Proofs

This e-companion contains the mathematical proofs of the results in the paper. Section EC.1 contains the proof of Theorems 1, 2, 3 and 4. The proofs frequently refer to a number of auxiliary lemmas, which are formulated and proven in Section EC.2.

EC.1. Proofs of Main Theorems

Proof of Theorem 1

Let $\beta \in B$. Consider the k-th selling season, and write $c(1) = c_{1+(k-1)S}$, $c(2) = c_{2+(k-1)S}$, …, $c(S) = c_{kS}$. The proof consists of two steps. In Step 1, we show that there is a $v_1(\beta) > 0$ such that if prices $\pi^*_{\beta(0)}(c(s), s)$ are used in state $(c(s), s)$, for all $s = 1, \ldots, S$, then there are $1 \leq s, s' \leq S$ with $|\pi^*_\beta(c(s), s) - \pi^*_\beta(c(s'), s')| > v_1(\beta)$ and $c(s)c(s') > 0$. Since π^*_β is continuous in β (Lemma EC.2(vi)), there is an open neighborhood $\mathcal{U}_\beta \subset B$ around β such that, if price $\pi^*_{\beta(s)}(c(s), s)$ is used in state $(c(s), s)$, for all $s = 1, \ldots, S$ and some sequence $(\beta(1), \ldots, \beta(S)) \in \mathcal{U}_\beta \cap B$, then there are $1 \leq s, s' \leq S$ such that $|\pi^*_{\beta(s)}(c(s), s) - \pi^*_{\beta(s')}(c(s'), s')| > v_1(\beta)/2$, $c(s) > 0$ and $c(s') > 0$. In Step 2 we show that $v_1(\beta)$ can be bounded from below by a constant $v_0 > 0$ independent of β. This proves (10). Equation (11) follows by application of Lemma EC.3.

Step 1: Occurence of price change. Define

$$\triangleleft = \{(c, s) \mid S + 1 - C \leq s \leq S, S + 1 - s \leq c \leq C\}. \tag{EC.1}$$

See Figure EC.1 for an illustration of $\triangleleft$ in the state space $\mathcal{X}$. Notice that since $(C, 1) \notin \triangleleft$ (by the assumption $C < S$), the path $(c(s), s)_{1 \leq s \leq S}$ may or may not hit $\triangleleft$. We show that, in both cases, at least two different selling prices occur on the path $(c(s), s)_{1 \leq s \leq S}$. Let $p^*_{a,\beta}$ be as in Lemma EC.1 and shorthand write $f^*_{a,\beta} = f_{a,\beta}(p^*_{a,\beta})$.

<u>Case 1</u>. The path $(c(s), s)_{1 \leq s \leq S}$ hits $\triangleleft$. Then there is an s such that $(c(s), s) \in \triangleleft$ and $(c(s), s-1) \in (L\triangleleft)$, where we define

$$(L\triangleleft) = \{(1, S-1), (2, S-2), \ldots, (C-1, S-C+1), (C, S-C)\}$$

as the set of points immediately left to $\triangleleft$ in Figure EC.1. The following two properties imply that a price change occurs when the path $(c(s), s)_{1 \leq s \leq S}$ hits $\triangleleft$.

(P.1) If $(c, s) \in \triangleleft$ then $\pi^*_\beta(c, s) = p^*_{0,\beta}$, $\Delta V_\beta(c, s+1) = 0$, and $V_\beta(c, s) = (S - s + 1) \cdot f^*_{0,\beta}$.

(P.2) If $(c, s) \in (L\triangleleft)$, then $\Delta V_\beta(c, s+1) \geq h(\beta_0 + \beta_1 p_h)^{c-1} f^*_{0,\beta}$ and
$\pi^*_\beta(c, s) \geq p^*_{h(\beta_0+\beta_1 p_h)^{c-1} f^*_{0,\beta}, \beta} > p^*_{0,\beta}$.

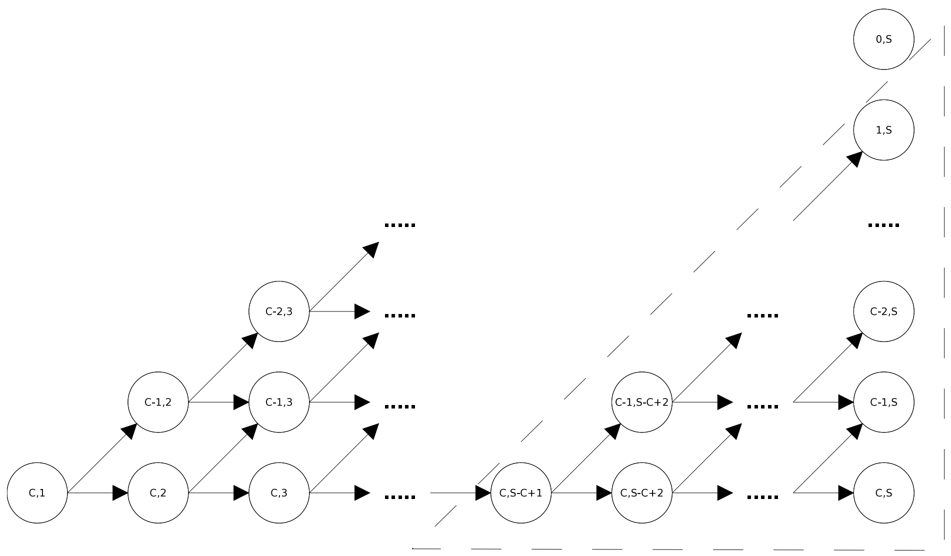

Figure EC.1 Schematic picture of $\triangleleft$

Proof of (P.1): Backward induction on s. If $s = S$ and $(c,s) \in \triangleleft$, then the assertions follow immediately. Let $s < S$. Then $\Delta V_\beta(c,s+1) = V_\beta(c,s+1) - V_\beta(c-1,s+1) = 0$, $\pi^*_\beta(c,s) = p^*_{0,\beta}$ and $V_\beta(c,s) = \max_{p\in[p_l,p_h]} ph(\beta_0 + \beta_1 p) + V_\beta(c,s+1) = (S-s+1)\cdot V_\beta(1,S)$, by (3) and the induction hypothesis. This proves (P.1).

Proof of (P.2). Induction on c. If $c=1$ and $(c,s) \in (L\triangleleft)$, then $(c,s) = (1,S-1)$, $\Delta V_\beta(1,S) = f^*_{0,\beta}$ and $\pi^*_\beta(1,S-1) = p^*_{\Delta V_\beta(1,S)} = p^*_{f^*_{0,\beta}} > p^*_{0,\beta}$, since by Lemma EC.1(ii) $p^*_{a,\beta}$ is strictly increasing in a. If $c > 1$ and $(c,s) \in (L\triangleleft)$ then $(c,s) = (c,S-c)$, and the induction hypothesis, together with the optimality of $\pi^*_\beta(c,S-c+1)$, implies

$$
\begin{aligned}
&\Delta V_\beta(c,S-c+1) = V_\beta(c,S-c+1) - V_\beta(c-1,S-c+1)\\
=&(\pi^*_\beta(c,S-c+1) - \Delta V_\beta(c,S-c+2))h(\beta_0+\beta_1\pi^*_\beta(c,S-c+1)) + V_\beta(c,S-c+2)\\
-&(\pi^*_\beta(c-1,S-c+1) - \Delta V_\beta(c-1,S-c+2))h(\beta_0+\beta_1\pi^*_\beta(c-1,S-c+1))\\
-&V_\beta(c-1,S-c+2)\\
\geq&\Delta V_\beta(c-1,S-c+2)h(\beta_0+\beta_1\pi^*_\beta(c-1,S-c+1))\\
\geq&h(\beta_0+\beta_1 p_h)^{c-1} f^*_{0,\beta},
\end{aligned}
$$

and $\pi^*_\beta(c,S-c) = p^*_{\Delta V_\beta(c,S-c+1),\beta} \geq p^*_{\Delta V_\beta(c,S-c+1),\beta} > p^*_{0,\beta}$, using again Lemma EC.1(ii). This proves (P.2), and concludes Case 1.

<u>Case 2</u>. The path $(c(s),s)_{1\leq s\leq S}$ does not hit $\triangleleft$. Then there is an $1 \leq s \leq S-2$ such that $c(s) = 2$ and $c(s+1) = 1$. We show by backward induction that

$$\Delta V_\beta(2,s) - \Delta V_\beta(1,s+1) \leq (V_\beta(1,1) - p_h)\cdot h(\beta_0+\beta_1 p_h), \tag{EC.2}$$

for all $2 \leq s \leq S-1$, and that $V_\beta(1,1) < p_h$. This implies $\Delta V_\beta(2, s+1) < \Delta V_\beta(1, s+2)$. By Lemma EC.1(ii) this implies

$$\pi^*_\beta(2,s) = p^*_{\Delta V_\beta(2,s+1),\beta} < p^*_{\Delta V_\beta(1,s+2),\beta} = \pi^*_\beta(1,s+1), \tag{EC.3}$$

and thus a price change occurs.

Let $2 \leq s \leq S-1$. The optimality of $\pi^*_\beta(1,s)$ and $\pi^*_\beta(1,s+1)$ implies

$$\begin{aligned}
&\Delta V_\beta(2,s) - \Delta V_\beta(1,s+1) \\
\leq &(\pi^*_\beta(2,s) - \Delta V_\beta(2,s+1))h(\beta_0 + \beta_1 \pi^*_\beta(2,s)) + V_\beta(2,s+1) \\
-&(\pi^*_\beta(1,s) - \Delta V_\beta(1,s+1))h(\beta_0 + \beta_1 \pi^*_\beta(1,s)) - V_\beta(1,s+1) \\
-&(\pi^*_\beta(2,s) - \Delta V_\beta(1,s+2))h(\beta_0 + \beta_1 \pi^*_\beta(2,s)) - V_\beta(1,s+2) \\
= &\Big[\Delta V_\beta(2,s+1) - \Delta V_\beta(1,s+2)\Big]\Big[1 - h(\beta_0 + \beta_1(\pi^*_\beta(2,s))\Big] \\
-&(\pi^*_\beta(1,s) - \Delta V_\beta(1,s+1))h(\beta_0 + \beta_1 \pi^*_\beta(1,s)) \\
\leq &-(p_h - V_\beta(1,s+1))h(\beta_0 + \beta_1 p_h) \\
\leq &-(p_h - V_\beta(1,1))h(\beta_0 + \beta_1 p_h).
\end{aligned}$$

Here $\Big[\Delta V_\beta(2,s+1) - \Delta V_\beta(1,s+2)\Big] \leq 0$ follows from the induction hypothesis if $s < S-1$, and follows from $V_\beta(2,S) - V_\beta(1,S) = 0$ if $s = S-1$. The last inequality follows from Lemma EC.1(ii). This proves (EC.2), and concludes Case 2.

