0.5)$ over 90 benchmark problems and the two supports $\mathcal{Q}_{\mathcal{B}}$ and $\mathcal{Q}_{\mathcal{F}}$.

Problem	Budgets ($\mathcal{Q}_{\mathcal{B}}$)		Factor model ($\mathcal{Q}_{\mathcal{F}}$)		Problem	Budgets ($\mathcal{Q}_{\mathcal{B}}$)		Factor model ($\mathcal{Q}_{\mathcal{F}}$)		Problem	Budgets ($\mathcal{Q}_{\mathcal{B}}$)		Factor model ($\mathcal{Q}_{\mathcal{F}}$)	
	Opt	t (sec) [LB]	Opt	t (sec) [LB]		Opt	t (sec) [LB]	Opt	t (sec) [LB]		Opt	t (sec) [LB]	Opt	t (sec) [LB]
A-n32-k5	748*	0	748*	0	B-n39-k5	532*	0	529*	0	F-n45-k4	718*	20	714*	2
A-n33-k5	642*	6	631*	0	B-n41-k6	796*	5	791*	3	F-n72-k4	232*	0	232*	0
A-n33-k6	717*	1,885	710*	33	B-n43-k6	681*	26	680*	6	F-n135-k7	1,149	[1,085]	1,086	[1,082]
A-n34-k5	715*	0	702*	0	B-n44-k7	835*	20	835*	3	M-n101-k10	809*	8,957	804*	21,329
A-n36-k5	755*	38	766*	4,085	B-n45-k5	701	[693]	680*	0	M-n121-k7	1,054	[961]	1,014	[957]
A-n37-k5	650*	2	648*	0	B-n45-k6	660*	442	657*	441	M-n151-k12	1,219	[935]	1,154	[928]
A-n37-k6	892*	3,795	892	[889]	B-n50-k7	679*	0	699*	0	P-n16-k8	439*	0	439*	0
A-n38-k5	704*	103	693*	2	B-n50-k8	1,228	[1,163]	1,221	[1,188]	P-n19-k2	195*	0	195*	0
A-n39-k5	777*	14,212	772*	2,086	B-n51-k7	962	[939]	928*	8	P-n20-k2	208*	0	208*	0
A-n39-k6	787*	8	786*	5	B-n52-k7	675*	2	670*	0	P-n21-k2	208*	0	208*	0
A-n44-k6	917	[885]	892*	56,840	B-n56-k7	623*	20	623*	0	P-n22-k2	213*	0	213*	0
A-n45-k6	896*	7,173	891*	172	B-n57-k7	1,055	[1,026]	1,052	[1,024]	P-n22-k8	537*	0	557*	0
A-n46-k7	888*	24,072	883*	864	B-n57-k9	1,540*	14,067	1,539*	15,499	P-n23-k8	504	[496]	503	[492]
A-n48-k7	1,033	[1,012]	1,034	[1,009]	B-n63-k10	1,411	[1,370]	1,405	[1,377]	P-n40-k5	447*	6	447*	5
A-n53-k7	974	[966]	967*	15,571	B-n64-k9	803*	0	803*	1	P-n45-k5	501*	894	494*	9
A-n54-k7	1,119	[1,065]	1,106	[1,063]	B-n66-k9	1,339	[1,198]	1,210	[1,205]	P-n50-k7	539	[531]	537	[531]
A-n55-k9	1,030	[1,010]	1,007*	2,736	B-n67-k10	1,007	[982]	1,009	[981]	P-n50-k8	592	[574]	588	[573]
A-n60-k9	1,298	[1,221]	1,269	[1,221]	B-n68-k9	1,215	[1,169]	1,191	[1,173]	P-n50-k10	656	[635]	658	[631]
A-n61-k9	983	[957]	977	[956]	B-n78-k10	1,131	[1,126]	1,130	[1,124]	P-n51-k10	712	[678]	698	[672]
A-n62-k8	1,278	[1,162]	1,320	[1,156]	E-n22-k4	373*	0	373*	0	P-n55-k7	549	[542]	544	[540]
A-n63-k9	1,752	[1,415]	1,728	[1,412]	E-n23-k3	563*	0	544*	0	P-n55-k8	572	[563]	568	[559]
A-n63-k10	1,262	[1,186]	1,229	[1,187]	E-n30-k3	475*	0	492*	0	P-n55-k10	674	[638]	663	[636]
A-n64-k9	1,535	[1,240]	1,500	[1,244]	E-n33-k4	814*	12	814*	4	P-n55-k15	900	[841]	877	[834]
A-n65-k9	1,106	[1,091]	1,094*	8,205	E-n51-k5	516*	429	516*	221	P-n60-k10	712	[683]	712	[677]
A-n69-k9	1,119	[1,057]	1,107	[1,050]	E-n76-k7	661	[645]	672	[643]	P-n60-k15	957	[875]	935	[864]
A-n80-k10	1,986	[1,561]	1,945	[1,560]	E-n76-k8	758	[680]	706	[682]	P-n65-k10	769	[729]	768	[724]
B-n31-k5	651*	0	651*	0	E-n76-k10	887	[755]	842	[750]	P-n70-k10	818	[755]	790	[749]
B-n34-k5	768	[749]	748*	5	E-n76-k14	1,068	[895]	1,089	[884]	P-n76-k4	590*	1,518	590*	1,352
B-n35-k5	883*	0	883*	0	E-n101-k8	846	[765]	786	[766]	P-n76-k5	616	[609]	615	[607]
B-n38-k6	729*	0	729*	0	E-n101-k14	1,171	[971]	1,140	[964]	P-n101-k4	673*	51	673*	97

*Optimally solved instances are highlighted with an asterisk, and the adjacent entries report the solution times t. If optimality could not be guaranteed after 24h CPU, we present the best solution found thus far and the lower bound at termination (in brackets).

EC.5. Propositions and Corollaries with Proofs

Proposition 1. *For any support $\mathcal{Q}$, we have*

$$\overline{\mathcal{CVRP}} \preceq \mathcal{RVRP} \preceq \overline{\mathcal{RVRP}}, \tag{2}$$

where '$\preceq$' refers to the ordering of optimal objective values between the problems. Moreover, the optimal objective values of the three problems coincide if $\mathcal{Q}$ is rectangular.

Proof. The first inequality in (2) follows from the fact that

$$\max_{q\in\mathcal{Q}} \min_{\mathbf{R}\in\mathcal{R}(q)} c(\mathbf{R}) \quad \leq \quad \max_{q\in\mathcal{Q}} \min_{\mathbf{R}\in\mathcal{R}(\mathcal{Q})} c(\mathbf{R}) \quad = \quad \min_{\mathbf{R}\in\mathcal{R}(\mathcal{Q})} c(\mathbf{R}),$$

where the first and last optimization problems are $\overline{\mathcal{CVRP}}$ and $\mathcal{RVRP}$, respectively. The inequality holds because $\mathcal{R}(\mathcal{Q}) \subseteq \mathcal{R}(q)$ for any $q \in \mathcal{Q}$, and the equality follows from the fact that $\min_{\mathbf{R}\in\mathcal{R}(\mathcal{Q})} c(\mathbf{R})$ does not depend on q.

