Appendix

Proof of Proposition 1. For any given initial reference r_1, if we adopt the T season optimal policies for the first T seasons and the optimal single season optimal policy for the last season for the $T+1$ season problem, we can show that $V_1^{T+1}(r_1) - V_1^T(r_1) \geq \beta^T \inf_{0 \leq r_{T+1} \leq p} \left\{ V_{T+1}^{T+1}(r_{T+1}) \right\} \geq 0$. Furthermore, $V_t^T(r_t)$ is bounded above as long as $\mathbb{E}(\xi_t) < \infty$ and $\beta < 1$. Thus, there exists a $V(r_t)$ such that $\lim_{T \to \infty} V_t^T(r_t) = V(r_t)$.

Suppose that $u(r)$ is a bounded solution that satisfies the optimality equation. Then by (1),

$$u(r_t) \geq \mathbb{E}\Big[p \min\{q^*(r_t), d_1(r_t)\xi_t\} - cq^*(r_t) + s^*(\xi_t|r_t) \cdot \min\left\{ d_2(s^*(\xi_t|r_t)|r_t)\xi_t, [q^*(r_t) - d_1(r_t)\xi_t]^+ \right\} \\ + \beta u\big(\theta s^*(\xi_t|r_t) + (1-\theta)r_t\big)\Big].$$

Thus,

$$V(r_t) - u(r_t) \leq \beta \mathbb{E}\big[V(\theta s^*(\xi_t|r_t) + (1-\theta)r_t) - u(\theta s^*(\xi_t|r_t) + (1-\theta)r_t)\big] \leq \beta \sup_{r_{t+1}} \{|V(r_{t+1}) - u(r_{t+1})|\}.$$

Similarly, $u(r_t) - V(r_t) \leq \beta \sup_{r_{t+1}} \{|V(r_{t+1}) - u(r_{t+1})|\}$. Therefore, $\sup_r |V(r) - u(r)| \leq \beta \sup_r \{|V(r) - u(r)|\}$, which implies $\sup_r |V(r) - u(r)| = 0$ and $V(r)$ is unique.

Let $r_t^*(\xi_t)$, $t = 2, 3, \cdots$, be the subsequent references under the optimal decisions when the initial reference is r_1. If we adopt the same policy at a higher initial reference r_1', the subsequent references $r_t'(\xi_t) \geq r_t^*(\xi_t)$, $t = 2, 3, \cdots$, and the total profit will be no less than $V(r_t)$. Thus, $V(r)$ increases in r. □

Before proving Proposition 2, we need the following lemma. For $d(p,r) \leq z \leq d(\underline{s}(r), r)$, let $\Pi(z,r) = pd_1(r) + \hat{s}(z,r)d_2(\hat{s}(z,r)|r)$ be the unconstrained single season revenue function when the markdown price is set to sell all the inventory, i.e., $\hat{s}(z,r)$ is such that $d(\hat{s}(z,r), r) = z$.

Lemma 1. *Under Assumption 1,* $\Pi(z,r)$ *is locally concave when* $\hat{s}(z,r) \geq \underline{s}(r)$.

Proof. It is sufficient to show that the Hessian $\begin{pmatrix} \frac{\partial^2 \Pi(z,r)}{\partial z^2} & \frac{\partial^2 \Pi(z,r)}{\partial z \partial r} \\ \frac{\partial^2 \Pi(z,r)}{\partial r \partial z} & \frac{\partial^2 \Pi(z,r)}{\partial r^2} \end{pmatrix}$ is negative semi-definite when $\hat{s}(z,r) \geq \underline{s}(r)$.

After some straightforward but tedious algebra, we can simplify the determinant of the Hessian as

$$
\begin{aligned}
&\frac{\partial^2\Pi(z,r)}{\partial r^2}\cdot\frac{\partial^2\Pi(z,r)}{\partial z^2}-\frac{\partial^2\Pi(z,r)}{\partial z\partial r}\frac{\partial^2\Pi(z,r)}{\partial r\partial z}\\
&=\left(\frac{\partial\hat{s}(z,r)}{\partial z}\right)^2\left[(p-s)d_1''(r)-\frac{\partial^2 d(s,r)}{\partial r^2}\frac{d_2(s|r)}{\frac{\partial d_2(s|r)}{\partial s}}\right]_{s=\hat{s}(z,r)}\cdot\left[\frac{\partial^2\{sd_2(s|r)\}}{\partial s^2}-\frac{\frac{\partial^2 d_2(s|r)}{\partial s^2}}{\frac{\partial d_2(s|r)}{\partial s}}\frac{\partial\{sd_2(s|r)\}}{\partial s}\right]_{s=\hat{s}(z,r)}\\
&\quad-\left(\frac{\partial\hat{s}(z,r)}{\partial z}\right)^2\left[\frac{\partial^2\{sd_2(s|r)\}}{\partial s\partial r}-\frac{\frac{\partial^2 d_2(s|r)}{\partial s\partial r}}{\frac{\partial d_2(s|r)}{\partial s}}\frac{\partial\{sd_2(s|r)\}}{\partial s}\right]^2_{s=\hat{s}(z,r)}\\
&=\Bigg\{\left[\frac{-s\frac{\partial d_2(s|r)}{\partial s}}{d_2(s|r)}(p-s)d_1''(r)+s\frac{\partial^2 d(s,r)}{\partial r^2}\right]\left[\frac{-s\frac{\partial d_2(s|r)}{\partial s}}{d_2(s|r)}\frac{\partial^2\{sd_2(s|r)\}}{\partial s^2}+\frac{s\frac{\partial^2 d_2(s|r)}{\partial s^2}}{d_2(s|r)}\frac{\partial\{sd_2(s|r)\}}{\partial s}\right]\\
&\qquad-\left[\frac{-s\frac{\partial d_2(s|r)}{\partial s}}{d_2(s|r)}\frac{\partial^2\{sd_2(s|r)\}}{\partial s\partial r}+\frac{s\frac{\partial^2 d_2(s|r)}{\partial s\partial r}}{d_2(s|r)}\frac{\partial\{sd_2(s|r)\}}{\partial s}\right]^2\Bigg\}_{s=\hat{s}(z,r)}\cdot\left(\frac{\partial\hat{s}(z,r)}{\partial z}\right)^2\left(\frac{d_2(s|r)}{-s\frac{\partial d_2(s|r)}{\partial s}}\right)^2_{s=\hat{s}(z,r)}\\
&=\Bigg\{\left[(p-s)d_1''(r)+s\frac{\partial^2 d(s,r)}{\partial r^2}-\frac{\frac{\partial\{sd_2(s|r)\}}{\partial s}}{d_2(s|r)}(p-s)d_1''(r)\right]\left[\frac{\partial^2\{sd_2(s|r)\}}{\partial s^2}-2\frac{\frac{\partial d_2(s|r)}{\partial s}}{d_2(s|r)}\frac{\partial\{sd_2(s|r)\}}{\partial s}\right]\\
&\qquad-\left[\frac{\partial^2\{sd_2(s|r)\}}{\partial s\partial r}-\frac{\frac{\partial d_2(s|r)}{\partial r}}{d_2(s|r)}\frac{\partial\{sd_2(s|r)\}}{\partial s}\right]^2\Bigg\}_{s=\hat{s}(z,r)}\cdot\left(\frac{\partial\hat{s}(z,r)}{\partial z}\right)^2\left(\frac{d_2(s|r)}{-s\frac{\partial d_2(s|r)}{\partial s}}\right)^2_{s=\hat{s}(z,r)}\\
&\geq\Bigg\{\left[(p-s)d_1''(r)+s\frac{\partial^2 d(s,r)}{\partial r^2}\right]\left[\frac{\partial^2\{sd_2(s|r)\}}{\partial s^2}\right]-\left[\frac{\partial^2\{sd_2(s|r)\}}{\partial s\partial r}\right]^2\Bigg\}_{s=\hat{s}(z,r)}\cdot\left(\frac{\partial\hat{s}(z,r)}{\partial z}\right)^2\left(\frac{d_2(s|r)}{-s\frac{\partial d_2(s|r)}{\partial s}}\right)^2_{s=\hat{s}(z,r)}\geq 0.
\end{aligned}
$$