We have shown that, on any path $(c(s),s)_{1 \leq s \leq S}$ in $\mathcal{X}$ starting at $(C,1)$, the policy π^*_β induces a price-change. It follows that there exists a $v_1(\beta) > 0$ such that for all paths $(c(s),s)_{1 \leq s \leq S}$,

$$|\pi^*_\beta(c(s),s) - \pi^*_\beta(c(s'),s')| \geq v_1(\beta).$$

Step 2: Lower bound on magnitude of price change. Property (P.2) in the proof of Theorem 1 shows that a price change of magnitude

$$|\pi^*_\beta(c(s),s) - \pi^*_\beta(c(s+1),s+1)| \geq |p^*_{h(\beta_0+\beta_1 p_h)^{C-1} f^*_{0,\beta},\beta} - p^*_{0,\beta}|$$

occurs in Case 1, and equation (EC.2) shows that a price change of magnitude

$$|\pi^*_\beta(c(s),s) - \pi^*_\beta(c(s+1),s+1)| \geq |p^*_{\Delta V_\beta(2,s+1),\beta} - p^*_{\Delta V_\beta(1,s+2),\beta}|$$

occurs, with $|\Delta V_\beta(2,s+1) - \Delta V_\beta(1,s+2)| < (V_\beta(1,1) - p_h)h(\beta + \beta_1 p_h) < 0$ (since $V_\beta(1,1) - p_h = \max_{p \in [p_l,p_h]} ph(\beta_0 + \beta_1 p) - p_h < \max_{p \in [p_l,p_h]}(p - p_h) = 0$). The proof of Lemma EC.1(ii) implies that, for all $0 \leq a_0, a_1 < p_h$ and all $\beta \in B$,

$$|p^*_{a_0,\beta} - p^*_{a_1,\beta}| \geq |a_0 - a_1| \cdot \inf_{0 \leq a \leq p_h} \frac{-\beta_1 \dot{h}(\beta_0 + \beta_1 p^*_{a,\beta})}{|\ddot{f}_{a,\beta}(p^*_{a,\beta})|} > 0,$$

by application of the Mean Value Theorem. If we define

$$\mathcal{C} = \inf_{\beta \in B} \inf_{0 \leq a \leq p_h} \frac{-\beta_1 \dot{h}(\beta_0 + \beta_1 p^*_{a,\beta})}{|\ddot{f}_{a,\beta}(p^*_{a,\beta})|} > 0$$

then the magnitude of the price change $v_1(\beta)$ on the path $(c(s), s)_{1 \leq s \leq S}$, is bounded from below by

$$v_1(\beta) \geq v_0 := \mathcal{C} \cdot \min\left\{ \inf_{\beta \in B} h(\beta_0 + \beta_1 p_h)^{C-1} f^*_{0,\beta}, \quad \inf_{\beta \in B} |(V_\beta(1,1) - p_h) h(\beta + \beta_1 p_h)| \right\}.$$

Proof of Theorem 2

Consider the k-th selling season, for some arbitrary fixed integer $k \geq 2$. The prices generated by $\Phi(\epsilon)$ are based on the estimates $\hat{\beta}_t$, which are determined by the historical prices and demand realizations. Now, different demand realizations can lead to the same state (c, s) of the MDP. For example, a sale in the first period of a selling season and no sale in the second period leads to state $(C-2, 3)$, but this state is also reached if there is no sale in the first period and a sale in the second period of the selling season. These two "routes" may lead to different estimates $\hat{\beta}_t$, and to different pricing decisions in state $(C-2, 3)$. Thus, with $\Phi(\epsilon)$, the prices in the k-th selling season are not determined by a stationary policy for the Markov decision problem described in Section 2.3.

To be able to compare the optimal revenue in a selling season with that obtained by $\Phi(\epsilon)$, we define a new Markov decision problem, in which the states are sequences of demand realizations in the selling season. Conditionally on all prices and demand realizations from before the start of the selling season, $\Phi(\epsilon)$ is then a stationary deterministic policy for this new MDP: each state is associated with a unique price prescribed by $\Phi(\epsilon)$. This enables us to calculate bounds on the regret obtained in a single selling season.

We define this new MDP for any $\beta \in B$. The state space $\tilde{\mathcal{X}}$ consists of all sequences of possible demand realizations in the selling season:

$$\tilde{\mathcal{X}} = \{(x_1, \ldots, x_s) \in \{0,1\}^s \mid 0 \leq s \leq S, \sum_{i=1}^{s} x_i \leq C\},$$

where we denote the empty sequence by $(\emptyset)$. The action space is $[p_l, p_h]$. Using action p in state $(x_1, \ldots, x_s)$, with $0 \leq s < S$ induces a state transition from $(x_1, \ldots x_s)$ to $(x_1, \ldots, x_s, 1)$ with probability $h(\beta_0 + \beta_1 p) \mathbf{1}_{\sum_{i=1}^{s} x_i < C}$ (corresponding to a sale, and inducing immediate reward p), and from $(x_1, \ldots x_s)$ to $(x_1, \ldots, x_s, 0)$ with probability $1 - h(\beta_0 + \beta_1 p) \mathbf{1}_{\sum_{i=1}^{s} x_i < C}$ (corresponding to no sale, and inducing zero reward). In the terminal state $(x_1, \ldots, x_S)$, no transitions occur, no reward is received, and no actions are taken. Note that the actions in states with zero inventory do not impact the reward or transitions.

It is easily seen that the MDP described in Section 2.3 corresponds to the one described here, except that there states are aggregated: all states $(x_1,\dots,x_s)$ and $(x'_1,\dots,x'_{s'})$ with $s=s'$ and $\sum_{i=1}^{s} x_i = \sum_{i=1}^{s'} x'_i$ are there taken together in the state $(C-\sum_{i=1}^{s} x_i, s+1)$.

Let $\tilde{\pi} = \{\tilde{\pi}(x) \mid x=(x_1,\dots,x_s)\in\tilde{\mathcal{X}}, 0\le s<S\}$ be a stationary deterministic policy for this MDP with augmented state space (defining an action for all except the terminal states $(x_1,\dots,x_S)$), and let $\tilde{V}^{\tilde{\pi}}_{\beta}(x)$ be the corresponding value function, for $\beta\in B$. For non-terminal states $x=(x_1,\dots,x_s)\in\tilde{\mathcal{X}}$ we write $(x;1)=(x_1,\dots,x_s,1)$ and $(x;0)=(x_1,\dots,x_s,0)$. Then, for all non-terminal states $x=(x_1,\dots,x_s)\in\tilde{\mathcal{X}}$, $s<S$, and all $\beta\in B$, $\tilde{V}^{\tilde{\pi}}_{\beta}(x)$ satisfies the backward recursion

$$\begin{aligned}\tilde{V}^{\tilde{\pi}}_{\beta}(x) &= (\tilde{\pi}(x)\mathbf{1}_{\sum_{i=1}^{s} x_i<C} + \tilde{V}^{\tilde{\pi}}_{\beta}(x;1))h(\beta_0+\beta_1\tilde{\pi}(x))\mathbf{1}_{\sum_{i=1}^{s} x_i<C}\\ &\quad + \tilde{V}^{\tilde{\pi}}_{\beta}(x;0)(1-h(\beta_0+\beta_1\tilde{\pi}(x))\mathbf{1}_{\sum_{i=1}^{s} x_i<C}),\end{aligned}$$

and $\tilde{V}^{\tilde{\pi}}_{\beta}(x)=0$ for all terminal states x.

Let $\tilde{\pi}^*_{\beta}$ be the optimal policy corresponding to $\beta\in B$, and write $\tilde{V}_{\beta}(x)=\tilde{V}^{\tilde{\pi}^*_{\beta}}_{\beta}(x)$. Then

$$\tilde{V}_{\beta}(x) = \max_{p\in[p_l,p_h]}\left[p-\left(\tilde{V}_{\beta}(x;0)-\tilde{V}_{\beta}(x;1)\right)\right]h(\beta_0+\beta_1 p)+\tilde{V}_{\beta}(x;0), \tag{EC.4}$$

$$\tilde{\pi}^*_{\beta}(x) = \arg\max_{p\in[p_l,p_h]}\left[p-\left(\tilde{V}_{\beta}(x;0)-\tilde{V}_{\beta}(x;1)\right)\right]h(\beta_0+\beta_1 p), \tag{EC.5}$$

for all states $(x_1,\dots,x_s)$ with $\sum_{i=1}^{s} x_i<C$ and $s<S$, and $\tilde{V}_{\beta}(x)=0$ for all states $(x_1,\dots,x_s)$ with $\sum_{i=1}^{s} x_i = C$ or $s=S$. Analogous to Lemma EC.2, it is not difficult to show that $\tilde{\pi}^*_{\beta}(x)$ is uniquely defined and lies in (p_l,p_h), for all $\beta\in B$ and all states $(x_1,\dots,x_s)$ with $\sum_{i=1}^{s} x_i<C$ and $s<S$. In the notation of Lemma EC.1, this implies $(\tilde{V}_{\beta}(x;0)-\tilde{V}_{\beta}(x;1),\beta)\in\mathcal{U}_{AB}$..

Let $\mathcal{U}$ and v_0 be as in Theorem 1, ρ_1 as in Proposition 1, and choose $\rho\in(0,\rho_1)$ such that $\beta\in\mathcal{U}$ whenever $||\beta-\beta^{(0)}||\le\rho$. For all $l\in\mathbb{N}$, if $(l-1)S>T_\rho$ then $\hat{\beta}_t\in\mathcal{U}$ for all $t=1+(l-1)S,\dots,S(l-1)S$, and Theorem 1 implies $\lambda_{\min}(P_{lS})-\lambda_{\min}(P_{(l-1)S})\ge\frac{1}{8}v_0^2(1+p_h^2)^{-1}\ge\frac{1}{2}\epsilon^2(1+p_h^2)^{-1}$, using $v_0/2\ge\epsilon$. If $(l-1)S\le T_\rho$, then I) of the pricing strategy $\Phi(\epsilon)$ guarantees that there are $1\le s,s'\le S$ such that $|p_{s+(l-1)S}-p_{s'+(l-1)S}|\ge\epsilon$, and Lemma EC.3 then implies $\lambda_{\min}(P_{lS})-\lambda_{\min}(P_{(l-1)S})\ge\frac{1}{2}\epsilon^2(1+p_h^2)^{-1}$. It follows that $\lambda_{\min}(P_{lS})\ge l\cdot\frac{1}{2}\epsilon^2(1+p_h^2)^{-1}$ for all $l\in\mathbb{N}$, and thus for all $t>S$,

$$\lambda_{\min}(P_t)\ge\lambda_{\min}(P_{(SS_t-1)S})\ge(SS_t-1)\cdot\frac{1}{2}\epsilon^2(1+p_h^2)^{-1}\ge t\cdot\frac{1}{4S}\epsilon^2(1+p_h^2)^{-1},$$

using $SS_t-1\ge t\frac{(SS_t-1)}{S\cdot SS_t}\ge\frac{t}{2S}$. (Recall the definition $SS_t=1+\lfloor(t-1)/S\rfloor$). By application of Proposition 1 with $t_0=S$ and $L(t)=t\cdot\frac{1}{4S}\epsilon^2(1+p_h^2)^{-1}$, we have

$$T_\rho<\infty \text{ a.s., } E[T_\rho]<\infty, \text{ and } E[||\hat{\beta}_t-\beta^{(0)}||^2\mathbf{1}_{t>T_\rho}] = O(\log(t)/t). \tag{EC.6}$$

