This Appendix provides the proofs of theorems, lemmas and propositions in the main body of the paper, and gives additional supporting results needed for these proofs.

Appendix

Proof of Lemma 1. We first prove the sufficient condition. Let $x \in \mathcal{A}$. For notational convenience, we omit the dependence of $\nu(\mathcal{A})$ on $\mathcal{A}$. For $j = 1, \ldots, n-1$,

$$\begin{aligned}
[x\mathbf{D}]_{\nu_{j+1}} - [x\mathbf{D}]_{\nu_j} &= \sum_{i=1}^{n} x_{(i)} \left(d^{\nu}_{i,j+1} - d^{\nu}_{i,j}\right) \\
&= x_{(n)} \left(d^{\nu}_{n,j+1} - d^{\nu}_{n,j}\right) + \sum_{i=1}^{n-1} x_{(i)} \left(d^{\nu}_{i,j+1} - d^{\nu}_{i,j}\right) \\
&\geq x_{(n-1)} \sum_{i=n-1}^{n} \left(d^{\nu}_{i,j+1} - d^{\nu}_{i,j}\right) + \sum_{i=1}^{n-2} x_{(i)} \left(d^{\nu}_{i,j+1} - d^{\nu}_{i,j}\right) \\
&\geq x_{(n-2)} \sum_{i=n-2}^{n} \left(d^{\nu}_{i,j+1} - d^{\nu}_{i,j}\right) + \sum_{i=1}^{n-3} x_{(i)} \left(d^{\nu}_{i,j+1} - d^{\nu}_{i,j}\right) \\
&\geq \cdots \geq x_{(1)} \sum_{i=1}^{n} \left(d^{\nu}_{i,j+1} - d^{\nu}_{i,j}\right) \geq 0
\end{aligned}$$

where each inequality above follows from the inequality $\sum_{i=k}^{n} d^{\nu}_{i,j} \leq \sum_{i=k}^{n} d^{\nu}_{i,j+1}$ for $k = 1, \ldots, n$. This shows that $x\mathbf{D}$ is similarly ordered to x and $\mathbf{D}$ is order preserving with respect to $\mathcal{A}$.

Next, we prove the necessary condition. Let $m = \min_{i \in \{1,\ldots,n\}}\{z_i\}$. For $k = 1, \ldots, n$, choose $x \in \mathcal{A}$ such that $x_{(n)} = x_{(n-1)} = \cdots = x_{(k)} = m$ and $x_{(k-1)} = x_{(k-2)} = \cdots = x_{(1)} = 0$ if $k > 1$, and $x_{(n)} = x_{(n-1)} = \cdots = x_{(k)} = m$ if $k = 1$. Because $x\mathbf{D}$ is similarly ordered to x, it holds that for $j = 1, \ldots, n-1$,

$$0 \leq [x\mathbf{D}]_{\nu_{j+1}} - [x\mathbf{D}]_{\nu_j} = \sum_{i=1}^{n} x_{(i)} d^{\nu}_{i,j+1} - \sum_{i=1}^{n} x_{(i)} d^{\nu}_{i,j} = \left(\sum_{i=k}^{n} d^{\nu}_{i,j+1} - \sum_{i=k}^{n} d^{\nu}_{i,j}\right) m.$$

Since $m > 0$, the above inequality implies that $\sum_{i=k}^{n} d^{\nu}_{i,j+1} - \sum_{i=k}^{n} d^{\nu}_{i,j} \geq 0$ holds for $j = 1, \ldots, n-1, k = 1, \ldots, n$.

□

Proof of Lemma 2. First, we prove the sufficient condition in the first statement. Let $x, y \in \mathcal{A}$, $x \prec_w y$. For notational convenience, we omit the dependence of $\nu(\mathcal{A})$ on $\mathcal{A}$. For $k = 1, \ldots n$,

$$\begin{aligned}
\sum_{j=k}^{n} [x\mathbf{D}]_{\nu_j} - \sum_{j=k}^{n} [y\mathbf{D}]_{\nu_j} &= \sum_{j=k}^{n} \left(\sum_{i=1}^{n} x_{(i)} d^{\nu}_{i,j} - \sum_{i=1}^{n} y_{(i)} d^{\nu}_{i,j}\right) \\
&= \sum_{i=1}^{n} \left(x_{(i)} - y_{(i)}\right) \sum_{j=k}^{n} d^{\nu}_{1,j} + \sum_{i=2}^{n} \left(x_{(i)} - y_{(i)}\right) \sum_{j=k}^{n} \left(d^{\nu}_{2,j} - d^{\nu}_{1,j}\right) \qquad (6) \\
&\quad + \cdots + \left(x_{(n)} - y_{(n)}\right) \sum_{j=k}^{n} \left(d^{\nu}_{n,j} - d^{\nu}_{n-1,j}\right) \leq 0,
\end{aligned}$$

where the first equality follows from the fact that $\mathbf{D}$ is order preserving w.r.t. $\mathcal{A}$. The last inequality follows from $x \prec_w y$ along with the assumption that $\sum_{j=k}^n d^\nu_{i-1,j} \le \sum_{j=k}^n d^\nu_{i,j}$ for $i = 2, \ldots, n$. Hence, using the definition of weak submajorization, $x\mathbf{D} \prec_w y\mathbf{D}$.

We then prove the necessary condition in the first statement. Choose $x, y \in \mathcal{A}$ such that $0 \le x_{(t)} < y_{(t)}$, $y_{(t)} - x_{(t)} = x_{(t-1)} - y_{(t-1)}$ for some $t \in \{2, \ldots, n\}$ and $x_{(u)} = y_{(u)}$ for $u = 1, \ldots, n, u \notin \{t-1, t\}$. Clearly,

$$\sum_{i=t}^n x_{(i)} < \sum_{i=t}^n y_{(i)}, \quad \sum_{i=u}^n x_{(i)} = \sum_{i=u}^n y_{(i)} \text{ for } u = 1, \ldots, n, u \ne t, \tag{7}$$

and $x \prec_w y$. Because $\mathbf{D}$ is order and weak submajorization preserving w.r.t. $\mathcal{A}$, the inequality in (6) must hold. Using (7) the inequality in (6) may be simplified to

$$\sum_{i=t}^n \left(x_{(i)} - y_{(i)}\right) \sum_{j=k}^n \left(d^\nu_{tj} - d^\nu_{t-1,j}\right) \le 0,$$

for $k = 1, \ldots, n$. Because $\sum_{i=t}^n \left(x_{(i)} - y_{(i)}\right) < 0$, we obtain that $\sum_{j=k}^n \left(d^\nu_{tj} - d^\nu_{t-1,j}\right) \ge 0$ for $k = 1, \ldots, n$. This concludes the proof of the first statement.