In view of the second inequality in (2), we note that

$$\min_{\mathbf{R}\in\mathcal{R}(\mathcal{Q})} c(\mathbf{R}) \quad \leq \quad \min_{\mathbf{R}\in\mathcal{R}(\overline{\mathcal{Q}})} c(\mathbf{R}) \quad = \quad \min_{\mathbf{R}\in\mathcal{R}(\overline{q})} c(\mathbf{R}),$$

where $\overline{\mathcal{Q}}$ is the rectangular projection of $\mathcal{Q}$ and $\overline{q}$ is defined through $\overline{q}_i = \max\{q_i : q \in \mathcal{Q}\}$, $i \in V_C$. Here, the inequality holds because $\mathcal{R}(\overline{\mathcal{Q}}) \subseteq \mathcal{R}(\mathcal{Q})$, and the equality follows from the monotonicity of $\mathcal{R}$. Note that the last optimization problem corresponds to $\overline{\mathcal{RVRP}}$.

Assume now that the support $\mathcal{Q}$ is rectangular. In this case, $\overline{q} \in \mathcal{Q}$ and hence, the optimal objective value of $\overline{\mathcal{CVRP}}$ must be at least as large as the optimal objective value of $\overline{\mathcal{RVRP}}$. From our previous analysis we then conclude that the optimal objective values of $\overline{\mathcal{CVRP}}$, $\mathcal{RVRP}$ and $\overline{\mathcal{RVRP}}$ coincide. $\square$

Proposition 2. *For supports of the form $\mathcal{Q} = \left\{q \in [0, \overline{q}] : \sum_{i\in V_C} q_i \leq B\right\}$, $\mathcal{RVRP}$ reduces to the solution of a single instance of $\mathcal{CVRP}$.*

Proof. Since the assertion is trivially satisfied if $\overline{q} \not\leq Q\mathrm{e}$, we assume that $\overline{q} \leq Q\mathrm{e}$.

We distinguish the two cases $B \leq Q$ and $B > Q$. If $B \leq Q$, then all customers together can be served by a single vehicle. Since all customer demands are non-negative, we conclude that in fact any subset of customers can be served by a single vehicle. We thus obtain an optimal solution to $\mathcal{RVRP}$ by solving $\mathcal{CVRP}$ for the demand vector $q = 0$.

If $B > Q$, we claim that each robust feasible set of routes for the support $\mathcal{Q}$ is also robust feasible for the rectangular support $\overline{\mathcal{Q}}$. Assume to the contrary that for some robust feasible set of routes $\mathbf{R} \in \mathcal{R}(\mathcal{Q})$ there is a demand vector $q \in \overline{\mathcal{Q}} \setminus \mathcal{Q}$ such that $\sum_{i\in R_k} q_i > Q$ for some vehicle $k \in K$. Since

$q \in \overline{\mathcal{Q}} \setminus \mathcal{Q}$, we have $q \in [0, \overline{q}]$ and $\sum_{i \in V_C} q_i > B$. We consider the demand vector q' defined through $q'_i = (Q+B)\, q_i / \left(2 \sum_{i \in V_C} q_i\right)$ if $i \in R_k$ and $q'_i = 0$ otherwise. By construction, we have $q' \in [0, \overline{q}]$ and

$$\sum_{i \in V_C} q'_i \;\leq\; \frac{Q+B}{2\sum_{i \in V_C} q_i} \sum_{i \in V_C} q_i \;\leq\; B,$$

that is, q' is an element of the support $\mathcal{Q}$, but at the same time

$$\sum_{i \in R_k} q'_i \;=\; \frac{Q+B}{2\sum_{i \in V_C} q_i} \sum_{i \in R_k} q_i \;=\; \frac{Q+B}{2\sum_{i \in V_C} q_i} \sum_{i \in V_C} q_i \;=\; \frac{Q+B}{2} \;>\; Q,$$

that is, q' violates the capacity constraint of vehicle k. This contradicts the robust feasibility of $\mathbf{R}$, and we conclude that each robust feasible set of routes for the support $\mathcal{Q}$ is also robust feasible for the rectangular support $\overline{\mathcal{Q}}$. Hence, $\mathcal{Q}$ can be replaced with $\overline{\mathcal{Q}}$, and the resulting instance of $\mathcal{RVRP}$ is equivalent to the $\mathcal{CVRP}$ instance with demand vector $q = \overline{q}$ by virtue of Proposition 1. $\square$

PROPOSITION 3. *For any support $\mathcal{Q}$ and binary variables x_{ij}, $(i,j) \in A$, we have that*

(i) the robust RCI constraints (7) *are* ***necessary*** *to induce a robust feasible set of routes $\mathbf{R}(x)$;*

(ii) in conjunction with the degree constraints (4)*, the robust RCI constraints* (7) *are* ***sufficient*** *to induce a robust feasible set of routes $\mathbf{R}(x)$.*

Proof. As for the first assertion, note that a robust feasible set of routes $\mathbf{R}(x)$ satisfies

$$\begin{aligned}\left\lceil \max_{q \in \mathcal{Q}} \frac{1}{Q} \sum_{i \in S} q_i \right\rceil \;&\leq\; \left\lceil \frac{1}{Q} \sum_{k \in K} \max_{q \in \mathcal{Q}} \sum_{i \in S \cap R_k(x)} q_i \right\rceil \;\leq\; \left\lceil \frac{1}{Q} \sum_{\substack{k \in K: \\ S \cap R_k(x) \neq \emptyset}} Q \right\rceil \\ &=\; |\{k \in K \,:\, S \cap R_k(x) \neq \emptyset\}| \;\leq\; \sum_{i \in V \setminus S} \sum_{j \in S} x_{ij}.\end{aligned}$$

Here, the first inequality follows from the subadditivity property of the maximum operator, the fact that any set of routes forms a partition of V_C, and the assumption that $S \subseteq V_C$. The second inequality follows from the non-negativity of the customer demands and the definition of robust feasibility. The equality holds by construction, and the last inequality follows from (3) and the fact that each route $R_k(x)$ can enter the customer set S multiple times.