In addition, $v_0/2 > \epsilon$ implies that I) of the pricing strategy $\Phi(\epsilon)$ does not occur for all t with $(SS_t - 1)S > T_\rho$. In particular, if $(k-1)S > T_\rho$, then

$$p_{1+s+(k-1)S} = \tilde{\pi}^*_{\hat{\beta}_{s+(k-1)S}}(d_{1+(k-1)S}, d_{2+(k-1)S}, \ldots, d_{s+(k-1)S}), \tag{EC.7}$$

for all $1 \le s \le S-1$, and

$$p_{1+(k-1)S} = \tilde{\pi}^*_{\hat{\beta}_{(k-1)S}}(\emptyset). \tag{EC.8}$$

Let $H = (p_1, \ldots, p_{(k-1)S}, d_1, \ldots, d_{(k-1)S})$ denote the history of prices and demand up to and including time period $(k-1)S$. Conditionally on H, and given that $(k-1)S > T_\rho$, the parameter estimates $\hat{\beta}_{s+(k-1)S}$ in (EC.7) and (EC.8) are completely determined by the state $(d_{1+(k-1)S}, d_{2+(k-1)S}, \ldots, d_{s+(k-1)S})$. Thus, for each state $(x_1, \ldots, x_s)$ with $\sum_{i=1}^s x_i < C$ and $s < S$, there is a uniquely associated price prescribed by $\Phi(\epsilon)$. Consequently, there is a stationary deterministic policy, denoted by $\tilde{\pi}^H$, such that $p_{1+(k-1)S} = \tilde{\pi}^H(\emptyset)$ and

$$p_{1+s+(k-1)S} = \tilde{\pi}^H(x)$$

when $x = (d_{1+(k-1)S}, d_{2+(k-1)S}, \ldots, d_{s+(k-1)S})$, $1 \le s < S$, and $\sum_{i=1}^s d_{i+(k-1)S} < C$.

This enables us to bound the regret in the k-th selling season:

$$\begin{aligned}
&V_{\beta^{(0)}}(C,1) - \sum_{i=1+(k-1)S}^{kS} E[p_i d_i] \\
=&E\left[\left(\tilde{V}_{\beta^{(0)}}(\emptyset) - \sum_{i=1+(k-1)S}^{kS} p_i d_i\right)\mathbf{1}_{(k-1)S \le T_\rho}\right] \\
+&E\left[\left(\tilde{V}_{\beta^{(0)}}(\emptyset) - \sum_{i=1+(k-1)S}^{kS} p_i d_i\right)\mathbf{1}_{(k-1)S > T_\rho}\right] \\
\le&\tilde{V}_{\beta^{(0)}}(\emptyset) P((k-1)S \le T_\rho) \\
+&E\left[E\left[\left(\tilde{V}_{\beta^{(0)}}(\emptyset) - \sum_{i=1+(k-1)S}^{kS} p_i d_i\right)\mathbf{1}_{(k-1)S > T_\rho} \mid H\right]\right] \\
\le&\tilde{V}_{\beta^{(0)}}(\emptyset)\frac{E[T_\rho]}{(k-1)S} && \text{(EC.9)} \\
+&E\left[E\left[\left(\tilde{V}_{\beta^{(0)}}(\emptyset) - \tilde{V}^{\tilde{\pi}^H}_{\beta^{(0)}}(\emptyset)\right)\mathbf{1}_{(k-1)S > T_\rho} \mid H\right]\right]. && \text{(EC.10)}
\end{aligned}$$

The term (EC.9) is finite because $E[T_\rho] < \infty$. To obtain an upper bound on the term (EC.10), we need two sensitivity results:

(S.1) Write $\vec{d_s} = (d_{1+(k-1)S}, \ldots, d_{s+(k-1)S})$ for $1 \leq s \leq S-1$, and set $\vec{d_0} = (\emptyset)$. There is a $K_0 > 0$ such that, for all stationary deterministic policies $\tilde{\pi}$ and all $0 \leq s \leq S-1$,

$$
\begin{aligned}
&(\tilde{V}_{\beta^{(0)}}(\vec{d_s}) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d_s}))\mathbf{1}_{\sum_{i=1}^{s} d_{i+(k-1)S} < C}\mathbf{1}_{(k-1)S > T_\rho} \\
&\leq K_0 \sum_{\sigma=s}^{S-1} (\tilde{\pi}^*_{\beta^{(0)}}(\vec{d_\sigma}) - \tilde{\pi}(\vec{d_\sigma}))^2 \mathbf{1}_{\sum_{i=1}^{\sigma} d_{i+(k-1)S} < C} \cdot \mathbf{1}_{(k-1)S > T_\rho} \text{ a.s.}
\end{aligned}
$$

(S.2) There is a $K_3 > 0$ such that

$$
|\tilde{\pi}^*_{\beta}(x) - \tilde{\pi}^*_{\beta^{(0)}}(x)| \leq K_3 ||\beta - \beta^{(0)}||, \tag{EC.11}
$$

for all $\beta \in B$ with $||\beta - \beta^{(0)}|| \leq \rho$, and all states $(x_1, \ldots, x_s)$ with $\sum_{i=1}^{s} x_i < C$ and $s < S$.

The proof of these two sensitivity properties is given below.

Application of (S.1), (S.2), and (EC.6) now gives

$$
\begin{aligned}
&E\left[E\left[\left(\tilde{V}_{\beta^{(0)}}(\emptyset) - \tilde{V}^{\tilde{\pi}^H}_{\beta^{(0)}}(\emptyset)\right)\mathbf{1}_{(k-1)S > T_\rho} \mid H\right]\right] \\
\leq &E\left[E\left[K_0 \sum_{\sigma=0}^{S-1} (\tilde{\pi}^*_{\beta^{(0)}}(\vec{d_\sigma}) - \tilde{\pi}^H(\vec{d_\sigma}))^2 \mathbf{1}_{\sum_{i=1}^{\sigma} d_{i+(k-1)S} < C} \cdot \mathbf{1}_{(k-1)S > T_\rho} \mid H\right]\right] \\
= &E\left[K_0 \sum_{\sigma=0}^{S-1} (\tilde{\pi}^*_{\beta^{(0)}}(\vec{d_\sigma}) - \tilde{\pi}^*_{\hat{\beta}_{\sigma+(k-1)S}}(\vec{d_\sigma}))^2 \mathbf{1}_{\sum_{i=1}^{\sigma} d_{i+(k-1)S} < C} \cdot \mathbf{1}_{(k-1)S > T_\rho}\right] \\
\leq &E\left[K_0 K_3^2 \sum_{\sigma=0}^{S-1} \left|\left|\beta^{(0)} - \hat{\beta}_{\sigma+(k-1)S}\right|\right|^2 \mathbf{1}_{(k-1)S > T_\rho}\right] \\
\leq &K_4 \sum_{\sigma=0}^{S-1} \frac{\log(\sigma + (k-1)S)}{\sigma + (k-1)S},
\end{aligned}
$$

for some K_4 independent of k and S.

We thus have

$$
\begin{aligned}
&V_{\beta^{(0)}}(C,1) - \sum_{i=1+(k-1)S}^{kS} E[p_i \min\{d_i, c_i\}] \\
\leq &\tilde{V}_{\beta^{(0)}}(\emptyset) E[T_\rho] \frac{1}{(k-1)S} + K_4 \sum_{\sigma=0}^{S-1} \frac{\log(\sigma + (k-1)S)}{\sigma + (k-1)S} \\
\leq &K_5 \sum_{t=1+(k-1)S}^{kS} \frac{\log(t)}{t},
\end{aligned} \tag{EC.12}
$$

for some $K_5 > 0$, independent of k and S.

The proof of the theorem is complete by observing

$$
\begin{aligned}
\text{Regret}(\Phi(\epsilon), T) &= \sum_{k=1}^{T} \left[V_{\beta^{(0)}}(C,1) - \sum_{i=1+(k-1)S}^{kS} E[p_i d_i]\right] \\
&\leq V_{\beta^{(0)}}(C,1) + \sum_{k=2}^{T} K_5 \sum_{t=1+(k-1)S}^{kS} \frac{\log(t)}{t} \leq V_{\beta^{(0)}}(C,1) + K_5 \sum_{t=1+S}^{TS} \frac{\log(t)}{t} \\
&= O(\log^2(T)).
\end{aligned}
$$

Proof of (S.1)

Backward induction on s. Let $s = S-1$. If $\sum_{i=1}^{S-1} d_{i+(k-1)S} < C$ then Lemma EC.1(iii) implies

$$\begin{aligned} &\tilde{V}_{\beta^{(0)}}(\vec{d}_{S-1}) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_{S-1}) = \max_{p\in[p_l,p_h]} ph(\beta_0^{(0)}+\beta_1^{(0)}p) - \tilde{\pi}(\vec{d}_{S-1})h(\beta_0^{(0)}+\beta_1^{(0)}\tilde{\pi}(\vec{d}_{S-1})) \\ \le& K_0(\tilde{\pi}^*_{\beta^{(0)}}(\vec{d}_{S-1}) - \tilde{\pi}(\vec{d}_{S-1}))^2 \text{ a.s.,} \end{aligned}$$

and thus

$$\begin{aligned} &(\tilde{V}_{\beta^{(0)}}(\vec{d}_{S-1}) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_{S-1}))\cdot \mathbf{1}_{\sum_{i=1}^{S-1} d_{i+(k-1)S}<C}\cdot \mathbf{1}_{(k-1)S>T_\rho} \\ \le& K_0(\tilde{\pi}^*_{\beta^{(0)}}(\vec{d}_{S-1}) - \tilde{\pi}(\vec{d}_{S-1}))^2\cdot \mathbf{1}_{\sum_{i=1}^{S-1} d_{i+(k-1)S}<C}\cdot \mathbf{1}_{(k-1)S>T_\rho} \text{ a.s.} \end{aligned}$$