The proof for the second statement follows using similar arguments as above. To prove the sufficiency, we need to show that $\sum_{j=1}^k [x\mathbf{D}]_{\nu_j} - \sum_{j=1}^k [y\mathbf{D}]_{\nu_j} \ge 0$ for $k = 1, \ldots, n, x, y \in \mathcal{A}$. This can be done by expanding vector-matrix products and combining terms in a similar way as done for the inequality (6). Next, we show the necessity of the condition. Choose $x, y \in \mathcal{A}$ such that $x_{(t)} > y_{(t)} \ge 0$, $x_{(t)} - y_{(t)} = y_{(t+1)} - x_{(t+1)}$ for some $t \in \{1, \ldots, n-1\}$ and $x_{(u)} = y_{(u)}$ for $u = 1, \ldots, n, u \notin \{t, t+1\}$. Then we obtain

$$\sum_{i=1}^t \left(x_{(i)} - y_{(i)}\right) \sum_{j=1}^k \left(d^\nu_{tj} - d^\nu_{t+1,j}\right) \ge 0,$$

for $k = 1, \ldots, n$. We conclude the proof following similar arguments as in the proof of the first statement.

□

Proof of Lemma 3. For notational convenience, we set $\tilde{\mathbf{\Pi}} := (1+\gamma)\mathbf{\Pi}$ and $\tilde{c} := (1+\gamma)c - \gamma\ell$. As pointed out in Eisenberg and Noe (2001), the clearing payment vector p^* is obtained as the solution to the following optimization problem:

$$\max_x f(x), \quad \text{s.t.} \quad x(\mathbf{I} - \tilde{\mathbf{\Pi}}) \le \tilde{c}, \quad 0 \le x \le \ell,$$

where the objective function f is any real valued increasing function of the vector x. Multiplying both sides of the first constraint by $(1-\alpha)$, with $\alpha \in [0,1)$, will lead to an equivalent optimization problem. But this leads to

$$x[\mathbf{I} - (1-\alpha)\tilde{\mathbf{\Pi}} - \alpha\mathbf{I}] \le (1-\alpha)\tilde{c}.$$

That is, if we replace $\tilde{\mathbf{\Pi}}$ by the matrix $\mathbf{\Pi}_{\alpha,\gamma} := (1-\alpha)\tilde{\mathbf{\Pi}} + \alpha\mathbf{I}$ and $\tilde{c}$ by $c_{\alpha,\gamma} := (1-\alpha)\tilde{c}$, the clearing payment vector stays the same.

□

LEMMA A1. *Let $(\mathbf{\Pi}^a, \ell, c, \gamma)$ and $(\mathbf{\Pi}^b, \ell, c, \gamma)$ be two financial systems. If there exists α so that $\mathbf{\Pi}^a_{\alpha,\gamma} \prec \mathbf{\Pi}^b_{\alpha,\gamma}$ and $\mathbf{\Pi}^a_{\alpha,\gamma} \not\succ \mathbf{\Pi}^b_{\alpha,\gamma}$, then it must hold that $\mathbf{\Pi}^a_{\beta,\gamma} \not\succ \mathbf{\Pi}^b_{\beta,\gamma}$ for all $\beta \in [0,1), \beta \neq \alpha$.*

Proof. Assume the existence of $\mathbf{A} := \mathbf{\Pi}^a_{\alpha,\gamma}, \mathbf{B} := \mathbf{\Pi}^b_{\alpha,\gamma}, \mathbf{C} := \mathbf{\Pi}^a_{\beta,\gamma}, \mathbf{D} := \mathbf{\Pi}^b_{\beta,\gamma}$, $\beta \neq \alpha$, so that $\mathbf{A} \prec \mathbf{B}$ and $\mathbf{A} \not\succ \mathbf{B}$, but $\mathbf{C} \succ \mathbf{D}$. Denote by $\mathbf{X}^k$ the k-th row of the matrix $\mathbf{X}$. Because $\mathbf{A} \prec \mathbf{B}$ and $\mathbf{A} \not\succ \mathbf{B}$, there must exist g and i such that

$$\sum_{j=1}^{g} \mathbf{A}^i_{(j)} > \sum_{j=1}^{g} \mathbf{B}^i_{(j)}. \tag{8}$$

Moreover, since $\mathbf{C} \succ \mathbf{D}$, from the definition of majorization the following inequality must hold:

$$\sum_{j=1}^{k} \mathbf{C}^i_{(j)} \leq \sum_{j=1}^{k} \mathbf{D}^i_{(j)} \text{ for any } k = 1, \ldots, n. \tag{9}$$

Next, we show that Eq. (8) and Eq. (9) cannot hold simultaneously. Let h, m, w, z be such that $\mathbf{A}^i_{(h)} = \mathbf{B}^i_{(m)} = \alpha$ and $\mathbf{C}^i_{(w)} = \mathbf{D}^i_{(z)} = \beta$. We first discuss the implications of Eq. (8) using a case-by-case analysis based on g.

- $g > \max\{h, m\}$. Using the definition of relaxed equivalent version, we obtain

$$\begin{aligned} &\sum_{j=1, j\neq h}^{g} \mathbf{A}^i_{(j)} + \alpha = \sum_{j=1}^{g} \mathbf{A}^i_{(j)} > \sum_{j=1}^{g} \mathbf{B}^i_{(j)} = \sum_{j=1, j\neq m}^{g} \mathbf{B}^i_{(j)} + \alpha \\ \Rightarrow &\sum_{j=1}^{g} (1-\alpha)(1+\gamma)\mathbf{\Pi}^{a,i}_{(j)} > \sum_{j=1}^{g} (1-\alpha)(1+\gamma)\mathbf{\Pi}^{b,i}_{(j)}. \end{aligned} \tag{10}$$

- $h \geq g \geq m$. We obtain

$$\sum_{j=1}^{g-1} \mathbf{A}^i_{(j)} + \underbrace{\alpha}_{\geq \mathbf{A}^i_{(g)}} \geq \sum_{j=1}^{g} \mathbf{A}^i_{(j)} > \sum_{j=1}^{g} \mathbf{B}^i_{(j)} = \sum_{j=1, j\neq m}^{g} \mathbf{B}^i_{(j)} + \alpha$$

leading to the inequality (10).

- $h \leq g \leq m$. Eq. (8) implies that

$$\begin{aligned} &\sum_{j=1}^{m} \mathbf{A}^i_{(j)} = \underbrace{\sum_{j=1, j\neq h}^{g} \mathbf{A}^i_{(j)} + \alpha}_{\sum_{j=1}^{g} \mathbf{A}^i_{(j)}} + \underbrace{\sum_{j=g+1}^{m} \mathbf{A}^i_{(j)}}_{\geq (m-g)\alpha} > \sum_{j=1}^{g} \mathbf{B}^i_{(j)} + \underbrace{\sum_{j=g+1}^{m-1} \mathbf{B}^i_{(j)} + \alpha}_{\leq (m-g)\alpha} = \sum_{j=1}^{m} \mathbf{B}^i_{(j)} \\ \Rightarrow &\sum_{j=1}^{m} (1-\alpha)(1+\gamma)\mathbf{\Pi}^{a,i}_{(j)} > \sum_{j=1}^{m} (1-\alpha)(1+\gamma)\mathbf{\Pi}^{b,i}_{(j)}. \end{aligned} \tag{11}$$

- $g < \min\{h,m\}$. Eq. (8) directly leads to

$$\sum_{j=1}^{g+1}(1-\alpha)(1+\gamma)\mathbf{\Pi}^{a,i}_{(j)} > \sum_{j=1}^{g+1}(1-\alpha)(1+\gamma)\mathbf{\Pi}^{b,i}_{(j)}, \quad g < \min\{h,m\}. \tag{12}$$

Next, we discuss the implications of Eq. (9) and show that it leads to

$$\sum_{j=1}^{k}(1-\beta)(1+\gamma)\mathbf{\Pi}^{a,i}_{(j)} \le \sum_{j=1}^{k}(1-\beta)(1+\gamma)\mathbf{\Pi}^{b,i}_{(j)}, \qquad k = 1,\dots,n. \tag{13}$$

This is done via a case-by-case analysis based on k.

