Electronic Companion to "Optimal Control Policy for Capacitated Inventory Systems with Remanufacturing"

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In this online appendix, we provide proofs for all the technical results. Some preliminaries on lattice and submodularity are needed. Recall that a partially ordered set X is called a lattice if it contains the minimum and maximum of each pairs of its elements; if X' is a subset of a lattice X and X' contains the minimum and maximum (with respect to X) of each pair of the elements of X', then X' is called a sublattice of X. For any subset $S \subset \Re^2$, we let S_x (S_y) denote the section of S at x (y). That is,

$$S_x = \{y : (x,y) \in S\}, \qquad S_y = \{x : (x,y) \in S\}.$$

A main tool used in our study is $L^\natural$-convexity. In this paper, we use the following property of an $L^\natural$-convex function.

LEMMA EC.1. *Let $S \subset \Re^2$ be a lattice. If $f(x,y)$ is $L^\natural$-convex in S, then both $y^*(x) = \arg\min_{y \in S_x} f(x,y)$ and $x^*(y) = \arg\min_{x \in S_y} f(x,y)$ are increasing functions with slopes between zero and one.*

Proof of Lemma EC.1 By symmetry, we only need to prove the result for $y^*(x)$. The monotonicity of $y^*(x)$ follows from Theorem 8.1 of Porteus (2002). Thus we only prove that $y^*(x)$ has slopes no more than one, or equivalently, $x - y^*(x)$ is increasing in x.

By $L^\natural$-convexity, $f(x-y,x)$ is submodular in (x,y). From Lemma 8.4 of Porteus (2002) and that S is a lattice, it follows that S_x is ascending in x. Thus its minimizer, denoted by $\hat{y}(x)$, is increasing in x. On the other hand, since $y^*(x)$ is the minimizer of $f(x,y)$, we have $x - \hat{y}(x) = y^*(x)$. Hence, $x - y^*(x) = \hat{y}(x)$ is increasing in x. $\square$

Proof of Proposition 1.

We prove the $L^\natural$-convexity of H_t and V_t by induction on t. Since $V_{T+1}(x^0,x^1)\equiv 0$, it is clearly an $L^\natural$-convex function. Now we assume inductively that $V_{t+1}(x^0,x^1)$ is $L^\natural$-convex, and prove that V_t and H_t are both $L^\natural$-convex functions.

We first prove $H_t(y^0,y^1)$ is $L^\natural$-convex. According to (7), we have

$$\begin{aligned}H_t(y^0-\zeta,y^1-\zeta)=(s-r)(y^0-\zeta)+p(y^1-\zeta)+G_t(y^1-y^0)\\+\alpha\mathsf{E}[V_{t+1}(y^0-\zeta+R_t,y^1-\zeta-D_t+R_t)].\end{aligned}$$

Since $G_t(\cdot)$ is a convex function and $V_{t+1}(x^0,x^1)$ is $L^\natural$-convex, it is easily seen that $H_t(y^0-\zeta,y^1-\zeta)$ is submodular in (y^0,y^1,ζ), thus $H_t(y^0,y^1)$ is $L^\natural$-convex.

We next prove that $V_t(x^0,x^1)$ is $L^\natural$-convex, or equivalently, $V_t(x^0-\zeta,x^1-\zeta)$ is submodular in (x^0,x^1,ζ). When there are capacity constraints K_r, K_m, and K, by (6) we have

$$V_t(x^0-\zeta,x^1-\zeta)=\min_{\substack{(x^0-\zeta-K_r)^+\le y^0\le x^0-\zeta,\\ 0\le y^1-x^1+\zeta\le K_m,\\ 0\le y^1-x^1+x^0-y^0\le K}}\left\{H_t(y^0,y^1)\right\}+rx^0-px^1+(p-r)\zeta. \tag{EC.1}$$

According to Theorem 2.6.2 of Topkis (1998), it suffices to prove that $V_t(x^0-\zeta,x^1-\zeta)$ is submodular in (x^0,x^1) for any fixed ζ, submodular in (x^0,ζ) for any fixed x^1, and submodular in (x^1,ζ) for any fixed x^0. In what follows, we sequentially prove these three results.