Now let $0 \le s < S-1$. If $\sum_{i=1}^{s} d_{i+(k-1)S} = C$ then $\tilde{V}_{\beta^{(0)}}(\vec{d}_s) = \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_s) = 0$. If $\sum_{i=1}^{s} d_{i+(k-1)S} < C$, then, again using Lemma EC.1(iii),

$$\begin{aligned} &\tilde{V}_{\beta^{(0)}}(\vec{d}_s) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_s) \\ =& \max_{p\in[p_l,p_h]} \left[p - (\tilde{V}_{\beta^{(0)}}(\vec{d}_s;0) - \tilde{V}_{\beta^{(0)}}(\vec{d}_s;1))\right] h(\beta_0^{(0)}+\beta_1^{(0)}p) + \tilde{V}(\vec{d}_s;0) \\ &-\left[\tilde{\pi}(\vec{d}_s) - (\tilde{V}_{\beta^{(0)}}(\vec{d}_s;0) - \tilde{V}_{\beta^{(0)}}(\vec{d}_s;1))\right] h(\beta_0^{(0)}+\beta_1^{(0)}\tilde{\pi}(\vec{d}_s)) \\ &+\left[\tilde{\pi}(\vec{d}_s) - (\tilde{V}_{\beta^{(0)}}(\vec{d}_s;0) - \tilde{V}_{\beta^{(0)}}(\vec{d}_s;1))\right] h(\beta_0^{(0)}+\beta_1^{(0)}\tilde{\pi}(\vec{d}_s)) \\ &-\left[\tilde{\pi}(\vec{d}_s) - (\tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_s;0) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_s;1))\right] h(\beta_0^{(0)}+\beta_1^{(0)}\tilde{\pi}(\vec{d}_s)) - \tilde{V}^{\tilde{\pi}}(\vec{d}_s;0) \\ \le& K_0(\tilde{\pi}^*_{\beta^{(0)}}(\vec{d}_s) - \tilde{\pi}(\vec{d}_s))^2 \\ &+\left(\tilde{V}_{\beta^{(0)}}(\vec{d}_s;0) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_s;0)\right)\cdot\left(1 - h(\beta_0^{(0)}+\beta_1^{(0)}\tilde{\pi}(\vec{d}_s))\right) \\ &+\left(\tilde{V}_{\beta^{(0)}}(\vec{d}_s;1) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_s;1)\right)\cdot(h(\beta_0^{(0)}+\beta_1^{(0)}\tilde{\pi}(\vec{d}_s)) \\ \le& K_0(\tilde{\pi}^*_{\beta^{(0)}}(\vec{d}_s) - \tilde{\pi}(\vec{d}_s))^2 + \left[\tilde{V}_{\beta^{(0)}}(\vec{d}_{s+1}) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_{s+1})\right] \text{ a.s.,} \end{aligned}$$

and the induction hypothesis implies

$$\begin{aligned} &(\tilde{V}_{\beta^{(0)}}(\vec{d}_s) - \tilde{V}^{\tilde{\pi}}_{\beta^{(0)}}(\vec{d}_s))\mathbf{1}_{\sum_{i=1}^{s} d_{i+(k-1)S}<C}\mathbf{1}_{(k-1)S>T_\rho} \\ \le& K_0\sum_{\sigma=s}^{S-1}(\tilde{\pi}^*_{\beta^{(0)}}(\vec{d}_\sigma) - \tilde{\pi}(\vec{d}_\sigma))^2\mathbf{1}_{\sum_{i=1}^{\sigma} d_{i+(k-1)S}<C}\mathbf{1}_{(k-1)S>T_\rho} \text{ a.s.} \end{aligned}$$

Proof of (S.2)

Let $x = (x_1,\ldots,x_s) \in \tilde{\mathcal{X}}$ with $\sum_{i=1}^{s} x_i < C$ and $s < S$. We have

$$\begin{aligned} \tilde{\pi}^*_\beta(x) - \tilde{\pi}^*_{\beta^{(0)}}(x) &= p^*_{\tilde{V}_\beta(x;0)-\tilde{V}_\beta(x;1),\beta} - p^*_{\tilde{V}_{\beta^{(0)}}(x;0)-\tilde{V}_{\beta^{(0)}}(x;1),\beta^{(0)}} \\ &= p^*_{a,\beta} - p^*_{a^{(0)},\beta^{(0)}}, \end{aligned} \tag{EC.13}$$

in the notation of Lemma EC.1, with $a = \tilde{V}_\beta(x;0) - \tilde{V}_\beta(x;1)$ and $a^{(0)} = \tilde{V}_{\beta(0)}(x;0) - \tilde{V}_{\beta(0)}(x;1)$. Also we have that both $(a,\beta) \in \mathcal{U}_{AB}$ and $(a^{(0)}, \beta^{(0)}) \in \mathcal{U}_{AB}$, since $p_l < \tilde{\pi}^*_\beta(x), \tilde{\pi}^*_{\beta(0)}(x) < p_h$ (as noted above, in the remarks below equation (EC.5)). The set

$$\{\tilde{V}_\beta(x;0) - \tilde{V}_\beta(x;1) \mid \beta \in B, \left|\left|\beta - \beta^{(0)}\right|\right| \leq \rho, x \in \tilde{\mathcal{X}}\} \times \{\beta \in B \mid \left|\left|\beta - \beta^{(0)}\right|\right| \leq \rho\}$$

is compact and contained in $\mathcal{U}_{AB}$. Since $p^*_{a,\beta}$ is continuously differentiable in a and β on $\mathcal{U}_{AB}$, it follows by a first order Taylor expansion that

$$|p^*_{a,\beta} - p^*_{a^{(0)},\beta^{(0)}}| \leq K_6(|a - a^{(0)}| + ||\beta - \beta^{(0)}||), \tag{EC.14}$$

for a $K_6 > 0$ independent of a, $a^{(0)}$. It is not difficult to show by backward induction that for all $x \in \tilde{\mathcal{X}}$ there is a $K_x > 0$ such that, for all β with $||\beta - \beta^{(0)}|| \leq \rho$,

$$\left|\tilde{V}_\beta(x) - \tilde{V}_{\beta(0)}(x)\right| \leq K_x \left|\left|\beta - \beta^{(0)}\right|\right|. \tag{EC.15}$$

Combining (EC.13), (EC.14), and (EC.15), we obtain

$$\begin{aligned}
&|\tilde{\pi}^*_\beta(x) - \tilde{\pi}^*_{\beta(0)}(x)| \\
\leq &K_6(|a - a^{(0)}| + ||\beta - \beta^{(0)}||) \\
\leq &K_6(|\tilde{V}_\beta(x;0) - \tilde{V}_{\beta(0)}(x;0)| + |\tilde{V}_\beta(x;1) - \tilde{V}_{\beta(0)}(x;1)| + ||\beta - \beta^{(0)}||) \\
\leq &K_6(1 + 2\max_{x \in \tilde{\mathcal{X}}} K_x)||\beta - \beta^{(0)}||.
\end{aligned}$$

This proves (S.2).

Proof of Theorem 3

First note that h, B and $[p_l, p_h]$ satisfy the assumptions of Section 2.2.

The proof consists of three steps. In Step 1 we consider pricing strategies that select optimal prices whenever inventory is strictly below C. For these strategies we show that the regret in a single selling season is bounded from below by a term proportional to the mean squared estimation error at the end of the selling season, viz. equation (EC.19). In Step 2 we prove a lower bound on the expected value of this term, according to a probability density on $\beta^{(0)}$. The proof is analogous to the proof of the van Trees inequality in Gill and Levit (1995, Section 2). We cannot directly apply their result, however, because we are in a slightly different setting (discrete instead of continuous random variables, and non-deterministic number of observations upon which the estimate of β_0 is based); we therefore include a complete proof, resulting in equation (EC.24). In Step 3 we combine the results of Step 1 and Step 2 to show that the regret in the k-th single selling season is bounded from below by a term proportional to $1/k$; this implies that the total regret after T selling seasons is bounded from below by a term proportional to $\log T$.

Throughout the proof, let ψ be an arbitrary price strategy.

Step 1. Let $\beta = (\beta_0, \beta_1) \in B$ be arbitrary and fixed, and let $k \in \mathbb{N}$, $k \geq 2$. For shorthand notation, write

$$h_\beta(p) = h(\beta_0 + \beta_1 p).$$

Define

$$\varepsilon = \inf_{\beta \in B} \inf_{p \in [p_l, p_h]} \min\{h_\beta(p), 1 - h_\beta(p)\},$$

and note that $\varepsilon > 0$. Let ψ' be the pricing strategy that for all $t \in \mathbb{N}$ coincides with ψ if $c_t = C$, and that equals the optimal price $\pi^*_\beta(c_t, s_t)$ if $c_t < C$. Let p_i be the prices generated by ψ'. For $1 \leq j < S$ write $\vec{d}_j = (d_{1+(k-1)S}, \ldots, d_{j+(k-1)S}) \in \{0,1\}^j$ for the vector of demand realizations in the first j stages of season k, and set $\vec{d}_0 = 0$. For all $j \in \{1, \ldots, S-1\}$, we have the following inequality:

$$\begin{aligned}
& V(C,j) - E\Big[\sum_{i=j+(k-1)S}^{kS} p_i d_i \mid \vec{d}_{j-1} = 0\Big] \\
=& h_\beta(\pi^*_\beta(C,j)) \cdot (\pi^*_\beta(C,j) + V(C-1, j+1)) + (1 - h_\beta(\pi^*_\beta(C,j))) \cdot V(C, j+1) \\
& - E\Big[\sum_{i=j+(k-1)S+1}^{kS} p_i d_i \mid \vec{d}_{j-1} = 0, d_{j+(k-1)S} = 1\Big] P\left(d_{j+(k-1)S} = 1 \mid \vec{d}_{j-1} = 0\right) \\
& - E\Big[\sum_{i=j+(k-1)S+1}^{kS} p_i d_i \mid \vec{d}_{j-1} = 0, d_{j+(k-1)S} = 0\Big] P\left(d_{1+(k-1)S} = 0 \mid \vec{d}_{j-1} = 0\right) \\
& - E[p_{j+(k-1)S}\, h_\beta(p_{j+(k-1)S}) \mid \vec{d}_{j-1} = 0] \\
=\; & h_\beta(\pi^*_\beta(C,j)) \cdot (\pi^*_\beta(C,j) + V(C-1, j+1)) + (1 - h_\beta(\pi^*_\beta(C,j))) \cdot V(C, j+1) \\
& - E\Big[h_\beta(p_{j+(k-1)S}) \cdot (p_{j+(k-1)S} + V(C-1, j+1)) + (1 - h_\beta(p_{j+(k-1)S})) \cdot V(C, j+1) \mid \vec{d}_{j-1} = 0\Big] \\
& + E\Big[h_\beta(p_{j+(k-1)S}) \mid \vec{d}_{j-1} = 0\Big] \cdot \left(V(C-1, j+1)) - E\Big[\sum_{i=j+(k-1)S+1}^{kS} p_i d_i \mid \vec{d}_{j-1} = 0, d_{j+(k-1)S} = 1\Big]\right) \\
& + E\Big[1 - h_\beta(p_{j+(k-1)S}) \mid \vec{d}_{j-1} = 0\Big] \cdot \left(V(C, j+1)) - E\Big[\sum_{i=j+(k-1)S+1}^{kS} p_i d_i \mid \vec{d}_{j-1} = 0, d_{j+(k-1)S} = 0\Big]\right) \\
\geq & \varepsilon \cdot \left(V(C, j+1) - E\Big[\sum_{i=j+(k-1)S+1}^{kS} p_i d_i \mid \vec{d}_j = 0\Big]\right).
\end{aligned} \tag{EC.16}$$