The first inequality follows as $\frac{\partial\{sd_2(s|r)\}}{\partial s}\Big|_{s=\hat{s}(z,r)}\leq 0$ and $\frac{\partial^2\{sd_2(s|r)\}}{\partial s\partial r}\geq\frac{\frac{\partial d_2(s|r)}{\partial r}}{d_2(s|r)}\frac{\partial\{sd_2(s|r)\}}{\partial s}$ by part 1 of Assumption 1, and the second inequality holds by part 2 of Assumption 1. Furthermore, it is easy to see that the diagonal elements in the Hessian are non-positive. Therefore, the Hessian is negative semi-definite and $\Pi(z,r)$ is locally concave when $\hat{s}(z,r)\geq\underline{s}(r)$. □

Proof of Proposition 2. Let

$$
\begin{aligned}
\phi_t^T(s_t,\xi_t,q_t,r_t)&=s_t\cdot\min\left\{d_2(s_t|r_t)\xi_t,[q_t-d_1(r_t)\xi_t]^+\right\}+\beta V_{t+1}^T(\theta s_t+(1-\theta)r_t),\\
\hat{\phi}_t^T(\xi_t,q_t,r_t)&=\max_{\underline{s}(r_t)\leq s_t\leq p}\left\{\phi_t^T(s_t,\xi_t,q_t,r_t)\right\}=\phi_t^T\left(s_t^T(\xi_t,q_t,r_t),\xi_t,q_t,r_t\right)
\end{aligned}
$$

be the total profit from stage 2 of season t onwards for a given (s_t,ξ_t,q_t,r_t) and under the optimal markdown price for a given (ξ_t,q_t,r_t), respectively. Note that $V_{T+1}^T(\cdot)\equiv 0$ is concave. Now assume that $V_{t+1}^T(\cdot)$ is concave and we prove that the proposition holds at t. As $T\to\infty$, $V_t^T(\cdot)$ converges to $V(\cdot)$ by Proposition 1 and $V(\cdot)$ is also concave. Thus, the proposition holds as $T\to\infty$.

Part 3: The structure of the markdown price $s_t^T(\xi_t,q_t,r_t)$ for any given q_t.

For any given q_t, the stage 2 profit $s_t\cdot\min\{d_2(s_t|r_t)\xi_t,[q_t-d_1(r_t)\xi_t]^+\}$ is concave in s_t since $s_td_2(s_t|r_t)$ is concave in s_t and hence $\phi_t^T(s_t,\xi_t,q_t,r_t)$ is also concave in s_t. When $\xi_t \geq \frac{q_t}{d_1(r_t)}$, all the inventory is sold at stage 1 and no discount needs to be offered at stage 2, i.e., $s_t^T(\xi_t,q_t,r_t) = p$. When $\frac{q_t}{d(p,r_t)} \leq \xi_t < \frac{q_t}{d_1(r_t)}$, $d_2(s_t|r_t)\xi_t \geq q_t - d_1(r_t)\xi_t$ for any s_t and the retailer will sell all its inventory regardless of the markdown price. Thus, the retailer will maximize the future profit by not marking down, i.e., $s_t^T(\xi_t,q_t,r_t) = p$. When $\xi_t < \frac{q_t}{d(p,r_t)}$, the sales at stage 2 is bounded by the available inventory and it is never optimal for the retailer to set the markdown price such that $d_2(s_t|r_t)\xi_t > q_t - d_1(r_t)\xi_t$. Thus, the stage 2 problem is equivalent to solving the following maximization problem,

$$\max_{\underline{s}(r_t)\leq s_t\leq p} \quad s_t d_2(s_t|r_t)\xi_t + \beta V_{t+1}^T(\theta s_t + (1-\theta)r_t) \tag{4}$$

$$\text{s.t.} \quad d_2(s_t|r_t)\xi_t \leq [q_t - d_1(r_t)\xi_t]^+. \tag{5}$$

If there are multiple solutions, we choose the largest one. Since

$$\frac{\partial^2\{s_t d_2(s_t|r_t)\xi_t + \beta V_{t+1}^T(\theta s_t + (1-\theta)r_t)\}}{\partial \xi_t \partial s_t} = \frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t} \leq 0,$$

the objective function (4) is submodular in (ξ_t, s_t) and hence the markdown price that maximizes the objective function decreases in ξ_t. As ξ_t increases, the right hand side of (5) decreases and the left hand side increases. Thus, there exists $\xi_t^T(q_t,r_t)$ such that $s_t^T(\xi_t,q_t,r_t)$ decreases for $\xi_t \leq \xi_t^T(q_t,r_t)$, and increases for $\xi_t > \xi_t(q_t,r_t)$. The constraint is binding at $s_t^T(\xi_t^T(q_t,r_t),q_t,r_t)$. Thus, the optimal markdown decision as a function of ξ_t can be summarized as follows.

- When $\xi_t < \frac{\theta\beta V_{t+1}^{T\prime}(\theta p + (1-\theta)r_t)}{-\frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t}\Big|_{s_t=p}}$, the total market size is so small that the objective function (4) is monotonically increasing in s_t. Thus, the retailer has no incentive to mark down and $s_t^T(\xi_t,q_t,r_t) = p$. There is unsold inventory at the end of the season.

- When $\frac{\theta\beta V_{t+1}^{T\prime}(\theta p + (1-\theta)r_t)}{-\frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t}\Big|_{s_t=p}} \leq \xi_t < \xi_t^T(q_t,r_t)$, $s_t^T(\xi_t,q_t,r_t)$ is the solution to

  $$\frac{\partial \phi_t^T(s_t,\xi_t,q_t,r_t)}{\partial s_t} = \frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t}\xi_t + \theta\beta V_{t+1}^{T\prime}(\theta s_t + (1-\theta)r_t) = 0, \tag{6}$$

  independent of q_t, and there is unsold inventory at the end of the season.

- When $\xi_t^T(q_t,r_t) \leq \xi_t < \frac{q_t}{d(p,r_t)}$, $s_t^T(\xi_t,q_t,r_t)$ is the solution to $d_2(s_t|r_t)\xi_t = q_t - d_1(r_t)\xi_t$ and there is no unsold inventory. In this region, $\frac{\partial \phi_t^T(s_t,\xi_t,q_t,r_t)}{\partial s_t}\Big|_{s_t = s_t^T(\xi_t,q_t,r_t)} \leq 0$.

