Proofs of Statements and Model Extensions

Our e-companion provides mathematical proofs for the theorems and propositions stated in our paper in Section EC.1, and an extension for our model that considers infinite-horizon and Markov-modulated settings in Section EC.2.

EC.1. Proofs

Proof of Theorem 1. Consider first the case when the shared capacity in period t is deterministic and equals κ. Clearly, only production quantities $s_1(\kappa)$ and $s_2(\kappa)$ matter – the trajectory of production (i.e., scheduling) is irrelevant. To reflect this, we refer to the scheduling problem with deterministic capacity in the current period as *capacity allocation problem.* We denote by $J_t(\tilde{y},\xi,\kappa)$ the intermediate cost function. Defining $z_i = \tilde{y}_i - s_i(\kappa)$ as the stage 1 product i inventory carried to the next period, the capacity allocation problem reduces to the following:

$$J_t(\tilde{y},\xi,\kappa) = \min_{(z_1,z_2)} \Big\{ hz + p[\xi - \tilde{y} + z] + \beta V_{t+1}(z) \tag{EC.1}$$

$$s.t \quad \underline{s}_i(\kappa|\tilde{y},\xi) \le \tilde{y}_i - z_i \le \bar{s}_i(\kappa|\tilde{y},\xi), \; i = 1,2 \tag{EC.2}$$

$$(\tilde{y}_1 - z_1) + (\tilde{y}_2 - z_2) \le \kappa \Big\} \tag{EC.3}$$

The decision problem is expressed in terms of the inventory to carry over to the next period. (EC.1) replaces (2), (EC.2) replaces (3), and (EC.3) replaces (4) and (5) in the original formulation (the trajectory of production is ignored). The optimal solution should satisfy the following condition.

Optimality Condition: If the shared capacity is not binding, $\min(\tilde{y}_1,\xi_1) + \min(\tilde{y}_2,\xi_2) \le \kappa$, then $s_i(\kappa) = \min(\tilde{y}_i,\xi_i)$, implying $s_i(\kappa) = \underline{s}_i(\kappa|\tilde{y},\xi) = \bar{s}_i(\kappa|\tilde{y},\xi)$ and $z_i = (\tilde{y}_i - \xi_i)^+$. Otherwise, all the shared capacity must be utilized, $s_1(\kappa) + s_2(\kappa) = \kappa$, implying that $z_1 + z_2 = \tilde{y}_1 + \tilde{y}_2 - \kappa$.

Due to $\mathbb{A}^2_{t+1}$, i.e., $p_i + h_i + \beta\Delta_{x_i} V_{t+1}(z) \ge 0$, the function minimized in (EC.1) is non-decreasing in z_1 and z_2, hence the optimality condition is justified. Recall that $\gamma = p + h$. When the shared capacity is binding, we replace $z_2 = \tilde{y}_1 + \tilde{y}_2 - \kappa - z_1$, leading to optimization in single variable.

$$J_t(\tilde{y},\xi,\kappa) = p(\xi - \tilde{y}) + \gamma_2(\tilde{y}_1 + \tilde{y}_2 - \kappa) + \min_{z_1} \Big\{ (\gamma_1 - \gamma_2) z_1 + \beta V_{t+1}(z_1, \tilde{y}_1 + \tilde{y}_2 - \kappa - z_1) \tag{EC.4}$$
$$s.t \quad \tilde{y}_1 - \bar{s}_1(\kappa|\tilde{y},\xi) \le z_1 \le \tilde{y}_1 - \underline{s}_1(\kappa|\tilde{y},\xi) \Big\}$$

Define $f_t(z_1,\omega) := (\gamma_1 - \gamma_2) z_1 + \beta V_{t+1}(z_1, \omega - z_1)$ as the function to be minimized in (EC.4). Since $V_{t+1}(x)$ is jointly convex, $f_t(z_1,\omega)$ is convex in z_1 for $z_1 \in [0,\omega]$ and for any $\omega \ge 0$. To obtain the optimal capacity allocation, let $\zeta_1(\omega) = \arg\min_{0 \le z_1 \le \omega} f_t(z_1,\omega)$. In case of multiple optimal solutions, we choose the smallest one. As a consequence, the optimal capacity allocation satisfies $z_1 = \zeta_1(\tilde{y}_1 + \tilde{y}_2 - \kappa)|_{[\tilde{y}_1 - \bar{s}_1(\kappa|\tilde{y},\xi), \tilde{y}_1 - \underline{s}_1(\kappa|\tilde{y},\xi)]}$. Defining $\zeta_2(\omega) = \omega - \zeta_1(\omega)$ and by a symmetric argument, we also have $z_2 = \zeta_2(\tilde{y}_1 + \tilde{y}_2 - \kappa)|_{[\tilde{y}_2 - \bar{s}_2(\kappa|\tilde{y},\xi), \tilde{y}_2 - \underline{s}_2(\kappa|\tilde{y},\xi)]}$, which implies $s_i(\kappa) = \{\tilde{y}_i - \zeta_i(\tilde{y}_1 + \tilde{y}_2 - \kappa)\}|_{[\underline{s}_i(\kappa|\tilde{y},\xi), \bar{s}_i(\kappa|\tilde{y},\xi)]}$ for each $i = 1, 2$. Before proceeding to the case with stochastic shared capacity, we show that $0 \le \zeta_i'(\omega) \le 1$. First, note that due to the optimality we have:

$$\left.\frac{\partial f_t(z_1,\omega)}{\partial z_1}\right|_{z_1=\zeta_1(\omega)} = 0 \;=\; (\gamma_1 - \gamma_2) + \beta\Delta_{x_1}V_{t+1}(\zeta_1(\omega),\zeta_2(\omega)) - \beta\Delta_{x_2}V_{t+1}(\zeta_1(\omega),\zeta_2(\omega)) \tag{EC.5}$$

To evaluate $\zeta_1'(\omega)$, we take the derivative of both sides of (EC.5) and obtain

$$\zeta_1'(\omega) = \frac{(\Delta_{x_2x_2} - \Delta_{x_1x_2})V_{t+1}(\zeta_1(\omega),\zeta_2(\omega))}{(\Delta_{x_1x_1} + \Delta_{x_2x_2} - 2\Delta_{x_1x_2})V_{t+1}(\zeta_1(\omega),\zeta_2(\omega))} \tag{EC.6}$$

$0 \le \zeta_1'(\omega) \le 1$ follows due to the inductional assumptions on $V_{t+1}(x)$. Since $\zeta_2(\omega) = \omega - \zeta_1(\omega)$, we immediately have $0 \le \zeta_2'(\omega) \le 1$ and $\zeta_1'(\omega) + \zeta_2'(\omega) = 1$.

Next, consider stochastic shared capacity. First, we relax (3) and (4), that is, we ignore $s_1'(\kappa) + s_2'(\kappa) \le 1$ and $s_i'(\kappa) \ge 0$. Since choice of $s_i(\kappa)$ for a given κ is now decoupled from any other capacity $\kappa' \ne \kappa$, the problem reduces to independently minimizing the objective function for each realization of κ, which becomes the deterministic capacity case. Thus, the solution to the relaxed problem is given by $s_i(\kappa) = \left\{\tilde{y}_i - \zeta_i(\tilde{y}_1 + \tilde{y}_2 - \kappa)\right\}|_{[\underline{s}_i(\kappa|\tilde{y},\xi), \bar{s}_i(\kappa|\tilde{y},\xi)]}$, which satisfies ignored conditions (3) and (4), hence, it is feasible for the original problem with stochastic shared capacity. (3) follows from $\zeta_1'(\omega) + \zeta_2'(\omega) = 1$ and (4) follows from $0 \le \zeta_i'(\omega) \le 1$. Hence, $s_i(\kappa)$ is indeed optimal for the original problem with stochastic shared capacity. Clearly, the optimal schedule is independent of the distribution of the current-period shared capacity. As a consequence of the optimal policy, we have $C_t(\tilde{y},\xi) = \mathbb{E}_\kappa J_t(\tilde{y},\xi,\kappa)$: the firm incurs $J_t(\tilde{y},\xi,\kappa)$ for each realization of κ. □

Proof of Proposition 1. From the proof of Theorem 1, we have $C_t(\tilde{y},\xi) = \mathbb{E}_\kappa J_t(\tilde{y},\xi,\kappa)$. It is sufficient to show that the derivatives of $J_t(\tilde{y},\xi,\kappa)$ satisfy the desired properties, as expectation operator preserves them. It is useful to first establish the convexity of $J_t(\tilde{y},\xi,\kappa)$. An equivalent formulation for the *capacity allocation problem* expresses all constraints as linear in the state and decision variables.

$$J_t(\tilde{y},\xi,\kappa) = \min_{(z_1,z_2)} \Big\{ hz + p[\xi - \tilde{y} + z] + \beta V_{t+1}(z) \tag{EC.7}$$

$$s.t \quad 0 \le z_i \le \tilde{y}_i \;\; i = 1,2 \tag{EC.8}$$

$$\tilde{y}_i - z_i \le \xi_i \;\; i = 1,2 \tag{EC.9}$$

$$z_1 + z_2 \ge \tilde{y}_1 + \tilde{y}_2 - \kappa \Big\} \tag{EC.10}$$

Function minimized in (EC.7) is convex due to inductional assumptions, the set of constraints, (EC.8)–(EC.10) is convex due to linearity. Hence, convexity of $J_t(\tilde{y},\xi,\kappa)$ follows from Heyman and Sobel (1984), Property B–4, page 525. Next, we investigate the properties of $J_t(\tilde{y},\xi,\kappa)$. Due to stationarity of holding and penalty costs, it is always beneficial to satisfy demand, if there exists sufficient capacity. This simplifies the analysis below.

Case 1: $\underline{\xi_1+\xi_2 \le \kappa}$. If the total demand does not exceed the available shared capacity, then the demands are satisfied subject to the availability of the inventories of stage-1 products, and the cost that is incurred for any $(\tilde{y}_1, \tilde{y}_2)$ is $J_t(\tilde{y}, \xi, \kappa) = h(\tilde{y}-\xi)^+ + p(\xi-\tilde{y})^+ + \beta V_{t+1}(\tilde{y}-\xi)^+$.

First, we analyze the first-order derivatives. By Theorem 23.1 of Rockafellar (1997), the one-sided directional partial derivatives of the convex function $J_t(\tilde{y}, \xi, \kappa)$ exist. Therefore, whenever the first-order derivatives are not continuous, we use the right-hand derivatives below.

$$\begin{aligned}\Delta_{\tilde{y}_i} J_t(\tilde{y}, \xi, \kappa) &= [h_i + \beta \Delta_{x_i} V_{t+1}(\tilde{y}-\xi)^+] 1_{\tilde{y}_i \ge \xi_i} - p_i 1_{\tilde{y}_i < \xi_i} \\ &\ge -p_i 1_{\tilde{y}_i \ge \xi_i} - p_i 1_{\tilde{y}_i < \xi_i} \;=\; -p_i\end{aligned}$$

Second, we analyze the second-order derivatives. The function $J_t(\tilde{y}, \xi, \kappa)$ obviously preserves all the second-order properties that the function $V_{t+1}(x)$ has. This can be verified in a straightforward way through considering four subregions, $\tilde{y}_i \ge (<) \xi_i$. For example, for $\tilde{y}_i \ge \xi_i$ for $i = 1, 2$, second-order properties hold as $\Delta_{\tilde{y}_i \tilde{y}_i} J_t(\tilde{y}, \xi, \kappa) = \beta \Delta_{x_i x_i} V_{t+1}(\tilde{y}-\xi) \ge \beta \Delta_{x_1 x_2} V_{t+1}(\tilde{y}-\xi) = \Delta_{\tilde{y}_1 \tilde{y}_2} J_t(\tilde{y}, \xi, \kappa) \ge 0$. Other subregions can be analyzed similarly. We also need to verify these properties on the boundaries, $\tilde{y}_1 = \xi_1$ and $\tilde{y}_2 = \xi_2$. Observe that the first-order derivatives are not continuous on the boundaries. Denoting by Δ^+ the right-hand derivative operator, we instead show that the following properties hold on the boundaries.

(i) $\Delta^+_{\tilde{y}_i} J_t(\tilde{y}, \xi, \kappa)$ is (weakly) increasing in $\tilde{y}_1$ and $\tilde{y}_2$.

(ii) $\Delta^+_{\tilde{y}_1} J_t(\tilde{y}, \xi, \kappa) - \Delta^+_{\tilde{y}_2} J_t(\tilde{y}, \xi, \kappa)$ is (weakly) increasing in $\tilde{y}_1$ and (weakly) decreasing in $\tilde{y}_2$.

Property (i) replicates $\Delta_{\tilde{y}_i \tilde{y}_i} J_t(\tilde{y}, \xi, \kappa) \ge 0$ and $\Delta_{\tilde{y}_1 \tilde{y}_2} J_t(\tilde{y}, \xi, \kappa) \ge 0$, and Property (ii) replicates $\Delta_{\tilde{y}_i \tilde{y}_i} J_t(\tilde{y}, \xi, \kappa) \ge \Delta_{\tilde{y}_1 \tilde{y}_2} J_t(\tilde{y}, \xi, \kappa)$. These properties are sufficient for the preservation of the second-order properties over the boundaries. (For further details, see Shaoxiang, 2004, Proposition 1.)