- $k > \max\{w,z\}$. We obtain

$$\sum_{j=1, j\neq w}^{k} \mathbf{C}^i_{(j)} + \beta = \sum_{j=1}^{k}\mathbf{C}^i_{(j)} \le \sum_{j=1}^{k}\mathbf{D}^i_{(j)} = \sum_{j=1, j\neq z}^{k}\mathbf{D}^i_{(j)} + \beta$$

hence implying the inequality (13).

- $k < \min\{w,z\}$. Eq. (9) directly leads to inequality (13).
- $w \ge k \ge z$. Eq. (9) implies the following inequality

$$\sum_{j=1}^{z-1}\mathbf{C}^i_{(j)} + \underbrace{\sum_{j=z}^{k-1}\mathbf{C}^i_{(j)}}_{\le (k-z)\beta} + \beta \le \sum_{j=1}^{z-1}\mathbf{D}^i_{(j)} + \beta + \underbrace{\sum_{j=z+1}^{k}\mathbf{D}^i_{(j)}}_{\ge (k-z)\beta},$$

which further leads to the inequality (13).

- $w \le k \le z$. We obtain

$$\sum_{j=1, j\neq w}^{k}\mathbf{C}^i_{(j)} + \beta = \sum_{j=1}^{k}\mathbf{C}^i_{(j)} \le \sum_{j=1}^{k}\mathbf{D}^i_{(j)} \le \sum_{j=1}^{k-1}\mathbf{D}^i_{(j)} + \underbrace{\beta}_{\ge \mathbf{D}^i_{(k)}},$$

which again leads to the inequality (13).

Setting $k = g$ in Eq. (13) shows that Eq. (10) and Eq. (13) cannot hold simultaneously. Setting $k = m$, we obtain that Eq. (11) and Eq. (13) cannot hold simultaneously. Setting $k = g+1$, we obtain that Eq. (12) and Eq. (13) cannot hold simultaneously. This ends the proof.

□

Proof of Lemma 4. We prove the first statement by showing that $\underline{\ell}\mathbf{\Pi} + c \ge \underline{\ell}$. The asset value of the node with the smallest liability is given by

$$\begin{aligned}\sum_{i=1}^{n}\underline{\ell}_{(i)}\pi^{\mu}_{i,1} + c_{(1)} \ge c_{(1)} &\ge \mathbb{1}_{c_{(1)}<\ell_{(1)}}\left[c_{(1)} + \gamma\left(c_{(1)} - \ell_{(1)}\right)\right] + \mathbb{1}_{c_{(1)}\ge\ell_{(1)}}\ell_{(1)} \\ &\ge \mathbb{1}_{c_{(1)}<\ell_{(1)}}\left[\ell_{(1)} - \max\left\{\ell_{(1)} - [(1+\gamma)c - \gamma\ell]_{(1)}, 0\right\}\right] + \mathbb{1}_{c_{(1)}\ge\ell_{(1)}}\ell_{(1)} \\ &\ge \mathbb{1}_{c_{(1)}<\ell_{(1)}}\underline{\ell}_{(1)} + \mathbb{1}_{c_{(1)}\ge\ell_{(1)}}\underline{\ell}_{(1)} \ge \underline{\ell}_{(1)},\end{aligned}$$

where $\mathbb{1}_A$ denotes the indicator function of the event A. Because $(\mathbf{\Pi}, \ell, c, \gamma)$ is unbalancing, by definition it satisfies the inequalities (5). Combining those with the above inequality leads to

$$\sum_{i=1}^{n} \underline{\ell}_{(i)} \pi^{\mu}_{i,n} + c_{(n)} - \underline{\ell}_{(n)} \geq \cdots \geq \sum_{i=1}^{n} \underline{\ell}_{(i)} \pi^{\mu}_{i,2} + c_{(2)} - \underline{\ell}_{(2)} \geq \sum_{i=1}^{n} \underline{\ell}_{(i)} \pi^{\mu}_{i,1} + c_{(1)} - \underline{\ell}_{(1)} \geq 0,$$

which proves the first statement.

Given ϵ, we can choose the entries of $\mathbf{\Pi}$ to be small enough so that $\sum_{i=1}^{n}(\underline{\ell}_{(i)} + \epsilon_{\mu_i})\pi^{\mu}_{i,1} + c_{(1)} < \underline{\ell}_{(1)} + \epsilon_{\mu_1}$ and $c_{(j+1)} - c_{(j)} \geq \underline{\ell}_{(j+1)} - \underline{\ell}_{(j)} + \max_{k=1,\dots,n}\left\{\sum_{l=1}^{n} \underline{\ell}_{(l)} \pi^{\mu}_{lk}\right\}$ for $j = 1, \dots, n-1$ (recall here that μ has been defined above Definition 8). Such a system $(\mathbf{\Pi}, \ell, c, \gamma)$ satisfies the inequalities (5), but $(\underline{\ell} + \epsilon)\mathbf{\Pi} + c \not\geq \underline{\ell} + \epsilon$. Hence, this system is unbalancing but $p^*(\mathbf{\Pi}, \underline{\ell} + \epsilon, c, \gamma) \not\geq \underline{\ell} + \epsilon$.

□

For any vector $x \in \mathbb{R}^n$, define

$$\Delta x := \left(0,\, x_{(2)} - x_{(1)},\, x_{(3)} - x_{(2)},\, \dots\, x_{(n)} - x_{(n-1)}\right),$$

i.e., the vector whose components are the increments from one component to the next rank ordered component in the original vector x.