To prove the second assertion, we show that if x satisfies the degree constraints (4) and the robust RCI constraints (7), then $\mathbf{R}(x)$ is a robust feasible set of routes (*i.e.*, it is feasible in $\mathcal{RVRP}$). To this end, we note that for any customer subset $S \subseteq V_C$, the right-hand side of (7) is strictly positive due to (C1) and (C2). Hence, the robust RCI constraints exclude subtours, and Observation 1 implies that x induces a set of routes $\mathbf{R}(x)$. We therefore only need to verify that $\sum_{i \in R_k(x)} q_i \leq Q$ for all $k \in K$ and $q \in \mathcal{Q}$. Assume that this is not the case for some vehicle k and consider the robust RCI constraint (7) for $S = R_k(x)$. The left-hand side of this constraint evaluates to 1, whereas the right-hand side exceeds 1. This contradicts the assumption that the solution satisfies the robust RCI constraints, and we thus conclude that the degree constraints and the robust RCI constraints are sufficient to guarantee robust feasibility of $\mathbf{R}(x)$. $\square$

PROPOSITION 4. *For any support $\mathcal{Q}$ and binary variables x_{ij}, $(i,j) \in A$, that satisfy the degree constraints* (4)*, the robust MTZ constraints* (9) *are satisfiable by functional decision variables $u : \mathcal{Q} \mapsto \mathbb{R}^n_+$ if and only if the constraints*

$$\left.\begin{array}{ll} \displaystyle\sum_{l \in V_C} (v_{jl} - v_{il})\, q_l + Q(1 - x_{ij}) \geq q_j & \forall i, j \in V_C,\, i \neq j \\ \displaystyle q_i \leq \sum_{l \in V_C} v_{il} q_l \leq Q & \forall i \in V_C \end{array}\right\} \forall q \in \mathcal{Q} \tag{10}$$

are satisfiable by decision variables $v \in \mathbb{R}^{n \times n}$.

Proof. If the constraints (10) are satisfied by variables $v \in \mathbb{R}^{n \times n}$, then the robust MTZ constraints (9) are satisfied by the functions $u : \mathcal{Q} \mapsto \mathbb{R}^n_+$ defined through $u_i(q) = \sum_{l \in V_C} v_{il} q_l$, $i \in V_C$.

Assume now that the robust MTZ constraints (9) are satisfied by the functional decision variables $u : \mathcal{Q} \mapsto \mathbb{R}^n_+$. By virtue of Observation 2, this implies that $\mathbf{R}(x)$ constitutes a robust feasible set of routes. Consider the functional decision variables $u' : \mathcal{Q} \mapsto \mathbb{R}^n_+$ defined through

$$u'_j(q) = \sum_{i=1}^{l} q_{R_{k,i}(x)} \qquad \text{for } j = R_{k,l}(x),\ k \in K,\ 1 \leq l \leq n_k.$$

Here, $u'_j(q)$ can be interpreted as the cumulative customer demands served by the vehicle that visits customer $j \in V_C$, immediately after it has left customer j, under the demand realization $q \in \mathcal{Q}$. By construction, u' exhibits a linear dependence on q and also satisfies the robust MTZ constraints (9). If we identify v_{jp} with the partial derivative of u'_j with respect to q_p, that is,

$$v_{jp} = \begin{cases} 1 & \text{if there is } k \in K \text{ and } 1 \leq l \leq n_k \text{ such that } p \in \{R_{k,1}(x), \ldots, R_{k,l}(x) = j\}, \\ 0 & \text{otherwise,} \end{cases} \tag{EC.5}$$

then v satisfies the constraints (10). $\square$

PROPOSITION 5. *For any support $\mathcal{Q}$ and binary variables x_{ij}, $(i,j) \in A$, that satisfy the degree constraints* (4)*, the constraints* (10) *are satisfiable by $v \in \mathbb{R}^{n \times n}$ if and only if the constraints*

$$\begin{array}{ll} x_{ij} \leq p_{ij} \leq \min\{1 - x_{ji}, 1 - x_{0j}\} & \forall i, j \in V_C,\, i \neq j \\ 1 - x_{ij} \geq \max\{p_{li} - p_{lj}, p_{lj} - p_{li}\} & \forall i, j \in V_C,\, i \neq j,\, \forall l \in V_C \setminus \{i, j\} \\ \displaystyle\sum_{j \in V_C} p_{ji} q_j \leq Q & \forall i \in V_C,\, q \in \mathcal{Q} \end{array} \tag{11}$$

are satisfiable by $p \in \{0,1\}^{n \times n}$, where $p_{ii} = 1$ for $i \in V_C$.

Proof. Assume that the constraints (10) are satisfiable. Then, Proposition 4 ensures that v_{jp} defined in (EC.5) satisfies (10), and one can verify that $p_{ij} = v_{ji}$ satisfies the constraints (11).

Assume now that $p \in \{0,1\}^{n\times n}$, $p_{ii} = 1$ for $i \in V_C$, satisfies the constraint set (11). One readily verifies that $v_{il} = p_{li}$ satisfies the second constraint in (10). We now show that the first constraint in (10) is also satisfied. To this end, assume that $x_{ij} = 1$ for some $(i,j) \in A$. Then the first constraint in (10) becomes

$$\sum_{l\in V_C} v_{jl} q_l \geq \sum_{l\in V_C} v_{il} q_l + q_j \qquad \forall\, q \in \mathcal{Q}.$$

This constraint is satisfied because *(i)* $v_{jj} = p_{jj} = 1$ while $v_{ij} = p_{ji} = 0$ since $x_{ij} = 1$, *(ii)* $v_{ji} = p_{ij} = 1$ since $x_{ij} = 1$ and $v_{ii} = p_{ii} = 1$, and *(iii)* $v_{jl} = p_{lj}$ coincides with $v_{il} = p_{li}$ for all $l \in V_C \setminus \{i,j\}$ due to the second constraint in (11). Assume now that $x_{ij} = 0$. Then the first constraint in (10) becomes

$$\sum_{l\in V_C} v_{jl} q_l - \sum_{l\in V_C} v_{il} q_l \geq -(Q - q_j) \qquad \forall\, q \in \mathcal{Q}. \tag{EC.6}$$

The non-negativity of p and the last constraint in (11) guarantee that the two summations on the left-hand side of this constraint are non-negative and bounded from above by Q for all $q \in \mathcal{Q}$. Since $v_{jj} = p_{jj} = 1$, the first summation is furthermore bounded from below by q_j, which implies satisfaction of constraint (EC.6). □

Proposition 6. *For any support $\mathcal{Q}$ and any solution (x,p), $x_{ij} \in \{0,1\}$, $(i,j) \in A$, $p \in \{0,1\}^{n\times n}$, $p_{ii} = 1$ for $i \in V_C$, that satisfies the degree constraints* (4) *and the constraints* (11)*, we have $p_{ij} = 1$ if and only if customers i and j are served by the same vehicle and customer i is served first.*

Proof. Fix two customers $i,j \in V_C$, $i \neq j$. We determine the value of p_{ij} in two cases.