For conciseness, denote by $J(\tilde{y}_1, \tilde{y}_2) = J_t(\tilde{y}, \xi, \kappa)$ and $z = (\tilde{y}-\xi)^+$. We have $\Delta^+_{\tilde{y}_i} J(\tilde{y}_1, \tilde{y}_2) = [h_i + \beta \Delta_{x_i} V_{t+1}(z)] 1_{\tilde{y}_i \ge \xi_i} - p_i 1_{\tilde{y}_i < \xi_i}$. $\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2)$ is increasing $\tilde{y}_1$, since $-p_1 \le h_1 + \beta \Delta_{x_1}(0, z_2)$ on the boundary $\tilde{y}_1 = \xi_1$, due to $\mathbb{A}^2_{t+1}$. $\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2)$ is increasing in $\tilde{y}_2$ for $\tilde{y}_1 < \xi_1$, as $\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2) = -p_1$ is constant. Let $\tilde{y}_1 \ge \xi_1$. In this case, $\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2) = h_1 + \beta \Delta_{x_1}(\tilde{y}_1 - \xi_1, 0)$ for $\tilde{y}_2 < \xi_2$ and $\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2) = h_1 + \beta \Delta_{x_1}(\tilde{y}_1 - \xi_1, \tilde{y}_2 - \xi)$ for $\tilde{y}_2 \ge \xi_2$ is increasing in $\tilde{y}_2$ since $\Delta_{x_1 x_2} V_{t+1}(x) \ge 0$. Similarly, we can show that $\Delta^+_{\tilde{y}_2} J(\tilde{y}_1, \tilde{y}_2)$ is increasing in $\tilde{y}_1$ and $\tilde{y}_2$, which completes property (i).

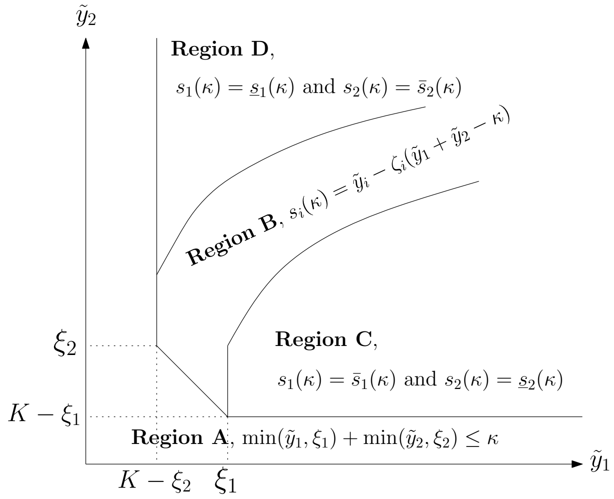
Next, we prove property (ii).

$$\begin{aligned}\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2) - \Delta^+_{\tilde{y}_2} J(\tilde{y}_1, \tilde{y}_2) = {} & [h_1 + \beta \Delta_{x_1} V_{t+1}(z)] 1_{\tilde{y}_1 \ge \xi_1} - p_1 1_{\tilde{y}_1 < \xi_1} \\ & - [h_2 + \beta \Delta_{x_2} V_{t+1}(z)] 1_{\tilde{y}_2 \ge \xi_2} + p_2 1_{\tilde{y}_2 < \xi_2}\end{aligned}$$

First, consider the boundary $\tilde{y}_1 = \xi_1$ with $\tilde{y}_2 < \xi_2$. $\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2) - \Delta^+_{\tilde{y}_2} J(\tilde{y}_1, \tilde{y}_2)$ is increasing in $\tilde{y}_1$, since over the boundary of $\tilde{y}_1 = \xi_1$, we have $-p_1 \le h_1 + \beta\Delta_{x_1}(0,0)$, and decreasing in $\tilde{y}_2$, since $-p_2 \le h_2 + \beta\Delta_{x_2}(0,0)$ over the boundary of $\tilde{y}_2 = \xi_2$. Next, consider the boundary of $\tilde{y}_1 = \xi_1$ with $\tilde{y}_2 \ge \xi_2$. In this case, $\Delta^+_{\tilde{y}_1} J(\tilde{y}_1, \tilde{y}_2) - \Delta^+_{\tilde{y}_2} J(\tilde{y}_1, \tilde{y}_2)$ is increasing in $\tilde{y}_1$ and decreasing in $\tilde{y}_2$, since $\Delta_{x_1} V_{t+1}(\tilde{y} - \xi) - \Delta_{x_2} V_{t+1}(\tilde{y} - \xi)$ is increasing in $\tilde{y}_1$ and decreasing in $\tilde{y}_2$ due to the inductional assumptions. The other case that considers the boundary of $\tilde{y}_2 = \xi_2$ can be handled similarly.

Case 2: $\underline{\xi_1 + \xi_2 > \kappa}$. The shared capacity is a binding resource, provided also that sufficient inventories exist. Figure EC.1 illustrates various regions induced by the optimal policy.

Figure EC.1 Regions Induced by Optimal Policy When $\xi_1 + \xi_2 > K$



Region A: $\underline{\min(\tilde{y}_1, \xi_1) + \min(\tilde{y}_2, \xi_2) \le \kappa}$. In this region, the realized capacity is not a binding resource. Although the total demand exceeds capacity, the amount of inventory does not allow for full utilization of the shared capacity. As a result, the cost that is incurred is $J_t(\tilde{y}, \xi, \kappa) = h(\tilde{y} - \xi)^+ + p(\xi - \tilde{y})^+ + \beta V_{t+1}(\tilde{y} - \xi)^+$, satisfying the desired properties, as discussed in Case 1.

Region B: $\underline{s_i(\kappa) = \tilde{y}_i - \zeta_i(\tilde{y}_1 + \tilde{y}_2 - \kappa)}$. Denote by $\zeta_i = \zeta_i(\tilde{y}_1 + \tilde{y}_2 - \kappa)$. Note that $\zeta_2 = \tilde{y}_1 + \tilde{y}_2 - \kappa - \zeta_1$. As a result, the following holds.

$$
\begin{aligned}
J_t(\tilde{y}, \xi, \kappa) &= h_1\zeta_1 + h_2\zeta_2 + p_1(\xi_1 - \tilde{y}_1 + \zeta_1) + p_2(\xi_2 - \tilde{y}_2 + \zeta_2) + \beta V_{t+1}(\zeta_1, \zeta_2) \\
&= h_1\zeta_1 + h_2(\tilde{y}_1 + \tilde{y}_2 - \kappa - \zeta_1) + p_1(\xi_1 - \tilde{y}_1 + \zeta_1) \\
&\quad + p_2(\xi_2 + \tilde{y}_1 + \kappa - \zeta_1) + \beta V_{t+1}(\zeta_1, \tilde{y}_1 + \tilde{y}_2 - \kappa - \zeta_1)
\end{aligned}
$$

Using the envelope theorem, we have $\Delta_{\tilde{y}_1} J_t(\tilde{y}, \xi, \kappa) = -p_1 + p_2 + h_2 + \beta\Delta_{x_2} V_{t+1}(\zeta_1, \zeta_2) \ \ge\ -p_1$. By a similar argument, we can show $\Delta_{\tilde{y}_2} J_t(\tilde{y}, \xi, \kappa) \ge -p_2$. Second-order conditions are derived as follows.

$$
\begin{aligned}
\Delta_{\tilde{y}_1\tilde{y}_1} J_t(\tilde{y}, \xi, \kappa) &= \beta\Delta_{x_1x_2} V_{t+1}(\zeta_1, \zeta_2)\zeta_1' + \beta\Delta_{x_2x_2} V_{t+1}(\zeta_1, \zeta_2)\zeta_2' \\
\Delta_{\tilde{y}_1\tilde{y}_2} J_t(\tilde{y}, \xi, \kappa) &= \beta\Delta_{x_1x_2} V_{t+1}(\zeta_1, \zeta_2)\zeta_1' + \beta\Delta_{x_2x_2} V_{t+1}(\zeta_1, \zeta_2)\zeta_2'
\end{aligned}
$$

Obviously, $\Delta_{\tilde{y}_1\tilde{y}_1} J_t(\tilde{y},\xi,\kappa) = \Delta_{\tilde{y}_1\tilde{y}_2} J_t(\tilde{y},\xi,\kappa) \geq 0$ due to inductional assumptions on $V_{t+1}(x)$ and the fact that the functions $\zeta_i'(k) \geq 0$. By symmetry, we have $\Delta_{\tilde{y}_2\tilde{y}_2} J_t(\tilde{y},\xi,\kappa) = \Delta_{\tilde{y}_1\tilde{y}_2} J_t(\tilde{y},\xi,\kappa) \geq 0$.

Region C: $\underline{s_1(\kappa) = \bar{s}_1(\kappa|\tilde{y},\xi) \text{ and } s_2(\kappa) = \underline{s}_2(\kappa|\tilde{y},\xi)}$. First, we prove that $\tilde{y}_1 \geq \min(\xi_1,\kappa)$ in this region. The optimal solution satisfies $s_1(\kappa) = \bar{s}_1(\kappa|\tilde{y},\xi)$ and $s_2(\kappa) = \underline{s}_2(\kappa|\tilde{y},\xi)$ if and only if $\bar{s}_1(\kappa|\tilde{y},\xi) < \tilde{y}_1 - \zeta_1(\tilde{y}_1 + \tilde{y}_2 - \kappa)$, since the optimal $s_i(\kappa)$ satisfies $s_i(\kappa) = [\tilde{y}_i - \zeta_i(\tilde{y}_1 + \tilde{y}_2 - \kappa)]|_{[\underline{s}_i(\kappa|\tilde{y},\xi),\bar{s}_i(\kappa|\tilde{y},\xi)]}$ for $i = 1,2$, by Theorem 1. To derive a contradiction, assume that $\tilde{y}_1 < \min(\xi_1,\kappa)$, in which case $\bar{s}_1(\kappa|\tilde{y},\xi) = \min(\tilde{y}_1,\xi_1,\kappa) = \tilde{y}_1$. Thus, either $\bar{s}_1(\kappa|\tilde{y},\xi) = \tilde{y}_1 < \tilde{y}_1 - \zeta_1(\tilde{y}_1 + \tilde{y}_2 - \kappa)$ or $\zeta_1(\tilde{y}_1 + \tilde{y}_2 - \kappa) < 0$. This is, however, a contradiction, since $\zeta_1(\omega) \geq 0$ by Theorem 1. As a consequence, $s_1(\kappa) = \min(\xi_1,\kappa)$ and $s_2(\kappa) = \kappa - \min(\xi_1,\kappa)$. Using $s_i = s_i(\kappa)$, we express the cost function as: $J_t(\tilde{y},\xi,\kappa) = h_1(\tilde{y}_1 - s_1) + p_1(\xi_1 - s_1) + h_2(\tilde{y}_2 - s_2) + p_2(\xi_2 - s_2) + \beta V_{t+1}(\tilde{y} - s)$. It is easy to check that $\Delta_{\tilde{y}_i} J_t(\tilde{y},\xi,\kappa) = h_i + \beta\Delta_{x_i} V_{t+1}(\tilde{y} - s) \geq -p_i$. Obviously, the second-order properties of $J_t(\tilde{y},\xi,\kappa)$ in this region are the same as these of $V_{t+1}(x)$. Hence, the second-order properties are preserved.

Region D: $\underline{s_1(\kappa) = \underline{s}_1(\kappa|\tilde{y},\xi) \text{ and } s_2(\kappa) = \bar{s}_2(\kappa|\tilde{y},\xi)}$. The proof follows the same reasoning as in Region C, except that product indices are switched. Hence, we omit the details.

We have shown that $J_t(\tilde{y},\xi,\kappa)$ is convex, $\Delta_{\tilde{y}_i} J_t(\tilde{y},\xi,\kappa) \geq -p_i$ and satisfies the second-order properties in $(\tilde{y}_1,\tilde{y}_2)$. Since $C_t(\tilde{y},\xi) = \mathbb{E}_\kappa J_t(\tilde{y},\xi,\kappa)$, we have that $C_t(\tilde{y},\xi)$ is also convex, $\Delta_{\tilde{y}_i} C_t(\tilde{y},\xi) \geq -p_i$ and satisfies the second-order properties in $(\tilde{y}_1,\tilde{y}_2)$.

Crossing from Region A to D, A to C, B to A and B to C does not cause problems, since the first derivatives are continuous. We only need to pay attention to the boundary between Regions A and B, however, the proof follows the same logic as in Case 1, by use of assumption $\mathbb{A}^2_{t+1}$. $\square$

Proof of Theorem 2. Since $G_t(\tilde{y}) = \mathbb{E}_\xi C_t(\tilde{y},\xi)$ and $C_t(\tilde{y},\xi)$ is convex and satisfies the second order properties in $(\tilde{y}_1,\tilde{y}_2)$ due to Proposition 1, $G_t(\tilde{y})$ is convex in $(\tilde{y}_1,\tilde{y}_2)$ and it satisfies the second-order properties, as the expectation operator preserves them.