We first prove that $V_t(x^0-\zeta,x^1-\zeta)$ is submodular in (x^0,x^1) for any fixed ζ. The main idea of the proof is to apply Theorem 1 of Chen et al. (2011) to show the preservation of submodularity under a minimization operation. To apply this theorem, we introduce two additional decision variables $z^0=x^0-y^0$ and $z^1=x^1-y^1$. Then, the minimization problem (EC.1) can be rewritten as

$$V_t(x^0-\zeta,x^1-\zeta)=\min_{(y^0,y^1,z^0,z^1)\in\mathcal{D}}\left\{H_t(y^0,y^1):y^i+z^i=x^i,i=0,1\right\}+rx^0-px^1+(p-r)\zeta, \tag{EC.2}$$

where

$$\mathcal{D}=\{(y^0,y^1,z^0,z^1)\ |y^0\ge 0,\ 0\le z^0-\zeta\le K_r,\ -K_m\le z^1-\zeta\le 0,\ 0\le z^0-z^1\le K\}.$$

Since $H_t(y^0, y^1)$ is $L^\natural$-convex, it is convex and submodular in (y^0, y^1). In addition, it can be easily checked that $\mathcal{D}$ is a closed convex sublattice. Thus, by applying Theorem 1 of Chen et al. (2011), $V_t(x^0 - \zeta, x^1 - \zeta)$ is submodular in (x^0, x^1).

We next prove that $V_t(x^0 - \zeta, x^1 - \zeta)$ is submodular in (x^0, ζ) for any fixed x^1. To this end, we introduce the following new decision variables:

$$\hat{y}^0 = y^0 - y^1 + x^1,\ \hat{y}^1 = x^1 - y^1,\ \hat{z}^0 = x^0 - \hat{y}^0,\ \hat{z}^1 = \zeta - \hat{y}^1.$$

Then, the minimization problem (EC.1) can be rewritten as

$$V_t(x^0 - \zeta, x^1 - \zeta) = \min_{(\hat{y}^0, \hat{y}^1, \hat{z}^0, \hat{z}^1) \in \hat{\mathcal{D}}} \left\{ H_t(\hat{y}^0 - \hat{y}^1, x^1 - \hat{y}^1) : \binom{\hat{y}^0 + \hat{z}^0}{\hat{y}^1 + \hat{z}^1} = \binom{x^0}{\zeta} \right\} + rx^0 - px^1 + (p - r)\zeta,$$

where

$$\hat{\mathcal{D}} = \{(\hat{y}^0, \hat{y}^1, \hat{z}^0, \hat{z}^1) \mid \hat{y}^0 \geq \hat{y}^1,\ 0 \leq \hat{z}^0 - \hat{z}^1 \leq K_r,\ 0 \leq \hat{z}^0 \leq K,\ 0 \leq \hat{z}^1 \leq K_m\}.$$

Since H_t is $L^\natural$-convex, $H_t(\hat{y}^0 - \hat{y}^1, x^1 - \hat{y}^1)$ is convex and submodular in $(\hat{y}^0, \hat{y}^1)$. In addition, it can be seen that $\hat{\mathcal{D}}$ is a closed convex sublattice. Thus, again by applying Theorem 1 of Chen et al. (2011), $V_t(x^0 - \zeta, x^1 - \zeta)$ is submodular in (x^0, ζ).

We finally prove that $V_t(x^0 - \zeta, x^1 - \zeta)$ is submodular in (x^1, ζ) for any fixed x^0. Similar to the previous proof, we introduce the following new decision variables:

$$\tilde{y}^0 = x^0 - y^0,\ \tilde{y}^1 = x^0 - y^0 + y^1,\ \tilde{z}^0 = \zeta - \tilde{y}^0,\ \tilde{z}^1 = x^1 - \tilde{y}^1.$$