- When $\xi_t \geq \frac{q_t}{d(p,r_t)}$, there is no need to mark down, i.e., $s_t^T(\xi_t,q_t,r_t) = p$.

Part 2: The concavity of $v_t^T(q_t, r_t)$ and the optimal structure of $q^*(r_t)$.

Note that $s_t^T(\xi_t, q_t, r_t)$ depends on q_t only when $\xi_t^T(q_t, r_t) \le \xi_t < \frac{q_t}{d(p,r_t)}$, in which case all the inventory is sold, i.e., $q_t - d_1(r_t)\xi_t = d_2(s_t^T(\xi_t, q_t, r_t)|r_t)\xi_t$ and $\frac{\partial \phi_t^T(s_t,\xi_t,q_t,r_t)}{\partial s_t}\Big|_{s_t=s_t^T(\xi_t,q_t,r_t)} \le 0$. Taking the derivative of

$$v_t^T(q_t, r_t) = \int_0^\infty [p\min\{d_1(r_t)\xi_t, q_t\} + \hat{\phi}_t^T(\xi_t, q_t, r_t)]dF(\xi_t) - cq_t$$

with respect to q_t yields

$$\begin{aligned}
\frac{\partial v_t^T(q_t, r_t)}{\partial q_t} &= p\bar{F}\left(\frac{q_t}{d(p,r_t)}\right) + \int_{\xi_t^T(q_t,r_t)}^{\frac{q_t}{d(p,r_t)}} \frac{\partial \hat{\phi}_t^T(\xi_t, q_t, r_t)}{\partial q_t} dF(\xi_t) - c \\
&= \int_{\xi_t^T(q_t,r_t)}^{\frac{q_t}{d(p,r_t)}} \frac{\partial s_t^T(\xi_t, q_t, r_t)}{\partial q_t} \frac{\partial \phi_t^T(s_t, \xi_t, q_t, r_t)}{\partial s_t}\Big|_{s_t=s_t^T(\xi_t,q_t,r_t)} dF(\xi_t) + p\bar{F}\left(\frac{q_t}{d(p,r_t)}\right) - c,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 v_t^T(q_t, r_t)}{\partial q_t^2} = &\int_{\xi_t^T(q_t,r_t)}^{\frac{q_t}{d(p,r_t)}} \left(\frac{\partial s_t^T(\xi_t, q_t, r_t)}{\partial q_t}\right)^2 \cdot \left\{\frac{\partial^2 \phi_t^T(s_t, \xi_t, q_t, r_t)}{\partial s_t^2} - \frac{\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t^2}}{\frac{\partial d_2(s_t|r_t)}{\partial s_t}} \frac{\partial \phi_t^T(s_t, \xi_t, q_t, r_t)}{\partial s_t}\right\}_{s_t=s_t^T(\xi_t,q_t,r_t)} dF(\xi_t) \\
&- \frac{1}{d(p,r_t)} f\left(\frac{q_t}{d(p,r_t)}\right)\left\{p - \left[\frac{\partial s_t^T(\xi_t, q_t, r_t)}{\partial q_t}\frac{\partial \phi_t^T(s_t, \xi_t, q_t, r_t)}{\partial s_t}\right]_{\xi_t=\frac{q_t}{d(p,r_t)}, s_t=p}\right\}.
\end{aligned}$$

The second line of $\frac{\partial^2 v_t^T(q_t,r_t)}{\partial q_t^2}$ is non-positive since

$$\begin{aligned}
&p - \left[\frac{\partial s_t^T(\xi_t, q_t, r_t)}{\partial q_t}\frac{\partial \phi_t^T(s_t, \xi_t, q_t, r_t)}{\partial s_t}\right]_{\xi_t=\frac{q_t}{d(p,r_t)}, s_t=p} \\
&= p - \left\{\frac{1}{\frac{\partial d_2(s_t|r_t)}{\partial s_t}\xi_t}\left[\left(s_t\frac{\partial d_2(s_t|r_t)}{\partial s_t} + d_2(s_t|r_t)\right)\xi_t + \theta\beta V_t^{T\prime}(\theta s_t + (1-\theta)r_t)\right]\right\}_{\xi_t=\frac{q_t}{d(p,r_t)}, s_t=p} \\
&= \frac{d(p,r_t)}{-q_t \left.\frac{\partial d_2(s_t|r_t)}{\partial s_t}\right|_{s_t=p}}\left[d_2(s_t|r_t)\frac{q_t}{d(p,r_t)} + \theta\beta V_t^{T\prime}(\theta p + (1-\theta)r_t)\right] \ge 0.
\end{aligned}$$

We now show that the first line of $\frac{\partial^2 v_t^T(q_t,r_t)}{\partial q_t^2}$ is negative by proving that the term in $\{\}$ is negative at $s_t = s_t^T(\xi_t, q_t, r_t)$ for any ξ_t, $\xi_t^T(q_t, r_t) < \xi_t < \frac{q_t}{d(p,r_t)}$. We first note that, for $\xi_t^T(y_t, r_t) < \xi_t < \frac{q_t}{d(p,r_t)}$, $\frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t}\Big|_{s_t=s_t^T(\xi_t,q_t,r_t)} < 0$ or $\frac{\partial d_2(s_t|r_t)}{\partial s_t}\Big|_{s_t=s_t^T(\xi_t,q_t,r_t)} < -\frac{d_2(s_t^T(\xi_t,q_t,r_t)|r_t)}{s_t^T(\xi_t,q_t,r_t)} \le 0$ and

$$\begin{aligned}
\left.\frac{\partial \phi_t^T(s_t, \xi_t, q_t, r_t)}{\partial s_t}\right|_{s_t=s_t^T(\xi_t,q_t,r_t)} &= \left[\frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t}\xi_t + \theta\beta V_{t+1}^{T\prime}(\theta s_t + (1-\theta)r_t)\right]_{s_t=s_t^T(\xi_t,q_t,r_t)} \\
&< \left[\frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t}\xi_t^T(q_t, r_t) + \theta\beta V_{t+1}^{T\prime}(\theta s_t + (1-\theta)r_t)\right]_{s_t=s_t^T(\xi_t,q_t,r_t)} \\
&\le \left.\frac{\partial \phi_t^T(s_t, \xi_t, q_t, r_t)}{\partial s_t}\right|_{s_t=s_t^T(\xi_t^T(q_t,r_t),q_t,r_t), \xi_t=\xi_t^T(q_t,r_t)} = 0,
\end{aligned}$$

where the last inequality follows from the concavity of $\phi_t^T(s_t,\xi_t,q_t,r_t)$ in s_t. Thus, if $\left.\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t^2}\right|_{s_t=s_t^T(\xi_t,q_t,r_t)} > 0$, then the term in $\{\}$ is negative at $s_t^T(\xi_t,q_t,r_t)$. Otherwise,