(a) Recall that $\bar{y}_1(y_2) = \arg\min_{y_1\geq 0} G_t(y)$, hence $\Delta_{y_1} G_t(\bar{y}_1(y_2),y_2) = 0$. Taking the derivative of both sides with respect to y_2, we obtain $\frac{d\bar{y}_1(y_2)}{dy_2} = -\frac{\Delta_{y_1y_2} G_t(\bar{y}_1(y_2),y_2)}{\Delta_{y_1y_1} G_t(\bar{y}_1(y_2),y_2)}$. It follows that $-1 \leq \frac{d\bar{y}_1(y_2)}{dy_2} \leq 0$ since $G_t(.)$ satisfies the second-order properties. By a similar argument, $-1 \leq \frac{d\bar{y}_2(y_1)}{dy_1} \leq 0$. *(b)* Recall that the feasible set of targets is $y \geq x$. Also, $\tilde{y} = y \wedge (x + K)$ denotes the inventories for realized capacity levels $K = (K_1,K_2)$. Obviously, $x \leq \tilde{y} \leq y$.

Region 1: $\underline{x_1 \geq \bar{y}_1(x_2) \text{ and } x_2 \geq \bar{y}_2(x_1)}$. For any capacity realization, we have $\tilde{y} \geq x$. As a consequence, $G_t(x_1,x_2) \leq G_t(\tilde{y}_1,x_2) \leq G_t(\tilde{y}_1,\tilde{y}_2)$ holds due to convexity, and the definitions of the curves $\bar{y}_1(.)$ and $\bar{y}_2(.)$. Consequently, $G_t(x_1,x_2) \leq \mathbb{E}_{K_1,K_2} G_t(y_1 \wedge (x_1 + K_1), y_2 \wedge (x_2 + K_2))$ for any $y \geq x$, hence, it is optimal not to produce any of the products.

Region 2: $\underline{x_1 < \bar{y}_1(x_2) \text{ and } x_2 \geq \bar{y}_2(x_1)}$. We first show that the optimal inventory targets would be $(\bar{y}_1(x_2),x_2)$ if the dedicated capacity realizations, K_1 and K_2, were known beforehand. First, we show that it is not optimal to produce product 2. Consider any $y_2 > x_2$ with $y_1 \geq x_1$. Since $\bar{y}_2(y_1)$ is non-increasing, it implies that $\bar{y}_2(x_1) \geq \bar{y}_2(\tilde{y}_1)$. And, since $x_2 \geq \bar{y}_2(x_1)$ by assumption, we have $x_2 \geq \bar{y}_2(\tilde{y}_1)$. Since $\tilde{y}_2 \geq x_2 \geq \bar{y}_2(\tilde{y}_1)$, we immediately have $G_t(\tilde{y}_1, x_2) \leq G_t\tilde{y}_1, \tilde{y}_2)$ by the definition of $\bar{y}_2(\tilde{y}_1)$ and due to convexity. Hence, not producing product 2 is optimal. Recall that $\bar{y}_1(x_2)$ is a minimizer of $G_t(y_1, x_2)$ in y_1. Due to convexity of $G_t(.)$, it follows that $\bar{y}_1(x_2)$ is also a minimizer of $G_t(y_1 \wedge (x_1 + K_1), x_2)$. Therefore, $(\bar{y}_1(x_2), x_2)$ is optimal. Note that the optimal inventory targets are independent of the capacity realizations and knowing the capacity levels in advance does not influence the optimal decisions. Therefore, inventory targets $(\bar{y}_1(x_2), x_2)$ are also optimal when capacities are not known *a priori.*

Region 3: $\underline{x_1 \geq \bar{y}_1(x_2) \text{ and } x_2 < \bar{y}_2(x_1)}$. We follow the same reasoning as for Region 2.

Region 4: $\underline{x_1 \leq \bar{y}_1(x_2) \text{ and } x_2 \leq \bar{y}_2(x_1)}$. We prove that the optimal inventory targets (y_1, y_2) satisfy $\bar{y}_i(y_{3-i}) \leq y_i \leq \bar{y}_i(x_{3-i})$ for each i:

- $\underline{y_i \leq \bar{y}_i(x_{3-i}) \text{ for each } i}$: Assume that one of the products, say product 1, violates the condition. Consider any $y_1 > \bar{y}_1(x_2)$ and $y_2 \geq x_2$. Since $\bar{y}_1(y_2)$ is non-increasing, we have $\bar{y}_1(x_2) \geq \bar{y}_1(y_2)$ for all $y_2 \geq x_2$, hence, $y_1 > \bar{y}_1(y_2)$. By the definition of $\bar{y}_1(y_2)$ and due to convexity, revising the production decision by keeping y_2 the same and setting $y_1 = \bar{y}_1(y_2)$, the firm could have incurred a lower cost for any realization of K_1 and K_2. (The details are identical to the one for Region 2.) Therefore, $y_i \leq \bar{y}_i(x_{3-i})$ for each i.
- $\underline{\bar{y}_i(y_{3-i}) \leq y_i \text{ for each } i}$: Assume that and one of the products, say product 1, violates the condition. Consider any $y_1 < \bar{y}_1(y_2)$ and $y_2 \geq x_2$. Since $\bar{y}_1(y_2)$ is non-increasing and $\tilde{y}_2 \leq y_2$, we have $\bar{y}_1(y_2) \leq \bar{y}_1(\tilde{y}_2)$. By the definition of $\bar{y}_1(y_2)$ and due to convexity, revising the production decision by keeping y_2 the same and setting $y_1 = \bar{y}_1(y_2)$, the firm could have incurred a lower cost for any realization of K_1 and K_2 (as argued for Region 2). Therefore, $\bar{y}_i(y_{3-i}) \leq y_i$ for each i.

The property above, derived for Region 4, has the following two consequences.

<u>Consequence 1:</u> Assume that producing product j is not optimal, $y_j = x_j$. Since $\bar{y}_i(y_j) \leq y_i \leq \bar{y}_i(x_j)$, the expression reduces to $y_i = \bar{y}_i(x_i)$.

<u>Consequence 2:</u> If $x_i < y_i^0$ for each i, then it is optimal to produce both products: $y_i > x_i$. To reach a contradiction, assume that it is not optimal to produce product 1. Then, by *Consequence 1*, inventory target for product 2 is $\bar{y}_2(x_1)$. However, a better solution can be obtained by setting $y_1 = \bar{y}_1(\bar{y}_2(x_1))$ and keeping $y_2 = \bar{y}_2(x_1)$, as shown for Region 4. Therefore, both products are produced in an optimal policy.

Consequence 2 always holds when $G_t(y)$ is strictly convex. If $G_t(y)$ is not strictly convex, there could be multiple optimal solutions, however, we focus on the one that satisfies *Consequence 2.* We next present an exact characterization of the optimal policy in Region 4. Define $EG(y;x) = \mathbb{E}_{K_1,K_2} G_t(y \wedge (x + K))$. The objective is to minimize $EG(y;x)$ in y for a given x with $y \geq x$.

$EG(y;x)$ is not necessarily convex in y, however, as we will show, the first-order conditions are sufficient for optimality. Denote by $f_i(.)$ and $F_i(.)$ the probability density function and the cumulative distribution function of the product i's capacity. We derive the first-order conditions as follows.

$$\begin{aligned}\Delta_{y_1} EG(y;x) &= [1-F_1(y_1-x_1)]\mathbb{E}_{K_2}\Delta_{y_1} G_t(y_1, y_2 \wedge (x_2+K_2)) \\ \Delta_{y_2} EG(y;x) &= [1-F_2(y_2-x_2)]\mathbb{E}_{K_1}\Delta_{y_2} G_t(y_1 \wedge (x_1+K_1), y_2)\end{aligned}$$

The following two steps show that $EG(y;x)$ is unimodal in y_1 for a given y_2 (and vice versa).

Step A1: The function $G_t(y_1, y_2 \wedge (x_2+K_2))$ is convex in y_1 for any K_2 due to the convexity of $G_t(.)$. Hence, $\mathbb{E}_{K_2} G_t(y_1, y_2 \wedge (x_2+K_2))$ is convex in y_1. Define $\check{y}_1(y_2;x) = \arg\min_{y_1 \ge 0} \mathbb{E}_{K_2} G_t(y_1, y_2 \wedge (x_2+K_2))$. For convenience, we treat x as a constant and write $\check{y}_2(y_1)$ instead of $\check{y}_2(y_1;x)$.

Step A2: For any K_1, $\check{y}_1(y_2)$ is also a minimizer of $\mathbb{E}_{K_2} G_t(y_1 \wedge (x_1+K_1), y_2 \wedge (x_2+K_2))$. Specifically, $\mathbb{E}_{K_2} G_t(y_1 \wedge (x_1+K_1), y_2 \wedge (x_2+K_2))$ is unimodal in y_1 and it is minimized at $y_1 = \check{y}_1(y_2)$ for any K_1. For $y_1 \le \check{y}_1(y_2)$, the function $\mathbb{E}_{K_2} G_t(y_1 \wedge (x_1+K_1), y_2 \wedge (x_2+K_2))$ is convex and decreasing in y_1. And for $y_1 \ge \check{y}_1(y_2)$, it is increasing (not necessarily convex). Thus, $EG(y;x) = \mathbb{E}_{K_1,K_2} G_t(y_1 \wedge (x_1+K_1), y_2 \wedge (x_2+K_2))$ is also unimodal in y_1 and it is minimized at $y_1 = \check{y}_1(y_2)$.

Steps A1–A2 show that $EG(y;x)$ is unimodal in y_i for given y_{3-i} and x. To establish the optimal solution, we show that $\check{y}_i(y_{3-i}) \ge \bar{y}_i(y_{3-i})$ for $i = 1,2$. Without loss of generality, consider $i = 1$.

$$\begin{aligned}\Delta_{y_1} EG(\bar{y}_1(y_2), y_2; x) &= [1-F_1(\bar{y}_1(y_2)-x_1)]\mathbb{E}_{K_2}\Delta_{y_1} G_t(\bar{y}_1(y_2), y_2 \wedge (x_2+K_2)) \\ &\le [1-F_1(\bar{y}_1(y_2)-x_1)]\mathbb{E}_{K_2}\Delta_{y_1} G_t(\bar{y}_1(y_2), y_2) \\ &= 0 \; = \; \Delta_{y_1} EG(\check{y}_1(y_2), y_2; x),\end{aligned}$$

where the inequality is due to second-order properties of G. Hence, $\check{y}_1(y_2) \ge \bar{y}_1(y_2)$. For $y_2 \le x_2$, the inequality holds with equality. Recall that x satisfies $x_i \le \bar{y}_i(x_{3-i})$ for each i (in Region 4). Consequently, the solution to the first-order conditions, denoted by $\hat{y} = \hat{y}(x)$ (the intersection of the curves $y_i = \check{y}_i(y_{3-i})$) is such that it is either feasible (i.e. $\hat{y} \ge x$), or $\hat{y}_i \ge x_i$ and $\hat{y}_{3-i} < x_{3-i}$ for some i. In other words, the first-order conditions never imply $\hat{y} < x$. Otherwise, a contradiction is reached, in which x would not belong to Region 4 using the fact that $\check{y}_i(y_{3-i}) \ge \bar{y}_i(y_{3-i})$. Consider three subcases.