In the first equality we use the recursive definition of $V(C,j)$, condition $E[\sum_{i=j+1+(k-1)S}^{kS} p_i d_i \mid \vec{d}_{j-1} = 0]$ on the event $\{d_{j+(k-1)S} = 1 \mid \vec{d}_{j-1} = 0\}$ and its complement, and use $E[p_{j+(k-1)S}\, d_{j+(k-1)S} \mid \vec{d}_{j-1} = 0] = E[p_{j+(k-1)S}\, h_\beta(p_{j+(k-1)S}) \mid \vec{d}_{j-1} = 0]$. In the second inequality we add and substract $E[h_\beta(p_{j+(k-1)S}) \cdot V(C-1, j+1) + (1 - h_\beta(p_{j+(k-1)S})) \cdot V(C, j+1) \mid \vec{d}_{j-1} = 0]$, and use $P(d_{j+(k-1)S} =$

$1 \mid \vec{d}_{j-1} = 0) = E[h_\beta(p_{j+(k-1)S}) \mid \vec{d}_{j-1} = 0]$. The inequality follows from the fact that $\pi^*_\beta(C,j)$, by definition, maximizes

$$h_\beta(p) \cdot (p + V(C-1, j+1)) + (1 - h_\beta(p)) \cdot V(C, j+1), \quad p \in [p_l, p_h],$$

and from

$$V(C-1, j+1)) - E\left[\sum_{i=j+(k-1)S+1}^{kS} p_i d_i \mid \vec{d}_{j-1} = 0, d_{j+(k-1)S} = 1\right] = 0,$$

which follows from the fact that, by definition of ψ', optimal prices are chosen whenever inventory is strictly lower than C.

Repeated application of (EC.16) gives

$$V(C,1) - E\left[\sum_{i=1+(k-1)S}^{kS} p_i d_i\right] \geq \varepsilon^{S-1} \cdot \left(V(C,S) - E\left[p_{kS} d_{kS} \mid \vec{d}_{S-1} = 0\right]\right). \tag{EC.17}$$

Write b_{kS} for p_{kS} multiplied by $-2\beta_1$, given that $\vec{d}_{S-1} = 0$:

$$b_{kS} = -2\beta_1 \cdot \psi'_{kS}(p_1, \ldots, p_{kS-1}, d_1, \ldots, d_{(k-1)S}, \underbrace{0, 0, \ldots, 0}_{S-1 \text{ zeros}}).$$

Note that b_{kS} (as well as all prices $p_{(k-1)S+1}, \ldots, p_{kS-1}$) only depends on prices $p_1, \ldots, p_{(k-1)S}$ and demand realizations $d_1, \ldots, d_{(k-1)S}$ up to selling season $k-1$; in particular, b_{kS} does not depend on the event $\{\vec{d}_{S-1} = 0\}$. Together with $\pi^*_\beta(C,S) = \beta_0/(-2\beta_1)$ this implies

$$\begin{aligned} V(C,S) - E\left[p_{kS} d_{kS} \mid \vec{d}_{S-1} = 0\right] &= \pi^*_\beta(C,S) h_\beta(\pi^*_\beta(C,S)) - E\left[p_{kS}\, h_\beta(p_{kS}) \mid \vec{d}_{S-1} = 0\right] \\ = -\beta_1 E\left[(\pi^*_\beta(C,S) - p_{kS})^2 \mid \vec{d}_{S-1} = 0\right] &= \frac{-1}{4\beta_1} E\left[(\beta_0 - b_{kS})^2\right]. \end{aligned} \tag{EC.18}$$

Combining (EC.17) and (EC.18) with $\inf_{\beta \in B} \frac{-1}{4\beta_1} = 1/3$ yields

$$V(C,1) - E\left[\sum_{i=1+(k-1)S}^{kS} p_i d_i\right] \geq \frac{1}{3} \varepsilon^{S-1} E\left[(\beta_0 - b_{kS})^2\right]. \tag{EC.19}$$

Step 2. In this step we view β as a random variable $\boldsymbol{\beta} = (\boldsymbol{\beta}_0, \boldsymbol{\beta}_1)$, drawn from B according to the probability density function λ, defined by

$$\lambda(\beta_0, \beta_1) = \frac{256}{3} \cos^2(8\pi\beta_0 - 11\pi/2), \quad (\beta_0, \beta_1) \in B.$$

We prove a lower bound on the term $E_\lambda\left[(\beta_0 - b_{kS})^2\right]$ in (EC.19), where the subscript λ emphasizes that we take expectation not only with respect to the distribution of the demand, but also with respect to λ. To this end, we need to introduce some notation. Let $D = (D_1, \ldots, D_{(k-1)S})$, with D_i equal to the (random) demand in period i, $1 \leq i \leq (k-1)S$. Note that, conditional on $\boldsymbol{\beta} = \beta$, D_i is

Bernoulli with mean $h_\beta(p_i)$ if $c_i > 0$, and $D_i = 0$ with probability one if $c_i = 0$. For $d \in \{0,1\}^{(k-1)S}$, let $c_i(d) = C - \sum_{j<i, SS_j = SS_i} d_j$ be the inventory available at the beginning of period i given the vector of demand realizations d, and write $c(d) = (c_1(d), \ldots, c_{(k-1)S}(d))$. Note that D takes values in $\mathcal{D} = \{d \in \{0,1\}^{(k-1)S} \mid d \leq c(d)\}$. The probability mass function $f(d \mid \beta)$ of D, conditional on $\boldsymbol{\beta} = \beta$, is given by

$$f(d \mid \beta) = \prod_{\substack{i=1,\\ c_i(d)>0}}^{(k-1)S} h_\beta(p_i(d))^{d_i}(1 - h_\beta(p_i(d))^{1-d_i},$$

where $p_i(d)$ is the price in period i, given demand realizations $d_1, \ldots, d_{i-1}$, under strategy ψ'. For $d \in \mathcal{D}$, write

$$b(d \mid \beta) = -2\beta_1 \psi'_{kS}(p_1(d), p_2(d), \ldots, p_{kS-1}, d_1, \ldots, d_{(k-1)S}, \underbrace{0, 0, \ldots, 0}_{S-1 \text{ zeros}})$$

for $-2\beta_1$ times the price in period kS given that demand is zero in periods $(k-1)S+1, \ldots, kS-1$ and demand in preceding selling seasons is given by d; all of this conditional on $\boldsymbol{\beta} = \beta$.

We now prove a lower bound on

$$E_\lambda[(b(D \mid \boldsymbol{\beta}) - \boldsymbol{\beta}_0)^2] = \int_{5/8}^{6/8} \int_{-3/4}^{-9/16} \sum_{d \in \mathcal{D}} f(d \mid \beta_0, \beta_1)\lambda(\beta_0, \beta_1)(b(d \mid \beta_0, \beta_1) - \beta_0)^2 \, d\beta_1 d\beta_0.$$

By the mean-value theorem, there is a $\tilde{\beta}_1 \in [-3/4, -9/16]$ such that

$$E_\lambda[(b(D \mid \boldsymbol{\beta}) - \boldsymbol{\beta}_0)^2] = \frac{3}{16} \int_{5/8}^{6/8} \sum_{d \in \mathcal{D}} f(d \mid \beta_0, \tilde{\beta}_1)\lambda(\beta_0, \tilde{\beta}_1)(b(d \mid \beta_0, \tilde{\beta}_1) - \beta_0)^2 \, d\beta_0.$$

For notational convenience we stop writing the dependence of f, λ, b on $\tilde{\beta}_1$. Applying Cauchy-Schwarz on the integral-sum and integrating by parts, we obtain

$$\begin{aligned}
&\frac{3}{16} \int_{5/8}^{6/8} \sum_{d \in \mathcal{D}} (b(d \mid \beta_0) - \beta_0)^2 f(d \mid \beta_0)\lambda(\beta_0) \, d\beta_0 \cdot \int_{5/8}^{6/8} \sum_{d \in \mathcal{D}} \Big(\frac{\partial}{\partial \beta_0} \log(f(d \mid \beta_0)\lambda(\beta_0))\Big)^2 f(d \mid \beta_0)\lambda(\beta_0) \, d\beta_0 \\
&\geq \frac{3}{16} \int_{5/8}^{6/8} \sum_{d \in \mathcal{D}} (b(d \mid \beta_0) - \beta_0)\Big(\frac{\partial}{\partial \beta_0} \log(f(d \mid \beta_0)\lambda(\beta_0))\Big) f(d \mid \beta_0)\lambda(\beta_0) \, d\beta_0 \\
&= \frac{3}{16} \sum_{d \in \mathcal{D}} b(d \mid \beta_0)\Big(f(d \mid 6/8)\lambda(6/8) - f(d \mid 5/8)\lambda(5/8)\Big) \\
&- \frac{3}{16} \sum_{d \in \mathcal{D}} \beta_0 \Big(f(d \mid 6/8)\lambda(6/8) - f(d \mid 5/8)\lambda(5/8)\Big) + \frac{3}{16} \int_{5/8}^{6/8} \sum_{d \in \mathcal{D}} f(d \mid \beta_0)\lambda(\beta_0) \, d\beta_0 = 1,
\end{aligned} \tag{EC.20}$$

since $\lambda(5/8) = \lambda(6/8) = 0$ and $\int_{5/8}^{6/8} \lambda(\beta_0) \, d\beta_0 = \int_{5/8}^{6/8} \lambda(\beta_0, \tilde{\beta}_1) \, d\beta_0 = 16/3$.

We have

$$\int_{5/8}^{6/8} \Big(\frac{\partial}{\partial \beta_0} \log(\lambda(\beta_0))\Big)^2 \lambda(\beta_0) \, d\beta_0 = \frac{4096\pi^2}{3}, \tag{EC.21}$$

and

$$E\left[\frac{\partial}{\partial\beta_0}\log f(D\mid\beta_0)\right]=\sum_{d\in\mathcal{D}}\frac{\partial}{\partial\beta_0}f(d\mid\beta_0)=0\quad\text{for all }\beta_0,\tag{EC.22}$$

and

$$E\left[\left(\frac{\partial}{\partial\beta_0}\log f(D\mid\beta_0)\right)^2\right]=E\left[\left(\sum_{\substack{i=1,\\ c_i(D)>0}}^{(k-1)S}\frac{D_i-h_{\beta_0}(p_i(D))}{h_{\beta_0}(p_i(D))(1-h_{\beta_0}(p_i(D)))}\right)^2\right]$$

$$=\sum_{\substack{i=1,\\ c_i(D)>0}}^{(k-1)S}\frac{1}{h_{\beta_0}(p_i(D))(1-h_{\beta_0}(p_i(D)))}\le\frac{(k-1)S}{\varepsilon^2},\tag{EC.23}$$

which follows from

$$E\left[\frac{D_i-h_{\beta_0}(p_i(D))}{h_{\beta_0}(p_i(D))(1-h_{\beta_0}(p_i(D)))}\cdot\frac{D_j-h_{\beta_0}(p_j(D))}{h_{\beta_0}(p_j(D))(1-h_{\beta_0}(p_j(D)))}\right]=0$$

for all $i\neq j$ with $c_i(D)>0$, $c_j(D)>0$, and

$$E\left[\left(\frac{D_i-h_{\beta_0}(p_i(D))}{h_{\beta_0}(p_i(D))(1-h_{\beta_0}(p_i(D)))}\right)^2\right]=\frac{1}{h_{\beta_0}(p_i(D))(1-h_{\beta_0}(p_i(D)))}$$

for all i with $c_i(D)>0$.

