Submitted to *Operations Research*
manuscript OPRE-2013-08-441.R2

Online Appendix for Unemployment Risks and Optimal Retirement in an Incomplete Market

Alain Bensoussan
Naveen Jindal School of Management, University of Texas at Dallas, Richardson, Texas 75080-3021, and Department of Systems Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong SAR, alain.bensoussan@utdallas.edu

Bong-Gyu Jang
Department of Industrial and Management Engineering, POSTECH, Pohang, Korea 37673, bonggyujang@postech.ac.kr

Seyoung Park
Department of Industrial and Management Engineering, POSTECH, Pohang, Korea 37673, and The Credit Finance Research Institute, The Credit Finance Association of Korea, Seoul, Korea 04521, seyoungpark86@gmail.com

We develop a new approach for solving the optimal retirement problem for an individual with an unhedgeable income risk. The income risk stems from a forced unemployment event, which occurs as an exponentially-distributed random shock. The optimal retirement problem is to determine an individual's optimal consumption and investment behaviors and optimal retirement time simultaneously. We introduce a new convex-duality approach for reformulating the original retirement problem and provide an iterative numerical method to solve it. Reasonably calibrated parameters say that our model can give an explanation for lower consumption and risky investment behaviors of individuals, and for relatively higher stock holdings of the poor. We also analyze the sensitivity of an individual's optimal behavior in changing her wealth level, investment opportunity, and the magnitude of preference for post-retirement leisure. Finally, we find that our model explains a counter-cyclical pattern of the number of unemployed job leavers.

Key words: dynamic programming/optimal control, investment, stochastic model applications
History: This paper was first submitted on August 6, 2013 and has been with the authors for 3 years for 2 revisions.

1. The Existence of a Solution

Recall that our main problem is formulated by the following: for $\lambda > \hat{\lambda}$,

$$
\begin{aligned}
&-\frac{1}{2}\theta^2\lambda^2 G''(\lambda) - \lambda G'(\lambda)(\theta^2+\beta+\delta-r) + rG(\lambda) + \frac{\delta}{\beta}\frac{G'(\lambda)}{G(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\lambda) > \frac{I_1}{r}\} = \frac{1}{\lambda},\\
&G(\hat{\lambda}) = \frac{1}{\beta\hat{\lambda}} + \frac{I_1 - I_2}{r},\\
&G(\infty) = 0.
\end{aligned}
\tag{1}
$$

We assume that the subjective discount rate β equals to the risk-free interest rate r. This assumption is only used when verifying the existence of a solution to (1). Actually, if one wants to relax it, it is necessary to take a restriction on the free boundary $\hat{\lambda}$ to verify the existence. By taking the assumption of $\beta = r$, we can show the existence of a solution to (1) for *any* $\hat{\lambda} > 0$. Most importantly, without the verification for the existence of a solution we could find a numerical solution satisfying (1) in a wide range of parameters. Specifically, we can always write a general solution G satisfying (17) as follows:

$$G(\lambda) = \frac{1}{\lambda(\beta+\delta)} + A(\lambda)\lambda^{-\alpha_\delta} + A^*(\lambda)\lambda^{-\alpha_\delta^*}$$

subject to

$$A'(\lambda)\lambda^{-\alpha_\delta} + (A^*(\lambda))'\lambda^{-\alpha_\delta^*} = 0,$$

where $\alpha_\delta > 0$ and $\alpha_\delta^* < 0$ are the two roots of the characteristic equation

$$F(\alpha;\delta) \equiv -\frac{1}{2}\theta^2\alpha(\alpha-1) + \alpha(\beta+\delta-r) + r = 0.$$

We suggest a Theorem concerning the existence for the solution.

THEOREM 1. *We assume that $\beta = r$. Then for any $\hat{\lambda} > 0$, there exists a solution to the problem formulated by equations in (1), which has the boundedness such that*

$$\int_{\hat{\lambda}}^{\infty} \lambda^{1-n}(G'(\lambda))^2 d\lambda \le C_n^*$$

and

$$\int_{\hat{\lambda}}^{\infty} \lambda^{3-n}(G''(\lambda))^2 d\lambda \le C_n^{**},$$

where n is a natural number, C_n^* and C_n^{**} are constants.