Then, the minimization problem (EC.1) can be rewritten as

$$V_t(x^0 - \zeta, x^1 - \zeta) = \min_{(\tilde{y}^0, \tilde{y}^1, \tilde{z}^0, \tilde{z}^1) \in \tilde{\mathcal{D}}} \left\{ H_t(x^0 - \tilde{y}^0, \tilde{y}^1 - \tilde{y}^0) : \binom{\tilde{y}^0 + \tilde{z}^0}{\tilde{y}^1 + \tilde{z}^1} = \binom{\zeta}{x^1} \right\} + rx^0 - px^1 + (p - r)\zeta,$$

where

$$\tilde{\mathcal{D}} = \{(\tilde{y}^0, \tilde{y}^1, \tilde{z}^0, \tilde{z}^1) \mid \tilde{y}^0 \leq x^0,\ 0 \leq \tilde{z}^0 - \tilde{z}^1 \leq K_m,\ -K_r \leq \tilde{z}^0 \leq 0,\ -K \leq \tilde{z}^1 \leq 0\}.$$

Since H_t is $L^\natural$-convex, $H_t(x^0 - \tilde{y}^0, \tilde{y}^1 - \tilde{y}^0)$ is convex and submodular in $(\tilde{y}^0, \tilde{y}^1)$. In addition, it can be seen that $\tilde{\mathcal{D}}$ is a closed convex sublattice. Applying Theorem 1 of Chen et al. (2011) again we obtain that $V_t(x^0 - \zeta, x^1 - \zeta)$ is submodular in (x^1, ζ). The proof is complete. $\square$

Proof of Proposition 2.

The proof is by induction on t. First, since $V_{T+1}(\cdot,\cdot)\equiv 0$, it follows from (7) that

$$H_T(y^0-\zeta,y^1-\zeta)=(s-r)y^0+py^1+G_T(y^1-y^0)-(p+s-r)\zeta.$$

Since $p>r$ and $s>0$, $H_T(y^0-\zeta,y^1-\zeta)$ is decreasing in ζ, so the lemma holds for $t=T$. Now assume inductively that the lemma holds for $t+1$. In what follows, we shall prove it also holds for t, which then completes the proof.

We first prove $V_{t+1}(x^0-\zeta,x^1-\zeta)-(p-r)\zeta$ is decreasing in ζ for any fixed x^0 and x^1. Denote $\phi=x^1+y^0-y^1$ and $\varphi=x^1-y^1$. By replacing y^0 and y^1 with ϕ and φ, (6) can be rewritten as

$$\begin{aligned}
V_{t+1}(x^0-\zeta,x^1-\zeta)-(p-r)\zeta &= \min_{\substack{\phi-x^0+\zeta\le\varphi\le\min\{\phi,\zeta,\phi-x^0+\zeta+K_r\}\\ x^0-K\le\phi\le x^0}} \left\{H_{t+1}(\phi-\varphi,x^1-\varphi)\right\}+rx^0-px^1\\
&= \min_{\substack{\varphi\le\min\{\phi,\zeta,\phi-x^0+\zeta+K_r\}\\ x^0-K\le\phi\le x^0}} \left\{H_{t+1}(\phi-\varphi,x^1-\varphi)\right\}+rx^0-px^1, \qquad \text{(EC.3)}
\end{aligned}$$

where the second equality follows from the result that $H_{t+1}(\phi-\varphi,x^1-\varphi)$ is decreasing in φ. Since the constraint in (EC.3) becomes less restrictive when ζ becomes larger, it follows that $V_{t+1}(x^0-\zeta,x^1-\zeta)-(p-r)\zeta$ is decreasing in ζ.

Now we prove $H_t(y^0-\zeta,y^1-\zeta)$ is decreasing in ζ. According (7), we have

$$\begin{aligned}
H_t(y^0-\zeta,y^1-\zeta) =&(s-r)y^0+py^1+G_t(y^1-y^0)\\
&+\alpha\mathsf{E}[V_{t+1}(y^0-\zeta+R_t,y^1-\zeta-D_t+R_t)-(p-r)\zeta]-\big(s+(1-\alpha)(p-r)\big)\zeta.
\end{aligned}$$

Since $V_{t+1}(x^0-\zeta,x^1-\zeta)-(p-r)\zeta$ is decreasing in ζ, $s>0$, $p>r$, and $0<\alpha\le 1$, $H_t(y^0-\zeta,y^1-\zeta)$ is decreasing in ζ. The proof is complete. □

Proof of Theorem 1.