Lemma A2. *Let $x, y \in \mathbb{R}^n_{\geq 0}$ such that x and y are similarly ordered. If $\Delta x \leq \Delta y$, then $\Delta(x \wedge y) \leq \Delta y$. Vice versa, if $\Delta x \geq \Delta y$, then $\Delta(x \wedge y) \geq \Delta y$.*

Proof. We first prove that $\Delta x \leq \Delta y \Rightarrow \Delta(x \wedge y) \leq \Delta y$. For $i = 1, \dots, n-1$,

$$\begin{aligned}
\Delta(x \wedge y)_{i+1} &= (x \wedge y)_{(i+1)} - (x \wedge y)_{(i)} \\
&= \min\left\{x_{(i+1)}, y_{(i+1)}\right\} - \min\left\{x_{(i)}, y_{(i)}\right\} \\
&= \begin{cases} x_{(i+1)} - x_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \leq y_{(i)} \\ x_{(i+1)} - y_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \geq y_{(i)} \\ y_{(i+1)} - y_{(i)} & \text{if } x_{(i+1)} \geq y_{(i+1)} \text{ and } x_{(i)} \geq y_{(i)} \end{cases} \\
&\leq y_{(i+1)} - y_{(i)} = \Delta y_{i+1},
\end{aligned}$$

where we have used the assumption that x is similarly ordered to y. Notice that the case $x_{(i+1)} \geq y_{(i+1)}$ and $x_{(i)} \leq y_{(i)}$ is not listed because it violates the assumption that $\Delta x \leq \Delta y$. Hence, $\Delta(x \wedge y) \leq \Delta y$.

We next show that $\Delta x \geq \Delta y \Rightarrow \Delta(x \wedge y) \geq \Delta y$. This holds because

$$\begin{aligned}
\Delta(x \wedge y)_{i+1} &= (x \wedge y)_{(i+1)} - (x \wedge y)_{(i)} \\
&= \min\left\{x_{(i+1)}, y_{(i+1)}\right\} - \min\left\{x_{(i)}, y_{(i)}\right\} \\
&= \begin{cases} x_{(i+1)} - x_{(i)} & \text{if } x_{(i+1)} \leq y_{(i+1)} \text{ and } x_{(i)} \leq y_{(i)} \\ y_{(i+1)} - x_{(i)} & \text{if } x_{(i+1)} \geq y_{(i+1)} \text{ and } x_{(i)} \leq y_{(i)} \\ y_{(i+1)} - y_{(i)} & \text{if } x_{(i+1)} \geq y_{(i+1)} \text{ and } x_{(i)} \geq y_{(i)} \end{cases} \\
&\geq y_{(i+1)} - y_{(i)} = \Delta y_{i+1},
\end{aligned}$$

where the second equality follows from the assumption that x is similarly ordered to y. The third equality does not include the case $x_{(i+1)} \le y_{(i+1)}$ and $x_{(i)} \ge y_{(i)}$ because such a case violates the assumption that $\Delta x \ge \Delta y$.

□

The next lemma shows that if a payment vector has smaller (larger) variation than the liability vector, such a relation is preserved if both vectors are multiplied by an order preserving relaxed equivalent version of the relative liability matrix.

LEMMA A3. *If $\mathbf{\Pi}_{\alpha,\gamma} := (\pi_{\alpha,\gamma,i,j})_{i,j=1}^n$ is order preserving w.r.t. $\mathcal{P}$, for any $p \in \mathcal{P}$ it must hold that*

(I) $\Delta p \le \Delta \ell$ implies that $\Delta(p\mathbf{\Pi}_{\alpha,\gamma}) \le \Delta(\ell \mathbf{\Pi}_{\alpha,\gamma})$.

(II) $\Delta p \ge \Delta \ell$ implies that $\Delta(p\mathbf{\Pi}_{\alpha,\gamma}) \ge \Delta\left(\left[\ell - \left(\ell_{(1)} - p_{(1)}\right)\right]\mathbf{\Pi}_{\alpha,\gamma}\right)$.

Proof. For $j = 1, \ldots, n-1, p \in \mathcal{P}$,

$$
\begin{aligned}
&[\Delta(\ell\mathbf{\Pi}_{\alpha,\gamma})]_{j+1} - [\Delta(p\mathbf{\Pi}_{\alpha,\gamma})]_{j+1} = \sum_{i=1}^n (\ell_{(i)} - p_{(i)})\left(\pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j}\right) \\
&= \left(\ell_{(1)} - p_{(1)}\right)\left(\pi^\mu_{\alpha,\gamma,1,j+1} - \pi^\mu_{\alpha,\gamma,1,j}\right) + \left[\left(\ell_{(1)} + \Delta\ell_2\right) - \left(p_{(1)} + \Delta p_2\right)\right]\left(\pi^\mu_{\alpha,\gamma,2,j+1} - \pi^\mu_{\alpha,\gamma,2,j}\right) + \ldots \\
&\quad + \left[\left(\ell_{(1)} + \sum_{k=2}^n \Delta\ell_k\right) - \left(p_{(1)} + \sum_{k=2}^n \Delta p_k\right)\right]\left(\pi^\mu_{\alpha,\gamma,n,j+1} - \pi^\mu_{\alpha,\gamma,n,j}\right) \\
&= \left(\ell_{(1)} - p_{(1)}\right)\left(\sum_{i=1}^n \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j}\right) + (\Delta\ell_2 - \Delta p_2)\left(\sum_{i=2}^n \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j}\right) + \ldots \\
&\quad + (\Delta\ell_n - \Delta p_n)\left(\pi^\mu_{\alpha,\gamma,n,j+1} - \pi^\mu_{\alpha,\gamma,n,j}\right).
\end{aligned}
$$

Applying Lemma 1 with $\mathcal{A} = \mathcal{P}$ we deduce that $\sum_{i=k}^n \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \ge 0$ for $k = 1, \ldots, n$. If $\Delta p \le \Delta\ell$, then $[\Delta(\ell\mathbf{\Pi}_{\alpha,\gamma})]_{i+1} - [\Delta(p\mathbf{\Pi}_{\alpha,\gamma})]_{i+1} \ge 0$ because $\ell \ge p$ and $\sum_{i=k}^n \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \ge 0$ for $k = 1, \ldots, n$. This proves (I). Vice versa, if $\Delta p \ge \Delta\ell$ then

$$
[\Delta(\ell\mathbf{\Pi}_{\alpha,\gamma})]_{j+1} - [\Delta(p\mathbf{\Pi}_{\alpha,\gamma})]_{j+1} - \left(\ell_{(1)} - p_{(1)}\right)\left(\sum_{i=1}^n \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j}\right) \le 0
$$

because $\sum_{i=k}^n \pi^\mu_{\alpha,\gamma,i,j+1} - \pi^\mu_{\alpha,\gamma,i,j} \ge 0$ for $k = 1, \ldots, n$. This proves (II).

□

The next lemma shows that if the financial system is balancing (unbalancing), the vector of asset values before clearing under the base (reduced) liability configuration has smaller (larger) variation than the liability vector. Moreover, if the payment vector has smaller (larger) variation than the liability vector, then the assets after payments are settled also have smaller (larger) variation than the liabilities.

LEMMA A4. *Let $(\mathbf{\Pi}, \ell, c, \gamma)$ be a financial system and $\mathbf{\Pi}_{\alpha,\gamma}$ be an α-relaxed equivalent version which is order preserving w.r.t. $\mathcal{P}$.*

(I) If $(\mathbf{\Pi},\ell,c,\gamma)$ is balancing, then

- $\Delta(\ell\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\leq\Delta\ell$.
- $\Delta p\leq\Delta\ell$ *implies that* $\Delta(p\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\leq\Delta\ell$ *for* $p\in\mathcal{P}$.