Case 1, $\hat{y} \ge x$: First, consider the case where the solution to the first-order conditions is feasible. We will show that $\Delta_{y_i y_i} EG(\hat{y};x) \ge \Delta_{y_1 y_2} EG(\hat{y};x) \ge 0$ for $i = 1,2$. In other words, whenever the first-order conditions are satisfied, then second-order properties are also satisfied. Obviously, second-order properties imply that the Hessian is positive, further implying the sufficiency of the first-order conditions for optimality. We obtain the second-order conditions as follows.

$$\Delta_{y_1 y_1} EG(y;x) = [1-F_1(y_1-x_1)]\mathbb{E}_{K_2}\Delta_{y_1 y_1} G_t(y_1, y_2 \wedge (x_2+K_2))$$

$$-f_1(y_1-x_1)\mathbb{E}_{K_2}\Delta_{y_1}G_t(y_1,y_2\wedge(x_2+K_2))$$
$$\Delta_{y_1y_2}EG(y;x) = [1-F_1(y_1-x_1)][1-F_2(y_2-x_2)]\Delta_{y_1y_2}G_t(y_1,y_2)$$

$\Delta_{y_1y_2}EG(y;x)\geq 0$ since $\Delta_{y_1y_2}G_t(y_1,y_2)\geq 0$. When the first-order conditions are satisfied, $\Delta_{y_1y_1}EG(\hat{y};x)$ reduces to $\Delta_{y_1y_1}EG(\hat{y};x)=[1-F_1(\hat{y}_1-x_1)]\mathbb{E}_{K_2}\Delta_{y_1y_1}G_t(\hat{y}_1,\hat{y}_2\wedge(x_2+K_2))$. Thus,

$$\begin{aligned}\mathbb{E}_{K_2}\Delta_{y_1y_1}G_t(\hat{y}_1,\hat{y}_2\wedge(x_2+K_2)) &= \int_0^{\hat{y}_2-x_2}\Delta_{y_1y_1}G_t(\hat{y}_1,x_2+K_2)f_2(k_2)dk_2+\int_{\hat{y}_2-x_2}^{\infty}\Delta_{y_1y_1}G_t(\hat{y}_1,\hat{y}_2)f_2(k_2)dk_2\\ &\geq \int_{\hat{y}_2-x_2}^{\infty}\Delta_{y_1y_1}G_t(\hat{y}_1,\hat{y}_2)f_2(k_2)dk_2 \;=\; [1-F_2(\hat{y}_2-x_2)]\Delta_{y_1y_1}G_t(\hat{y}_1,\hat{y}_2)\end{aligned}$$

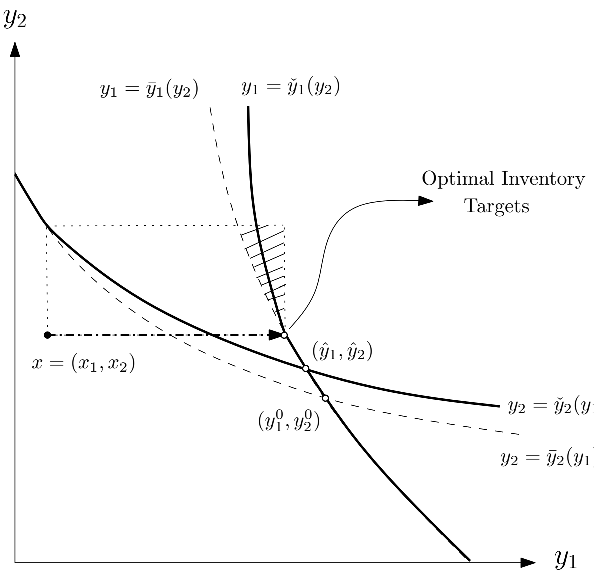
Multiplying both sides of the inequality by $1-F_1(\hat{y}_1-x_1)$, we obtain $\Delta_{y_1y_1}EG(\hat{y};x)\geq \Delta_{y_1y_2}EG(\hat{y};x)$.Similarly, $\Delta_{y_2y_2}EG(\hat{y};x)\geq\Delta_{y_1y_2}EG(\hat{y};x)$. As a result, the Hessian is positive when the first-order conditions are satisfied, hence, $y^*(x)=\hat{y}(x)$.

Case 2, $\hat{y}_1\geq x_1$, $\hat{y}_2<x_2$: We have either $x_1>y_1^0$ or $x_2>y_2^0$, since otherwise, producing both products would be optimal, and the first-order conditions would be feasible (*Consequence 2*). Since $\hat{y}_2<x_2$, we have $x_2>y_2^0$ (it can be shown by contradiction). From Step A1, $\mathbb{E}_{K_1}\Delta_{y_2}G_t(y_1\wedge(x_1+K_1),\check{y}_2)=0$. Taking the derivative of both sides with respect to y_1, we obtain the following.

$$\frac{d\check{y}_2}{dy_1}=-\frac{[1-F_1(y_1-x_1)]\Delta_{y_1y_2}G_t(y_1,\check{y}_2)}{[1-F_1(y_1-x_1)]\Delta_{y_1y_2}G_t(y_1,\check{y}_2)+\int_0^{y_1-x_1}\Delta_{y_2y_2}G_t(x_1+K_1,\check{y}_2)f_1(k_1)dk_1}$$

Due to the second-order properties of $G_t(y)$, $-1\leq\frac{d\check{y}_2}{dy_1}\leq 0$. Similarly, $-1\leq\frac{d\check{y}_1}{dy_2}\leq 0$. Figure EC.2 illustrates the shaded region, where the optimal inventory targets lie. The curves reflect the relationship $\check{y}_i(y_{3-i})\geq\bar{y}_i(y_{3-i})$, and that for $y_{3-i}\leq x_{3-i}$, the inequality becomes an equality.

Figure EC.2 Optimal Solution in Region 4, $\hat{y}_1\geq x_1$, $\hat{y}_2<x_2$



We show that it is optimal not to produce product 2. To reach a contradiction, consider any (y_1, y_2) (from the shaded region), satisfying $y_2 > x_2$ with $\bar{y}_2(y_1) \le y_2 \le \bar{y}_2(x_1)$, and assume that it is optimal. Due to unimodality of $EG(y;x)$ in y_1, we must have $y_1 = \check{y}_1(y_2)$ for optimality. In this case, revising the target for product 2 as $\max(\check{y}_2(y_1), x_2)$ improves the objective function due to unimodality of $EG(y;x)$ in y_2. Thus, we are able to construct a better solution, which is in contradiction with the optimality of (y_1, y_2). Hence, not ordering product 2 is optimal. As a result, product 1 target is set to be $\bar{y}_1(x_2)$ by *Consequence 1.*

Case 3, $\hat{y}_1 < x_1$, $\hat{y}_2 \ge x_2$: Similar analysis for Case 2 applies. $\square$

Proof of Proposition 2. Assume first that (x_1, x_2) belongs to the region, where it is optimal to produce both products. In this case, the optimal targets (y_1^*, y_2^*) satisfy the first-order conditions: $\mathbb{E}_{K_2}\Delta_{y_1}G_t(y_1^*, y_2^* \wedge (x_2+K_2)) = 0$ and $\mathbb{E}_{K_1}\Delta_{y_2}G_t(y_1^* \wedge (x_1+K_1), y_2^*) = 0$ Taking the derivatives of both sides of the two equalities with respect to x_1 and x_2, we obtain

$$
\begin{aligned}
0 &= \mathbb{E}_{K_2}\Delta_{y_1y_1}G_t(y_1^*, y_2^* \wedge (x_2+K_2))\frac{\partial y_1^*}{\partial x_1} + [1 - F_2(y_2^* - x_2)]\Delta_{y_1y_2}G_t(y_1^*, y_2^*)\frac{\partial y_2^*}{\partial x_1} \\
&:= a_{11}\frac{\partial y_1^*}{\partial x_1} + a_{12}\frac{\partial y_2^*}{\partial x_1} \\
0 &= [1 - F_1(y_1^* - x_1)]\Delta_{y_1y_2}G_t(y_1^*, y_2^*)\frac{\partial y_1^*}{\partial x_1} + \mathbb{E}_{K_1}\Delta_{y_2y_2}G_t(y_1^* \wedge (x_1+K_1), y_2^*)\frac{\partial y_2^*}{\partial x_1} \\
&\quad + \int_0^{y_1^*-x_1}\Delta_{y_1y_2}G_t(x_1+k_1, y_2^*)f_1(k_1)dk_1 \;:=\; a_{21}\frac{\partial y_1^*}{\partial x_1} + a_{22}\frac{\partial y_2^*}{\partial x_1} + b_1 \\
0 &= \mathbb{E}_{K_2}\Delta_{y_1y_1}G_t(y_1^*, y_2^* \wedge (x_2+K_2))\frac{\partial y_1^*}{\partial x_2} + [1 - F_2(y_2^* - x_2)]\Delta_{y_1y_2}G_t(y_1^*, y_2^*)\frac{\partial y_2^*}{\partial x_2} \\
&\quad + \int_0^{y_2^*-x_2}\Delta_{y_1y_2}G_t(y_1^*, x_2+k_2)f_2(k_2)dk_2 \;:=\; a_{11}\frac{\partial y_1^*}{\partial x_2} + a_{12}\frac{\partial y_2^*}{\partial x_2} + b_2 \\
0 &= [1 - F_1(y_1^* - x_1)]\Delta_{y_1y_2}G_t(y_1^*, y_2^*)\frac{\partial y_1^*}{\partial x_2} + \mathbb{E}_{K_1}\Delta_{y_2y_2}G_t(y_1^* \wedge (x_1+K_1), y_2^*)\frac{\partial y_2^*}{\partial x_2} \\
&:= a_{21}\frac{\partial y_1^*}{\partial x_2} + a_{22}\frac{\partial y_2^*}{\partial x_2}
\end{aligned}
$$

We have four equations with four unknowns. The solution of the above set of equations is straightforward, and given by $\frac{\partial y_1^*}{\partial x_1} = \frac{a_{12}b_1}{a_{11}a_{22}-a_{12}a_{21}}$, $\frac{\partial y_2^*}{\partial x_1} = -\frac{a_{11}b_1}{a_{11}a_{22}-a_{12}a_{21}}$, $\frac{\partial y_1^*}{\partial x_2} = -\frac{a_{22}b_2}{a_{11}a_{22}-a_{12}a_{21}}$, and $\frac{\partial y_2^*}{\partial x_2} = \frac{a_{21}b_2}{a_{11}a_{22}-a_{12}a_{21}}$. To establish the monotonicity properties, we prove $a_{22} \ge a_{21} + b_1$.

$$
\begin{aligned}
a_{22} &= \mathbb{E}_{K_1}\Delta_{y_2y_2}G_t(y_1^* \wedge (x_1+K_1), y_2^*) \\
&= \int_{y_1^*-x_1}^{\infty}\Delta_{y_2y_2}G_t(y_1^*, y_2^*)f_1(k_1)dk_1 + \int_0^{y_1^*-x_1}\Delta_{y_2y_2}G_t(x_1+k_1, y_2^*)f_1(k_1)dk_1 \\
&= [1 - F_1(y_1^* - x_1)]\Delta_{y_2y_2}G_t(y_1^*, y_2^*) + b_1 \;\ge\; [1 - F_1(y_1^* - x_1)]\Delta_{y_1y_2}G_t(y_1^*, y_2^*) + b_1 \;=\; a_{21} + b_1
\end{aligned}
$$

The inequality follows from the second-order properties. Thus, we have $0 \leq b_1 \leq a_{22} - a_{21}$. In the same manner, we have $0 \leq b_2 \leq a_{11} - a_{12}$ by symmetry of the argument. Obviously, $a_{11} \geq a_{12} \geq 0$ and $a_{22} \geq a_{21} \geq 0$. Therefore, $\frac{\partial y_1^*}{\partial x_1} \geq 0$, $\frac{\partial y_2^*}{\partial x_1} \leq 0$, $\frac{\partial y_1^*}{\partial x_2} \leq 0$, and $\frac{\partial y_2^*}{\partial x_2} \geq 0$ follow immediately. Thus,

$$\frac{\partial y_1^*}{\partial x_1} = \frac{a_{12} b_1}{a_{11}a_{22} - a_{12}a_{21}} \leq \frac{a_{12}(a_{22} - a_{21})}{a_{11}a_{22} - a_{12}a_{21}} \leq \frac{a_{11}(a_{22} - a_{21})}{a_{11}a_{22} - a_{12}a_{21}} \leq 1$$
$$\frac{\partial y_2^*}{\partial x_1} = -\frac{a_{11} b_1}{a_{11}a_{22} - a_{12}a_{21}} \geq -\frac{a_{11}(a_{22} - a_{21})}{a_{11}a_{22} - a_{12}a_{21}} \geq -1$$

$$\frac{\partial y_1^*}{\partial x_1} - \frac{\partial y_1^*}{\partial x_2} - 1 = \frac{a_{22}(b_2 - a_{11}) + a_{12}(b_1 + a_{21})}{a_{11}a_{22} - a_{12}a_{21}} \leq 0$$

The last inequality is due to the fact that $b_2 - a_{11} \leq -a_{12}$ and $b_1 + a_{21} \leq a_{22}$. Using a symmetric argument, we have $\frac{\partial y_1^*}{\partial x_2} \geq -1$, $\frac{\partial y_2^*}{\partial x_2} \leq 1$ and $\frac{\partial y_2^*}{\partial x_2} - \frac{\partial y_2^*}{\partial x_1} \leq 1$.

Assume now that (x_1, x_2) belongs to the region, where it is optimal to order only one product, say product 1. In this case, $y_1^* = \bar{y}_1(x_2)$ and $y_2^* = x_2$. From Theorem 2, part (a), we have $-1 \leq \frac{d\bar{y}_1(y_2)}{dy_2} \leq 0$. Hence, $-1 \leq \frac{\partial y_1^*}{\partial x_2} \leq 0$ and $\frac{\partial y_1^*}{\partial x_1} = 0$. In addition, $\frac{\partial y_2^*}{\partial x_1} = 0$ and $\frac{\partial y_2^*}{\partial x_2} = 1$. Hence, the desired monotonicity results hold. In the region, where it is not optimal to order, $y_i^* = x_i$, the desired monotonicity results hold trivially: $\frac{\partial y_i^*}{\partial x_i} = 1$ and $\frac{\partial y_i^*}{\partial x_{3-i}} = 0$. □

Proof of Proposition 3. Since $G_t(\tilde{y}) = \mathbb{E}_\xi C_t(\tilde{y}, \xi)$, we have that $\Delta_{\tilde{y}_i} G_t(\tilde{y}) \geq -p_i$ and $G_t(\tilde{y})$ satisfies the second order properties in $(\tilde{y}_1, \tilde{y}_2)$. In all of the cases considered below, it is easy to show that $\Delta_{x_i} V_t(x) \geq -p_i$, and as a consequence, $p_i + h_i + \beta \Delta_{x_i} V_t(x) \geq p_i + h_i - \beta p_i = (1-\beta)p_i + h_i \geq 0$, as desired. Therefore, we focus on the second-order properties. Consider (x_1, x_2) in each of the regions.