Plugging (EC.21), (EC.22) and (EC.23) into (EC.20), we obtain the following lower bound:

$$E_\lambda[(b(D\mid\boldsymbol{\beta})-\boldsymbol{\beta}_0)^2]\ge\frac{16}{3}\left(\int_{5/8}^{6/8}\sum_{d\in\mathcal{D}}\left(\frac{\partial}{\partial\beta_0}\log(f(d\mid\beta_0)\lambda(\beta_0))\right)^2 f(d\mid\beta_0)\lambda(\beta_0)\,d\beta_0\right)^{-1}$$

$$\ge\frac{1}{(k-1)S/\varepsilon^2+\frac{4096\pi^2}{3}}.\tag{EC.24}$$

Step 3. Combining (EC.19) and (EC.24), we obtain

$$\begin{aligned}\sup_{\beta^{(0)}\in B}\ \text{Regret}(\psi,T)&\ge E_\lambda[\text{Regret}(\psi,T)]\\ &\ge\sum_{k=2}^{T}E_\lambda\left[V(C,1)-E\left[\sum_{i=1+(k-1)S}^{kS}p_id_i\right]\right]\\ &\ge\sum_{k=2}^{T}\frac{1}{3}\varepsilon^{S-1}E_\lambda[(b(D_1,\ldots,D_{(k-1)S}\mid\boldsymbol{\beta})-\boldsymbol{\beta}_0)^2]\\ &\ge\sum_{k=2}^{T}\frac{1}{3}\varepsilon^{S-1}\frac{1}{(k-1)S/\varepsilon^2+\frac{4096\pi^2}{3}}\\ &\ge K_0\log(T),\end{aligned}$$

where we define

$$K_0=\frac{1}{3}\varepsilon^{S-1}\frac{1}{\max\{S/\varepsilon^2,\frac{4096\pi^2}{3}\}}\cdot\frac{1}{2\log(2)}.$$

This completes the proof.

Proof of Theorem 4

The proof of Theorem 2 shows, for the base case of non-overlapping selling seasons with non-varying capacity and season length, that the regret of the k-th selling season ($k \geq 2$) satisfies

$$V_{\beta^{(0)}}(C,1) - \sum_{i=1+(k-1)S}^{kS} E[p_i \min\{d_i, c_i\}] \leq K_5 \sum_{t=1+(k-1)S}^{kS} \frac{\log(t)}{t}, \tag{EC.25}$$

for some $K_5 > 0$ independent of k; viz. Equation (EC.12).

For any season j, estimation of $\beta^{(0)}$ to determine the prices during season j is based only on (i) sales data preceding t_j and (ii) sales data generated during season j (thus, all sales data generated in other seasons overlapping with season j is neglected). Because of this assumption, we can use the same MDP as the one constructed in the proof of Theorem 2 to show, analogous to (EC.25), that the regret of the j-th selling season ($t_j > 1$) satisfies

$$\begin{aligned} V_{\beta^{(0)},S_j}(C_j,1) - \sum_{s=1}^{S_j} E[p_{j,s} d_{j,s}] &\leq K_5 \sum_{s=1}^{S_j} \frac{\log(t_j - 1 + s)}{t_j - 1 + s} \\ &\leq K_5 S_j \log(t_j)/t_j \end{aligned}$$

for some constant $K_5 = K_5(C_j, S_j)$. The constant $K_j(C_j, S_j)$ may depend on C_j and S_j. However, $\tilde{K}_5 := \sup_{j\in\mathbb{N}} S_j K_5(C_j, S_j)$ does not depend on (C_j, S_j), and $\tilde{K}_5 < \infty$ because it is simply a supremum over a finite set (note that C_j and S_j are bounded). It follows that

$$\begin{aligned} \text{Regret}(\Phi'(\epsilon), T) &= \sum_{j: t_j - 1 + S_j < T} \left\{ V_{\beta^{(0)},S_j}(C_j,1) - \sum_{s=1}^{S_j} E[p_{j,s} d_{j,s}] \right\} \\ &\leq \sum_{j: t_j = 1} V_{\beta^{(0)},S_j}(C_j,1) + \sum_{j: t_j > 1, t_j - 1 + S_j < T} \tilde{K}_5 \log(t_j)/t_j \\ &\leq V_{\beta^{(0)},S_j}(C_j,1) \cdot \sup_{t\in\mathbb{N}} |\{j \in \mathbb{N} : t_j = t\}| + \sum_{i=2}^{T} \sum_{j: t_j = i, t_j - 1 + S_j < T} \tilde{K}_5 \log(t_j)/t_j \\ &\leq V_{\beta^{(0)},S_j}(C_j,1) \cdot \sup_{t\in\mathbb{N}} |\{j \in \mathbb{N} : t_j = t\}| + \sum_{i=2}^{T} \tilde{K}_5 \sup_{t\in\mathbb{N}} |\{j \in \mathbb{N} : t_j = t\}| \cdot \log(i)/i \\ &= O(\log^2(T)). \end{aligned}$$

EC.2. Auxiliary Lemmas

Lemma EC.1. *For all $a \in \mathbb{R}$ and $\beta \in B$, define the function $f_{a,\beta} : [p_l, p_h] \to \mathbb{R}$, $f(p) = (p - a)h(\beta_0 + \beta_1 p)$. Write $\dot{f}_{a,\beta}(p)$ and $\ddot{f}_{a,\beta}(p)$ for the first and second derivative of $f_{a,\beta}(p)$ with respect to p, and let $p^*_{a,\beta} = \arg\max_{p\in[p_l,p_h]} f_{a,\beta}(p)$. In addition, let*

$$\mathcal{U}_B = \left\{ (\beta_0, \beta_1) \in \times\mathbb{R} \times (-\infty, 0) \,\middle|\, 0 < h(z) < 1 \text{ and } \dot{h}(z) > 0, \text{ for all } z = \beta_0 + \beta_1 p,\ p \in [p_l, p_h] \right\}$$

and

$$\mathcal{U}_{AB} = \left\{(a,\beta_0,\beta_1) \in \mathbb{R} \times \mathcal{U}_B \,\middle|\, p^*_{a,\beta} \in (p_l,p_h)\right\}.$$

Then:

(i) For each $(a,\beta) \in \mathbb{R} \times \mathcal{U}_B$, $p^*_{a,\beta}$ *is uniquely defined, and for each* $(a,\beta) \in \mathcal{U}_{AB}$, $\dot{f}(p^*_{a,\beta}) = 0$ *and* $\ddot{f}(p^*_{a,\beta}) < 0$.

(ii) On $\mathcal{U}_{AB}$, $p^*_{a,\beta}$ *is continuously differentiable in* a *and* β, *strictly increasing in* a, *and nondecreasing in* β_0 *and* β_1. *On* $\mathbb{R} \times \mathcal{U}_B$, $p^*_{a,\beta}$ *is nondecreasing in* a, β_0 *and* β_1. *In addition, on* $\mathcal{U}_{AB}$, $f_{a,\beta}(p^*_{a,\beta})$ *is continuously differentiable in* a *and* β, *strictly decreasing in* a, *and strictly increasing in* β_0 *and* β_1.

(iii) There is a $K_0 > 0$ *such that* $f_{a,\beta}(p^*_{a,\beta}) - f_{a,\beta}(p) \le K_0(p - p^*_{a,\beta})^2$ *for all* $p \in [p_l,p_h]$ *and* $(a,\beta) \in \mathcal{U}_{AB}$.

Lemma EC.2. *Let* $\beta \in B$ *be arbitrary.*

(i) $\Delta V_\beta(c,s) \ge 0$ *for all* $1 \le c \le C$ *and* $1 \le s < S$.

(ii) $p_h \ge V_\beta(1,s) \ge V_\beta(1,s+1) \ge p_l$ *for all* $1 \le s < S$.

(iii) $\Delta V_\beta(c,s) \ge \Delta V(c+1,s)$ *for all* $1 \le c \le C$ *and* $1 \le s \le S$.

(iv) $V_{\beta'}(1,s) \le V_\beta(1,s)$, *for all* $1 \le s \le S$ *and all* $\beta' = (\beta'_0,\beta'_1) \in B$ *with* $\beta'_0 \le \beta_0$ *and* $\beta'_1 \le \beta_1$.

(v) $\pi^*_\beta(C,S) \le \pi^*_\beta(c,s) \le \pi^*_\beta(1,1)$, *for all* $1 \le c \le C$ *and* $1 \le s \le S$.

(vi) $\pi^*_\beta(c,s)$ *is continuous in* β, *for all* $1 \le c \le C$ *and* $1 \le s \le S$.

Lemma EC.3. *Let* $k \in \mathbb{N}$ *and* $\delta > 0$. *If there are* $s,s' \in \{1,\dots,S\}$ *such that* $|p_{s+(k-1)S} - p_{s'+(k-1)S}| \ge \delta$ *and* $c(s+(k-1)S)c(s'+(k-1)S) > 0$, *then* $\lambda_{\min}(P_{kS}) \ge \lambda_{\min}(P_{(k-1)S}) + \frac{1}{2}\delta^2(1+p_h^2)^{-1}$.

Proof of Lemma EC.1

(i) Let $(a,\beta) \in \mathbb{R} \times \mathcal{U}_B$. We have

$$\dot{f}(p) = h(\beta_0+\beta_1 p) + (p-a)\beta_1 \dot{h}(\beta_0+\beta_1 p)$$

and

$$\ddot{f}(p) = 2\beta_1 \dot{h}(\beta_0+\beta_1 p) + (p-a)\beta_1^2 \ddot{h}(\beta_0+\beta_1 p),$$

for all $p \in [p_l,p_h]$. Log-concavity of h implies

$$2 - \frac{h(z)\ddot{h}(z)}{\dot{h}(z)^2} = 1 + \frac{\dot{h}(z)^2 - h(z)\ddot{h}(z)}{h(z)^2} \cdot \frac{h(z)^2}{\dot{h}(z)^2} = 1 - \frac{h(z)^2}{\dot{h}(z)^2} \cdot \frac{\partial^2 \log(h(z))}{\partial z^2} > 0,$$

for all $z = \beta_0 + \beta_1 p$, $p \in [p_l, p_h]$. It follows that all $p \in [p_l, p_h]$ with $\dot{f}(p) = 0$ satisfy

$$\begin{aligned}\ddot{f}(p) &= 2\beta_1 \dot{h}(\beta_0 + \beta_1 p) + \frac{-h(\beta_0 + \beta_1 p)}{\beta_1 \dot{h}(\beta_0 + \beta_1 p)} \beta_1^2 \ddot{h}(\beta_0 + \beta_1 p) \\ &= \beta_1 \dot{h}(\beta_0 + \beta_1 p) \Big[2 - \frac{h(\beta_0 + \beta_1 p) \ddot{h}(\beta_0 + \beta_1 p)}{\dot{h}(\beta_0 + \beta_1 p)^2} \Big] < 0.\end{aligned}$$

This implies that either $f_{a,\beta}(p)$ is strictly monotone on $[p_l, p_h]$ (in which case the unique maximum of $f_{a,\beta}(p)$ is on the boundary of $[p_l, p_h]$, or $f_{a,\beta}(p)$ has a unique maximum $p^*_{a,\beta}$ on $[p_l, p_h]$ with $\ddot{f}(p^*_{a,\beta}) < 0$.