Customer i precedes customer j on the same route. In that case, there is $k \in K$, $1 \leq l \leq n_k$ and $t \in \mathbb{N} \setminus \{1\}$ such that $i = R_{k,l}(x)$ and $j = R_{k,l+t-1}(x)$. Let $\{i_s\}_{s=1}^{t}$ denote the customers served between i and j, that is, $i_1 = i$, $i_2 = R_{k,l+1}$, $\ldots$, $i_t = j$. Since $x_{i_1 i_2} = 1$, the first constraint in (11) guarantees that $p_{i_1 i_2} = 1$. Likewise, if $t \geq 3$, then $x_{i_s i_{s+1}} = 1$ for all $s = 2,\ldots,t-1$ guarantees that $p_{i_1 i_{s+1}} = 1$ due to the second constraint in (11). We thus conclude that $p_{ij} = p_{i_1 i_t} = 1$.

Customer i does not precede customer j on the same route. In that case, there is $k \in K$, $1 \leq l \leq n_k$, $t \in \mathbb{N}$ and $\{i_s\}_{s=1}^{t}$ such that $i_1 = R_{k,1}(x)$, $\ldots$, $i_t = R_{k,t}(x) = j$ and $i \notin \{i_1,\ldots,i_t\}$. Since $x_{0i_1} = 1$, the first constraint in (11) guarantees that $p_{ii_1} = 0$. Likewise, if $t \geq 2$, then $x_{i_s i_{s+1}} = 1$ for all $s = 1,\ldots,t-1$ guarantees that $p_{ii_{s+1}} = 0$ due to the second constraint in (11). We thus conclude that $p_{ij} = p_{ii_t} = 0$. □

Corollary 1. *For any support $\mathcal{Q}$ and binary variables $x_{ij} \in \{0,1\}$, $(i,j) \in A$, that satisfy the degree constraints* (4)*, the constraints* (11) *are satisfiable by binary variables $p \in \{0,1\}^{n\times n}$, $p_{ii} = 1$ for $i \in V_C$, if and only if they are satisfiable by continuous variables $p \in [0,1]^{n\times n}$, $p_{ii} = 1$ for $i \in V_C$.*

Proof. This follows from the fact that the proof of Proposition 6 does not use the assumption that the variable vector p is binary. $\square$

PROPOSITION 7. *For any support $\mathcal{Q}$ and binary variables x_{ij}, $(i,j) \in A$, that satisfy the degree constraints* (4)*, the robust 1CF constraints* (13) *are satisfiable by functional decision variables $f_{ij} : \mathcal{Q} \mapsto \mathbb{R}_+$, $(i,j) \in A$, if and only if the constraints*

$$\begin{aligned} &\sum_{\substack{i \in V: \\ i \neq j}} (g_{jil} - g_{ijl}) = \mathbb{I}_{[l=j]} && \forall\, j, l \in V_C \\ &q_i x_{ij} \leq \sum_{l \in V_C} g_{ijl} q_l \leq (Q - q_j)\, x_{ij} && \forall\, (i,j) \in A,\ \forall\, q \in \mathcal{Q} \end{aligned} \tag{14}$$

are satisfiable by decision variables $g_{ijl} \in \mathbb{R}$, $(i,j) \in A$ and $l \in V_C$.

Proof. Assume that the constraints (14) are satisfied by decision variables $g_{ijl} \in \mathbb{R}$, $(i,j) \in A$ and $l \in V_C$, and define the functional decision variables $f_{ij} : \mathcal{Q} \mapsto \mathbb{R}_+$ through $f_{ij}(q) = \sum_{l \in V_C} g_{ijl} q_l$, $(i,j) \in A$. For any $j \in V_C$, we obtain

$$\sum_{\substack{i \in V: \\ i \neq j}} [f_{ji}(q) - f_{ij}(q)] \;=\; \sum_{\substack{i \in V: \\ i \neq j}} \sum_{l \in V_C} [g_{jil} - g_{ijl}]\, q_l \;=\; q_j,$$

where the second equality follows from the first constraint in (14). Hence, f_{ij} satisfies the first robust 1CF constraint in (13). In the same way, one verifies that f_{ij} also satisfies the second constraint in (13). We thus conclude that the constraints (13) are satisfiable by functional decision variables f_{ij} whenever the constraints (14) are satisfiable by decision variables g_{ijl}.

Assume now that the robust 1CF constraints (13) are satisfied by the functional decision variables $f_{ij} : \mathcal{Q} \mapsto \mathbb{R}_+$, $(i,j) \in A$. By virtue of Observation 3, this implies that $\mathbf{R}(x)$ constitutes a robust feasible set of routes. Consider the functional decision variables $f'_{ij} : \mathcal{Q} \mapsto \mathbb{R}_+$, $(i,j) \in A$, defined through

$$f'_{ij}(q) = x_{ij} \sum_{s=1}^{l} q_{R_{k,s}(x)} \qquad \text{for } i = R_{k,l}(x),\ k \in K,\ 1 \leq l \leq n_k.$$

Here, $f'_{ij}(q)$ can be interpreted as the cumulative customer demands served by the vehicle that traverses the arc $(i,j) \in A$, immediately after it has left the (customer or depot) node $i \in V$, under the demand realization $q \in \mathcal{Q}$. By construction, f'_{ij} exhibits a linear dependence on q and also satisfies the robust 1CF constraints (13). If we identify g_{ijp} with the partial derivative of f'_{ij} with respect to q_p, that is,

$$g_{ijp} = \begin{cases} 1 & \text{if } x_{ij} = 1 \text{ and there is } k \in K \text{ and } 1 \leq l \leq n_k \text{ such that } p \in \{R_{k,1}(x), R_{k,2}(x), \ldots, R_{k,l}(x) = i\}, \\ 0 & \text{otherwise,} \end{cases}$$

then g satisfies the constraints (14). $\square$

PROPOSITION 8. *For any binary variables x_{ij}, $(i,j) \in A$, that satisfy the degree constraints* (4) *and the subtour constraints* (5)*, $z \in \{0,1\}^{n\times m}$ satisfies the* Vehicle Assignment (VA) constraints

$$\begin{aligned}
&1 - x_{ij} - x_{ji} \geq z_{ik} - z_{jk} && \forall\, i,j \in V_C,\ i \neq j,\ \forall\, k \in K \\
&z_{ik} + \sum_{l \in \{1,\ldots,k\}} z_{jl} \leq 3 - x_{0i} - x_{0j} && \forall\, i,j \in V_C,\ i < j,\ \forall\, k \in K \\
&\sum_{k \in K} z_{ik} = 1 && \forall\, i \in V_C
\end{aligned} \tag{15}$$

if and only if $z_{ik} = \mathbb{I}_{[i \in R_k(x)]}$, where $\mathbf{R}(x)$ satisfies $R_{1,1}(x) < R_{2,1}(x) < \ldots < R_{m,1}(x)$.