Proof. Let us assume that we have a solution to the problem formulated by equations in (1). Then G cannot have a maximum at $\lambda^* > \hat{\lambda}$. If such a point λ^* exists, then

$$G(\lambda^*) \ge \frac{1}{\beta\hat{\lambda}} + \frac{I_1 - I_2}{r}, \tag{2}$$

$$G'(\lambda^*) = 0, \quad \text{and} \quad G''(\lambda^*) < 0.$$

The first equation in (1) shows that

$$rG(\lambda^*) < \frac{1}{\lambda^*} < \frac{1}{\hat{\lambda}}.$$

However this is contradict to the inequality (2) because

$$rG(\lambda^*) \ge \frac{1}{\hat{\lambda}},$$

which is obtained from the assumption that $\beta = r$ and the relationship that $I_1 > I_2$. The function G cannot have a negative minimum at $\lambda^{**} > \hat{\lambda}$. If such a point λ^{**} exists, or equivalently,

$$G(\lambda^{**}) < 0, \tag{3}$$

$$G'(\lambda^{**}) = 0, \quad \text{and} \quad G''(\lambda^{**}) > 0,$$

then the first equation in (1) shows that

$$rG(\lambda^{**}) > \frac{1}{\lambda^{**}} > 0,$$

which is contradict to the inequality (3). Therefore, G satisfies

$$0 < G(\lambda) < \frac{1}{\beta\hat{\lambda}} + \frac{I_1 - I_2}{r} \tag{4}$$

for any $\lambda > \hat{\lambda}$. Because we know that

$$0 \leq \frac{\mathbf{1}\{G(\lambda) > \frac{I_1}{r}\}}{G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}} < \frac{r}{I_2}, \tag{5}$$

we can easily verify the boundeness of G. When we multiply the first equation in (1) by $\dfrac{G(\lambda) - G(\hat{\lambda})}{\lambda^{1+n}}$ for a natural number n and use integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\theta^2 \int_{\hat{\lambda}}^{\infty} G'(\lambda) \frac{d}{d\lambda}\Big\{\Big(G(\lambda) - G(\hat{\lambda})\Big)\lambda^{1-n}\Big\} d\lambda \\
&\quad + \int_{\hat{\lambda}}^{\infty} G'(\lambda) \frac{G(\lambda) - G(\hat{\lambda})}{\lambda^{1+n}} \Big[-\lambda(\theta^2 + \beta + \delta - r) + \frac{\delta}{\beta} \frac{\mathbf{1}\{G(\lambda) > \frac{I_1}{r}\}}{G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}} \Big] d\lambda \\
&\quad = \int_{\hat{\lambda}}^{\infty} \Big(\frac{1}{\lambda} - rG(\lambda)\Big) \frac{G(\lambda) - G(\hat{\lambda})}{\lambda^{1+n}} d\lambda.
\end{aligned}$$

By some calculations, we get

$$\begin{aligned}
&\frac{1}{2}\theta^2 \int_{\hat{\lambda}}^{\infty} (G'(\lambda))^2 \lambda^{1-n} d\lambda \\
&\quad + \int_{\hat{\lambda}}^{\infty} G'(\lambda) \frac{G(\lambda) - G(\hat{\lambda})}{\lambda^{1+n}} \Big[-\lambda\Big(\frac{1+n}{2}\theta^2 + \beta + \delta - r\Big) + \frac{\delta}{\beta} \frac{\mathbf{1}\{G(\lambda) > \frac{I_1}{r}\}}{G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}} \Big] d\lambda \\
&\quad = \int_{\hat{\lambda}}^{\infty} \Big(\frac{1}{\lambda} - rG(\lambda)\Big) \frac{G(\lambda) - G(\hat{\lambda})}{\lambda^{1+n}} d\lambda.
\end{aligned}$$

Using the inequalities (4), (5), and Cauchy-Schwartz inequality to deal with the second term of the left hand side we can show that

$$\int_{\hat{\lambda}}^{\infty} \lambda^{1-n} (G'(\lambda))^2 d\lambda \leq C_n^*, \tag{6}$$

where C_n^*, $n=1,2,3,...$, are constants. By multiplying the first equation in (1) by $\lambda^{-\frac{1+n}{2}}$, we also get

$$-\frac{1}{2}\theta^2\lambda^{\frac{3-n}{2}}G''(\lambda)-\lambda^{\frac{1-n}{2}}G'(\lambda)(\theta^2+\beta+\delta-r)+rG(\lambda)\lambda^{-\frac{1+n}{2}}+\frac{\delta}{\beta}G'(\lambda)\frac{\mathbf{1}\{G(\lambda)>\frac{I_1}{r}\}}{G(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}}\lambda^{-\frac{1+n}{2}}=\lambda^{-\frac{3+n}{2}}.$$

Therefore, utilizing the inequality (6) we can show that

$$\int_{\hat{\lambda}}^{\infty}\lambda^{3-n}(G''(\lambda))^2d\lambda\leq C_n^{**}, \tag{7}$$

where C_n^{**}, $n=1,2,3,...$, are constants.