$$
\begin{aligned}
&\frac{\partial^2\phi_t^T(s_t,\xi_t,q_t,r_t)}{\partial s_t^2} - \frac{\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t^2}}{\frac{\partial d_2(s_t|r_t)}{\partial s_t}}\frac{\partial\phi_t^T(s_t,\xi_t,q_t,r_t)}{\partial s_t}\\
&= \theta^2\beta V_{t+1}^{T''}(\theta s_t+(1-\theta)r_t) + \frac{\partial^2\{s_t d_2(s_t|r_t)\}}{\partial s_t^2}\xi_t - \frac{\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t^2}}{\frac{\partial d_2(s_t|r_t)}{\partial s_t}}\left[\frac{\partial\{s_t d_2(s_t|r_t)\}}{\partial s_t}\xi_t + \theta\beta V_{t+1}^{T\prime}(\theta s_t+(1-\theta)r_t)\right]\\
&\le \frac{1}{\frac{\partial d_2(s_t|r_t)}{\partial s_t}}\left\{2\left[\frac{\partial d_2(s_t|r_t)}{\partial s_t}\right]^2\xi_t - \frac{\partial^2 d_2(s_t|r_t)}{\partial s_t^2}\left[d_2(s_t|r_t)\xi_t + \theta\beta V_{t+1}^{T\prime}(\theta s_t+(1-\theta)r_t)\right]\right\}\\
&\le 2\frac{\partial d_2(s_t|r_t)}{\partial s_t}\xi_t
\end{aligned}
$$

which is negative at $s_t = s_t^T(\xi_t,q_t,r_t)$. Thus, the term in $\{\}$ is also negative at $s_t^T(\xi_t,q_t,r_t)$ and hence, $\frac{\partial^2 v_t^T(q_t,r_t)}{\partial q_t^2} < 0$, i.e., $v_t^T(q_t,r_t)$ is strictly concave in q_t and $q_t^{T*}(r_t)$ is the unique solution to $\left.\frac{\partial v_t^T(q_t,r_t)}{\partial q_t}\right|_{q_t=q_t^{T*}(r_t)} = 0$. Furthermore, $\frac{dq_t^{T*}(r_t)}{dr_t} = \left.\frac{\frac{\partial^2 v_t^T(q_t,r_t)}{\partial q_t\partial r_t}}{-\frac{\partial^2 v_t^T(q_t,r_t)}{\partial q_t^2}}\right|_{q_t=q_t^{T*}(r_t)}$ by the implicit function theorem.

Note that $\frac{\partial s_t^T(\xi_t,q_t,r_t)}{\partial q_t}\left.\frac{\partial\phi_t^T(s_t,\xi_t,q_t,r_t)}{\partial s_t}\right|_{s_t=s_t^T(\xi_t,q_t,r_t)} \ge 0$ and hence $p\bar{F}\left(\frac{q_t^{T*}(r_t)}{d(p,r_t)}\right) - c \le 0$. Thus, $\frac{q_t^{T*}(r_t)}{d(p,r_t)} \ge F^{-1}\left(\frac{p-c}{p}\right)$ for any r_t, which will be used in the proof of Proposition 5.

Part 1: The concavity of $V_t^T(r_t)$.

Suppose that $r_t^\ell < r_t^h$. At $r_t = \frac{r_t^\ell + r_t^h}{2}$, let the retailer orders $q_t(r_t) = \frac{q_t^{T*}(r_t^\ell)+q_t^{T*}(r_t^h)}{2}$ and set the markdown price at[1]

$$
s_t(\xi_t|r_t) = \begin{cases} p, & \text{if } d(p,r_t)\xi_t > q_t(r_t),\\ \frac{s_t^{T*}(\xi_t|r_t^\ell)+s_t^{T*}(\xi_t|r_t^h)}{2}, & \text{if } d(p,r_t)\xi_t \le q_t(r_t) \text{ but } d\left(\frac{s_t^{T*}(\xi_t|r_t^\ell)+s_t^{T*}(\xi_t|r_t^h)}{2}, r_t\right)\xi_t \le q_t(r_t)\\ & \quad \text{or } s_t^{T*}(\xi_t|r_t^\ell) \le s_t^{T*}(\xi_t|r_t^h),\\ d^{-1}\left(\frac{q_t(r_t)}{\xi_t}, r_t\right) > \frac{s_t^{T*}(\xi_t|r_t^\ell)+s_t^{T*}(\xi_t|r_t^h)}{2}, & \text{otherwise.} \end{cases}
$$

By Assumption 1, $\frac{-\frac{\partial d_2(s_t|r_t)}{\partial s_t}}{d_2(s_t|r_t)}$ decreases in r_t, which guarantees the supermodularity of $sd_2(s|r)$ and leads to

$$
d^{-1}\left(\frac{q_t(r_t)}{\xi_t}, r_t\right) \ge \frac{s_t^{T*}(\xi_t|r_t^\ell)+s_t^{T*}(\xi_t|r_t^h)}{2} \ge s_t^{T*}(\xi_t|r_t^h) \ge \underline{s}(r_t^h) \ge \underline{s}(r_t).
$$

Due to the monotonicity and concavity of $V_{t+1}^T(\cdot)$, the future profit at r_t is at least $\frac{V_{t+1}^T\left(\theta s_t^{T*}(\xi_t|r_t^\ell)+(1-\theta)r_t^\ell\right)+V_{t+1}^T\left(\theta s_t^{T*}(\xi_t|r_t^h)+(1-\theta)r_t^h\right)}{2}$ for any given ξ_t. Let $\pi(s,\xi,q,r)$ denote the single season revenue for given (s,ξ,q,r) and $\hat{\pi}(\xi_t,r)$ be the season t revenue under the optimal ordering and markdown prices from season t to T for given (ξ_t,r). Now consider the season t revenue $\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t)$.

[1] We define the inverse demand function as $d^{-1}(z,r) = s$ if $d(s,r) = z$ for $d(p,r) \le z \le d(\underline{s}(r),r)$.

1. If $d(p,r_t)\xi_t > q_t(r_t)$, $\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t) = pq_t(r_t) = \frac{pq_t^{T*}(r_t^\ell)}{2} + \frac{pq_t^{T*}(r_t^h)}{2} \geq \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h)$.

2. If $d(p,r_t)\xi_t \leq q_t(r_t)$, $d\left(\frac{s_t^{T*}(\xi_t|r_t^\ell)+s_t^{T*}(\xi_t|r_t^h)}{2}, r_t\right)\xi_t \leq q_t(r_t)$, then

$$\begin{aligned}
&\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t) = pd_1(r_t)\xi_t + s_t(\xi_t|r_t)d_2(s_t(\xi_t|r_t)|r_t)\xi_t \\
&\geq \frac{1}{2}\left[pd_1(r_t^\ell)\xi_t + s_t^{T*}(\xi_t|r_t^\ell)\cdot d_2(s_t^{T*}(\xi_t|r_t^\ell)|r_t^\ell)\xi_t\right] + \frac{1}{2}\left[pd_1(r_t^h)\xi_t + s_t^{T*}(\xi_t|r_t^h)\cdot d_2(s_t^{T*}(\xi_t|r_t^h)|r_t^h)\xi_t\right] \\
&\geq \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h)
\end{aligned}$$

due to the joint concavity of the unconstrained single season revenue function in (s,r).