(ii) Note that $\mathcal{U}_{AB}$ is an open set, and that, for each $(a, \beta) \in \mathcal{U}_{AB}$, the equation $\dot{f}_{a,\beta}(p) = 0$ has a unique solution $p^*_{a,\beta} \in (p_l, p_h)$ with $\ddot{f}_{a,\beta}(p^*_{a,\beta}) < 0$. By the Implicit Function Theorem (see e.g. Duistermaat and Kolk 2004), $p^*_{a,\beta}$ is continuously differentiable in a and β on $\mathcal{U}_{AB}$, with derivatives given by

$$\left.\frac{\partial p^*_{a,\beta}}{\partial a}\right|_{a,\beta} = \frac{-1}{\ddot{f}_{a,\beta}(p^*_{a,\beta})} \cdot \left.\frac{\partial \dot{f}_{a,\beta}(p)}{\partial a}\right|_{p^*_{a,\beta}}$$

and

$$\left.\frac{\partial p^*_{a,\beta}}{\partial \beta_i}\right|_{a,\beta} = \frac{-1}{\ddot{f}_{a,\beta}(p^*_{a,\beta})} \cdot \left.\frac{\partial \dot{f}_{a,\beta}(p)}{\partial \beta_i}\right|_{p^*_{a,\beta}}, \quad (i = 1, 2).$$

For all $(a, \beta) \in \mathcal{U}_{AB}$ we have $\ddot{f}_{a,\beta}(p^*_{a,\beta}) < 0$,

$$\left.\frac{\partial \dot{f}_{a,\beta}(p)}{\partial a}\right|_{p^*_{a,\beta}} = -\beta_1 \dot{h}(\beta_0 + \beta_1 p^*_{a,\beta}) > 0, \tag{EC.26}$$

$$\begin{aligned}\left.\frac{\partial}{\partial \beta_0} \dot{f}_{a,\beta}(p)\right|_{p^*_{a,\beta}} &= \dot{h}(\beta_0 + \beta_1 p^*_{a,\beta}) + (p^*_{a,\beta} - a)\beta_1 \ddot{h}(\beta_0 + \beta_1 p^*_{a,\beta}) \\ &= \dot{h}(\beta_0 + \beta_1 p^*_{a,\beta}) \left(1 - \frac{h(\beta_0 + \beta_1 p^*_{a,\beta}) \ddot{h}(\beta_0 + \beta_1 p^*_{a,\beta})}{\dot{h}(\beta_0 + \beta_1 p^*_{a,\beta})^2} \right) \geq 0,\end{aligned} \tag{EC.27}$$

and

$$\begin{aligned}\left.\frac{\partial}{\partial \beta_1} \dot{f}_{a,\beta}(p)\right|_{p^*_{a,\beta}} &= p^*_{a,\beta} \dot{h}(\beta_0 + \beta_1 p^*_{a,\beta}) + (p^*_{a,\beta} - a)\dot{h}(\beta_0 + \beta_1 p^*_{a,\beta}) + p^*_{a,\beta}(p^*_{a,\beta} - a)\beta_1 \ddot{h}(\beta_0 + \beta_1 p^*_{a,\beta}) \\ &= p^*_{a,\beta} \left.\frac{\partial}{\partial \beta_0} \dot{f}(p)\right|_{p^*_{a,\beta}} + \frac{1}{-\beta_1} h(\beta_0 + \beta_1 p^*_{a,\beta}) \geq 0,\end{aligned} \tag{EC.28}$$

using $(p^*_{a,\beta} - a)\beta_1 = -h(\beta_0 + \beta_1 p^*_{a,\beta}) / \dot{h}(\beta_0 + \beta_1 p^*_{a,\beta})$ and log-concavity of h. It follows that $p^*_{a,\beta}$ is strictly increasing in a and nondecreasing in β_0 and β_1.

If $(a,\beta)\in\mathbb{R}\times\mathcal{U}_B$ and $p^*_{a,\beta}=p_l$, then $\dot f_{a,\beta}(p_l)\le 0$. By (EC.26), $\frac{\partial}{\partial a}\dot f_{a,\beta}(p_l)\ge 0$, which implies $\dot f_{a',\beta}(p_l)\le 0$ and $p^*_{a',\beta}=p_l$ for all $a'\le a$. By (EC.27) and (EC.28),

$$\begin{aligned}\frac{\partial}{\partial\beta_0}\dot f_{a,\beta}(p_l)&=\dot h(\beta_0+\beta_1p_l)+(p_l-a)\beta_1\ddot h(\beta_0+\beta_1p_l)\\&\ge\frac{\dot h(\beta_0+\beta_1p_l)}{h(\beta_0+\beta_1p_l)}\cdot(\dot f_{a,\beta}(p_l)-h(\beta_0+\beta_1p_l))\cdot\frac{\ddot h(\beta_0+\beta_1p_l)h(\beta_0+\beta_1p_l)}{\dot h(\beta_0+\beta_1p_l)^2}\ge 0,\end{aligned}$$

and

$$\frac{\partial}{\partial\beta_1}\dot f_{a,\beta}(p_l)=p_l\frac{\partial}{\partial\beta_0}\dot f_{a,\beta}(p_l)+\frac{(\dot f_{a,\beta}(p_l)-h(\beta_0+\beta_1p_l))}{\beta_1}\ge 0,$$

using $\dot f_{a,\beta}(p_l)-h(\beta_0+\beta_1p_l)\le 0$ and log-concavity of h. This implies $\dot f_{a,\beta'}(p_l)\le 0$ and thus $p^*_{a,\beta'}=p_l$ for all $\beta'\in B$ with $\beta'\le\beta$ (in both coordinates). By similar arguments we can show that $p^*_{a',\beta'}=p_h$ if $p^*_{a,\beta}=p_h$, $a'\ge a$ and $\beta'\ge\beta$. These observations, combined with the fact that $p^*_{a,\beta}$ is nondecreasing in a, β_0 and β_1 if $p^*_{a,\beta}\in(p_l,p_h)$, show that $p^*_{a,\beta}$ is nondecreasing in a, β_0 and β_1 on $\mathbb{R}\times\mathcal{U}_B$.

The assertions on $f_{a,\beta}(p^*_{a,\beta})$ follow by observing that

$$\left.\frac{\partial f_{a,\beta}(p^*_{a,\beta})}{\partial a}\right|_{a,\beta}=\left.\frac{\partial f_{a,\beta}(p)}{\partial p}\right|_{p^*_{a,\beta}}\cdot\left.\frac{\partial p^*_{a,\beta}}{\partial a}\right|_{a,\beta}+\left.\frac{\partial f_{a,\beta}(p)}{\partial a}\right|_{p^*_{a,\beta}}=0-h(\beta_0+\beta_1p^*_{a,\beta})<0,$$

$$\begin{aligned}\left.\frac{\partial f_{a,\beta}(p^*_{a,\beta})}{\partial\beta_0}\right|_{a,\beta}&=\left.\frac{\partial f_{a,\beta}(p)}{\partial p}\right|_{p^*_{a,\beta}}\cdot\left.\frac{\partial p^*_{a,\beta}}{\partial\beta_0}\right|_{a,\beta}+\left.\frac{\partial f_{a,\beta}(p)}{\partial\beta_0}\right|_{p^*_{a,\beta}}\\&=0+(p^*_{a,\beta}-a)\dot h(\beta_0+\beta_1p^*_{a,\beta})=\frac{-h(\beta_0+\beta_1p^*_{a,\beta})}{\beta_1}>0,\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial f_{a,\beta}(p^*_{a,\beta})}{\partial\beta_1}\right|_{a,\beta}&=\left.\frac{\partial f_{a,\beta}(p)}{\partial p}\right|_{p^*_{a,\beta}}\cdot\left.\frac{\partial p^*_{a,\beta}}{\partial\beta_1}\right|_{a,\beta}+\left.\frac{\partial f_{a,\beta}(p)}{\partial\beta_1}\right|_{p^*_{a,\beta}}\\&=0+(p^*_{a,\beta}-a)p^*_{a,\beta}\dot h(\beta_0+\beta_1p^*_{a,\beta})=\frac{-h(\beta_0+\beta_1p^*_{a,\beta})p^*_{a,\beta}}{\beta_1}>0,\end{aligned}$$

again using $(p^*_{a,\beta}-a)\beta_1=-h(\beta_0+\beta_1p^*_{a,\beta})/\dot h(\beta_0+\beta_1p^*_{a,\beta})$.

(iii) Let $K_0=\sup_{(a,\beta,p)\in\mathcal{U}_{AB}\times[p_l,p_h]}-\ddot f_{a,\beta}(p)/2$. Since $\ddot f_{a,\beta}(p)$ is defined and continuous on the closure of $\mathcal{U}_{AB}\times[p_l,p_h]$ (which is compact) and since $\ddot f_{a,\beta}(p^*_{a,\beta})<0$ for all $(a,\beta)\in\mathcal{U}_{AB}$, it follows that $0<K_0<\infty$. A Taylor expansion then implies

$$f_{a,\beta}(p)\ge f_{a,\beta}(p^*_{a,\beta})-K_0(p-p^*_{a,\beta})^2,$$

using $\dot f_{a,\beta}(p^*_{a,\beta})=0$ for all $(a,\beta)\in\mathcal{U}_{AB}$.

Proof of Lemma EC.2

Throughout the proof, fix $\beta \in B$.