(II) If $(\mathbf{\Pi},\ell,c,\gamma)$ is unbalancing, then

- $\Delta\left(\underline{\ell}\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma}\right)\geq\Delta\ell$.
- $\Delta p\geq\Delta\ell$ *implies that* $\Delta(p\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\geq\Delta\ell$ *for* $p\in\mathcal{P}$ *such that* $p\geq[(1+\gamma)c-\gamma\ell]\wedge\ell$.

Proof. (I)

$$\begin{aligned}
[\Delta(\ell\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})]_j &= [\Delta\left(\alpha\ell+(1-\alpha)\left[\ell(1+\gamma)\mathbf{\Pi}+(1+\gamma)c-\gamma\ell\right]\right)]_j\\
&=\alpha(\ell_{(j)}-\ell_{(j-1)})\\
&\quad+(1-\alpha)\left[\sum_{i=1}^{n}\ell_{(i)}(1+\gamma)\left(\pi^{\mu}_{i,j}-\pi^{\mu}_{i,j-1}\right)+(1+\gamma)\left(c_{(j)}-c_{(j-1)}\right)-\gamma\left(\ell_{(j)}-\ell_{(j-1)}\right)\right]\\
&\leq\alpha(\ell_{(j)}-\ell_{(j-1)})+(1-\alpha)\left[(1+\gamma)(\ell_{(j)}-\ell_{(j-1)})-\gamma(\ell_{(j)}-\ell_{(j-1)})\right]=[\Delta\ell]_j\,,
\end{aligned}$$

for $j=2,\dots,n$. The second equality holds because $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving w.r.t. $\mathcal{P}$ and $(1+\gamma)c-\gamma\ell$ is similarly ordered to ℓ. The inequality follows from the fact that the system is balancing.

Next, by Lemma A3 (I), $\Delta p\leq\Delta\ell$ implies that

$$\Delta(p\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\leq\Delta(\ell\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\leq\Delta\ell,$$

where the first inequality holds because $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving w.r.t. $\mathcal{P}$, and $c_{\alpha,\gamma}$ and p are similarly ordered to ℓ.

(II)

$$\begin{aligned}
[\Delta(\underline{\ell}\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})]_j &= [\Delta\left(\alpha\underline{\ell}+(1-\alpha)\left[\underline{\ell}(1+\gamma)\mathbf{\Pi}+(1+\gamma)c-\gamma\ell\right]\right)]_j\\
&=\alpha(\ell_{(j)}-\ell_{(j-1)})\\
&\quad+(1-\alpha)\left[\sum_{i=1}^{n}\underline{\ell}_{(i)}(1+\gamma)\left(\pi^{\mu}_{i,j}-\pi^{\mu}_{i,j-1}\right)+(1+\gamma)\left(c_{(j)}-c_{(j-1)}\right)-\gamma\left(\ell_{(j)}-\ell_{(j-1)}\right)\right]\\
&\geq\alpha\left(\ell_{(j)}-\ell_{(j-1)}\right)+(1-\alpha)\left[(1+\gamma)\left(\underline{\ell}_{(j)}-\underline{\ell}_{(j-1)}\right)-\gamma\left(\ell_{(j)}-\ell_{(j-1)}\right)\right]\\
&=\alpha\left(\ell_{(j)}-\ell_{(j-1)}\right)+(1-\alpha)\left[(1+\gamma)\left(\ell_{(j)}-\ell_{(j-1)}\right)-\gamma\left(\ell_{(j)}-\ell_{(j-1)}\right)\right]=[\Delta\ell]_j\,,
\end{aligned}$$

for $j=2,\dots,n$. The second equality follows from the fact that $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving w.r.t. $\mathcal{P}$, the fact that both $(1+\gamma)c-\gamma\ell$ and $\underline{\ell}$ are similarly ordered to ℓ, and the equality $\Delta\underline{\ell}=\Delta\ell$. The inequality comes from the fact that the system is unbalancing.

Next, by Lemma A3 (II), $\Delta p\geq\Delta\ell$ implies

$$\Delta(p\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\geq\Delta\left((\ell-(\ell_{(1)}-p_{(1)}))\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\right)\geq\Delta(\underline{\ell}\mathbf{\Pi}_{\alpha,\gamma}+c_{\alpha,\gamma})\geq\Delta\ell,$$

where the first inequality follows because $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving w.r.t. $\mathcal{P}$, and $c_{\alpha,\gamma}$, p, and $\ell - (\ell_{(1)} - p_{(1)})$ are similarly ordered to ℓ. The second inequality follows from the inequality

$$\ell_{(1)} - p_{(1)} \leq \ell_{(1)} - [[(1+\gamma)c - \gamma\ell] \wedge \ell]_{(1)} = \max\left\{\ell_{(1)} - [(1+\gamma)c - \gamma\ell]_{(1)}, 0\right\},$$

the inequality $\Delta((\ell - (\ell_{(1)} - p_{(1)}))\mathbf{\Pi}_{\alpha,\gamma}) \geq \Delta(\underline{\ell}\mathbf{\Pi}_{\alpha,\gamma})$ which is implied by Lemma 1 (choosing $\mathbf{D} = \mathbf{\Pi}_{\alpha,\gamma}$ and $\mathcal{A} = \mathcal{P}$ therein), and using the assumption that $p \geq [(1+\gamma)c - \gamma\ell] \wedge \ell$.

□

The following lemma gives some useful properties of the vector sequence converging to the clearing payment vector.

LEMMA A5. *Let* $(\mathbf{\Pi}, \ell, c, \gamma)$ *be a financial system. Suppose there exists* $\alpha \in [0,1)$ *such that* $\mathbf{\Pi}_{\alpha,\gamma}$ *is order preserving w.r.t. to* $\mathcal{P}$. *Define the vector valued function* $F(p; \mathbf{\Pi}_{\alpha,\gamma}, \ell, c_{\alpha,\gamma}) := \ell \wedge (p\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma})$, *and a sequence of vectors* $\{f_u\}_{u=0}^{\infty}$ *given by* $f_u := F(f_{u-1})$ *and* $f_0 := \ell$. *The following statements hold:*

(I) f_u is similarly ordered to ℓ and $\{f_u\}_{u=0}^{\infty}$ decreasingly converges to $p^*(\mathbf{\Pi}, \ell, c, \gamma)$.

(II) If $(\mathbf{\Pi}, \ell, c, \gamma)$ is balancing, then $\Delta f_u \leq \Delta\ell$ and $\Delta(f_u\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta\ell$.

(III) If $(\mathbf{\Pi}, \ell, c, \gamma)$ is unbalancing, then $\Delta f_u \geq \Delta\ell$ and $\Delta(f_u\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta\ell$.

Proof. (I) It has been proven in Eisenberg and Noe (2001), Lemma 5, that f_u decreasingly converges to $p^*(\mathbf{\Pi}, \ell, c, \gamma)$. The statement that f_u is similarly ordered to ℓ follows from the fact that $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving and from the assumption that $c_{\alpha,\gamma}$ is similarly ordered to ℓ.