3. If $d(p,r_t)\xi_t \leq q_t(r_t)$, $d\left(\frac{s_t^{T*}(\xi_t|r_t^\ell)+s_t^{T*}(\xi_t|r_t^h)}{2}, r_t\right)\xi_t > q_t(r_t)$ and $s_t^{T*}(\xi_t|r_t^\ell) \leq s_t^{T*}(\xi_t|r_t^h)$, then $d_2(s_t^{T*}(\xi_t|r_t^\ell)|r_t^\ell) \geq d_2(s_t^{T*}(\xi_t|r_t^h)|r_t^h)$.

    (a) If $d(s_t^{T*}(\xi_t|r_t^\ell),r_t^\ell)\xi_t > q_t^{T*}(r_t^\ell)$, then $p = s_t^{T*}(\xi_t|r_t^\ell) \leq s_t^{T*}(\xi_t|r_t^h)$ and hence $s_t(\xi_t|r_t) = p$. We have $\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t) = pq_t(r_t) \geq \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h)$.

    (b) If $d(s_t^{T*}(\xi_t|r_t^\ell),r_t^\ell)\xi_t \leq q_t^{T*}(r_t^\ell)$,

$$\begin{aligned}
\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t) &= pd_1(r_t)\xi_t + s_t(\xi_t|r_t)[q_t(r_t) - d_1(r_t)\xi_t] \\
&= [p - s_t(\xi_t|r_t)]d_1(r_t)\xi_t + s_t(\xi_t|r_t)q_t(r_t) \\
&\geq [p - s_t(\xi_t|r_t)]\cdot\frac{\min\{d_1(r_t^\ell)\xi_t, q_t^{T*}(r_t^\ell)\} + \min\{d_1(r_t^h)\xi_t, q_t^{T*}(r_t^h)\}}{2} \\
&\quad + \frac{s_t^{T*}(\xi_t|r_t^\ell) + s_t^{T*}(\xi_t|r_t^h)}{2}\cdot\frac{q_t^{T*}(r_t^\ell) + q_t^{T*}(r_t^h)}{2}.
\end{aligned}$$

- If $d(s_t^{T*}(\xi_t|r_t^h),r_t^h)\xi_t > q_t^{T*}(r_t^h)$, then

$$q_t^{T*}(r_t^h) - d_1(r_t^h)\xi_t < d_2(s_t^{T*}(\xi_t|r_t^h)|r_t^h)\xi_t \leq d_2(s_t^{T*}(\xi_t|r_t^\ell)|r_t^\ell)\xi_t \leq q_t^{T*}(r_t^\ell) - d_1(r_t^\ell)\xi_t$$

  and

$$\begin{aligned}
\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t) &\geq \frac{1}{4}[s_t^{T*}(\xi_t|r_t^h) - s_t^{T*}(\xi_t|r_t^\ell)]\left\{[q_t^{T*}(r_t^\ell) - d_1(r_t^\ell)\xi_t] - [q_t^{T*}(r_t^h) - d_1(r_t^h)\xi_t]^+\right\} \\
&\quad + \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h) \geq \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h).
\end{aligned}$$

- If $d(s_t^{T*}(\xi_t|r_t^h),r_t^h)\xi_t \leq q_t^{T*}(r_t^h)$, then $q_t^{T*}(r_t^h) - d_1(r_t^h)\xi_t \geq d_2(s_t^{T*}(\xi_t|r_t^h)|r_t^h)\xi_t$, $q_t^{T*}(r_t^\ell) - d_1(r_t^\ell)\xi_t \geq d_2(s_t^{T*}(\xi_t|r_t^\ell)|r_t^\ell)\xi_t$, and

$$\begin{aligned}
\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t) &\geq \frac{1}{4}[s_t^{T*}(\xi_t|r_t^h) - s_t^{T*}(\xi_t|r_t^\ell)]\cdot\left[d_2(s_t^{T*}(\xi_t|r_t^\ell)|r_t^\ell) - d_2(s_t^{T*}(\xi_t|r_t^h)|r_t^h)\right] \\
&\quad + \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h) \geq \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h).
\end{aligned}$$

4. If $d(p,r_t)\xi_t \le q_t(r_t) < d\left(\frac{s_t^{T*}(\xi_t|r_t^\ell)+s_t^{T*}(\xi_t|r_t^h)}{2}, r_t\right)\xi_t$ and $s_t^{T*}(\xi_t|r_t^\ell) > s_t^{T*}(\xi_t|r_t^h)$, then

$$d^{-1}\left(\frac{\min\left\{d(s_t^{T*}(\xi_t,r_t^\ell),r_t^\ell)\xi_t, q_t^{T*}(r_t^\ell)\right\}+\min\left\{d(s_t^{T*}(\xi_t,r_t^h),r_t^h)\xi_t, q_t^{T*}(r_t^h)\right\}}{2\xi_t}, r_t\right) \ge d^{-1}\left(\frac{q_t(r_t)}{\xi_t}, r_t\right) \ge \underline{s}(r_t).$$

By Lemma 1, $pd_1(r)\xi + d^{-1}(\frac{q}{\xi}|r)[q-d_1(r)\xi]^+$ is concave on the line segment between the points $\left(\min\left\{d(s_t^{T*}(\xi_t,r_t^\ell),r_t^\ell)\xi_t, q_t^{T*}(r_t^\ell)\right\}, r_t^\ell\right)$ and $\left(\min\left\{d(s_t^{T*}(\xi_t,r_t^h),r_t^h)\xi_t, q_t^{T*}(r_t^h)\right\}, r_t^h\right)$. Thus,

$$\begin{aligned}
&\pi(s_t(\xi_t|r_t),\xi_t,q_t(r_t),r_t) = pd_1(r_t)\xi_t + d^{-1}\left(\frac{q_t(r_t)}{\xi_t}, r_t\right)[q_t(r_t)-d_1(r_t)\xi_t]^+ \\
&\ge pd_1(r_t)\xi_t + d^{-1}\left(\frac{\min\left\{d(s_t^{T*}(\xi_t,r_t^\ell),r_t^\ell)\xi_t, q_t^{T*}(r_t^\ell)\right\}+\min\left\{d(s_t^{T*}(\xi_t,r_t^h),r_t^h)\xi_t, q_t^{T*}(r_t^h)\right\}}{2\xi_t}, r_t\right) \\
&\qquad \cdot\left[\frac{\min\left\{d(s_t^{T*}(\xi_t,r_t^\ell),r_t^\ell)\xi_t, q_t^{T*}(r_t^\ell)\right\}+\min\left\{d(s_t^{T*}(\xi_t,r_t^h),r_t^h)\xi_t, q_t^{T*}(r_t^h)\right\}}{2} - d_1(r_t)\xi_t\right]^+ \\
&\ge \frac{1}{2}\left\{p\min\{d_1(r_t^\ell)\xi_t, q_t^{T*}(r_t^\ell)\} + d^{-1}\left(\frac{\min\left\{d(s_t^{T*}(\xi_t,r_t^\ell),r_t^\ell)\xi_t, q_t^{T*}(r_t^\ell)\right\}}{\xi_t}\middle| r_t^\ell\right)\left[q_t^{T*}(r_t^\ell)-d_1(r_t^\ell)\xi_t\right]^+\right\} \\
&\quad + \frac{1}{2}\left\{p\min\{d_1(r_t^h)\xi_t, q_t^{T*}(r_t^h)\} + d^{-1}\left(\frac{\min\left\{d(s_t^{T*}(\xi_t,r_t^h),r_t^h)\xi_t, q_t^{T*}(r_t^h)\right\}}{\xi_t}\middle| r_t^h\right)\left[q_t^{T*}(r_t^h)-d_1(r_t^h)\xi_t\right]^+\right\} \\
&= \frac{1}{2}\hat{\pi}(\xi_t,r_t^\ell) + \frac{1}{2}\hat{\pi}(\xi_t,r_t^h).
\end{aligned}$$