Proof. If the routing decision x satisfies the degree constraints (4) and the subtour constraints (5), then x induces a set of routes $\mathbf{R}(x)$ by virtue of Observation 1. One readily verifies that if $\mathbf{R}(x)$ satisfies $R_{1,1}(x) < R_{2,1}(x) < \ldots < R_{m,1}(x)$, then the binary vector $z \in \{0,1\}^{n\times m}$ defined through $z_{ik} = \mathbb{I}_{[i \in R_k(x)]}$ satisfies the constraints (15).

It remains to be shown that any binary vector $z \in \{0,1\}^{n\times m}$ which satisfies the constraints (15) also satisfies $z_{ik} = \mathbb{I}_{[i \in R_k(x)]}$ for the unique set of routes $\mathbf{R}(x)$ that satisfies $R_{1,1}(x) < R_{2,1}(x) < \ldots < R_{m,1}(x)$. We prove this assertion in two steps. First, we show that if two customers $i,j \in V_C$ are served by the same vehicle in $\mathbf{R}(x)$, that is, if there is $k \in K$ such that $i,j \in R_k(x)$, then there is a vehicle $k' \in K$ such that $z_{il} = z_{jl} = \mathbb{I}_{[l=k']}$ for all $l \in K$. Subsequently, we prove that if two customers $i,j \in V_C$, $i < j$, are the first to be served on their respective routes in $\mathbf{R}(x)$, that is, if $x_{0i} = x_{0j} = 1$, then there are two vehicles $k,k' \in K$ such that $z_{il} = \mathbb{I}_{[l=k]}$, $z_{jl} = \mathbb{I}_{[l=k']}$ and $k < k'$. The combination of both statements then proves the proposition.

In view of the first assertion, assume that the two customers $i,j \in V_C$ are served by the same vehicle in $\mathbf{R}(x)$, that is, that $i,j \in R_k(x)$ for some vehicle $k \in K$. By virtue of Observation 1, this implies that there is a sequence of customers $\{i_1,\ldots,i_t\} \subseteq V_C$ such that $(i_1,i_t) = (i,j)$ or $(i_1,i_t) = (j,i)$, as well as $x_{i_s i_{s+1}} = 1$ for all $s = 1,\ldots,t-1$. The first constraint in (15) then requires that $z_{i_s l} = z_{i_{s+1} l}$ for all $s = 1,\ldots,t-1$ and $l \in K$, that is, in particular $z_{il} = z_{jl}$ for all $l \in K$. The last constraint in (15) implies that for any customer $i \in V_C$ there is a vehicle $k' \in K$ such that $z_{il} = \mathbb{I}_{[l=k']}$ for all $l \in K$. Taken together, our previous observations ensure that there is a vehicle $k' \in K$ such that $z_{il} = z_{jl} = \mathbb{I}_{[l=k']}$ for all $l \in K$.

As for the second assertion, assume that the two customers $i,j \in V_C$, $i < j$, are the first to be served on their respective routes in $\mathbf{R}(x)$, that is, that $x_{0i} = x_{0j} = 1$. Due to the third constraint in (15), there are two vehicles $k,k' \in K$ such that $z_{il} = \mathbb{I}_{[l=k]}$ and $z_{jl} = \mathbb{I}_{[l=k']}$. Assume now that $k' \leq k$ and consider the second constraint in (15) associated with i, j and k. The left-hand side of this constraint is greater or equal to $z_{ik} + z_{jk'} = 2$, whereas its right-hand side evaluates to $3 - x_{0i} - x_{0j} = 1$. This is a contradiction, and we conclude that $k < k'$, which proves our second assertion. $\square$

Proposition 9. *Assume that the binary variables x_{ij}, $(i,j) \in A$, satisfy the degree constraints* (4) *and the subtour constraints* (5)*. Then the VA constraints* (15) *are satisfiable by a variable vector $z \in \{0,1\}^{n\times m}$ if and only if they are satisfiable by a variable vector $z \in [0,1]^{n\times m}$.*

Proof. Since $\{0,1\}^{n\times m} \subseteq [0,1]^{n\times m}$, the constraints (15) are satisfiable by a variable vector $z \in [0,1]^{n\times m}$ whenever they are satisfiable by a variable vector $z \in \{0,1\}^{n\times m}$.

Let us now assume that the constraints (15) are satisfied by the variable vector $z \in [0,1]^{n\times m}$. We want to show that $z \in \{0,1\}^{n\times m}$, that is, that z is indeed binary. To this end, we will first show that if two customers $i,j \in V_C$ are served by the same vehicle in $\mathbf{R}(x)$, that is, if there is $k \in K$ such that $i,j \in R_k(x)$, then $z_{il} = z_{jl}$ for all $l \in K$. In particular, this implies that all components of z are binary if and only if the components z_{il} corresponding to the first customers $i \in \{R_{k,1}(x)\}_{k\in K}$ on each route $k \in K$ are binary for all $l \in K$. Next, we show that if $i = R_{k,1}(x)$ for some vehicle $k \in K$, then $z_{il} = 0$ for all $l > k$. Finally, we show that if $i = R_{k,1}(x)$ for some vehicle $k \in K$, then $z_{il} = 0$ also for all $l < k$. Together with the last constraint in (15), the previous two assertions ensure that for any route $k \in K$, the first customer $i = R_{k,1}(x)$ satisfies $z_{il} = 1$ if $l = k$ and $z_{il} = 0$ otherwise. In other words, the components z_{il} corresponding to the first customers $i \in \{R_{k,1}(x)\}_{k\in K}$ on each route $k \in K$ are indeed binary for all $l \in K$, which concludes the proof.

In view of the first assertion, we use a similar argument as in the proof of Proposition 8. Assume that the two customers $i,j \in V_C$ are served by the same vehicle in $\mathbf{R}(x)$, that is, that $i,j \in R_k(x)$ for some vehicle $k \in K$. By virtue of Observation 1, this implies that there is a sequence of customers $\{i_1,\ldots,i_t\} \subseteq V_C$ such that $(i_1,i_t) = (i,j)$ or $(i_1,i_t) = (j,i)$, as well as $x_{i_s i_{s+1}} = 1$ for all $s = 1,\ldots,t-1$. The first constraint in (15) then requires that $z_{i_s l} = z_{i_{s+1} l}$ for all $s = 1,\ldots,t-1$ and $l \in K$, that is, in particular $z_{il} = z_{jl}$ for all $l \in K$.