Note that the model with only a total capacity K is a special case of the model with only capacities K_r and K satisfying $K_r=K$. Therefore, to prove Theorem 1, it suffices to verify that Theorem 2 reduces to Theorem 1 when $K=K_r$. Since Theorem 2 (i) is clearly consistent with Theorem 1

(i) when $K_r = K$, it remains to verify that Theorem 2 (ii) is consistent with Theorem 1 (ii) when $K_r = K$. To this end, we consider the following two cases, $0 \le x_t^0 \le K$ and $x_t^0 > K$, separately.

Case 1. If $0 \le x_t^0 \le K_r = K$, then Theorem 2 (ii-1) states that it is optimal to use the remaining capacity $K - x_t^0$ to raise the total inventory level to as close to x_t^{1*} as possible when $x_t^1 \le x_t^{1*}$ and it is optimal not to manufacture when $x_t^1 \ge x_t^{1*}$. These two statements are consistent with those specified in Theorem 1 (ii-1) and Theorem 1 (ii-2), respectively.

Case 2. If $x_t^0 \ge K = K_r$, then Theorem 2 (ii) states that it is optimal not to manufacture, which is also consistent with that specified in Theorem 1 (ii-1).

In summary, we have shown that, for the model with only a total capacity K, the optimal policies are given in Theorem 1. The proof is complete. □

Proof of Theorem 2.

For the model with only capacities K_r and K with $K_r \le K$, the constraints for decisions (y_t^0, y_t^1) must satisfy

$$(x_t^0 - K_r)^+ \le y_t^0 \le x_t^0,\ \ y_t^1 \ge x_t^1,\ \ 0 \le (y_t^1 - x_t^1) + (x_t^0 - y_t^0) \le K.$$

Let $\phi_t = x_t^1 + y_t^0 - y_t^1$, then these constraints are transformed to

$$(x_t^0 - K_r)^+ \le y_t^0 \le x_t^0,\ y_t^0 \ge \phi_t,\ \ x_t^0 - K \le \phi \le x_t^0.$$

Thus, the optimality equation (6) can be rewritten as

$$V_t(x_t^0, x_t^1) = \min_{\substack{\max\{\phi,(x_t^0-K_r)^+\}\le y_t^0 \le x_t^0 \\ x_t^0 - K \le \phi \le x_t^0}} \left\{ H_t(y_t^0, x_t^1 + y_t^0 - \phi_t) \right\} + r x_t^0 - p x_t^1. \tag{EC.4}$$

From Proposition 2, $H_t(y_t^0, x_t^1 + y_t^0 - \phi)$ is increasing in y_t^0, thus the optimal y_t^0 is achieved at $y_t^0 = \max\{\phi_t, (x_t^0 - K_r)^+\}$. Substituting this into (EC.4) yields

$$V_t(x_t^0, x_t^1) = \min_{x_t^0 - K \le \phi_t \le x_t^0} \left\{ H_t(\max\{\phi_t, (x_t^0 - K_r)^+\}, x_t^1 + ((x_t^0 - K_r)^+ - \phi_t)^+) \right\} + r x_t^0 - p x_t^1. \tag{EC.5}$$

For each period t, define $(x_t^{0*}, x_t^{1*}) = \arg\min_{x^0 \ge 0, x^1 \in \Re}\{H_t(x^0, x^1)\}$. By Proposition 2, $H_t(\delta - \zeta, \xi - \zeta)$ is decreasing ζ and thus $x_t^{0*} = 0$. In addition, for any $\delta \ge 0$ and $\xi \in \Re$, we define

$$\delta_t^*(\xi) = \arg\min_{\delta \geq 0} \{H_t(\delta, \xi)\} \text{ and } \xi_t^*(\delta) = \arg\min_{\xi \in \Re} \{H_t(\delta, \xi)\}.$$

Then, $\delta_t^*(x_t^{1*}) = 0$ and $\xi_t^*(0) = x_t^{1*}$. In addition, by Proposition 1 and Lemma EC.1, both $\delta_t^*(\cdot)$ and $\xi_t^*(\cdot)$ are increasing functions with slopes between zero and one.