- Region 1: In this case, $V_t(x_1, x_2) = G_t(x_1, x_2)$, hence it satisfies the second-order properties.
- Region 2: In this case, $V_t(x_1, x_2) = \mathbb{E}_{K_1} G_t(\bar{y}_1(x_2) \wedge (x_1 + K_1), x_2)$. To show that $V_t(x_1, x_2)$ satisfies the second-order properties in this region, it suffices to show that the function $G_t(\bar{y}_1(x_2) \wedge (x_1 + K_1), x_2)$ satisfies the second-order properties for any given K_1. When that is the case, expectation with respect to K_1 preserves the second-order properties.

When $x_1 + K_1 \leq \bar{y}_1(x_2)$, we have $G_t(\bar{y}_1(x_2) \wedge (x_1 + K_1), x_2) = G_t(x_1 + K_1, x_2)$, which preserves the second-order properties. When $x_1 + K_1 > \bar{y}_1(x_2)$, we have $G_t(\bar{y}_1(x_2) \wedge (x_1 + K_1), x_2) = G_t(\bar{y}_1(x_2), x_2)$. Below, we show that the function $G_t(\bar{y}_1(x_2), x_2)$ satisfies the second-order properties in (x_1, x_2). Obviously, $\frac{\partial G_t(\bar{y}_1(x_2), x_2)}{\partial x_1} = 0$, hence, $\frac{\partial^2 G_t(\bar{y}_1(x_2), x_2)}{\partial x_1^2} = \frac{\partial^2 G_t(\bar{y}_1(x_2), x_2)}{\partial x_1 \partial x_2} = 0$. The first-order derivative with respect to x_2 is obtained using the envelope theorem. The second-order derivative is obtained using the expression for $\bar{y}_1'(x_2)$ from Theorem 2, part(a).

$$\frac{\partial G_t(\bar{y}_1(x_2), x_2)}{\partial x_2} = \Delta_{y_2} G_t(\bar{y}_1(x_2), x_2)$$

$$\begin{aligned}\frac{\partial^2 G_t(\bar{y}_1(x_2),x_2)}{\partial x_2^2} &= \Delta_{y_1y_2}G_t(\bar{y}_1(x_2),x_2)\bar{y}_1'(x_2)+\Delta_{y_2y_2}G_t(\bar{y}_1(x_2),x_2)\\ &= \frac{(\Delta_{y_1y_1}\Delta_{y_2y_2}-\Delta^2_{y_1y_2})G_t(\bar{y}_1(x_2),x_2)}{\Delta_{y_1y_1}G_t(\bar{y}_1(x_2),x_2)} \ \geq \ 0\end{aligned}$$

Therefore, $V_t(x_1,x_2)$ satisfies the second-order properties in Region 2 (hence in Region 3).

• Region 4: We first derive the first-order derivative of the optimal cost function. The envelope theorem is useful in obtaining the following result.

$$\begin{aligned}V_t(x_1,x_2) &= \mathbb{E}_{K_1,K_2}G_t(y_1^*\wedge(x_1+K_1),y_2^*\wedge(x_2+K_2))\\ \Delta_{x_1}V_t(x_1,x_2) &= \int_0^{y_1^*-x_1}\mathbb{E}_{K_2}\Delta_{y_1}G_t(x_1+k_1,y_2^*\wedge(x_2+K_2))f_1(k_1)dk_1\end{aligned}$$

Next, we derive the second-order derivatives.

$$\begin{aligned}\Delta_{x_1x_1}V_t(x_1,x_2) = &\int_0^{y_1^*-x_1}\int_0^{y_2^*-x_2}\Delta_{y_1y_1}G_t(x_1+k_1,x_2+k_2)f_1(k_1)f_2(k_2)dk_2dk_1\\ &+\int_0^{y_1^*-x_1}\int_{y_2^*-x_2}^{\infty}\Delta_{y_1y_1}G_t(x_1+k_1,y_2^*)f_1(k_1)f_2(k_2)dk_2dk_1\\ &+\left[\int_0^{y_1^*-x_1}\int_{y_2^*-x_2}^{\infty}\Delta_{y_1y_2}G_t(x_1+k_1,y_2^*)f_1(k_1)f_2(k_2)dk_2dk_1\right]\frac{\partial y_2^*}{\partial x_1}\end{aligned}$$

$$\begin{aligned}\Delta_{x_1x_2}V_t(x_1,x_2) = &\int_0^{y_1^*-x_1}\int_0^{y_2^*-x_2}\Delta_{y_1y_2}G_t(x_1+k_1,x_2+k_2)f_1(k_1)f_2(k_2)dk_2dk_1\\ &+\left[\int_0^{y_1^*-x_1}\int_{y_2^*-x_2}^{\infty}\Delta_{y_1y_2}G_t(x_1+k_1,y_2^*)f_1(k_1)f_2(k_2)dk_2dk_1\right]\frac{\partial y_2^*}{\partial x_2}\end{aligned}$$

$\Delta_{x_1x_2}V_t(x_1,x_2)\geq 0$ due to the second-order properties of $G_t(.)$ and $\frac{\partial y_2^*}{\partial x_2}\geq 0$, as shown in Proposition 2. Next, we prove that $\Delta_{x_1x_1}V_t(x_1,x_2)\geq\Delta_{x_1x_2}V_t(x_1,x_2)$. From Property (iii) of, Proposition 2 we have $\frac{\partial y_2^*}{\partial x_1}-\frac{\partial y_2^*}{\partial x_2}\geq -1$. Thus,

$$\begin{aligned}(\Delta_{x_1x_1}-\Delta_{x_1x_2})V_t(x_1,x_2) = &\int_0^{y_1^*-x_1}\int_0^{y_2^*-x_2}(\Delta_{y_1y_1}-\Delta_{y_1y_2})G_t(x_1+k_1,x_2+k_2)f_1(k_1)f_2(k_2)dk_2dk_1\\ &+\int_0^{y_1^*-x_1}\int_{y_2^*-x_2}^{\infty}\Delta_{y_1y_1}G_t(x_1+k_1,y_2^*)f_1(k_1)f_2(k_2)dk_2dk_1\\ &+\left[\int_0^{y_1^*-x_1}\int_{y_2^*-x_2}^{\infty}\Delta_{y_1y_2}G_t(x_1+k_1,y_2^*)f_1(k_1)f_2(k_2)dk_2dk_1\right]\left(\frac{\partial y_2^*}{\partial x_1}-\frac{\partial y_2^*}{\partial x_2}\right)\geq 0\end{aligned}$$

By a similar argument, we have $\Delta_{x_2x_2}V_t(x_1,x_2)\geq\Delta_{x_1x_2}V_t(x_1,x_2)\geq 0$, hence, $V_t(x_1,x_2)$ satisfies the second-order properties in this region. Finally, crossing from one region to another does not cause any problem since the first derivatives are continuous. □

Proof of Proposition 4. We use Theorem 2 and explicitly derive the optimal targets in each region. We show that the optimal targets (y_1^*, y_2^*) from Theorem 2 and the targets $(y_1(\tau_2(x_2)), y_2(\tau_1(x_1)))$, claimed in this proposition, either coincide, or lead to the same production decisions. That is, if it is optimal to order product i, $y_i^* > x_i$, then $y_i^* \wedge (x_i + K_i) = \bar{y}_i(\tau_{3-i}(x_{3-i})) \wedge (x_i + K_i)$. Otherwise, if ordering product i is not optimal, $y_i^* = x_i$, then $\bar{y}_i(\tau_{3-i}(x_{3-i})) \le x_i$. Recall that the functions $\bar{y}_i(.)$ are non-increasing with $-1 \le \bar{y}_i'(.) \le 0$ from Theorem 2, which is used below.

Region 1: $\underline{x_1 \ge \bar{y}_1(x_2) \text{ and } x_2 \ge \bar{y}_2(x_1)}$. From Theorem 2, it is not optimal to produce any of the products. Below, we show for product 1 that the initial inventory x_1 exceeds the target defined by $\bar{y}_1(\tau_2(x_2))$ in Region 1. (Similar argument applies to product 2, hence omitted.)

- If $\tau_2(x_2) = x_2 + K_2$, then, $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(x_2 + K_2) \le \bar{y}_1(x_2) \le x_1$.
- If $\tau_2(x_2) = y_2^0 (< x_2)$, then $x_1 > y_1^0$ in Region 1, because otherwise $x_2 > y_2^0$, and $\tau_2(x_2) = x_2$. Thus, $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(y_2^0) = y_1^0 < x_1$.
- Finally, if $\tau_2(x_2) = x_2$, then, $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(x_2) \le x_1$.

Region 2: $\underline{x_1 < \bar{y}_1(x_2) \text{ and } x_2 \ge \bar{y}_2(x_1)}$. From Theorem 2, it is not optimal to produce product 2, while product 1 target equals $\bar{y}_1(x_2)$. In Region 2, we have $x_1 \le y_1^0$ and $x_2 \ge y_2^0$. So, $\tau_1(x_1)$ equals either y_1^0 or $x_1 + K_1$, while $\tau_2(x_2) = x_2$. As a result, $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(x_2)$. Product 1 target is indeed as given by Theorem 2. For product 2, consider the following two cases.

- If $\tau_1(x_1) = x_1 + K_1$, then, $\bar{y}_2(\tau_1(x_1)) = \bar{y}_2(x_1 + K_1) \le \bar{y}_2(x_1) \le x_2$.
- If $\tau_1(x_1) = y_1^0$, then, $\bar{y}_2(\tau_1(x_1)) = \bar{y}_2(y_1^0) = y_2^0 \le x_2$, likewise.

Region 3: $\underline{x_1 \ge \bar{y}_1(x_2) \text{ and } x_2 < \bar{y}_2(x_1)}$. Same as Region 2.

Region 4: $\underline{x_1 \le \bar{y}_1(x_2) \text{ and } x_2 \le \bar{y}_2(x_1)}$. Recall that $\Delta_{y_1} G_t(\bar{y}_1(y_2), y_2) = 0$ and $\Delta_{y_2} G_t(y_1, \bar{y}_2(y_1)) = 0$ by definition. Also, $\bar{y}_1(y_2^0) = y_1^0$ and $\bar{y}_2(y_1^0) = y_2^0$. From Theorem 2, targets satisfy: $\Delta_{y_1} G_t(y_1, y_2 \wedge (x_2 + K_2)) = 0$ and $\Delta_{y_2} G_t(y_1 \wedge (x_1 + K_1), y_2) = 0$.

- Let $\tau_i(x_i) = y_i^0$ for each i, i.e., $y_i^0 - K_i \le x_i \le y_i^0$. Then, $\bar{y}_i(\tau_{3-i}(x_{3-i})) = \bar{y}_i(y_{3-i}^0) = y_i^0$. Clearly, $(y_1, y_2) = (y_1^0, y_2^0)$, satisfies the first order conditions: $\Delta_{y_1} G_t(y_1^0, y_2^0 \wedge (x_2 + K_2)) = \Delta_{y_1} G_t(y_1^0, y_2^0) = 0$ and $\Delta_{y_2} G_t(y_1^0 \wedge (x_1 + K_1), y_2^0) = \Delta_{y_2} G_t(y_1^0, y_2^0) = 0$, hence it must be optimal.
- Let $\tau_i(x_i) = x_i + K_i$ for each i, i.e., $x_i + K_i < y_i^0$. We have $\bar{y}_1(x_2 + K_2) \ge \bar{y}_1(y_2^0) = y_1^0 \ge x_1 + K_1$, since $\bar{y}_1(.)$ is non-increasing. Similarly, $\bar{y}_2(x_1 + K_1) \ge x_2 + K_2$. For $(y_1, y_2) = (\bar{y}_1(x_2 + K_2), \bar{y}_2(x_1 + K_1))$, first-order conditions are satisfied: $\Delta_{y_1} G_t(\bar{y}_1(x_2 + K_2), \bar{y}_2(x_1 + K_1) \wedge (x_2 + K_2)) = \Delta_{y_1} G_t(\bar{y}_1(x_2 + K_2), x_2 + K_2) = 0$ and $\Delta_{y_2} G_t(\bar{y}_1(x_2 + K_2) \wedge (x_1 + K_1), \bar{y}_2(x_1 + K_1)) = \Delta_{y_2} G_t(x_1 + K_1, \bar{y}_2(x_1 + K_1)) = 0$.
- Let $\tau_1(x_1) = x_1 + K_1$ and $\tau_2(x_2) = y_2^0$. Then, $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(y_2^0) = y_1^0$ and $\bar{y}_2(\tau_1(x_1)) = \bar{y}_2(x_1 + K_1)$. Define $z_2 = \bar{y}_2(x_1 + K_1) \wedge (x_2 + K_2)$ and $\acute{y}_1 = \bar{y}_1(z_2)$. Below, we show that $(y_1, y_2) = (\acute{y}_1, \bar{y}_2(x_1 + K_1))$ satisfies first-order conditions, and $\acute{y}_1 \ge x_1 + K_1$ (i.e., $\acute{y}_1$ is not reachable), establishing the optimality

of targets $\bar{y}_i(\tau_{3-i}(x_{3-i}))$. Indeed, since $\acute{y}_1$ is not reachable, it can be replaced by $\bar{y}_1(\tau_2(x_2))$, which is also not reachable ($\bar{y}_1(\tau_2(x_2)) = y_1^0 \geq x_1 + K_1$).