Consider now the second assertion. Since the assertion is trivially satisfied for $m = 1$, we assume that $m > 1$. For any vehicle $k \in K$, we define $\mathcal{I}_k = \{R_{1,1}(x),\ldots,R_{k,1}(x)\}$ as the set of all customers that are served first by one of the vehicles $1,\ldots,k$. We want to show that $z_{ik} = 0$ for all $k = 2,\ldots,m$ and $i \in \mathcal{I}_{k-1}$. We achieve this by induction over k. For $k = m$, consider the second constraint in (15) associated with k, $i \in \mathcal{I}_{m-1}$ and $j = R_{m,1}(x)$. Summing over all these constraints gives

$$\sum_{i\in\mathcal{I}_{m-1}} z_{im} + (m-1)\sum_{l\in K} z_{jl} \,\le\, m-1.$$

The second term on the left-hand side of this inequality evaluates to $m-1$ due to the third constraint in (15). From the non-negativity of z we thus conclude that $z_{im} = 0$ for all $i \in \mathcal{I}_{m-1}$. Assume now that for some vehicle $k \in \{2,\ldots,m-1\}$, we have $z_{il} = 0$ for all $l \in \{k+1,\ldots,m\}$ and $i \in \mathcal{I}_{l-1}$. We want to show that $z_{ik} = 0$ for all $i \in \mathcal{I}_{k-1}$. To this end, consider the second constraint in (15) associated with k, $i \in \mathcal{I}_{k-1}$ and $j = R_{k,1}(x)$. Summing over all these constraints gives

$$\sum_{i\in\mathcal{I}_{k-1}} z_{ik} + (k-1)\sum_{l\in\{1,\ldots,k\}} z_{jl} \,\le\, k-1.$$

The second term on the left-hand side of this inequality evaluates to $k-1$. To see this, we note that $j \in \mathcal{I}_k$, and hence we have $z_{jl} = 0$ for all $l = k+1, \ldots, m$ due to the induction hypothesis. Thus, the second term on the left-hand side is equivalent to $(k-1)\sum_{l \in K} z_{jl}$, and this term evaluates to $k-1$ due to the third constraint in (15). From the non-negativity of z we thus conclude that $z_{ik} = 0$ for all $i \in \mathcal{I}_{k-1}$.

Consider now the third assertion. Again, this assertion is trivially satisfied for $m = 1$, and we can assume that $m > 1$. For any vehicle $k \in K$, we define $\mathcal{J}_k = \{R_{k,1}(x), \ldots, R_{m,1}(x)\}$ as the set of all customers that are served first by one of the vehicles $k, \ldots, m$. We want to show that $z_{jk} = 0$ for all $k = 1, \ldots, m-1$ and $j \in \mathcal{J}_{k+1}$. We again achieve this by induction over k. For $k = 1$, consider the second constraint in (15) associated with k, $i = R_{1,1}(x)$ and $j \in \mathcal{J}_2$. Summing over all these constraints gives

$$(m-1)\, z_{i1} + \sum_{j \in \mathcal{J}_2} z_{j1} \;\leq\; m-1.$$

The first term on the left-hand side of this inequality evaluates to $m-1$. To see this, we note that $z_{il} = 0$ for all $l \in \{2, \ldots, m\}$ due to the second assertion in this proof, and hence the third constraint in (15) implies that $z_{i1} = 1$. From the non-negativity of z we thus conclude that $z_{j1} = 0$ for all $j \in \mathcal{J}_2$. Assume now that for some vehicle $k \in \{2, \ldots, m-1\}$, we have $z_{jl} = 0$ for all $l \in \{1, \ldots, k-1\}$ and $j \in \mathcal{J}_{l+1}$. We want to show that $z_{jk} = 0$ for all $j \in \mathcal{J}_{k+1}$. To this end, consider the second constraint in (15) associated with k, $i = R_{k,1}(x)$ and $j \in \mathcal{J}_{k+1}$. Summing over all these constraints gives

$$(m-k)\, z_{ik} + \sum_{\substack{j \in \mathcal{J}_{k+1},\\ l \in \{1, \ldots, k\}}} z_{jl} \;\leq\; m-k.$$

The first term on the left-hand side of this inequality evaluates to $m-k$. To see this, we note that $z_{il} = 0$ for all $l \in \{k+1, \ldots, m\}$ due to the second assertion in this proof and $z_{il} = 0$ for all $l \in \{1, \ldots, k-1\}$ due to the induction hypothesis. Hence, the third constraint in (15) implies that $z_{ik} = 1$. From the non-negativity of z we conclude that $z_{jl} = 0$ for all $j \in \mathcal{J}_{k+1}$ and $l \in \{1, \ldots, k\}$, that is, in particular $z_{jk} = 0$ for all $j \in \mathcal{J}_{k+1}$. □

PROPOSITION 10. *Assume that the sets $\{B_l\}_{l=1}^L$ in (17) are disjoint, that is, $B_l \cap B_{l'} = \emptyset$ for $l \neq l'$. Then, for any customer subset $S \subseteq V_C$, the maximum of $\sum_{i \in S} q_i$ over $\mathcal{Q}$ from (17) is given by*

$$\sum_{i \in S} \underline{q}_i + \sum_{l=1}^{L} \min\left\{ b_l - \sum_{i \in B_l} \underline{q}_i,\; \sum_{i \in S \cap B_l} \left(\overline{q}_i - \underline{q}_i\right) \right\} + \sum_{i \in S \setminus \bigcup_{l=1}^{L} B_l} \left(\overline{q}_i - \underline{q}_i\right).$$

Proof. If we introduce variables v_i for the excess demand of customer $i \in V_C$ beyond $\underline{q}_i$, then the maximization of $\sum_{i \in S} q_i$ over the polytope defined in (17) can be reformulated as

$$\max_{v \in \mathbb{R}^n_+} \left\{ \sum_{i \in S} \left(\underline{q}_i + v_i\right) \,:\, v \leq \overline{q} - \underline{q},\; \sum_{i \in B_l} \left(\underline{q}_i + v_i\right) \leq b_l \text{ for } l = 1, \ldots, L \right\}.$$

Strong linear programming duality implies that this problem is equivalent to

$$\min_{\substack{\eta\in\mathbb{R}^n_+,\\ \theta\in\mathbb{R}^L_+}} \left\{ \sum_{i\in S} \underline{q}_i + \sum_{i\in V_C} (\overline{q}_i - \underline{q}_i)\,\eta_i + \sum_{l=1}^{L} \left(b_l - \sum_{i\in B_l} \underline{q}_i \right) \theta_l \,:\, \eta_i + \sum_{l=1}^{L} \mathbb{I}_{[i\in B_l]}\theta_l \geq 1 \;\; \forall i\in S \right\}. \qquad \text{(EC.7)}$$