□

Proof of Proposition 3. Under the optimal policy $s_t = s^*(\xi_t|r_t)$, $\{r_t : t = 1,2,\cdots\}$ has the Markov property and hence is a Markov process on a compact state space. Furthermore, ξ_t is a continuous random variable on $(0,\infty)$ and $s^*(\xi_t|r_t)$ is continuous. According to Exercise 11.4 in Stokey et al. (1989), the transition function $r_{t+1} = \theta s^*(\xi_t|r_t) + (1-\theta)r_t$ satisfies condition D and hence Theorem 11.10 can be applied. As a result, there exists a unique invariant measure $\Lambda(r)$ (we call it steady-state distribution) and the process is ergodic. □

Proof of Proposition 4. By part 2 of Assumption 2,

$$0 \le \frac{\partial d(s^*(\xi_t|r_t),r_t)}{\partial r_t} = d_1'(r_t) + \left.\frac{\partial d_2(s_t|r_t)}{\partial r_t}\right|_{s_t=s^*(\xi_t|r_t)} + \frac{ds^*(\xi_t|r_t)}{dr_t}\cdot\left.\frac{\partial d_2(s_t|r_t)}{\partial s_t}\right|_{s_t=s^*(\xi_t|r_t)},$$

and hence

$$\frac{ds^*(\xi_t|r_t)}{dr_t} \le \left.\frac{d_1'(r_t)+\frac{\partial d_2(s_t|r_t)}{\partial r_t}}{-\frac{\partial d_2(s_t|r_t)}{\partial s_t}}\right|_{s_t=s^*(\xi_t|r_t)} \le 1,$$

where the last inequality follows from part 1 of Assumption 2.

Thus, $\mathbb{E}\left[\frac{ds^*(\xi_t|r_t)}{dr_t}\right] \le 1$ and $\mathbb{E}\left[s^*(\xi_t|r_t)-r_t|r_t\right]$ decreases in r_t, which implies that $\mathbb{E}\left[r_{t+1}^*|r_t\right]$ as a function of r_t crosses the 45 degree line at most once and from above the line as $\mathbb{E}\left[r_{t+1}^*|r_t=\underline{s}\right] > \underline{s}$. Therefore, there exists Γ such that $\mathbb{E}[r_{t+1}^*|r_t] \ge r_t$ for $r_t \le \Gamma$ and $\mathbb{E}[r_{t+1}^*|r_t] < r_t$ for $r_t > \Gamma$, and r_t is mean-reverting. □

Proof of Proposition 5. We only need to consider the case where the reference is p in season 1. If a retailer never marks down, then $r_t = p$ for all t and the retailer will order $\tilde{q} = d(p,p)F^{-1}\left(\frac{p-c}{p}\right)$, a newsvendor solution.

Consider the marginal gain at stage 2 of season 1 as a function of the order quantity q_1 and demand realization ξ_1 if the retailer marks down

$$-\left.\frac{\partial\{s_1 \min\{d_2(s_1,p)\xi_1, q_1 - d_1(p)\xi_1\}\}}{\partial s_1}\right|_{s_1=p} = \begin{cases} -\xi_1 \left.\frac{\partial\{s_1 d_2(s_1,p)\}}{\partial s_1}\right|_{s_1=p}, & \text{if } \xi_1 < \frac{q_1}{d(p,p)}, \\ -[q_1 - d_1(p)\xi_1] \leq 0, & \text{if } \frac{q_1}{d(p,p)} \leq \xi_1 < \frac{q_1}{d_1(p)}, \\ 0, & \text{if } \xi_1 \geq \frac{q_1}{d_1(p)}. \end{cases}$$

Thus, the retailer will never mark down if $\left.\frac{\partial\{s_1 d_2(s_1,p)\}}{\partial s_1}\right|_{s_1=p} \geq 0$ as markdown will only lower future profit, which is equivalent to $\underline{s}(p) = p$. Now suppose that $\left.\frac{\partial\{s_1 d_2(s_1,p)\}}{\partial s_1}\right|_{s_1=p} < 0$, in which case the maximum marginal gain from a markdown is less than $-\frac{\tilde{q}}{d(p,p)} \left.\frac{\partial\{s_1 d_2(s_1,p)\}}{\partial s_1}\right|_{s_1=p}$. The retailer will never mark down if the minimum marginal loss of the future revenue is larger than the maximum marginal gain at stage 2 in season 1.

Since the reference in season t, $t = 2,3,\cdots$, is given by

$$r_t = \theta s_{t-1} + \theta(1-\theta)s_{t-2} + \cdots + \theta(1-\theta)^{t-2}s_1 + (1-\theta)^{t-1}p, \tag{7}$$

a markdown in season 1 reduces the references for all subsequent seasons. Let $S_t(s_1,\xi)$ be the set of the optimal decisions from season 2 to t, given the markdown price in season 1 is s_1 and a sample path of the demand up to season t. Consider the marginal profit loss in season t from a markdown in season 1, $s_1 < p$, as a function of ξ_t. When $\xi_t < \xi_t^\infty(q^*(r_t), r_t)$, there is unsold inventory in season t and the marginal loss

$$\begin{aligned} &\left.\frac{\partial\left\{pd_1(r_t)\xi_t - cq^*(r_t) + s^*(\xi_t|r_t)d_2(s^*(\xi_t|r_t),r_t)\xi_t\right\}}{\partial s_1}\right|_{s_1=p} \\ &\geq \left(\frac{\partial\left\{pd_1(r_t)\xi_t + s_t d_2(s_t|r_t)\xi_t\right\}}{\partial r_t} \cdot \left.\frac{\partial r_t}{\partial s_1}\right|_{s_1=p}\right)_{\substack{(q_i,s_i)\in S_t(s_1,\xi) \\ i=1,\cdots,t}} \\ &= \theta(1-\theta)^{t-2}\left[pd_1'(r_t)\xi_t + s_t\frac{\partial d_2(s_t|r_t)}{\partial r_t}\xi_t\right]_{\substack{(q_i,s_i)\in S_t(s_1,\xi) \\ i=1,\cdots,t}} \\ &\geq \theta(1-\theta)^{t-2}\left[pd_1'(p)\xi_t + s_t\xi_t\left.\frac{\partial d_2(s_t|r_t)}{\partial r_t}\right|_{r_t=p}\right]_{\substack{(q_i,s_i)\in S_t(s_1,\xi) \\ i=1,\cdots,t}}. \end{aligned}$$

The first inequality follows as the marginal profit loss is at least that if the optimal decisions $S_t(s_1,\xi)$ are adopted even when the retailer does not mark down in season 1, while the second and third inequalities follow from (7) and the concavity of $d_1(r)$ and $d(s,r)$.