First, we prove that $\acute{y}_1 \geq x_1 + K_1$. Since $x_1 + K_1 \leq y_1^0$, and $z_2 \leq \bar{y}_2(x_1 + K_1)$ by definition, the point $(x_1 + K_1, z_2)$ belongs to Region 4. From the definition of Region 4, $\acute{y}_1 = \bar{y}_1(z_2) \geq x_1 + K_1$. The first-order conditions are satisfied: $\Delta_{y_1} G_t(\acute{y}_1, \bar{y}_2(x_1 + K_1) \wedge (x_2 + K_2)) = \Delta_{y_1} G_t(\bar{y}_1(z_2), z_2) = 0$, and $\Delta_{y_2} G_t(\acute{y}_1 \wedge (x_1 + K_1), \bar{y}_2(x_1 + K_1)) = \Delta_{y_2} G_t(x_1 + K_1, \bar{y}_2(x_1 + K_1)) = 0$.

Since $\acute{y}_1 \geq x_1 + K_1$, product 1 target is not reachable. And, since $y_1^0 \geq x_1 + K_1$, replacing product 1 target with y_1^0 does not alter the ending inventories, hence it is optimal. In addition, we have $\bar{y}_2(\tau_1(x_1)) = \bar{y}_2(x_1 + K_1)$ as desired.

• The case when $\tau_1(x_2) = x_2 + K_2$ and $\tau_1(x_1) = y_1^0$ is similar as above.

• Let $\tau_i(x_i) = x_i$ for some i, the only case not analyzed for Region 4. In Region 4, we cannot have $\tau_i(x_i) = x_i$ for both i. Without loss of generality, we analyze the case when $\tau_2(x_2) = x_2$, i.e., $x_2 > y_2^0$, in which case, we must have $x_1 \leq y_1^0$ (otherwise, (x_1, x_2) would fall into Region 1). Hence, $\tau_1(x_1)$ equals either $x_1 + K_1$ or y_1^0.

∘ Start with $\tau_1(x_1) = y_1^0$. First, we derive the optimal targets based on Theorem 2. We can show that (y_1^0, y_2^0) is a solution to the first-order conditions, but it is not a feasible production decision, since $x_2 > y_2^0$. Using Theorem 2, product 2 is not produced, while product 1 target is $\bar{y}_1(x_2)$. Comparing to the targets given by $\bar{y}_i(\tau_{3-i}(x_{3-i}))$, we have $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(x_2)$ as desired, and $\bar{y}_2(\tau_1(x_1)) = \bar{y}_2(y_1^0) = y_2^0 < x_2$, which is not feasible, as desired.

∘ Next, let $\tau_1(x_1) = x_1 + K_1$. Define $z_2 = \bar{y}_2(x_1 + K_1) \wedge (x_2 + K_2)$ and $\acute{y}_1 = \bar{y}_1(z_2)$. We show that $(y_1, y_2) = (\acute{y}_1, \bar{y}_2(x_1 + K_1))$ satisfies the first-order conditions. Since $x_1 + K_1 \leq y_1^0$ and $z_2 \leq \bar{y}_2(x_1 + K_1)$ by definition, the point $(x_1 + K_1, z_2)$ belongs to Region 4. Thus, it must satisfy $\acute{y}_1 = \bar{y}_1(z_2) \geq x_1 + K_1$. The first-order conditions are satisfied: $\Delta_{y_1} G_t(\acute{y}_1, \bar{y}_2(x_1 + K_1) \wedge (x_2 + K_2)) = \Delta_{y_1} G_t(\bar{y}_1(z_2), z_2) = 0$ and $\Delta_{y_2} G_t(\acute{y}_1 \wedge (x_1 + K_1), \bar{y}_2(x_1 + K_1)) = \Delta_{y_2} G_t(x_1 + K_1, \bar{y}_2(x_1 + K_1)) = 0$. Using Theorem 2, the optimal targets are as follows. If $\bar{y}_2(x_1 + K_1) \geq x_2$, then the optimal target is $(\acute{y}_1, \bar{y}_2(x_1 + K_1))$. Otherwise, product 2 target is not feasible, hence, product 1 target must be $\bar{y}_1(x_2)$, while product 2 should not be produced.

We now compare the optimal targets from Theorem 2 to the targets given by $\bar{y}_i(\tau_{3-i}(x_{3-i}))$. Consider first the case that product 2 is not produced, $\bar{y}_2(x_1 + K_1) \leq x_2$. Since $\tau_2(x_2) = x_2$, product 1 target is $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(x_2)$ as desired. Since $\tau_1(x_1) = x_1 + K_1$, product 2 target satisfies $\bar{y}_2(\tau_1(x_1)) = \bar{y}_2(x_1 + K_1) \leq x_2$, as desired. Consider next the case of $\bar{y}_2(x_1 + K_1) \geq x_2$, that is, the optimal targets are $(\acute{y}_1, \bar{y}_2(x_1 + K_1))$. In this case, $\bar{y}_2(\tau_1(x_1)) = \bar{y}_2(x_1 + K_1)$, as desired. For product 1, since $\acute{y}_1 \geq x_1 + K_1$, it suffices to show that $\bar{y}_1(\tau_2(x_2)) = \bar{y}_1(x_2) \geq x_1 + K_1$. First, notice that $\bar{y}_2(\bar{y}_1(x_2)) \leq x_2$ when $x_2 \geq y_2^0$ due to the monotonic behavior of the curves $\bar{y}_i(.)$ (follows from a simple tatonnement argument). Since $\bar{y}_2(x_1 + K_1) \geq x_2$, we have $\bar{y}_2(\bar{y}_1(x_2)) \leq \bar{y}_2(x_1 + K_1)$. Since $\bar{y}_2(.)$ is non-increasing, $\bar{y}_1(x_2) \geq x_1 + K_1$, as desired. $\square$

Proof of Proposition 5. Assume that the myopic policy is optimal in period $t+1$ and onwards. Let $y^0=(y_1^0,y_2^0)=\arg\min_{y\geq 0} G_{t+1}(y)$ be the optimal base-stock levels in period $t+1$ (and onwards). Then, the following must hold for $V_{t+1}(x)$.

$$V_{t+1}(x) = \begin{cases} G_{t+1}(y^0) & \text{if } \ x \leq y^0 \\ \\ G_{t+1}(x) & \text{otherwise} \end{cases}$$

We first derive the scheduling policy in period t when the stock levels are below y^0. From the proof of Proposition 1, we have the following formulation. (Note that $\tilde{y}$ is replaced by y, as no capacity uncertainties exist for the dedicated production lines.)

$$\begin{aligned} J_t(y,\epsilon,\kappa) = \min_{(z_1,z_2)} \Big\{ & hz + p[\epsilon - y + z] + \beta V_{t+1}(z) \\ s.t \quad & 0 \leq z_i \leq y_i \ \ i=1,2 \\ & y_i - z_i \leq \epsilon_i \ \ i=1,2 \\ & z_1 + z_2 \geq y_1 + y_2 - \kappa \Big\} \end{aligned}$$

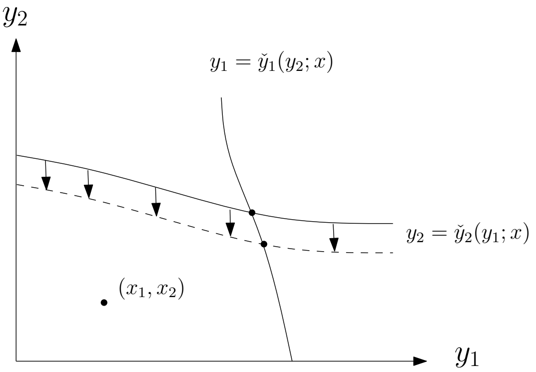
If $y\leq y^0$, then, any feasible z satisfies $z\leq y$, and hence, $z\leq y^0$. In this case, the above optimization problem reduces to a linear program for $y\leq y^0$, for which, the optimal solution can readily be obtained: $z_1^*=y_1-\min(y_1,\epsilon_1,\kappa)$ and $z_2^*=y_2-\min(y_1,\epsilon_1,\kappa-y_1+z_1^*)$. This leads to the following scheduling functions: $s_1^*(\kappa|\tilde{y},\xi)=\min(y_1,\epsilon_1,\kappa)$ and $s_2^*(\kappa|\tilde{y},\xi)=\min(y_2,\epsilon_2,\kappa-s_1^*(\kappa|\tilde{y},\xi))$, implying that the calibration of product 1 is prioritized.

The next step is to derive the production policy in period t. Let $y\leq y^0$. For any realization of ϵ and κ, we have $J_t(y,\epsilon,\kappa)=J_T(y,\epsilon,\kappa)+\beta V_{t+1}(y^0)$, hence, $G_t(y)=G_T(y)+\beta V_{t+1}(y^0)$. As a consequence, the optimal stock levels in period t are myopic, when restrict our attention to $y\leq y^0$. But, due to convexity of $G_t(y)$, it must also be optimal when all possible y's are considered. □

Proof of Proposition 6. The proof is for product 1. Refer to Theorem 2 for the description of the inventory policy. The optimal targets do not depend on dedicated capacities in Regions 1, 2, and 3. Hence, the optimal policy remains the same. For initial inventories in Region 4, we inspect how the minimizers of function $EG(y;x)$ change. Obviously, the minimizer $\check{y}_1(y_2;x)$ is not influenced by the capacity increase in product 1, since $\mathbb{E}_{K_2}\Delta_{y_1}G_t(\check{y}_1(y_2;x),y_2\wedge(x_2+K_2))=0$. Below, we argue that the curve $\check{y}_2(y_1;x)$ shifts downwards, as shown in Figure EC.3.

The function $G_t(y_1\wedge(x_1+K_1),y_2)$ is supermodular in (y_2,K_1). Hence, the value of $\mathbb{E}_{K_1}\Delta_{y_2}G_t(y_1\wedge(x_1+K_1),y_2)$ is increased due to Lemma EC.1. This implies that the minimizer $\check{y}_2(y_1;x)$ decreases, that is, the curve $\check{y}_2(y_1;x)$ shifts downwards. Recall that the curves $\check{y}_1(y_2;x)$ and $\check{y}_2(y_1;x)$ have negative slopes slopes, greater than or equal to -1. Hence, the point of intersection of these curves shifts such that product 1 target is increased and product target 2 is decreased. □

Figure EC.3 Sensitivity of the Production Targets to K_1



Proof of Proposition 7. We show that $\Delta_{\tilde{y}_i\kappa}J_t(\tilde{y},\xi,\kappa) \le 0$ and $(\Delta_{\tilde{y}_j\kappa}\Delta_{\tilde{y}_1\tilde{y}_2} - \Delta_{\tilde{y}_j\tilde{y}_j}\Delta_{\tilde{y}_i\kappa})J_t(\tilde{y},\xi,\kappa) \ge 0$ for $i=1,2$ and $j=3-i$. We consider only $i=1$ and the results are valid for $i=2$. We follow the same sequence as in the proof of Proposition 1.

Region A: $\underline{\min(\tilde{y}_1,\xi_1)+\min(\tilde{y}_2,\xi_2)\le\kappa}$. In this region, we have $\Delta_{\tilde{y}_i\kappa}J_t(\tilde{y},\xi,\kappa)=0$ for $i=1,2$. Hence, the desired properties hold by equality.

Region B: $\underline{s_i(\kappa)=\tilde{y}_i-\zeta_i(\tilde{y}_1+\tilde{y}_2-\kappa)}$.

$$\begin{aligned}
\Delta_{\tilde{y}_1\kappa}J_t(\tilde{y},\xi,\kappa) &= -\beta\Delta_{x_1x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_1' - \beta\Delta_{x_2x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_2' \\
\Delta_{\tilde{y}_2\kappa}J_t(\tilde{y},\xi,\kappa) &= -\beta\Delta_{x_1x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_1' - \beta\Delta_{x_2x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_2' \\
\Delta_{\tilde{y}_2\tilde{y}_2}J_t(\tilde{y},\xi,\kappa) &= \beta\Delta_{x_1x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_1' + \beta\Delta_{x_2x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_2' \\
\Delta_{\tilde{y}_1\tilde{y}_2}J_t(\tilde{y},\xi,\kappa) &= \beta\Delta_{x_1x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_1' + \beta\Delta_{x_2x_2}V_{t+1}(\zeta_1,\zeta_2)\zeta_2'
\end{aligned}$$

Because of the inductional assumptions, we have $\Delta_{\tilde{y}_1\kappa}J_t(\tilde{y},\xi,\kappa)\le 0$, and due to above expressions, we have $(\Delta_{\tilde{y}_2\kappa}\Delta_{\tilde{y}_1\tilde{y}_2} - \Delta_{\tilde{y}_2\tilde{y}_2}\Delta_{\tilde{y}_1\kappa})J_t(\tilde{y},\xi,\kappa)=0$.