By condition (C1), the support $\mathcal{Q}$ is nonempty, which implies that the objective coefficients in (EC.7) are non-negative, that is, the objective value of (EC.7) is non-decreasing in η and θ. An inspection of the constraints reveals that there is always an optimal solution to (EC.7) that satisfies all constraints as equalities. In particular, from the objective coefficients of η and θ we conclude that we obtain an optimal solution (η^*,θ^*) if for each $l=1,\ldots,L$, we set $\theta^*_l=1$ if $b_l - \sum_{i\in B_l}\underline{q}_i < \sum_{i\in S\cap B_l}(\overline{q}_i - \underline{q}_i)$ and $\theta^*_l = 0$ otherwise, as well as $\eta^*_i = \left[\mathbb{I}_{[i\in S]} - \sum_{l=1}^{L}\mathbb{I}_{[i\in B_l]}\theta^*_l\right]^+$. This solution attains the objective value stated in the assertion. □

Proposition 11. *For a customer subset $S\subseteq V_C$, assume that $f_1,\ldots,f_F$ represents an ordering of the factors ξ_f in* (18b) *according to non-increasing marginal demands*

$$\sum_{i\in S}\Phi_{if_1} \geq \sum_{i\in S}\Phi_{if_2} \geq \ldots \geq \sum_{i\in S}\Phi_{if_F}.$$

Then, the maximum of $\sum_{i\in S} q_i$ over the polytope defined in (18) *is given by*

$$\sum_{i\in S} q^0_i + \min\left\{ \sum_{f=1}^{F} \left| \sum_{i\in S}\Phi_{if} - \lambda \right| + \beta F\,|\lambda| \,:\, \lambda\in\left\{0,\, \sum_{i\in S}\Phi_{if_{\ell^+}},\, \sum_{i\in S}\Phi_{if_{\ell^-}}\right\}\right\}, \qquad (19)$$

where $\ell^+ = \lceil (1+\beta)F/2\rceil$ and $\ell^- = \max\{\lceil(1-\beta)F/2\rceil, 1\}$.

Proof. The maximum of $\sum_{i\in S} q_i$ over the polytope defined in (18) is given by

$$\max_{\xi\in\mathbb{R}^F}\left\{ \sum_{i\in S}\left[q^0_i + \sum_{f=1}^{F}\Phi_{if}\xi_f \right] \,:\, -\mathrm{e}\leq\xi\leq+\mathrm{e},\; -\beta F \leq \mathrm{e}^\top\xi \leq +\beta F \right\}.$$

Strong linear programming duality implies that this problem is equivalent to

$$\min_{\substack{\eta^+,\eta^-\in\mathbb{R}^F_+,\\ \theta^+,\theta^-\in\mathbb{R}_+}} \left\{ \sum_{i\in S} q^0_i + \sum_{f=1}^{F}(\eta^+_f + \eta^-_f) + \beta F(\theta^+ + \theta^-) \,:\, (\eta^+_f - \eta^-_f) + (\theta^+ - \theta^-) = \sum_{i\in S}\Phi_{if} \text{ for } f=1,\ldots,F \right\}.$$

Since the objective coefficients of all decision variables are non-negative, there is always an optimal solution that satisfies the complementarity conditions $\eta^+_f\cdot\eta^-_f = 0$, $f=1,\ldots,F$, and $\theta^+\cdot\theta^-=0$. We can therefore replace the non-negative variables η^+_f, η^-_f, θ^+ and θ^- with unrestricted variables η_f and θ to obtain the equivalent optimization problem

$$\min_{\substack{\eta\in\mathbb{R}^F,\\ \theta\in\mathbb{R}}} \left\{ \sum_{i\in S} q^0_i + \sum_{f=1}^{F}|\eta_f| + \beta F\,|\theta| \,:\, \eta_f + \theta = \sum_{i\in S}\Phi_{if} \text{ for } f=1,\ldots,F \right\}.$$

Any feasible solution (η, θ) to this problem satisfies $\eta_f = \sum_{i \in S} \Phi_{if} - \theta$, so that the problem is equivalent to the following one-dimensional nonlinear program.

$$\min_{\theta \in \mathbb{R}} \;\; \phi(\theta) \;=\; \sum_{i \in S} q_i^0 + \sum_{f=1}^{F} \left| \sum_{i \in S} \Phi_{if} - \theta \right| + \beta F \left| \theta \right|.$$

This problem has a convex, piecewise linear objective function ϕ that is minimized over the real line. Since $\phi(\theta) \longrightarrow +\infty$ for $\theta \longrightarrow \pm\infty$, ϕ is minimized at one of its breakpoints $\theta \in \{0\} \cup \left\{ \sum_{i \in S} \Phi_{if} \right\}_{f=1}^{F}$. Note that the expression (19) evaluates to the minimum of $\phi(\theta)$ over the three breakpoints $\theta \in \left\{ 0, \sum_{i \in S} \Phi_{if_{\ell+}}, \sum_{i \in S} \Phi_{if_{\ell-}} \right\}$. Thus, (19) constitutes an upper bound on the minimum of ϕ. In the following, we argue that (19) also constitutes a lower bound on the minimum of ϕ. This is trivially the case if ϕ is minimized at the breakpoint $\theta^* = 0$. Assume now that ϕ is minimized at the breakpoint $\theta^* = \sum_{i \in S} \Phi_{if_\ell}$, $\ell \in \{1, \ldots, F\}$, where we use the ordering $f_1, \ldots, f_F$ of factors described in the assertion, and that this breakpoint satisfies $\theta^* > 0$. Then, ϕ has the following directional derivatives at θ^*:

$$\begin{aligned}
\lim_{h \uparrow 0} \frac{\phi(\theta^* + h) - \phi(\theta^*)}{h} \;&=\; \left| \left\{ f \in \{1, \ldots, F\} : \sum_{i \in S} \Phi_{if} < \theta^* \right\} \right| - \left| \left\{ f \in \{1, \ldots, F\} : \sum_{i \in S} \Phi_{if} \geq \theta^* \right\} \right| + \beta F \\
&\leq\; (1 + \beta) F - 2\ell \\
\lim_{h \downarrow 0} \frac{\phi(\theta^* + h) - \phi(\theta^*)}{h} \;&=\; \left| \left\{ f \in \{1, \ldots, F\} : \sum_{i \in S} \Phi_{if} \leq \theta^* \right\} \right| - \left| \left\{ f \in \{1, \ldots, F\} : \sum_{i \in S} \Phi_{if} > \theta^* \right\} \right| + \beta F \\
&\geq\; (1 + \beta) F - 2\ell + 2.
\end{aligned}$$