(i) If $s = S$ then $\Delta V_\beta(c,S) = 0$ for $c > 1$ or $c = 0$, and $V_\beta(1,S) = \max_{p\in[p_l,p_h]} ph(\beta_0 + \beta_1 p) \geq 0$. If $s < S$ then by backward induction on s,

$$\begin{aligned}
\Delta V_\beta(c,s) &= (\pi^*_\beta(c,s) - \Delta V_\beta(c,s+1))h(\beta_0 + \beta_1\pi^*_\beta(c,s)) + V_\beta(c,s+1) \\
&\quad - (\pi^*_\beta(c-1,s) - \Delta V_\beta(c-1,s+1))h(\beta_0 + \beta_1\pi^*_\beta(c-1,s)) - V_\beta(c-1,s+1) \\
&\geq (\pi^*_\beta(c-1,s) - \Delta V_\beta(c,s+1))h(\beta_0 + \beta_1\pi^*_\beta(c-1,s)) + V_\beta(c,s+1) \\
&\quad - (\pi^*_\beta(c-1,s) - \Delta V_\beta(c-1,s+1))h(\beta_0 + \beta_1\pi^*_\beta(c-1,s)) - V_\beta(c-1,s+1) \\
&= \Delta V_\beta(c,s+1))(1 - h(\beta_0 + \beta_1\pi^*_\beta(c-1,s))) \\
&\quad + \Delta V_\beta(c-1,s+1))h(\beta_0 + \beta_1\pi^*_\beta(c-1,s)) \geq 0.
\end{aligned}$$

(ii) By backward induction on s it follows that $V(1,s) \in [p_l,p_h]$ for all $1 \leq s \leq S$. This implies

$$\begin{aligned}
V_\beta(1,s) &= h(\beta_0 + \beta_1\pi(1,s))\pi(1,s) + (1 - h(\beta_0 + \beta_1\pi(1,s)))V(1,s+1) \\
&\geq h(\beta_0 + \beta_1 V(1,s+1))V(1,s+1) + (1 - h(\beta_0 + \beta_1 V(1,s+1)))V(1,s+1) \\
&= V(1,s+1),
\end{aligned}$$

for all $1 \leq s < S$.

(iii) The proof is by backward induction on s, and mimics the proof of Proposition 5.2, page 238 of Talluri and van Ryzin (2004). For $s = S$ the assertion follows immediately from $\Delta V_\beta(1,s) \geq 0$ and $\Delta V_\beta(c,s) = 0$ for all $2 \leq c \leq C$. Now assume the assertion is true for $s+1$, and consider stage s, $1 \leq s < S$. For $2 \leq c \leq C$ we have

$$\begin{aligned}
&\Delta V_\beta(c+1,s) - \Delta V_\beta(c,s) \\
=&(\pi^*_\beta(c+1,s) - \Delta V_\beta(c+1,s+1)h(\beta_0 + \beta_1\pi^*_\beta(c+1,s)) + V_\beta(c+1,s+1) \\
&-(\pi^*_\beta(c,s) - \Delta V_\beta(c,s+1)h(\beta_0 + \beta_1\pi^*_\beta(c,s)) - V_\beta(c,s+1) \\
&-(\pi^*_\beta(c,s) - \Delta V_\beta(c,s+1)h(\beta_0 + \beta_1\pi^*_\beta(c,s)) - V_\beta(c,s+1) \\
&+(\pi^*_\beta(c-1,s) - \Delta V_\beta(c-1,s+1)h(\beta_0 + \beta_1\pi^*_\beta(c-1,s)) + V_\beta(c-1,s+1) \\
\leq&(1 - h(\beta_0 + \beta_1\pi^*_\beta(c+1,s)))(\Delta V_\beta(c+1,s+1) - \Delta V_\beta(c,s+1)) \\
&+h(\beta_0 + \beta_1\pi^*_\beta(c-1,s)))(\Delta V_\beta(c,s+1) - \Delta V_\beta(c-1,s+1)) \\
\leq&0,
\end{aligned}$$

using the optimality of $\pi^*_\beta(c,s)$ and the induction hypothesis, and for $c=1$ we have

$$\begin{aligned}&\Delta V_\beta(2,s)-\Delta V_\beta(1,s)\\ \leq&(\pi^*_\beta(2,s)-\Delta V_\beta(2,s+1)h(\beta_0+\beta_1\pi^*_\beta(2,s))+V_\beta(2,s+1)\\ &-(\pi^*_\beta(1,s)-\Delta V_\beta(1,s+1)h(\beta_0+\beta_1\pi^*_\beta(1,s))-V_\beta(1,s+1)\\ \leq&(1-h(\beta_0+\beta_1\pi^*_\beta(2,s)))(\Delta V_\beta(2,s+1)-\Delta V_\beta(1,s+1))\leq 0,\end{aligned}$$

using the induction hypothesis.

(iv) The proof is by backward induction on s.

First observe that for all fixed $p\in[p_l,p_h]$ and $0\leq a\leq p$,

$$\frac{\partial}{\partial\beta_0}\{ph(\beta_0+\beta_1 p)+a(1-h(\beta_0+\beta_1 p))\}=(p-a)\dot{h}(\beta_0+\beta_1 p)\geq 0 \tag{EC.29}$$

and

$$\frac{\partial}{\partial\beta_1}\{ph(\beta_0+\beta_1 p)+a(1-h(\beta_0+\beta_1 p))\}=p(p-a)\dot{h}(\beta_0+\beta_1 p)\geq 0. \tag{EC.30}$$

The optimality of $\pi^*_\beta(1,S)$, together with (EC.29) and (EC.30) applied to $p=\pi^*_\beta(1,S)$ and $a=0$, implies for any $\beta'=(\beta'_0,\beta'_1)\in B$ with $\beta'_0\leq\beta_0$ and $\beta'_1\leq\beta_1$,

$$\begin{aligned}V_\beta(1,S)=&\pi^*_\beta(1,S)h(\beta_0+\beta_1\pi^*_\beta(1,S))\geq\pi^*_{\beta'}(1,S)h(\beta_0+\beta_1\pi^*_{\beta'}(1,S))\\ \geq&\pi^*_{\beta'}(1,S)h(\beta'_0+\beta'_1\pi^*_{\beta'}(1,S))=V_{\beta'}(1,S).\end{aligned}$$

Let $1\leq s<S$, and assume the assertion is true for $s+1,\dots,S$. Since $\Delta V_\beta(1,s+1)=V_\beta(1,s+1)\in(p_l,p_h)$ by (ii), it follows that $\pi^*_\beta(1,s)\geq V_\beta(1,s+1)$. The optimality of $\pi^*_\beta(1,s)$ and the induction hypothesis, together with equations (EC.29) and (EC.30) applied to $p=\pi^*_\beta(1,s)$ and $a=V_\beta(1,s+1)$, imply for any $\beta'=(\beta'_0,\beta'_1)\in B$ with $\beta'_0\leq\beta_0$ and $\beta'_1\leq\beta_1$,

$$\begin{aligned}V_\beta(1,s)=&\pi^*_\beta(1,s)h(\beta_0+\beta_1\pi^*_\beta(1,s))+V_\beta(1,s+1)\cdot(1-h(\beta_u+\beta_u\pi^*_\beta(1,s)))\\ \geq&\pi^*_{\beta'}(1,s)h(\beta_0+\beta_1\pi^*_{\beta'}(1,s))+V_{\beta'}(1,s+1)\cdot(1-h(\beta_0+\beta_1\pi^*_{\beta'}(1,s)))\\ \geq&\pi^*_{\beta'}(1,s)h(\beta'_0+\beta'_1\pi^*_{\beta'}(1,s))+V_{\beta'}(1,s+1)\cdot(1-h(\beta'_0+\beta'_1\pi^*_{\beta'}(1,s)))\\ =&V_{\beta'}(1,s).\end{aligned}$$

(v) For all $1\leq s<S$ and $1\leq c\leq C$, we have by (repeated) application of (i) and (ii),

$$0\leq\Delta V_\beta(c,s+1)\leq\Delta V_\beta(1,s+1)\leq\Delta V_\beta(1,2).$$

Let $p^*_{a,\beta}$ be as in Lemma EC.1. Since

$$\pi^*_\beta(C,S)=p^*_{0,\beta},\quad \pi^*_\beta(c,s)=p^*_{\Delta V_\beta(c,s+1),\beta},\quad \text{and } \pi^*_\beta(1,1)=p^*_{\Delta V_\beta(1,2),\beta},$$

it follows from Lemma EC.1(ii) that $\pi^*_\beta(C,S) \leq \pi^*_\beta(c,s) \leq \pi^*_\beta(1,1)$.

For $s = S$, note that $\pi^*_\beta(C,S) = \pi^*_\beta(c,s) = \pi^*_\beta(1,S) = p^*_{0,\beta} \leq p^*_{\Delta V_\beta(1,2)} = \pi^*_\beta(1,1)$, for all $1 \leq c \leq C$.

(vi) Let $1 \leq c \leq C$ and $1 \leq s \leq S$. Since $\pi^*_\beta(c,s) = p^*_{\Delta V_\beta(c,s+1),\beta}$ and $\pi^*_\beta(c,s) \in [\pi^*_\beta(C,S), \pi^*_\beta(1,1)] \subset (p_l, p_h)$ (Lemma EC.2(v) and equation (4)), we have $(\Delta V_\beta(c,s+1), \beta) \in \mathcal{U}_{AB}$. The continuity assertion then follows from Lemma EC.1(ii).

Proof of Lemma EC.3

For any 2×2 positive definite matrix A with eigenvalues $0 < \lambda_1 \leq \lambda_2$, we have $\lambda_2 \leq \lambda_1 + \lambda_2 = \mathrm{tr}(A)$, $\det(A) = \lambda_1\lambda_2$, and consequentially $\lambda_1 = \det(A)/\lambda_2 \geq \det(A)/\mathrm{tr}(A)$. For $a, b \leq p_h$ we thus have

$$\lambda_{\min}\begin{pmatrix} 2 & a+b \\ a+b & a^2+b^2 \end{pmatrix} \geq \frac{2a^2+2b^2-(a+b)^2}{2+a^2+b^2} \geq \frac{(a-b)^2}{2(1+p_h^2)}.$$

Since $\lambda_{\min}(P_t) \geq \lambda_{\min}(P_r) + \lambda_{\min}(P_{r'})$ for all $r, r', t \in \mathbb{N}$ with $r + r' = t$ (Bhatia 1997, Corollary III.2.2, page 63), we have

$$\begin{aligned}
\lambda_{\min}(P_{kS}) &\geq \lambda_{\min}(P_{(k-1)S}) + \lambda_{\min}\left(\sum_{1 \leq i \leq S, i \notin \{s,s'\}} \begin{pmatrix} 1 \\ p_{i+(k-1)S} \end{pmatrix} (1, p_{i+(k-1)S}) \mathbf{1}_{c_{i+(k-1)S} > 0} \right) \\
&\quad + \lambda_{\min}\left(\begin{pmatrix} 1 \\ p_{s+(k-1)S} \end{pmatrix} (1, p_{s+(k-1)S}) \mathbf{1}_{c_{s+(k-1)S}>0} + \begin{pmatrix} 1 \\ p_{s'+(k-1)S} \end{pmatrix} (1, p_{s'+(k-1)S}) \mathbf{1}_{c_{s'+(k-1)S}>0} \right) \\
&\geq \lambda_{\min}(P_{(k-1)S}) + \frac{\left(p_{s+(k-1)S} - p_{s'+(k-1)S}\right)^2}{2(1+p_h^2)} \\
&\geq \lambda_{\min}(P_{(k-1)S}) + \frac{\delta^2}{2(1+p_h^2)}.
\end{aligned}$$

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