Instead of considering different regions separately, we take a unified approach and show that the firm's optimal policies have the following simple and unified mathematical expressions:

$$y_t^{0*}(x_t^0, x_t^1) = \min\{x_t^0, \max\{(x_t^0 - K_r)^+, \delta_t^*(x_t^1)\}\}; \tag{EC.6}$$

$$y_t^{1*}(x_t^0, x_t^1) = \max\{x_t^1, \min\{x_t^1 + K - \min\{x_t^0, K_r\}, \xi_t^*((x_t^0 - K_r)^+)\}\}. \tag{EC.7}$$

Note that $H_t(\cdot,\cdot)$ is a convex function due to its $L^\natural$-convexity from Proposition 1 and that the constraint set

$$\{(x^0, y^0, \phi_t) \mid \max\{\phi_t, (x_t^0 - K_r)^+\} \leq y_t^0 \leq x_t^0\}$$

is a convex set. Thus, it follows that

$$H_t(\max\{\phi_t, (x_t^0 - K_r)^+\}, x_t^1 + ((x_t^0 - K_r)^+ - \phi_t)^+) = \min_{\max\{\phi_t, (x_t^0 - K_r)^+\} \leq y_t^0 \leq x_t^0} \left\{H_t(y_t^0, x_t^1 + y_t^0 - \phi_t)\right\}$$

is convex in ϕ_t. In addition, we have

$$\begin{aligned} &H_t(\max\{\phi_t, (x_t^0 - K_r)^+\}, x_t^1 + ((x_t^0 - K_r)^+ - \phi_t)^+) \\ &= \begin{cases} H_t((x_t^0 - K_r)^+, x_t^1 + (x_t^0 - K_r)^+ - \phi_t), & \text{if } \phi_t \leq (x_t^0 - K_r)^+; \\ H_t(\phi_t, x_t^1), & \text{if } \phi_t \geq (x_t^0 - K_r)^+. \end{cases} \end{aligned} \tag{EC.8}$$

Note that $H_t((x_t^0 - K_r)^+, x_t^1 + (x_t^0 - K_r)^+ - \phi_t)$ is convex in ϕ_t and achieves its minimum value at $\phi_t = x_t^1 + (x_t^0 - K_r)^+ - \xi_t^*((x_t^0 - K_r)^+)$; and $H_t(\phi_t, x_t^1)$ is also convex in ϕ_t and achieves its minimum value at $\phi_t = \delta_t^*(x_t^1)$. In what follows, we consider two cases separately and verify that the optimal manufacturing/remanufacturing policies are given by (EC.6) and (EC.7) in each case.

First, if $\delta_t^*(x_t^1) > (x_t^0 - K_r)^+$, by (EC.8), $H_t(\max\{\phi_t, (x_t^0 - K_r)^+\}, x_t^1 + ((x_t^0 - K_r)^+ - \phi_t)^+)$ achieves its minimum value at $\phi_t = \delta_t^*(x_t^1)$, so the optimal ϕ_t^* in (EC.5) is equal to $\max\{x_t^0 - K, \min\{x_t^0, \delta_t^*(x_t^1)\}\}$. Therefore, when $\delta_t^*(x_t^1) > (x_t^0 - K_r)^+$, we have

$$y_t^{0*}(x_t^0, x_t^1) = \max\{\phi_t^*, (x_t^0 - K_r)^+\} = \min\{x_t^0, \delta_t^*(x_t^1)\};$$

$$y_t^{1*}(x_t^0, x_t^1) = x_t^1 + ((x_t^0 - K_r)^+ - \phi_t^*)^+ = x_t^1.$$

Since $\xi_t^*(\delta)$ is increasing in δ, it holds that $\xi_t^*(\delta_t^*(x_t^1)) > \xi_t^*((x_t^0 - K_r)^+)$, and since $\delta_t^*(x_t^1)$ is increasing in x_t^1 and $\delta_t^*(x_t^{1*}) = 0$, $\delta_t^*(x_t^1) > (x_t^0 - K_r)^+$ implies $x_t^1 > x_t^{1*}$. In addition, because

$$\xi_t^*(\delta_t^*(x_t^{1*})) = \xi_t^*(0) = x_t^{1*}$$

and both $\delta_t^*(\cdot)$ and $\xi_t^*(\cdot)$ are increasing functions with slopes no more than 1, we have

$$x_t^1 \geq \xi_t^*(\delta_t^*(x_t^1)) \geq \xi_t^*((x_t^0 - K_r)^+).$$

Hence it is seen that (EC.6) and (EC.7) hold when $\delta_t^*(x_t^1) > (x_t^0 - K_r)^+$.