Since the first-order optimality conditions are necessary and sufficient for ϕ, we can assume that $\phi(\theta^*)$ has a non-positive left derivative and a positive right derivative. This is the case if and only if $\theta^* = \sum_{i \in S} \Phi_{if_{\ell+}}$, and in that case the value of $\phi(\theta^*)$ is given by (19) evaluated at $\lambda = \sum_{i \in S} \Phi_{if_{\ell+}}$. Similarly, one can show that the minimum of ϕ is given by (19) evaluated at $\lambda = \sum_{i \in S} \Phi_{if_{\ell-}}$ and $\lambda = 0$ if the optimal breakpoint $\theta^* = \sum_{i \in S} \Phi_{if_\ell}$ of ϕ satisfies $\theta^* < 0$ and $\theta^* = 0$, respectively. $\square$

Proposition EC.1. *Assume that the probability distribution $\mathbb{P}$ has mean $\mu \in \mathbb{R}^n_+$ and a positive definite covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$, and that the set defined in (C3) contains the support of q, that is, that $\mathbb{P}(Wq \leq h) = 1$. Then, every feasible solution to any of the $\mathcal{RVRP}$ formulations from Section 4 induces a set of routes $\mathbf{R}$ that satisfies the chance constraint* (EC.1) *if we replace the set $\mathcal{Q}$ in the $\mathcal{RVRP}$ formulation with*

$$\mathcal{Q}(\epsilon) = \left\{ q \in \mathbb{R}^n_+ : \max \left\{ \frac{1}{\sqrt{n}} \left\| \Sigma^{-1/2}(q - \mu) \right\|_1, \left\| \Sigma^{-1/2}(q - \mu) \right\|_\infty \right\} \leq \sqrt{\frac{1 - \epsilon}{\epsilon}}, \; Wq \leq h \right\}.$$

Proof. Consider first Formulation (VA), the robust Vehicle Assignment formulation presented in Section 4.4. The uncertain customer demands enter this formulation only through the constraints

$$\sum_{i \in V_C} q_i z_{ik} \leq Q \qquad\qquad \forall\, k \in K, \, \forall\, q \in \mathcal{Q},$$

where $z_{ik} = 1$ if $i \in R_k(x)$ and $z_{ik} = 0$ otherwise. It follows from Theorem 1 in El Ghaoui et al. (2003) that every feasible solution in (VA) induces a set of routes $\mathbf{R}$ that satisfies the chance constraint (EC.1) if we replace the set $\mathcal{Q}$ with

$$\mathcal{Q}'(\epsilon) = \left\{ q \in \mathbb{R}^n_+ : \left\| \Sigma^{-1/2}(q-\mu) \right\|_2 \leq \sqrt{\frac{1-\epsilon}{\epsilon}} \right\}.$$

Likewise, since $\mathbb{P}(Wq \leq h) = 1$, every feasible solution in (VA) induces a set of routes $\mathbf{R}$ that satisfies the chance constraint (EC.1) if we use the set $\mathcal{Q}$ defined in (C3). We can now apply Corollary 2.8 and Theorem 2.9 in Chen et al. (2010) to conclude that every feasible solution in (VA) induces a set of routes $\mathbf{R}$ that satisfies the chance constraint (EC.1) if we replace the set $\mathcal{Q}$ with

$$\mathcal{Q}''(\epsilon) = \left\{ q \in \mathbb{R}^n_+ : \left\| \Sigma^{-1/2}(q-\mu) \right\|_2 \leq \sqrt{\frac{1-\epsilon}{\epsilon}},\ Wq \leq h \right\}.$$

Note that $\mathcal{Q}''(\epsilon) \subseteq \mathcal{Q}(\epsilon)$ for all $\epsilon \in (0,1]$, which implies that every feasible solution in (VA) induces a set of routes $\mathbf{R}$ that satisfies the chance constraint (EC.1) if we replace the set $\mathcal{Q}$ with $\mathcal{Q}(\epsilon)$. The result now follows from the fact that for a given set $\mathcal{Q}$, all of the $\mathcal{RVRP}$ formulations from Section 4 are equivalent. □

COROLLARY EC.1. *For decision variables $\zeta \in \mathcal{Z}$ and functions $f : \mathcal{Z} \mapsto \mathbb{R}^n$, $b : \mathcal{Z} \mapsto \mathbb{R}$, the constraint*

$$q^\top f(\zeta) \leq b(\zeta) \qquad \forall q \in \mathcal{Q}(\epsilon) \tag{EC.2}$$

is satisfied if and only if there exist $\psi^+, \psi^-, \omega^+, \omega^- \in \mathbb{R}^n_+$, $\vartheta \in \mathbb{R}_+$ and $\tau \in \mathbb{R}^r_+$ such that

$$\begin{aligned} &\left(\Sigma^{-1/2}\mu\right)^\top \left(\psi^+ - \psi^- + \omega^+ - \omega^-\right) + \sqrt{\frac{1-\epsilon}{\epsilon}} \left(\mathrm{e}^\top \psi^+ + \mathrm{e}^\top \psi^- + \sqrt{n}\vartheta\right) + h^\top \tau \leq b(\zeta) \\ &\Sigma^{-1/2}\left(\psi^+ - \psi^- + \omega^+ - \omega^-\right) + W^\top \tau \geq f(\zeta) \\ &\vartheta\, \mathrm{e} \geq \omega^+ + \omega^-. \end{aligned} \tag{EC.3}$$

Proof. For $\Delta \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ satisfies $\|x\|_1 \leq \Delta$ if and only if there is a vector $y \in \mathbb{R}^n_+$ such that $-y \leq x \leq +y$ and $\mathrm{e}^\top y \leq \Delta$. Likewise, for $\Delta \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ satisfies $\|x\|_\infty \leq \Delta$ if and only if $-\Delta\,\mathrm{e} \leq x \leq +\Delta\,\mathrm{e}$. Hence, the support $\mathcal{Q}(\epsilon)$ from Proposition EC.1 is the projection of the lifted support

$$\begin{aligned} \mathcal{Q}'(\epsilon) = \Bigg\{ (q,u) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : -u \leq \Sigma^{-1/2}(q-\mu) \leq +u,\ \mathrm{e}^\top u \leq \sqrt{\frac{1-\epsilon}{\epsilon}} n, \\ -\sqrt{\frac{1-\epsilon}{\epsilon}}\mathrm{e} \leq \Sigma^{-1/2}(q-\mu) \leq +\sqrt{\frac{1-\epsilon}{\epsilon}}\mathrm{e},\ Wq \leq h \Bigg\} \end{aligned}$$

onto its first component, and the constraint (EC.2) is equivalent to

$$q^\top f(z) \leq b(z) \qquad \forall (q,u) \in \mathcal{Q}'(\epsilon).$$

Standard robust optimization results (see Section 4.2.1) now imply that this constraint is satisfied if and only if the constraint set (EC.3) is satisfiable by some $\psi^+, \psi^-, \omega^+, \omega^- \in \mathbb{R}^n_+$, $\vartheta \in \mathbb{R}_+$ and $\tau \in \mathbb{R}^r_+$. $\square$

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