Region C: $\underline{s_1(\kappa)=\bar{s}_1(\kappa|\tilde{y},\xi) \text{ and } s_2(\kappa)=\underline{s}_2(\kappa|\tilde{y},\xi)}$. We consider two subregions.

Region C1: First, let $\xi_1\le\tilde{y}_1$.

$$\begin{aligned}
\Delta_{\tilde{y}_1\kappa}J_t(\tilde{y},\xi,\kappa) &= -\beta\Delta_{x_1x_2}V_{t+1}(\tilde{y}_1-\xi_1,\tilde{y}_2+\xi_1-\kappa) \\
\Delta_{\tilde{y}_2\kappa}J_t(\tilde{y},\xi,\kappa) &= -\beta\Delta_{x_2x_2}V_{t+1}(\tilde{y}_1-\xi_1,\tilde{y}_2+\xi_1-\kappa) \\
\Delta_{\tilde{y}_2\tilde{y}_2}J_t(\tilde{y},\xi,\kappa) &= \beta\Delta_{x_2x_2}V_{t+1}(\tilde{y}_1-\xi_1,\tilde{y}_2+\xi_1-\kappa) \\
\Delta_{\tilde{y}_1\tilde{y}_2}J_t(\tilde{y},\xi,\kappa) &= \beta\Delta_{x_1x_2}V_{t+1}(\tilde{y}_1-\xi_1,\tilde{y}_2+\xi_1-\kappa)
\end{aligned}$$

Because of the inductional assumptions, we have $\Delta_{\tilde{y}_1\kappa}J_t(\tilde{y},\xi,\kappa)\le 0$, and from the above expressions, we have $(\Delta_{\tilde{y}_2\kappa}\Delta_{\tilde{y}_1\tilde{y}_2} - \Delta_{\tilde{y}_2\tilde{y}_2}\Delta_{\tilde{y}_1\kappa})J_t(\tilde{y},\xi,\kappa)=0$.

Region C2: Next, let $\xi_1>\tilde{y}_1$.

$$\Delta_{\tilde{y}_1\kappa}J_t(\tilde{y},\xi,\kappa) = -\beta\Delta_{x_2x_2}V_{t+1}(0,\tilde{y}_1+\tilde{y}_2-\kappa)$$

$$\Delta_{\tilde{y}_2\kappa} J_t(\tilde{y},\xi,\kappa) = -\beta \Delta_{x_2x_2} V_{t+1}(0,\tilde{y}_1+\tilde{y}_2-\kappa)$$
$$\Delta_{\tilde{y}_2\tilde{y}_2} J_t(\tilde{y},\xi,\kappa) = \beta \Delta_{x_2x_2} V_{t+1}(0,\tilde{y}_1+\tilde{y}_2-\kappa)$$
$$\Delta_{\tilde{y}_1\tilde{y}_2} J_t(\tilde{y},\xi,\kappa) = \beta \Delta_{x_2x_2} V_{t+1}(0,\tilde{y}_1+\tilde{y}_2-\kappa)$$

Because of the inductional assumptions, we have $\Delta_{\tilde{y}_1\kappa} J_t(\tilde{y},\xi,\kappa) \le 0$, and due to above expressions, we have $(\Delta_{\tilde{y}_2\kappa}\Delta_{\tilde{y}_1\tilde{y}_2} - \Delta_{\tilde{y}_2\tilde{y}_2}\Delta_{\tilde{y}_1\kappa}) J_t(\tilde{y},\xi,\kappa) = 0$.

Region D: $\underline{s_1(\kappa) = \underline{s}_1(\kappa|\tilde{y},\xi) \text{ and } s_2(\kappa) = \bar{s}_2(\kappa|\tilde{y},\xi)}$. Same as Region C.

Now we are ready to evaluate the effect of increase in the shared capacity. First, assume that the shared capacity is uncertain. From Theorem 1, $C_t(\tilde{y},\xi) = \mathbb{E}_\kappa J_t(\tilde{y},\xi,\kappa)$. Using Lemma EC.1 (stated in Appendix EC.3), the derivatives of $C_t(\tilde{y},\xi)$ with respect to $\tilde{y}_1$ and $\tilde{y}_2$ are decreased when the shared capacity is stochastically increased. Since $G_t(\tilde{y}) = \mathbb{E}_\xi C_t(\tilde{y},\xi)$, the derivatives of $G_t(\tilde{y})$ with respect to $\tilde{y}_1$ and $\tilde{y}_2$ decrease, as well. As a result, the minimizers of $G_t(\tilde{y})$ are increased, effectively increasing the sum of the inventory targets by Proposition EC.2. When the shared capacity is deterministic, $G_t(\tilde{y}) = \mathbb{E}_\xi J_t(\tilde{y},\xi,\kappa)$, where κ is now a fixed parameter. Since $\Delta_{\tilde{y}_i\kappa} J_t(\tilde{y},\xi,\kappa) \le 0$ and $(\Delta_{\tilde{y}_j\kappa}\Delta_{\tilde{y}_1\tilde{y}_2} - \Delta_{\tilde{y}_j\tilde{y}_j}\Delta_{\tilde{y}_i\kappa}) J_t(\tilde{y},\xi,\kappa) \ge 0$ for $i = 1,2$ and $j = 3-i$, the targets are increased for both products due to Proposition EC.2. □

Proof of Proposition 8. We only prove for product 1. Following the same procedure in Proposition 7, we can verify that $\Delta_{\tilde{y}_1\xi_1} J_t(\tilde{y},\xi,\kappa) \le 0$, $\Delta_{\tilde{y}_1\xi_1} J_t(\tilde{y},\xi,\kappa) \le \Delta_{\tilde{y}_2\xi_1} J_t(\tilde{y},\xi,\kappa)$, $(\Delta_{\tilde{y}_2\xi_1}\Delta_{\tilde{y}_1\tilde{y}_2} - \Delta_{\tilde{y}_2\tilde{y}_2}\Delta_{\tilde{y}_1\xi_1}) J_t(\tilde{y},\xi,\kappa) \ge 0$ and $(\Delta_{\tilde{y}_1\xi_1}\Delta_{\tilde{y}_1\tilde{y}_2} - \Delta_{\tilde{y}_1\tilde{y}_1}\Delta_{\tilde{y}_2\xi_1}) J_t(\tilde{y},\xi,\kappa) \le 0$. Recall that $G_t(\tilde{y}) = \mathbb{E}_{\xi,\kappa} J_t(\tilde{y},\xi,\kappa)$. First observe that the minimizers of $G_t(.)$ behave as follows. As ξ_1 is stochastically increased, $\bar{y}_1(\tilde{y}_2)$ increases. As a consequence, in Region 2, where only product 1 is produced, product 1 target increases, while product 2 target remains the same. In Region 3, product 1 target remains the same. To analyze Region 4, we need to consider global minimizers (y_1^0, y_2^0). Due to Proposition EC.2, part *(iii)*, y_1^0 increases, while y_2^0 decreases. □

Proof of Proposition 9. We only consider sensitivity to p_1. Sensitivity to p_2, h_1 and h_2 are similar. Following the same procedure in Proposition 7, we can verify that $\Delta_{\tilde{y}_1p_1} J_t(\tilde{y},\xi,\kappa) \le 0$ and $\Delta_{\tilde{y}_2p_1} J_t(\tilde{y},\xi,\kappa) \ge 0$. Since $G_t(\tilde{y}) = \mathbb{E}_{\xi,\kappa} J_t(\tilde{y},\xi,\kappa)$, and due to Proposition EC.2, we conclude that product 1 target increases, while product 2 target decreases. □

EC.2. Model Extensions

In this section, we provide model extensions to address infinite-horizon and Markov-modulated settings.

Infinite Time Horizon: Assuming that all parameters are stationary, and with a discount factor of $0 \le \beta < 1$, the optimality equations are stated below.

$$\textbf{Phase One :} \qquad V^*(x) = \min_{y \ge x} \mathbb{E}_{K,\xi} C^*(y \wedge (x+K), \xi), \tag{EC.11}$$

$$\textbf{Phase Two :} \quad C^*(y,\xi) = \min_{s_i(.)} \Big\{ \mathbb{E}_\kappa \left\{ h[y - s(\kappa)] + p[\xi - s(\kappa)] + \beta V^*(y - s(\kappa)) \right\} \tag{EC.12}$$

$$s.t \quad \underline{s}_i(\kappa|y,\xi) \le s_i(\kappa) \le \bar{s}_i(\kappa|y,\xi), \tag{EC.13}$$

$$s_1'(\kappa) + s_2'(\kappa) \le 1, \tag{EC.14}$$

$$s_i'(\kappa) \ge 0 \ ; \qquad \forall \kappa \ge 0,\ i = 1,2 \Big\} \tag{EC.15}$$

The following proposition establishes the infinite-horizon solution and shows that the optimal policy structure is the same as its finite-horizon counterpart.

PROPOSITION EC.1. *There exists a function $V^*(x) = \lim_{t\to\infty} V_t(x)$, which solves (EC.11-EC.15) and satisfies the second-order properties, and $p_i + h_i + \beta \Delta_{x_i} V^*(x) \ge 0$ for $i = 1,2$.*

Proof of Proposition EC.1. Consider the following reformulation based on Theorem 1 and Proposition 1. Notice that periods are now indexed backwards in time, and $V_0(x) = x$.

$$\textbf{Stage One :} \qquad V_t(x) = \min_{y \ge x} \mathbb{E}_{K,\kappa,\xi} J_t(y \wedge (x+K), \xi, \kappa), \tag{EC.16}$$

$$\textbf{Stage Two :} \quad J_t(\tilde{y},\xi,\kappa) = \min_{(z_1,z_2)} \Big\{ hz + p[\xi - \tilde{y} + z] + \beta V_{t-1}(z) \tag{EC.17}$$

$$s.t \quad 0 \le z_i \le \tilde{y}_i \ \ i = 1,2 \tag{EC.18}$$

$$\tilde{y}_i - z_i \le \xi_i \ \ i = 1,2 \tag{EC.19}$$

$$z_1 + z_2 \ge \tilde{y}_1 + \tilde{y}_2 - \kappa \Big\} \tag{EC.20}$$

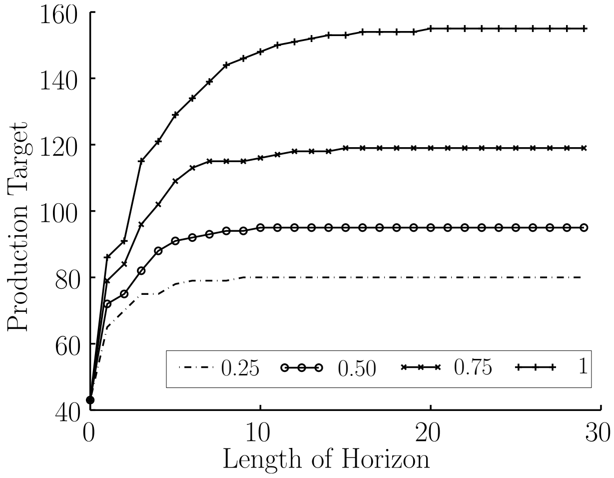
We focus our discussion on the infinite-horizon version of (EC.16)–(EC.20). To establish the infinite-horizon solution, we verify the conditions (a)–(d) of Theorem 8-14 in Heyman and Sobel (1984). Conditions (b) and (d) are straightforward. Condition (b) holds, since single-period costs are non-negative for any initial state and any feasible decision. Condition (d) holds, since continuity is a consequence of convexity. To show that condition (a) holds, it suffices to show that $V_t(x)$ is bounded above by a function $B(x)$ for all t, as $V_t(x) \le V_{t+1}(x)$ due to $V_0(x) = 0$. This is straightforward by considering a suboptimal policy, which sets $y = x$ and $s_i(\kappa) = 0$ at the current period t and onwards, (i.e., no production and no calibration of final products). Define $B_t(x)$ as the cost of such policy. Define $B(x) = \lim_{t\to\infty} B_t(x) = \frac{1}{1-\beta}\Big(h_1 x_1 + h_2 x_2 + p_1 \mathbb{E}[\xi_1] + p_2 \mathbb{E}[\xi_2]\Big)$. Obviously, $V_t(x) \le B_t(x) \le B(x)$, hence condition (a) holds.

To show condition (c), we show that the set of feasible decisions is compact. This is obvious for stage two decisions, as the linear constraints define a closed and bounded region. For stage one, there is no loss of optimality by replacing $y \ge x$ with $x \le y \le \max(U,x)$ with $U_i < \infty$ sufficiently large (see Example 8–33 in Heyman and Sobel, 1984). Therefore, compactness of the feasible set in stage one can be claimed without loss of optimality, hence condition (c) holds.