Let us now move on the existence of a solution to the problem specified by equations in (1). We begin by proving the existence of a solution of a locally smoothed region of the problem given by (1). Then we consider the following problem:

$$-\frac{1}{2}\theta^2\lambda^2G_\epsilon''(\lambda)-\lambda G_\epsilon'(\lambda)(\theta^2+\beta+\delta-r)+rG_\epsilon(\lambda)+\frac{\delta}{\beta}G_\epsilon'(\lambda)\frac{\left(G_\epsilon(\lambda)-\frac{I_1}{r}\right)^+}{\left(G_\epsilon(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}\right)\left(G_\epsilon(\lambda)-\frac{I_1}{r}+\epsilon\right)}=\frac{1}{\lambda},$$
$$G_\epsilon(\hat{\lambda})=\frac{1}{\beta\hat{\lambda}}+\frac{I_1-I_2}{r},\quad G_\epsilon(\infty)=0, \tag{8}$$

where

$$\left(G_\epsilon(\lambda)-\frac{I_1}{r}\right)^+=\max\left\{G_\epsilon(\lambda)-\frac{I_1}{r},0\right\}.$$

To prove the existence of a solution to the problem formulated by (8), we rely on a fixed point argument. Specifically, we use the Leray-Schauder fixed point theorem. We consider the Hilbert space $\mathcal{H}_n$ of measurable function ψ defined on $(\hat{\lambda},\infty)$ such that

$$\mathcal{H}_n=\left\{\psi\Big|\int_{\hat{\lambda}}^{\infty}\frac{\psi^2(\lambda)}{\lambda^{1+n}}d\lambda<\infty\right\}.$$

We also consider the set $\mathcal{W}_n$ in $\mathcal{H}_n$ as

$$\mathcal{W}_n=\Big\{\psi\in\mathcal{H}_n\Big|0\leq\psi\leq\frac{1}{\beta\hat{\lambda}}+\frac{I_1-I_2}{r},$$
$$\int_{\hat{\lambda}}^{\infty}(\psi'(\lambda))^2\lambda^{1-n}d\lambda\leq C_n^*,\int_{\hat{\lambda}}^{\infty}\lambda^{3-n}(\psi''(\lambda))^2d\lambda\leq C_n^{**}\Big\}.$$

where C_n^* and C_n^{**} are the constants defined in Theorem 1. Then the set $\mathcal{W}_n$ is a convex and compact subset of $\mathcal{H}_n$. This is because if $\psi\in\mathcal{W}_n$, then ψ is C^1 and ψ' is bounded. Therefore, if $\psi_m\in\mathcal{W}_n$, then ψ_m, ψ_m' remain bounded. We know that for a constant $\tilde{C}$,

$$\int_{\hat{\lambda}}^{\infty}\frac{\psi_m^2(\lambda)}{\lambda^{1+n}}d\lambda\leq\tilde{C}$$

and that

$$\int_{\hat{\lambda}}^{\infty}(\psi_m'(\lambda))^2\lambda^{1-n}d\lambda\leq C_n^*.$$

Then we can consider a subsequence ψ_m such that

$$\psi_m \to \psi \quad \text{in} \quad \mathcal{H}_n \quad \text{weakly},$$

$$\lambda\psi'_m \to \lambda\psi' \quad \text{in} \quad \mathcal{H}_n \quad \text{weakly},$$

where ψ_m converges pointwise to ψ. Because ψ_m is bounded,

$$\psi_m \to \psi \quad \text{in} \quad \mathcal{H}_n \quad \text{strongly}.$$

Therefore, $\mathcal{W}_n$ is obviously non-empty and convex.

For any $\psi \in \mathcal{H}_n$, we consider a mapping $\Gamma(\psi) : \mathcal{W}_n \to \mathcal{W}_n$ such that

$$\begin{aligned}&-\frac{1}{2}\theta^2\lambda^2\chi''(\lambda) - \lambda\chi'(\lambda)(\theta^2+\beta+\delta-r) + r\chi(\lambda) + \frac{\delta}{\beta}\chi'(\lambda)\frac{\left(\psi(\lambda)-\frac{I_1}{r}\right)^+}{\left(\psi(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}\right)\left(\psi(\lambda)-\frac{I_1}{r}+\epsilon\right)} = \frac{1}{\lambda},\\&\chi(\hat{\lambda}) = \frac{1}{\beta\hat{\lambda}} + \frac{I_1-I_2}{r}, \quad \chi(\infty)=0,\end{aligned} \tag{9}$$