When $\xi_t^\infty(q^*(r_t), r_t) \le \xi_t < \frac{q^*(r_t)}{d(p,r_t)}$, all the remaining inventory $q^*(r_t) - d_1(r_t)\xi_t$ is sold in stage 2. Following a similar argument, we have that the marginal loss in season t

$$\begin{aligned}
&\left.\frac{\partial\left\{pd_1(r_t)\xi_t - cq^*(r_t) + s^*(\xi_t|r_t)[q^*(r_t) - d_1(r_t)\xi_t]\right\}}{\partial s_1}\right|_{s_1=p} \\
&\ge \left(\frac{\partial\left\{pd_1(r_t)\xi_t + s_t[q_t - d_1(r_t)\xi_t]\right\}}{\partial r_t} \cdot \left.\frac{\partial r_t}{\partial s_1}\right|_{s_1=p}\right)_{\substack{(q_i,s_i)\in S_t(s_1,\xi)\\ i=1,\cdots,t}} \\
&= \theta(1-\theta)^{t-2}\left[(p-s_t)d_1'(r_t)\xi_t\right]_{\substack{(q_i,s_i)\in S_t(s_1,\xi)\\ i=1,\cdots,t}} \ge 0.
\end{aligned}$$

When $\xi_t \ge \frac{q^*(r_t)}{d(p,r_t)}$, all $q_t^*(r_t)$ will be sold in stage 1 and the marginal loss in season t becomes

$$\left.\frac{\partial\left\{(p-c)q_t^*(r_t)\right\}}{\partial s_1}\right|_{s_1=p} \ge \left(\left.\frac{\partial\{pq_t\}}{\partial r_t}\right|_{q_t=q^*(r_t)} \cdot \left.\frac{\partial r_t}{\partial s_1}\right|_{s_1=p}\right)_{\substack{(q_i,s_i)\in S_t(s_1,\xi)\\ i=1,\cdots,t}} = 0.$$

Let $\underline{s} = \min\limits_{0\le r\le p}\{\underline{s}(r)\}$. Then, it is never optimal for the retailer to offer a discount price lower than $\underline{s}$ and hence, the reference will never be lower than $\underline{s}$. Note that

$$\begin{aligned}
\xi_t^\infty(q^*(r_t), r_t) &= \frac{q^*(r_t)}{d\left(s^*\left(\xi_t^\infty(q^*(r_t), r_t)\,\middle|\, r_t\right), r_t\right)} \ge \frac{q^*(r_t)}{d(\underline{s}(r_t), r_t)} = \frac{q^*(r_t)}{d(p,r_t)}\frac{d(p,r_t)}{d(\underline{s}(r_t), r_t)} \ge \frac{q^*(r_t)}{d(p,r_t)} \cdot \frac{d(p,r_t)}{d(\underline{s}, r_t)} \\
&\ge \frac{q^*(r_t)}{d(p,r_t)} \cdot \min_{\underline{s}\le r_t\le p}\left\{\frac{d(p,r_t)}{d(\underline{s}, r_t)}\right\} \ge F^{-1}\left(\frac{p-c}{p}\right) \cdot \frac{d(p,\underline{s})}{d(\underline{s},\underline{s})}
\end{aligned}$$

where the last inequality follows from the fact that $\frac{q^*(r_t)}{d(p,r_t)} > F^{-1}\left(\frac{p-c}{p}\right)$. The total marginal loss of a markdown in season 1 at s_1 is at least

$$\begin{aligned}
&\sum_{t=2}^\infty \beta^{t-1}\theta(1-\theta)^{t-2}\mathbb{E}\left\{\left[\left(pd_1'(p)\xi_t + s_t\xi_t \left.\frac{\partial d_2(s_t|r_t)}{\partial r_t}\right|_{r_t=p}\right) \cdot \mathbf{1}\{\xi_t < \xi_t^\infty(q_t, r_t)\}\right]_{\substack{(q_i,s_i)\in S_t(s_1,\xi)\\ i=1,\cdots,t}}\right\} \\
&\ge \sum_{t=2}^\infty \beta^{t-1}\theta(1-\theta)^{t-2}\mathbb{E}\left\{\left[pd_1'(p) + \min_{\underline{s}<s\le p}\left\{s\left.\frac{\partial d_2(s|r)}{\partial r}\right|_{r=p}\right\}\right]\xi_t \cdot \mathbf{1}\left\{\xi_t < F^{-1}\left(\frac{p-c}{p}\right) \cdot \frac{d(p,\underline{s})}{d(\underline{s},\underline{s})}\right\}\right\} \\
&\ge \frac{\theta}{\beta^{-1} - (1-\theta)}\left[pd_1'(p) + \min_{\underline{s}<s\le p}\left\{s\left.\frac{\partial d_2(s|r)}{\partial r}\right|_{r=p}\right\}\right]\int_0^{F^{-1}\left(\frac{p-c}{p}\right)\frac{d(p,\underline{s})}{d(\underline{s},\underline{s})}} x\,dF(x).
\end{aligned}$$

The retailer should never mark down if this minimum marginal future loss is greater than the maximum marginal gain in season t, $-\frac{\tilde{q}}{d(p,p)}\left.\frac{\partial\{s_1 d_2(s_1,p)\}}{\partial s_1}\right|_{s_1=p}$, which yields (2). □

Proof of Proposition 6. Let $\bar{q}(r_t)$ and $\bar{\pi}(q_t, r_t)$ be the optimal single season profit of the myopic retailer for given (q_t, r_t). Consider $\frac{\partial v(q_t,r_t)}{\partial q_t} = \frac{\partial\bar{\pi}(q_t,r_t)}{\partial q_t} + \beta\mathbb{E}\left[\frac{\partial V(\theta s_t^\infty(\xi_t,q_t,r_t)+(1-\theta)r_t)}{\partial q_t}\right]$. Since $s_t^\infty(\xi_t, q_t, r_t)$ decreases in q_t and $V(r)$ increases in r, $\frac{\partial\bar{\pi}(q_t,r_t)}{\partial q_t} \ge \frac{\partial v(q_t,r_t)}{\partial q_t}$ and hence $\bar{q}(r_t) \ge q^*(r_t)$. Given ξ_t, the myopic retailer's problem is to maximize $s_t \cdot \min\{d_2(s_t|r_t)\xi_t, [q_t - d_1(r_t)\xi_t]^+\}$, while the retailer's optimal decisions are obtained by maximizing $s_t \cdot \min\{d_2(s_t|r_t)\xi_t, [q_t - d_1(r_t)\xi_t]^+\} + \beta V(\theta s_t + (1-\theta)r_t)$. It is obvious that $\bar{s}(\xi_t, r_t) \leq s_t^\infty(\xi_t, \bar{q}(r_t), r_t)$ since $V(r_t)$ is increasing in r_t. Therefore, $\bar{s}(\xi_t, r_t) \leq s_t^\infty(\xi_t, \bar{q}(r_t), r_t) \leq s_t^\infty(\xi_t, q^*(r_t), r_t) = s^*(\xi_t|r_t)$ for any ξ_t. □