On the other hand, if $\delta_t^*(x_t^1) \leq (x_t^0 - K_r)^+$, then by (EC.8),

$$H_t(\max\{\phi_t, (x_t^0 - K_r)^+\}, x_t^1 + ((x_t^0 - K_r)^+ - \phi_t)^+)$$

is increasing in ϕ_t when $\phi_t \geq (x_t^0 - K_r)^+$, and achieves its minimum value at

$$\phi_t = (x_t^0 - K_r)^+ + \min\{x_t^1 - \xi_t^*((x_t^0 - K_r)^+), 0\}.$$

Thus, the optimal ϕ_t^* in (EC.5) is equal to

$$\min\{(x_t^0 - K_r)^+, \max\{x_t^0 - K, x_t^1 + (x_t^0 - K_r)^+ - \xi_t^*((x_t^0 - K_r)^+)\}.$$

Therefore, when $\delta_t^*(x_t^1) \leq (x_t^0 - K_r)^+$, we have

$$\begin{aligned}
y_t^{0*}(x_t^0, x_t^1) &= \max\{\phi_t^*, (x_t^0 - K_r)^+\} = (x_t^0 - K_r)^+;\\
y_t^{1*}(x_t^0, x_t^1) &= x_t^1 + ((x_t^0 - K_r)^+ - \phi_t^*)^+\\
&= \max\{x_t^1, \min\{x_t^1 + K - \min\{x_t^0, K_r\}, \xi_t^*((x_t^0 - K_r)^+)\}\}.
\end{aligned}$$

This shows that (EC.6) and (EC.7) also hold when $\delta_t^*(x_t^1) \leq (x_t^0 - K_r)^+$.

Summarizing the above two cases, we have shown that the firm's optimal policies can be mathematically expressed by (EC.6) and (EC.7).

To complete the proof, it remains to show that the firm's optimal policies expressed in (EC.6) and (EC.7) are consistent with those in Theorem 2. To see this, we first consider the optimal remanufacturing policy. If $x_t^1 \leq x_t^{1*}$, then by the monotonicity of $\delta_t^*(x_t^1)$ we have $\delta_t^*(x_t^1) \leq \delta_t^*(x_t^{1*}) = 0$, so

(EC.6) is simplified to $y_t^{0*}(x_t^0, x_t^1) = (x_t^0 - K_r)^+$ and it corresponds to case i-1); and if $x_t^1 \geq x_t^{1*}$, then (EC.6) corresponds to case i-2). Thus, (EC.6) is consistent with the firm's optimal remanufacturing policy in Theorem 2.

We next consider the optimal manufacturing policy. First, if $0 \leq x_t^0 \leq K_r$, since $\xi_t^*(0) = x_t^{1*}$, then (EC.7) is simplified to

$$y_t^{1*}(x_t^0, x_t^1) = \max\{x_t^1, \min\{x_t^1 + (K - x_t^0), x_t^{1*}\}\}.$$

In this case, if $x_t^1 \leq x_t^{1*}$, then $y_t^{1*}(x_t^0, x_t^1) = \min\{x_t^1 + (K - x_t^0), x_t^{1*}\}$ and it is optimal to use the remaining capacity $K - x_t^0$ to raise the total inventory level to as close to x_t^{1*} as possible; and otherwise, $y_t^{1*}(x_t^0, x_t^1) = x_t^1$ and it is optimal not to manufacture. Thus, this first case corresponds to case ii-1) in Theorem 2. Second, if $x_t^0 \geq K_r$, then (EC.7) becomes

$$y_t^{1*}(x_t^0, x_t^1) = \max\{x_t^1, \min\{x_t^1 + K - K_r, \xi_t^*(x_t^0 - K_r)\}\}.$$