(II) We prove that $\Delta f_u \leq \Delta\ell$ and $\Delta(f_u\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta\ell$ by induction. For $u = 0$, we know that $f_0 = \ell$. Clearly, $\Delta f_0 \leq \Delta\ell$. From the assumptions that $(\mathbf{\Pi}, \ell, c, \gamma)$ is balancing and $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving w.r.t. $\mathcal{P}$, it follows that $\Delta(\ell\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta\ell$ from Lemma A4 (I). Hence, $\Delta(f_0\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta\ell$. Next, we prove the statement for $u+1$ assuming that it holds for u. Since $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving w.r.t. $\mathcal{P}$, $f_u \in \mathcal{P}$, and $c_{\alpha,\gamma}$ is similarly ordered to ℓ, it follows that $f_u\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}$ is similarly ordered to ℓ. Hence $f_{u+1} = \ell \wedge (f_u\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma})$ is similarly ordered to ℓ. Using the induction hypothesis that $\Delta(f_u\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta\ell$ and Lemma A2, we obtain

$$\Delta f_{u+1} = \Delta\left[\ell \wedge (f_u\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma})\right] \leq \Delta\ell.$$

Since $(\mathbf{\Pi}, \ell, c, \gamma)$ is balancing, $\mathbf{\Pi}_{\alpha,\gamma}$ is order preserving w.r.t. $\mathcal{P}$, and $f_{u+1} \in \mathcal{P}$, we can apply Lemma A4 (I) and deduce that $\Delta(f_{u+1}\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}) \leq \Delta\ell$. This concludes the induction.

(III) Using the fact that $\{f_u\}$ is a decreasing sequence converging to p^* and $p^* \geq [(1+\gamma)c - \gamma\ell] \wedge \ell$, we deduce that $f_u \geq [(1+\gamma)c - \gamma\ell] \wedge \ell$. Then, applying Lemma A4 (II), we can use similar arguments as in (II) to conclude the proof.

□

Proof of Proposition 1. Letting $u \to \infty$ in Lemma A5 (I) leads to p^* being similarly ordered to ℓ, hence yielding (I). Moreover, we obtain $\Delta p^* \le \Delta\ell$ for balancing financial systems and $\Delta p^* \ge \Delta\ell$ for unbalancing financial systems from (II) and (III) in Lemma A5. It then follows that

$$\ell_{[i]} - p^*_{[i]} \ge \ell_{[i]} - p^*_{[i]} - ([\Delta\ell]_{n-i+1} - [\Delta p^*]_{n-i+1}) = \ell_{[i+1]} - p^*_{[i+1]}, \qquad i = 1,\dots,n-1$$

for balancing financial systems, and

$$\ell_{(i)} - p^*_{(i)} \ge \ell_{(i)} - p^*_{(i)} + ([\Delta\ell]_{i+1} - [\Delta p^*]_{i+1}) = \ell_{(i+1)} - p^*_{(i+1)}, \qquad i = 1,\dots,n-1$$

for unbalancing financial systems.

□

The following lemma provides sufficient conditions under which the minimum operation preserves the weak majorization relation. This is needed in the following proofs, given that the clearing payment is given by the minimum between two vectors.

LEMMA A6. *Let $x, y, z \in \mathbb{R}^n_{\ge 0}$ such that x and y are similarly ordered to z.*

(I) If $z_{[i]} \le a_{[i]}$ implies $z_{[k]} \le a_{[k]}$ for $k > i$, $a \in \{x, y\}$, then $x \prec_w y$ implies $(x \wedge z) \prec_w (y \wedge z)$.

(II) If $z_{(i)} \le a_{(i)}$ implies $z_{(k)} \le a_{(k)}$ for $k > i$, $a \in \{x, y\}$, then $x \prec^w y$ implies $(x \wedge z) \prec^w (y \wedge z)$.

Proof. (I) Because x and y are similarly ordered to z, clearly, $(x \wedge z)$ and $(y \wedge z)$ are similarly ordered to z. Hence, proving $(x \wedge z) \prec_w (y \wedge z)$ is equivalent to show that

$$\sum_{i=1}^{k} \min\left\{x_{[i]}, z_{[i]}\right\} \le \sum_{i=1}^{k} \min\left\{y_{[i]}, z_{[i]}\right\} \text{ for } k = 1,\dots,n. \tag{14}$$

Let $m_x = \min\{i = 1,\dots,n | z_{[i]} \le x_{[i]}\}$ and $m_y = \min\{i = 1,\dots,n | z_{[i]} \le y_{[i]}\}$. It must hold that for $k = 1,\dots,m_y - 1$,

$$\sum_{i=1}^{k} \min\left\{x_{[i]}, z_{[i]}\right\} \le \sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} = \sum_{i=1}^{k} \min\left\{y_{[i]}, z_{[i]}\right\}, \tag{15}$$

where the second inequality follows from $x \prec_w y$. Moreover, for $k = m_y,\dots,n$,

$$\begin{aligned} &\sum_{i=1}^{k} \min\left\{x_{[i]}, z_{[i]}\right\} \\ &\quad = \mathbb{1}_{m_x < m_y} \left(\sum_{i=1}^{m_y-1} \min\left\{x_{[i]}, z_{[i]}\right\} + \sum_{i=m_y}^{k} z_{[i]} \right) + \mathbb{1}_{m_x \ge m_y} \left(\sum_{i=1}^{m_y-1} x_{[i]} + \sum_{i=m_y}^{k} \min\left\{x_{[i]}, z_{[i]}\right\} \right) \\ &\quad \le \sum_{i=1}^{m_y-1} y_{[i]} + \sum_{i=m_y}^{k} z_{[i]} = \sum_{i=1}^{k} \min\left\{y_{[i]}, z_{[i]}\right\}, \end{aligned} \tag{16}$$

where $\mathbb{1}_A$ denotes the indicator function of the event A. The above inequality follows from $x \prec_w y$. Using inequalities (15) and (16), we obtain the inequality in (14).