where

$$\chi \equiv \Gamma(\psi).$$

Because

$$0 \le \frac{\left(\psi(\lambda)-\frac{I_1}{r}\right)^+}{\left(\psi(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}\right)\left(\psi(\lambda)-\frac{I_1}{r}+\epsilon\right)} < \frac{r}{I_2},$$

it is easy to show that the inequalities (4), (6), (7) are valid for χ regardless of ψ. As a result, $\chi \in \mathcal{W}_n$ and Γ maps $\mathcal{W}_n$ into $\mathcal{W}_n$. The map Γ is continuous for the $\mathcal{H}_n$ norm. If $\psi_m \to \psi$ in $\mathcal{H}_n$, then we can consider a subsequence that converges pointwise. Moreover, $\chi_m \equiv \Gamma(\psi_m)$ is in $\mathcal{W}_n$, accordingly we can extract a subsequence such that

$$\chi_m \to \chi \quad \text{pointwise},$$

$$\chi'_m \to \chi' \quad \text{pointwise},$$

$$\lambda^{\frac{3-n}{2}}\chi''_m \to \lambda^{\frac{3-n}{2}}\chi'' \quad \text{in} \quad \mathcal{H}_n \quad \text{weakely}.$$

Considering the problem given by (9) with ψ_m, χ_m, it is easy to check that $\chi = \Gamma(\psi)$ by taking the limit. By utilizing the Leray-Schauder fixed point theorem, we can conclude that the map Γ has a fixed point, as a result, the point becomes a solution to the problem specified by (8).

The solution to the problem formulated by (8) is in $\mathcal{W}_n$. Therefore, as $\epsilon \to 0$, we can consider a subsequence such that

$$\begin{aligned}&G_\epsilon(\lambda) \to G(\lambda) \quad \text{pointwise},\\&G'_\epsilon(\lambda) \to G'(\lambda) \quad \text{pointwise},\\&\lambda^{\frac{3-n}{2}}G''_\epsilon(\lambda) \to \lambda^{\frac{3-n}{2}}G''(\lambda) \quad \text{in} \quad \mathcal{H}_n \quad \text{weakly}.\end{aligned} \tag{10}$$

We define

$$\iota_\epsilon(\lambda) \equiv G'_\epsilon(\lambda) \frac{\Big(G_\epsilon(\lambda) - \frac{I_1}{r}\Big)^+}{\Big(G_\epsilon(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}\Big)\Big(G_\epsilon(\lambda) - \frac{I_1}{r} + \epsilon\Big)}.$$

Then

$$|\iota_\epsilon(\lambda)| \le |G'_\epsilon(\lambda)| \frac{r}{I_2}.$$

Therefore, we obtain that

$$\int_{\hat{\lambda}}^\infty |\iota_\epsilon(\lambda)|^2 \lambda^{1-n} d\lambda \le C_n^*$$

and subsequently,

$$\int_{\hat{\lambda}}^\infty |\iota_\epsilon(\lambda)|^2 d\lambda \le \tilde{D},$$

where $\tilde{D}$ is a constant. By considering a new subsequence we can argue that

$$\iota_\epsilon(\lambda) \to \iota(\lambda) \;\text{ in }\; L^2(\hat{\lambda}, \infty) \;\text{ weakly.}$$

We rewrite ι_ϵ as

$$\begin{aligned}
\iota_\epsilon(\lambda) &= G'_\epsilon(\lambda) \frac{\mathbf{1}\{G(\lambda) > \frac{I_1}{r}\}}{\frac{I_2}{r} - \epsilon} \Big(\frac{\frac{I_2}{r}}{G_\epsilon(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}} - \frac{\epsilon}{G_\epsilon(\lambda) - \frac{I_1}{r} + \epsilon} \Big) \\
&= \frac{1}{\frac{I_2}{r} - \epsilon} \Big[\frac{I_2}{r} \frac{d}{d\lambda} \ln \Big\{ \Big(G_\epsilon(\lambda) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r} \Big\} - \epsilon \frac{d}{d\lambda} \ln \Big\{ \Big(G_\epsilon(\lambda) - \frac{I_1}{r}\Big)^+ + \epsilon \Big\} \Big].
\end{aligned}$$

If we take a function $\varphi \in C_0^\infty(\hat{\lambda}, \infty)$, then we can obtain

$$\begin{aligned}
\int_{\hat{\lambda}}^\infty \iota_\epsilon(\lambda)\varphi(\lambda) d\lambda = &-\frac{\frac{I_2}{r}}{\frac{I_2}{r} - \epsilon} \int_{\hat{\lambda}}^\infty \varphi'(\lambda) \ln \Big\{ \Big(G_\epsilon(\lambda) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r} \Big\} d\lambda \\
&+ \frac{\epsilon}{\frac{I_2}{r} - \epsilon} \int_{\hat{\lambda}}^\infty \varphi'(\lambda) \ln \Big\{ \Big(G_\epsilon(\lambda) - \frac{I_1}{r}\Big)^+ + \epsilon \Big\} d\lambda.
\end{aligned}$$