To show that $V^*(x)$ satisfies the second-order properties, we follow a similar procedure as in Hu et al. (2008) and restate the second-order conditions as follows: for any $\xi > 0$, $\Delta_{x_1} V^*(x_1+\xi, x_2) \geq \Delta_{x_1} V^*(x_1, x_2+\xi)$ and $\Delta_{x_2} V^*(x_1, x_2+\xi) \geq \Delta_{x_2} V^*(x_1+\xi, x_2)$. Since $V_t(x)$ satisfies the second-order conditions, it suffices to show that $\Delta_{x_i} V_t(x)$ converges to $\Delta_{x_i} V^*(x)$. From Theorem 8-14 of Heyman and Sobel (1984), $V_t(x)$ converges uniformly to $V^*(x)$, hence $\Delta_{x_i} V^*(x) = \lim_{t\to\infty} \Delta_{x_i} V_t(x)$. □

Proposition EC.1 implies that it is possible to solve a finite-horizon problem for a sufficiently large horizon length and obtain the numerical solution to an infinite-horizon problem in a practical manner. Figure EC.4 illustrates convergence of the optimal inventory targets with initial inventory (0,0), as a test case.

Figure EC.4 Inventory targets increase in the length of horizon



Note: Identical products configuration with $p = 0.99$ and $h = 0.01$, $\mathbb{E}[\xi_i] = 24$, $cv(\xi_i) = 0.8$, $\mathbb{E}[\kappa] = 60$, $cv(\kappa) = 0.50$, $\mathbb{E}[K_i] = 24$, legend shows different values for $cv(K_i)$.

Markov-Modulated Process: Our model assumed that demands and capacities are independent across periods. This assumption can be relaxed employing a Markov chain driving the demand and capacity distributions. This enables the original model to capture the real-life situations involving for example, seasonal demand, or the possibility that capacity loss in a given period is an indicator of potential capacity loss in the next few periods to come.

Suppose that there are N states of the world with a probability transition matrix $P = [p_{\omega,\omega'}]$. When the state of the world is ω, dedicated capacity distributions are $K_\omega = (K_{1\omega}, K_{2\omega})$, shared capacity distribution is κ_ω and the demand distributions are given by $\xi_\omega = (\xi_{1\omega}, \xi_{2\omega})$. With this, the model is reformulated below.

$$\textbf{Phase One :} \qquad V_t(x;\omega) = \min_{y \geq x} \mathbb{E}_{K_\omega, \xi_\omega} C_t(y \wedge (x + K_\omega), \xi_\omega; \omega),$$

$$\textbf{Phase Two :} \quad C_t(y,\xi;\omega) = \min_{s_i(.)} \Big\{ \mathbb{E}_{\kappa_\omega,\omega'} \{h[y - s(\kappa_\omega)] + p[\xi - s(\kappa_\omega)] + \beta V_{t+1}(y - s(\kappa_\omega);\omega')|\omega\}$$
$$s.t \quad \underline{s}_i(\kappa|y,\xi) \le s_i(\kappa) \le \bar{s}_i(\kappa|y,\xi),$$
$$s_1'(\kappa) + s_2'(\kappa) \le 1,$$
$$s_i'(\kappa) \ge 0 \; ; \qquad \forall \kappa \ge 0, \; i = 1,2 \Big\}$$

The revised formulation has the same optimal policy structure, however, the optimal policy itself depends on the state of the world ω. For example, if one production process i has lost full capacity for a number of periods, then product $3-i$ follows a modified base-stock policy, which is a function of the existing stock of product i.

EC.3. Technical Results Needed for Sensitivity Analysis

Our sensitivity results depend on the following technical results below. The first result is a special case of Theorem 3.10.1 of Topkis (1978).

LEMMA EC.1 **(Topkis)**. *For a submodular function $\phi(x,\xi)$ (i.e., $\Delta_{x\xi}\phi(x,\xi) \le 0$), let $x^* = \arg\min_x \mathbb{E}[\phi(x,\xi)]$. If ξ is increased stochastically, then x^* increases.*

Lemma EC.1 establishes the monotonicity of the minimizer of $\mathbb{E}[\phi(x,\xi)]$. Analogous conclusions can be drawn when $\phi(.)$ is supermodular. Next, we consider optimizing a function in two variables.

PROPOSITION EC.2. *Let function $\phi(y;\gamma)$ satisfy the second-order properties in $y = (y_1, y_2)$ (Definition 2) and let $(y_1^*, y_2^*) = \arg\min_{y_1,y_2} \mathbb{E}_\gamma[\phi(y;\gamma)]$. If γ increases stochastically, then:*
i) y_i^* increases if $\Delta_{y_i\gamma}\phi(y;\gamma) \le 0$ and $\Delta_{y_i\gamma}\phi(y;\gamma) \le \Delta_{y_{3-i}\gamma}\phi(y;\gamma)$.
ii) y_i^* increases and y_{3-i}^* decreases if $\Delta_{y_i\gamma}\phi(y;\gamma) \le 0 \le \Delta_{y_{3-i}\gamma}\phi(y;\gamma)$.
iii) $y_1^* + y_2^*$ increases if $\Delta_{y_i\gamma}\phi(y;\gamma) \le 0$ for each $i = 1,2$. Both y_1^* and y_2^* increase if, in addition, $(\Delta_{y_{3-i}\gamma}\Delta_{y_1y_2} - \Delta_{y_{3-i}y_{3-i}}\Delta_{y_i\gamma})\phi(y;\gamma) \ge 0$ for each $i = 1,2$.

Proof of Proposition EC.2. Before we prove the main result of this proposition, consider a function $\psi(y;\lambda)$ that satisfies the second-order properties in (y_1, y_2). Let $\hat{y}(\lambda) = \arg\min_y \psi(y;\lambda)$. This is the deterministic counter-part of our problem. $\hat{y}(\lambda)$ satisfies the following set of equations.

$$\Delta_{y_1}\psi(\hat{y}(\lambda);\lambda) = 0 \tag{EC.21}$$

$$\Delta_{y_2}\psi(\hat{y}(\lambda);\lambda) = 0 \tag{EC.22}$$

Taking the derivatives of the equations (EC.21) and (EC.22) with respect to λ and rearranging the terms, we obtain the following expressions.

$$\hat{y}_i'(\lambda) = \left(\frac{\Delta_{y_{3-i}\lambda}\Delta_{y_1y_2} - \Delta_{y_{3-i}y_{3-i}}\Delta_{y_i\lambda}}{\Delta_{y_1y_1}\Delta_{y_2y_2} - \Delta^2_{y_1y_2}} \right) \psi(\hat{y}(\lambda);\lambda) \tag{EC.23}$$

$$\hat{y}_1'(\lambda)+\hat{y}_2'(\lambda) = -\left(\frac{\Delta_{y_1\lambda}(\Delta_{y_2y_2}-\Delta_{y_1y_2})+\Delta_{y_2\lambda}(\Delta_{y_1y_1}-\Delta_{y_1y_2})}{\Delta_{y_1y_1}\Delta_{y_2y_2}-\Delta_{y_1y_2}^2}\right)\psi(\hat{y}(\lambda);\lambda) \qquad \text{(EC.24)}$$

Using expressions (EC.23) and (EC.24), as well as the second-order properties, we can readily verify properties 1–3 for $\psi(y;\lambda)$ below, which are the deterministic counter-parts of properties (i)–(iii).

1. If $\Delta_{y_i\lambda}\psi(y;\lambda)\le 0$ and $\Delta_{y_i\lambda}\psi(y;\lambda)\le \Delta_{y_{3-i}\lambda}\psi(y;\lambda)$, then $\hat{y}_i'(\lambda)\ge 0$.
2. If $\Delta_{y_i\lambda}\psi(y;\lambda)\le 0\le \Delta_{y_{3-i}\lambda}\psi(y;\lambda)$, then $\hat{y}_i'(\lambda)\ge 0$ and $\hat{y}_{3-i}'(\lambda)\le 0$.
3. If $\Delta_{y_i\lambda}\psi(y;\lambda)\le 0$ for each $i=1,2$, then $\hat{y}_1'(\lambda)+\hat{y}_2'(\lambda)\ge 0$. If, in addition, $(\Delta_{y_{3-i}\lambda}\Delta_{y_1y_2}-\Delta_{y_{3-i}y_{3-i}}\Delta_{y_i\lambda})\psi(y;\lambda)\ge 0$ for each $i=1,2$, then $\hat{y}_i'(\lambda)\ge 0$ for both $i=1,2$.

Consider now the original optimization problem $(y_1^*,y_2^*)=\arg\min_{y_1,y_2}\mathbb{E}_\gamma[\phi(y;\gamma)]$. Consider two random variables with $\gamma_1\le_{s.t}\gamma_2$. Denote by $F_1(.)$ and $F_2(.)$ the cumulative probability distribution functions of γ_1 and γ_2 respectively. Due to stochastic dominance, $F_1^{-1}(u)\le F_2^{-1}(u)$ for all $0\le u\le 1$.

For a constant parameter $0\le\lambda\le 1$, define a random variable $\gamma(\lambda)$ such that its inverse cumulative distribution function is given by $F^{-1}(u;\lambda)=(1-\lambda)F_1^{-1}(u)+\lambda F_2^{-1}(u)$ for all $0\le u\le 1$. As a result, the inverse functions satisfy $F_1^{-1}(u)\le F^{-1}(u;\lambda)\le F_2^{-1}(u)$ for all $0\le u,\lambda\le 1$. In other words, $\gamma_1\le_{s.t}\gamma(\lambda)\le_{s.t}\gamma_2$. Also note that $\gamma(\lambda_1)\le_{s.t}\gamma(\lambda_2)$ for $0\le\lambda_1\le\lambda_2\le 1$.

To show that the monotonicity properties hold when γ is stochastically increased from $\gamma=\gamma_1$ to $\gamma=\gamma_2$, it suffices to show that the monotonicity results hold for $\gamma=\gamma(\lambda)$ when λ is increased marginally. For that purpose, define $\psi(y;\lambda)=\mathbb{E}_{\gamma(\lambda)}[\phi(y;\gamma(\lambda))]$. First, observe that $\psi(y;\lambda)$ satisfies the second-order properties. Therefore, we can use the results obtained above.

We prove only the second part of property (iii), which is the least straightforward case. All other properties can be established similarly. We show that $(\Delta_{y_2\lambda}\Delta_{y_1y_2}-\Delta_{y_2y_2}\Delta_{y_1\lambda})\psi(y;\lambda)\ge 0$, hence $\hat{y}_1'(\lambda)\ge 0$. First, we derive $\Delta_\lambda\psi(y;\lambda)$ as follows. (Below, we change the variable of integration. We let $x=F^{-1}(u;\lambda)$.)

$$\begin{aligned}\psi(y;\lambda) &= \mathbb{E}_{\gamma(\lambda)}[\phi(y;\gamma(\lambda))] \;=\; \int \phi(y;x)dF(x;\lambda)\\ &= \int_0^1 \phi(y;F^{-1}(u;\lambda))du\\ &= \int_0^1 \phi(y;(1-\lambda)F_1^{-1}(u)+\lambda F_2^{-1}(u))du\end{aligned}$$

$$\Delta_\lambda\psi(y;\lambda) = \int_0^1 \Delta_\gamma\phi(y;F^{-1}(u;\lambda))[F_2^{-1}(u;\lambda)-F_1^{-1}(u;\lambda)]du$$

The above expression is useful in deriving $\Delta_{y_i\lambda}\psi(y;\lambda)$ for each i. As a result, we can immediately derive $(\Delta_{y_2\lambda}\Delta_{y_1y_2}-\Delta_{y_2y_2}\Delta_{y_1\lambda})\psi(y;\lambda)$ as follows.

$$(\Delta_{y_2\lambda}\Delta_{y_1y_2}-\Delta_{y_2y_2}\Delta_{y_1\lambda})\psi(y;\lambda) = \int_0^1 \Big(\Delta_{y_2\gamma}\phi(y;F^{-1}(u;\lambda))\Delta_{y_1y_2}\mathbb{E}_{\gamma(\lambda)}[\phi(y;\gamma(\lambda))]$$

$$- \Delta_{y_1\gamma}\phi(y; F^{-1}(u;\lambda))\Delta_{y_2y_2}\mathbb{E}_{\gamma(\lambda)}[\phi(y;\gamma(\lambda))]\Big) \dots$$

$$[F_2^{-1}(u;\lambda) - F_1^{-1}(u;\lambda)]du$$

To show that $(\Delta_{y_2\lambda}\Delta_{y_1y_2} - \Delta_{y_2y_2}\Delta_{y_1\lambda})\psi(y;\lambda) \geq 0$, it suffices to show $\Delta_{y_2\gamma}\phi(y;\gamma)\Delta_{y_1y_2}\mathbb{E}_{\gamma(\lambda)}[\phi(y;\gamma(\lambda))] - \Delta_{y_1\gamma}\phi(y;\gamma)\Delta_{y_2y_2}\mathbb{E}_{\gamma(\lambda)}[\phi(y;\gamma(\lambda))] \geq 0$, since $F_2^{-1}(u;\lambda) \geq F_1^{-1}(u;\lambda)$. But, this immediately follows, since $(\Delta_{y_2\gamma}\Delta_{y_1y_2} - \Delta_{y_2y_2}\Delta_{y_1\gamma})\phi(y;\gamma) \geq 0$. $\square$