(II) Because x and y are similarly ordered to z, clearly, $(x \wedge z)$ and $(y \wedge z)$ are similarly ordered to z. Hence, proving $(x \wedge z) \prec^w (y \wedge z)$ is equivalent to show

$$\sum_{i=1}^{k} \min\left\{x_{(i)}, z_{(i)}\right\} \geq \sum_{i=1}^{k} \min\left\{y_{(i)}, z_{(i)}\right\} \text{ for } k = 1, \dots, n. \tag{17}$$

Let $m_x = \min\{i = 1, \dots, n | z_{(i)} \leq x_{(i)}\}$ and $m_y = \min\{i = 1, \dots, n | z_{(i)} \leq y_{(i)}\}$. For $k = 1, \dots, m_x - 1$, we must have

$$\sum_{i=1}^{k} \min\left\{x_{(i)}, z_{(i)}\right\} = \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \geq \sum_{i=1}^{k} \min\left\{y_{(i)}, z_{(i)}\right\}, \tag{18}$$

where the first inequality follows from $x \prec^w y$. Moreover, for $k = m_x, \dots, n$,

$$\begin{aligned}
\sum_{i=1}^{k} \min\left\{x_{(i)}, z_{(i)}\right\} &= \sum_{i=1}^{m_x - 1} x_{(i)} + \sum_{i=m_x}^{k} z_{(i)} \\
&\geq \mathbb{1}_{m_x < m_y} \left(\sum_{i=1}^{m_x - 1} y_{(i)} + \sum_{i=m_x}^{k} \min\left\{y_{(i)}, z_{(i)}\right\} \right) + \mathbb{1}_{m_x \geq m_y} \left(\sum_{i=1}^{m_x - 1} \min\left\{y_{(i)}, z_{(i)}\right\} + \sum_{i=m_x}^{k} z_{(i)} \right) \\
&= \sum_{i=1}^{k} \min\left\{y_{(i)}, z_{(i)}\right\},
\end{aligned} \tag{19}$$

where the inequality follows from $x \prec^w y$. Using inequalities (18) and (19), we obtain the inequality (17).

□

Proof of Theorem 1. Recall from Lemma A5 that a sequence of vectors $\{f_u(\mathbf{\Pi}_{\alpha,\gamma})\}_{u=0}^{\infty}$, where

$$F(p, \mathbf{\Pi}_{\alpha,\gamma}; \ell, c_{\alpha,\gamma}) := \ell \wedge (p\mathbf{\Pi}_{\alpha,\gamma} + c_{\alpha,\gamma}), \quad f_u(\mathbf{\Pi}_{\alpha,\gamma}) := F\left(f_{u-1}(\mathbf{\Pi}_{\alpha,\gamma}), \mathbf{\Pi}_{\alpha,\gamma}\right), \quad \text{and } f_0(\mathbf{\Pi}_{\alpha,\gamma}) := \ell,$$

converges to the clearing payment vector p^*. Hence, proving that $p^*(\mathbf{\Pi}^a_{\alpha,\gamma}, \ell, c_{\alpha,\gamma}) \prec_w p^*(\mathbf{\Pi}^b_{\alpha,\gamma}, \ell, c_{\alpha,\gamma})$ is equivalent to showing that $f_u(\mathbf{\Pi}^a_{\alpha,\gamma}) \prec_w f_u(\mathbf{\Pi}^b_{\alpha,\gamma})$ for $u = 1, 2, \dots$. For brevity, we denote hereafter $f_u(\mathbf{\Pi}^z_{\alpha,\gamma})$ by f^z_u and $p^*(\mathbf{\Pi}^z_{\alpha,\gamma}, \ell, c_{\alpha,\gamma})$ by p^{z*}.

Next, we prove that $f^a_u \prec_w f^b_u$ by induction. Without loss of generality, we take $\mathbf{\Pi}^a_{\alpha,\gamma}$ to be weak submajorization preserving w.r.t. $\mathcal{P}$. (If it were the case that $\mathbf{\Pi}^b_{\alpha,\gamma}$ is weak submajorization preserving w.r.t. $\mathcal{P}$, we would obtain the same result and the proof would proceed in a symmetric fashion by interchanging the roles of a and b.) For $u = 0$, we have $f^a_0 = \ell \prec_w \ell = f^b_0$. Assume $f^a_u \prec_w f^b_u$. Then we want to prove the statement for $u + 1$. First, notice that we have

$$\left(f^a_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma}\right) \prec_w \left(f^b_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma}\right) \prec_w \left(f^b_u \mathbf{\Pi}^b_{\alpha,\gamma} + c_{\alpha,\gamma}\right).$$

The first inequality follows from the assumption that $\mathbf{\Pi}^a_{\alpha,\gamma}$ is order and weak submajorization preserving, the fact that f^a_u, f^b_u are similarly ordered to ℓ by Lemma A5, and Assumption 2 that $c_{\alpha,\gamma}$ is similarly ordered to ℓ. The second inequality follows from the majorization inequality $\mathbf{\Pi}^a_{\alpha,\gamma} \prec \mathbf{\Pi}^b_{\alpha,\gamma}$, along with the fact that $\mathbf{\Pi}^a_{\alpha,\gamma}$ and $\mathbf{\Pi}^b_{\alpha,\gamma}$ are both order preserving w.r.t. $\mathcal{P}$. For $z \in \{a,b\}$, since $\Delta(f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma}) \le \Delta\ell$ (by Lemma A5 (II)), $\ell_{[i]} \le (f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma})_{[i]}$ we must have that $\ell_{[k]} \le (f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma})_{[k]}$ for $k > i$. Moreover, $f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma}$ is similarly ordered to ℓ. Using Lemma A6 (I), we deduce

$$f^a_{u+1} = \left[\ell \wedge (f^a_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma})\right] \prec_w \left[\ell \wedge (f^b_u \mathbf{\Pi}^b_{\alpha,\gamma} + c_{\alpha,\gamma})\right] = f^b_{u+1}$$

by taking $x = f^a_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma},\ y = f^b_u \mathbf{\Pi}^b_{\alpha,\gamma} + c_{\alpha,\gamma}$, and $z = \ell$. This concludes the proof that $f^a_u \prec_w f^b_u$ for $u = 1, 2, \ldots$.

By definition of weak submajorization, this means that

$$\sum_{i=1}^k f^b_{u,[i]} - \sum_{i=1}^k f^a_{u,[i]} \ge 0 \text{ for } k = 1, \ldots, n.$$

Letting $u \to \infty$, the above inequality leads to

$$\sum_{i=1}^k p^{b*}_{[i]} - \sum_{i=1}^k p^{a*}_{[i]} \ge 0 \text{ for } k = 1, \ldots, n, \quad \text{hence,} \quad p^{a*} \prec_w p^{b*}.$$

Together with the above inequalities, Proposition 1 (II) implies that

$$\sum_{i=1}^k [\ell - p^{a*}]_{[i]} = \sum_{i=1}^k \ell_{[i]} - p^{a*}_{[i]} \ge \sum_{i=1}^k \ell_{[i]} - p^{b*}_{[i]} = \sum_{i=1}^k [\ell - p^{b*}]_{[i]}$$

for $k = 1, \ldots, n$, or equivalently $s(\mathbf{\Pi}^a, \ell, c, \gamma) \succ_w s(\mathbf{\Pi}^b, \ell, c, \gamma)$.

□

Proof of Theorem 2. Similarly to the proof for Theorem 1, proving that $p^*(\mathbf{\Pi}^a_{\alpha,\gamma}, \ell, c_{\alpha,\gamma}) \prec^w p^*(\mathbf{\Pi}^b_{\alpha,\gamma}, \ell, c_{\alpha,\gamma})$ is equivalent to showing that $f_u(\mathbf{\Pi}^a_{\alpha,\gamma}) \prec^w f_u(\mathbf{\Pi}^b_{\alpha,\gamma})$ for $u = 1, 2, \ldots$. For brevity, we denote hereafter $f_u(\mathbf{\Pi}^z_{\alpha,\gamma})$ by f^z_u and $p^*(\mathbf{\Pi}^z_{\alpha,\gamma}, \ell, c_{\alpha,\gamma})$ by p^{z*}.