Proof of Proposition 7. With deterministic demand, the retailer will order exactly the amount to be sold, i.e., $q^*(r_t) = d(s^*(r_t), r_t)$ and there is only one decision variable in each season. Thus, the single season profit function becomes $(p-c)d_1(r_t) + (s_t - c)d_2(s_t|r_t)$, which is concave in s_t. Next, we show that the single season profit function is jointly concave in (s_t, r_t) when $\frac{\partial\{(s_t-c)d_2(s_t|r_t)\}}{\partial s_t} \leq 0$. By Assumption 1.(2), the single season revenue function is jointly concave, i.e.,

$$\left[pd_1''(r_t) + s_t\frac{\partial^2 d_2(s_t|r_t)}{\partial r_t^2}\right]\left[s_t\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t^2} + 2\frac{\partial d_2(s_t|r_t)}{\partial s_t}\right] - \left[s_t\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t \partial r_t} + \frac{\partial d_2(s_t|r_t)}{\partial r_t}\right]^2 \geq 0.$$

Multiplying both sides by $\left(\frac{s_t-c}{s_t}\right)^2$, we have

$$\begin{aligned}
0 &\leq \left(\frac{s_t-c}{s_t}\right)^2\left[pd_1''(r_t) + s_t\frac{\partial^2 d_2(s_t|r_t)}{\partial r_t^2}\right]\left[s_t\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t^2} + 2\frac{\partial d_2(s_t|r_t)}{\partial s_t}\right] - \left(\frac{s_t-c}{s_t}\right)^2\left[s_t\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t \partial r_t} + \frac{\partial d_2(s_t|r_t)}{\partial r_t}\right]^2 \\
&= \left[\left(p - \frac{p}{s_t}c\right)d_1''(r_t) + (s_t-c)\frac{\partial^2 d_2(s_t|r_t)}{\partial r_t^2}\right]\left[\frac{(s_t-c)\partial^2 d_2(s_t|r_t)}{\partial s_t^2} + 2\frac{s_t-c}{s_t}\frac{\partial d_2(s_t|r_t)}{\partial s_t}\right] - \left[(s_t-c)\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t \partial r_t} + \frac{s_t-c}{s_t}\frac{\partial d_2(s_t|r_t)}{\partial r_t}\right]^2 \\
&\leq \left[(p-c)d_1''(r_t) + (s_t-c)\frac{\partial^2 d_2(s_t|r_t)}{\partial r_t^2}\right]\left[\frac{(s_t-c)\partial^2 d_2(s_t|r_t)}{\partial s_t^2} + 2\frac{\partial d_2(s_t|r_t)}{\partial s_t}\right] - \left[(s_t-c)\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t \partial r_t} + \frac{\partial d_2(s_t|r_t)}{\partial r_t}\right]^2
\end{aligned}$$

which is exactly the determinant of the Hessian of the single season profit function. The last inequality follows as $d_1''(r_t) \leq 0$, $\frac{\partial d_2(s_t|r_t)}{\partial s_t} \leq 0$, $\frac{\partial d_2(s_t|r_t)}{\partial r_t} \leq 0$, and $\frac{-\frac{\partial d_2(s_t|r_t)}{\partial s_t}}{d_2(s_t|r_t)}$ decreases in r_t, which implies that $\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t \partial r_t}d_2(s_t|r_t) \geq \frac{\partial d_2(s_t|r_t)}{\partial r_t}\frac{\partial d_2(s_t|r_t)}{\partial s_t}$ and hence

$$(s_t-c)\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t \partial r_t} + \frac{s_t-c}{s_t}\frac{\partial d_2(s_t|r_t)}{\partial r_t} \geq (s_t-c)\frac{\partial^2 d_2(s_t|r_t)}{\partial s_t \partial r_t} + \frac{\partial d_2(s_t|r_t)}{\partial r_t} \geq \frac{\frac{\partial d_2(s_t|r_t)}{\partial r_t}}{d_2(s_t|r_t)} \cdot \frac{\partial\{(s_t-c)d_2(s_t|r_t)\}}{\partial s_t} \geq 0,$$

and by Assumption 1. Thus, the Hessian is negative semi-definite and the single season profit function is also jointly concave in (s_t, r_t) for $\frac{\partial\{(s_t-c)d_2(s_t|r_t)\}}{\partial s_t} \leq 0$. Following a similar argument as in the proof of Propostion 2, we can show the concavity of $V(r_t)$.

We now prove the monotonicity of the convergence of the reference price. Since $r_{t+1} = \theta s_t + (1-\theta)r_t$, rewrite $V(r_t)$ as

$$V(r_t) = \max_{\theta \underline{s}(r_t) + (1-\theta)r_t \leq r_{t+1} \leq \theta p + (1-\theta)r_t} \left\{u(r_t, r_{t+1}) + \beta V(r_{t+1})\right\},$$

and

$$u(r_t, r_{t+1}) = (p-c)d_1(r_t) + \left[\frac{r_{t+1} - (1-\theta)r_t}{\theta} - c\right]d_2\left(\frac{r_{t+1} - (1-\theta)r_t}{\theta}\Big| r_t\right).$$

Since

$$\frac{\partial^2 u(r_t, r_{t+1})}{\partial r_t \partial r_{t+1}} = \frac{1}{\theta}\left[\frac{\partial^2\{(s_t - c)d_2(s_t|r_t)\}}{\partial s_t \partial r_t} - \frac{(1-\theta)}{\theta}\frac{\partial^2\{(s_t - c)d_2(s_t|r_t)\}}{\partial s_t^2}\right]_{s_t = \frac{r_{t+1} - (1-\theta)r_t}{\theta}} \geq 0,$$

as the unconstrained single season profit function $(p-c)d_1(r_t) + (s_t - c)d_2(s_t|r_t)$ is concave and $\frac{\partial^2\{(s_t-c)d_2(s_t|r_t)\}}{\partial s_t \partial r_t} \geq \frac{\frac{\partial d_2(s_t|r_t)}{\partial r_t}}{d_2(s_t|r_t)} \times \frac{\partial\{(s_t-c)d_2(s_t|r_t)\}}{\partial s_t} \geq 0$ by Assumption 1, $u(r_t, r_{t+1})$ is supermodular. By Theorem 2.8.2 in Topkis (1998), $r^*_{t+1}(r_t) = \theta s^*(r_t) + (1-\theta)r_t$ increases in r_t and the reference price converges monotonically for a given initial reference. □

Proof of Proposition 8. Let

$$\begin{aligned} e(r) &= \frac{1}{\beta}\left[\frac{\partial u(r_t, r_{t+1})}{\partial r_{t+1}} + \beta \frac{\partial u(r_t, r_{t+1})}{\partial r_t}\right]_{r_t = r_{t+1} = r} \\ &= (p-c)d_1'(r) + (r-c)\left.\frac{\partial d_2(s|r)}{\partial r}\right|_{s=r} + \left(1 + \frac{1-\beta}{\beta\theta}\right)\left((r-c)\left.\frac{\partial d_2(s|r)}{\partial s}\right|_{s=r} + d_2(r|r)\right). \end{aligned}$$

The proof follows because a boundary point 0 or p is a steady state if $e(0) \leq 0$ or $e(p) \geq 0$, respectively, and an interior point r is a steady state if it satisfies the Euler equation of problem (3) $e(r) = 0$. Since $d_1'(0) > 0$ and $-c\left.\frac{\partial d_2(s|0)}{\partial s}\right|_{s=0} + d_2(0|0) \geq 0$, the lower bound 0 cannot be a steady state. □

References

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