In this case, if $x_t^1 \leq \xi_t^*(x_t^0 - K_r)$, then $y_t^{1*}(x_t^0, x_t^1) = \min\{x_t^1 + K - K_r, \xi_t^*(x_t^0 - K_r)\}$ and it is optimal to use the remaining capacity $K - K_r$ to raise the total inventory level to as close to $\xi_t^*(x_t^0 - K_r)$ as possible; and otherwise, $y_t^{1*}(x_t^0, x_t^1) = x_t^1$ and it is optimal not to manufacture. Thus, this second case corresponds to case ii-2) in Theorem 2. Therefore, (EC.7) is consistent with the firm's optimal manufacturing policy in Theorem 2. The proof is complete. □

Proof of Theorem 3.

We start the proof by defining the control parameters and functions and proving their properties. For each period t, define $(x_t^{0*}, x_t^{1*}) = \arg\min_{x^0 \geq 0, x^1 \in \Re}\{H_t(x^0, x^1)\}$; and for $\delta \geq 0$ and $\xi \in \Re$, define

$$\delta_t^*(\xi) = \arg\min_{\delta \geq 0} \{H_t(\delta, \xi)\} \text{ and } \xi_t^*(\delta) = \arg\min_{\xi \in \Re} \{H_t(\delta, \xi)\}.$$

Then, it directly follows from the above definition that $x_t^{0*} = \delta_t^*(x_t^{1*}) = \arg\min_{\delta \geq 0} \{H_t(\delta, \xi_t^*(\delta))\}$ and $x_t^{1*} = \xi_t^*(x_t^{0*}) = \arg\min_{\xi \in \Re} \{H_t(\delta_t^*(\xi), \xi)\}$. In addition, from (7) and Proposition 1, $H_t(\delta, \xi)$ is $L^\natural$-convex. Thus, it follows from Lemma EC.1 that both $\delta_t^*(\cdot)$ and $\xi_t^*(\cdot)$ are increasing functions with slopes between zero and one.

We now verify the firm's optimal policies in Theorem 3. Note that the firm's optimal policies specified in Theorem 3 have the following unified mathematical expressions:

$$y_t^{0*}(x_t^0, x_t^1) = \min\{x_t^0, \max\{x_t^0 - K_r, \delta_t^*(\max\{x_t^1, \min\{x_t^1 + K_m, x_t^{1*}\}\})\}\}; \tag{EC.9}$$

$$y_t^{1*}(x_t^0, x_t^1) = \max\{x_t^1, \min\{x_t^1 + K_m, \xi_t^*(\min\{x_t^0, \max\{x_t^0 - K_r, x_t^{0*}\}\})\}\}. \tag{EC.10}$$

Thus, to complete the proof of the theorem, we only need to prove that the firm's optimal policies can be mathematically expressed by (EC.9) and (EC.10).

We first prove the firm's optimal manufacturing policy can be expressed by (EC.10). For convenience, we denote $\tilde{H}_t(x_t^0, y_t^1) = \min_{(x_t^0 - K_r)^+ \le y_t^0 \le x_t^0}\{H_t(y_t^0, y_t^1)\}$. Then, the optimality equation (6) can be rewritten as

$$V_t(x_t^0, x_t^1) = \min_{x_t^1 \le y_t^1 \le x_t^1 + K_m} \{\tilde{H}_t(x_t^0, y_t^1)\} + r x_t^0 - p x_t^1. \tag{EC.11}$$

Since $H_t(\cdot,\cdot)$ is a jointly convex function and the constraint is a convex set, it follows that $\tilde{H}_t(x_t^0, y_t^1)$ is convex in y_t^1. Thus, it follows from (EC.11) that

$$y_t^{1*} = \max\Big\{x_t^1, \min\big\{x_t^1 + K_m, \arg\min_{y_t^1 \in \Re}\{\tilde{H}_t(x_t^0, y_t^1)\}\big\}\Big\}.$$