Next, we prove $f^a_u \prec^w f^b_u$ by induction. Without loss of generality, we take $\mathbf{\Pi}^a_{\alpha,\gamma}$ to be weak supermajorization preserving w.r.t. $\mathcal{P}$. (If $\mathbf{\Pi}^b_{\alpha,\gamma}$ were to be weak supermajorization preserving w.r.t. $\mathcal{P}$, we would obtain the same result and the proof would proceed in a symmetric fashion by interchanging the roles of a and b.) For $u = 0$, by definition, $f^a_0 = \ell \prec^w \ell = f^b_0$. Assume $f^a_u \prec^w f^b_u$. Then we want to prove the statement for $u + 1$. First, we notice that the following majorization inequalities hold:

$$\left(f^a_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma}\right) \prec^w \left(f^b_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma}\right) \prec^w \left(f^b_u \mathbf{\Pi}^b_{\alpha,\gamma} + c_{\alpha,\gamma}\right),$$

where the first inequality follows from the assumption that $\mathbf{\Pi}^a_{\alpha,\gamma}$ is order and weak supermajorization preserving w.r.t. $\mathcal{P}$ and the fact that f^a_u and f^b_u are similarly ordered to ℓ by Lemma A5 and $c_{\alpha,\gamma}$

is similarly ordered to ℓ in light of Assumption 2; the second inequality is due to that $\mathbf{\Pi}^a_{\alpha,\gamma} \prec \mathbf{\Pi}^b_{\alpha,\gamma}$, and $\mathbf{\Pi}^a_{\alpha,\gamma}$ and $\mathbf{\Pi}^b_{\alpha,\gamma}$ are order preserving w.r.t. $\mathcal{P}$. For $z \in \{a,b\}$, because $\Delta(f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma}) \geq \Delta\ell$ (by Lemma A5 (III)), $\ell_{(i)} \leq (f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma})_{(i)}$ must imply that $\ell_{(k)} \leq (f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma})_{(k)}$ for $k > i$. Moreover, $f^z_u \mathbf{\Pi}^z_{\alpha,\gamma} + c_{\alpha,\gamma}$ is similarly ordered to ℓ. Applying Lemma A6 (II) with $x = f^a_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma}$, $y = f^b_u \mathbf{\Pi}^b_{\alpha,\gamma} + c_{\alpha,\gamma}$, and $z = \ell$, we deduce

$$f^a_{u+1} = \left[\ell \wedge (f^a_u \mathbf{\Pi}^a_{\alpha,\gamma} + c_{\alpha,\gamma})\right] \prec^w \left[\ell \wedge (f^b_u \mathbf{\Pi}^b_{\alpha,\gamma} + c_{\alpha,\gamma})\right] = f^b_{u+1}.$$

This concludes the proof that $f^a_u \prec^w f^b_u$ for $u = 1, 2, \ldots$.

By definition of weak supermajorization,

$$\sum_{i=1}^{k} f^b_{u,(i)} - \sum_{i=1}^{k} f^a_{u,(i)} \leq 0 \text{ for } k = 1, \ldots, n.$$

Letting $u \to \infty$, the above inequality leads to

$$\sum_{i=1}^{k} p^{b*}_{(i)} - \sum_{i=1}^{k} p^{a*}_{(i)} \leq 0 \text{ for } k = 1, \ldots, n, \quad \text{hence,} \quad p^{a*} \prec^w p^{b*}.$$

Together with the above inequalities, Proposition 1 (III) leads to

$$\sum_{i=1}^{k} [\ell - p^{a*}]_{[i]} = \sum_{i=1}^{k} \ell_{(i)} - p^{a*}_{(i)} \leq \sum_{i=1}^{k} \ell_{(i)} - p^{b*}_{(i)} = \sum_{i=1}^{k} [\ell - p^{b*}]_{[i]}$$

for $k = 1, \ldots, n$, or equivalently $s(\mathbf{\Pi}^a, \ell, c, \gamma) \prec_w s(\mathbf{\Pi}^b, \ell, c, \gamma)$.

□

Proof of Proposition 2. Because $(\mathbf{\Pi}, \ell, c, \gamma)$ is unbalancing, it must hold that for $j = 1, \ldots, n-1$,

$$\begin{aligned}
&\left[\sum_{i=1}^{n} \underline{\ell}_{(i)} \pi^{\mu}_{i,j+1} + c_{(j+1)}\right] - \left[\sum_{i=1}^{n} \underline{\ell}_{(i)} \pi^{\mu}_{i,j} + c_{(j)}\right] \geq \underline{\ell}_{(j+1)} - \underline{\ell}_{(j)} = \ell_{(j+1)} - \ell_{(j)} \\
\Rightarrow &\left[\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,j+1} + c_{(j+1)}\right] - \left[\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,j} + c_{(j)}\right] \\
&\qquad - \max\left\{\ell_{(1)} - [(1+\gamma)c - \gamma\ell]_{(1)}, 0\right\} \left(\sum_{i=1}^{n} \pi^{\mu}_{i,j+1} - \sum_{i=1}^{n} \pi^{\mu}_{i,j}\right) \geq \ell_{(j+1)} - \ell_{(j)} \\
\Rightarrow &\left[\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,j+1} + c_{(j+1)}\right] - \left[\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,j} + c_{(j)}\right] \geq \ell_{(j+1)} - \ell_{(j)},
\end{aligned}$$

where the last inequality follows from the assumption that $\sum_{i=1}^{n} \pi^{\mu}_{i,j+1} - \sum_{i=1}^{n} \pi^{\mu}_{i,j}$ is nonnegative. Moreover, since $(\mathbf{\Pi}, \ell, c, \gamma)$ is also balancing, we deduce

$$\left[\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,j+1} + c_{(j+1)}\right] - \left[\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,j} + c_{(j)}\right] = \ell_{(j+1)} - \ell_{(j)} \tag{20}$$

for $j = 1, \ldots, n-1$. By the assumption of the lemma, at least one node repays its liabilities in full, hence it must hold that

$$\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,k} + c_{(k)} \geq \sum_{i=1}^{n} p^*_{\mu_i} \pi^{\mu}_{i,k} + c_{(k)} \geq \ell_{(k)} \text{ for some } k \in \{1, \ldots, n\},$$

where we recall that $p^*_{\mu_i}$ is the clearing payment made by node j if $\ell_{(i)} = \ell_j$. Together with Eq. (20), the above inequality implies

$$\left[\sum_{i=1}^{n} \ell_{(i)} \pi^{\mu}_{i,j} + c_{(j)}\right] - \ell_{(j)} \geq 0 \text{ for } j = 1, \ldots, n,.$$

But this means that $p^* = \ell = \ell \wedge (\ell \mathbf{\Pi} + c)$. Hence, it must hold that $s(\mathbf{\Pi}, \ell, c, \gamma) = \mathbf{0}$.

□

References

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