Therefore, to verify the firm's optimal manufacturing policy in (EC.10), it suffices to show that

$$\arg\min_{y_t^1 \in \Re}\{\tilde{H}_t(x_t^0, y_t^1)\} = \xi_t^*(\min\{x_t^0, \max\{x_t^0 - K_r, x_t^{0*}\}\});$$

or equivalently,

$$\min_{y_t^1 \in \Re}\{\tilde{H}_t(x_t^0, y_t^1)\} = \tilde{H}_t(x_t^0, \xi_t^*(\min\{x_t^0, \max\{x_t^0 - K_r, x_t^{0*}\}\})). \tag{EC.12}$$

By the definitions of $\tilde{H}_t(\cdot,\cdot)$, $\xi_t^*(\cdot)$ and x_t^{0*} and the convexity of $H_t(y_t^0, \xi_t^*(y_t^0))$ in y_t^0, it is easy to verify that

$$\begin{aligned}\min_{y_t^1 \in \Re}\{\tilde{H}_t\left(x_t^0, y_t^1\right)\} &= \min_{\substack{y_t^1 \in \Re \\ (x_t^0 - K_r)^+ \le y_t^0 \le x_t^0}} \{H_t(y_t^0, y_t^1)\} = \min_{(x_t^0 - K_r)^+ \le y_t^0 \le x_t^0}\{H_t(y_t^0, \xi_t^*(y_t^0))\} \\ &= H_t(\min\{x_t^0, \max\{x_t^0 - K_r, x_t^{0*}\}\}, \xi_t^*(\min\{x_t^0, \max\{x_t^0 - K_r, x_t^{0*}\}\}))\end{aligned}$$

$$\geq \min_{(x_t^0-K_r)^+\leq y_t^0\leq x_t^0}\{H_t(y_t^0,\xi_t^*(\min\{x_t^0,\max\{x_t^0-K_r,x_t^{0*}\}\}))\}$$

$$=\tilde{H}_t(x_t^0,\xi_t^*(\min\{x_t^0,\max\{x_t^0-K_r,x_t^{0*}\}\}))$$

$$\geq \min_{y_t^1\in\Re}\{\tilde{H}_t\left(x_t^0,y_t^1\right)\}.$$

Thus, all the inequalities above must hold as equalities and (EC.12) is established. Therefore, the firm's optimal manufacturing policy can be expressed by (EC.10).

We next prove the firm's optimal remanufacturing policy can be expressed by (EC.9). Similar to the above proof, we denote $\hat{H}_t(x_t^1,y_t^0)=\min_{x_t^1\leq y_t^1\leq x_t^1+K_m}\{H_t(y_t^0,y_t^1)\}$. Then, the optimality equation (6) can be rewritten as

$$V_t(x_t^0,x_t^1)=\min_{(x_t^0-K_r)^+\leq y_t^0\leq x_t^0}\{\hat{H}_t(x_t^1,y_t^0)\}+rx_t^0-px_t^1. \tag{EC.13}$$

Since, from Proposition 1, $H_t(\cdot,\cdot)$ is a convex function, $\hat{H}_t(x_t^1,y_t^0)$ is convex in y_t^0. Thus, it follows from (EC.13) that

$$y_t^{0*}=\min\Big\{x_t^0,\max\big\{x_t^0-K_r,\arg\min_{y_t^0\geq 0}\{\hat{H}_t(x_t^1,y_t^0)\}\big\}\Big\}.$$

Notice that $x_t^{0*}=\delta_t^*(x_t^{1*})$. Thus, to complete the proof, it remains to show that

$$\arg\min_{y_t^0\geq 0}\{\hat{H}_t(x_t^1,y_t^0)\}=\delta_t^*(\max\{x_t^1,\min\{x_t^1+K_m,x_t^{1*}\}\}).$$

Since the remaining proof is very similar to that of showing (EC.12), we omit the details for brevity. The proof is complete. □

References

[1] Chen, X., P. Hu, S. He. 2011. Preservation of supermodularity in two dimensional parametric optimization problems and its applications. *Working paper.*

[2] Porteus, E. L. 2002. *Foundations of Stochastic Inventory Theory.* Stanford University Press, Stanford, CA.

[3] Topkis, D. 1998. *Supermodularity and Complementarity.* Princeton University Press, Princeton, NJ.