Because $G_\epsilon(\lambda)$ is bounded and converges pointwise to $G(\lambda)$, we can get

$$\int_{\hat{\lambda}}^\infty \iota_\epsilon(\lambda)\varphi(\lambda) d\lambda \to -\int_{\hat{\lambda}}^\infty \varphi'(\lambda) \ln \Big\{ \Big(G(\lambda) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r} \Big\} d\lambda \;\text{ as }\; \epsilon \to 0.$$

The right hand side in the above is equivalent to

$$\int_{\hat{\lambda}}^\infty \varphi(\lambda) G'(\lambda) \frac{\mathbf{1}\{G(\lambda) > \frac{I_1}{r}\}}{G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}} d\lambda.$$

Hence, we can conclude that

$$\iota(\lambda) = G'(\lambda) \frac{\mathbf{1}\{G(\lambda) > \frac{I_1}{r}\}}{G(\lambda) - \frac{I_1}{r} + \frac{I_2}{r}}.$$

As a result, we can lead to the limit in (8) and obtain that $G(\lambda)$ satisfies (1).

It remains to check that whether or not G satisfies the condition $G(\infty) = 0$. We first note that

$$-G'_\epsilon(\lambda) = \int_{\hat{\lambda}}^\infty G''_\epsilon(\lambda) d\lambda.$$

Next we have that

$$\frac{d}{d\lambda}\Big((G'_\epsilon(\lambda))^2\lambda^{2-n}\Big) = (2-n)\lambda^{1-n}(G'_\epsilon(\lambda))^2 + 2\lambda^{2-n}G'_\epsilon(\lambda)G''_\epsilon(\lambda)$$
$$= (2-n)\lambda^{1-n}(G'_\epsilon(\lambda))^2 + 2\lambda^{\frac{1-n}{2}}G'_\epsilon(\lambda)^{\frac{3-n}{2}}G''_\epsilon(\lambda).$$

Utilizing the inequalities (6), (7), we get

$$(G'_\epsilon(\lambda))^2\lambda^{2-n} \le D_n^*,$$

where D_n^*, $n=1,2,3,...$, are constants. Therefore, we obtain

$$|G'_\epsilon(\lambda)| \le \frac{D_n^*}{\lambda^{1-\frac{n}{2}}}.$$

From the problem (8), we deduce that

$$\Big| -\frac{1}{2}\theta^2\lambda^2 G''_\epsilon(\lambda) - \lambda G'_\epsilon(\lambda)(\theta^2+\beta+\delta-r) + rG_\epsilon(\lambda)\Big| \le \frac{1}{\lambda} + \frac{\delta r D_n^*}{\beta I_2 \lambda^{1-\frac{n}{2}}}.$$

Consider the solution to the following problem:

$$-\frac{1}{2}\theta^2\lambda^2 H''_n(\lambda) - \lambda H_n(\lambda)(\theta^2+\beta+\delta-r) + rH_n(\lambda) = \frac{1}{\lambda} + \frac{\delta r D_n^*}{\beta I_2 \lambda^{1-\frac{n}{2}}},$$
$$H_n(\hat{\lambda}) = \frac{1}{\beta\hat{\lambda}} + \frac{I_1 - I_2}{r}, \quad H_n(\infty) = 0.$$

Following the maximum principle we can argue that

$$-H_n(\lambda) \le G_\epsilon(\lambda) \le H_n(\lambda).$$

Hence, we conclude that

$$-H_n(\lambda) \le G(\lambda) \le H_n(\lambda)$$

and that $G(\infty) = 0$. **Q.E.D.**

2. The Derivation of Equations

In this section, we provide the details of the derivation of important equations. The first equation to be derived is

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{U_1\big(c(t)\big) + \delta U_2\big(X(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big], \tag{11}$$

where

$$U_2(z) = \frac{1}{\beta}\Big[\ln\Big\{\beta\Big(z+\frac{I_2}{r}\Big)\Big\} + \frac{1}{\beta}\Big(r+\frac{\theta^2}{2} - \beta(1-\ln K)\Big)\Big], \quad \text{for} \quad z>0.$$

For the next, we will derive the following:

$$G(\lambda) = \frac{1}{\lambda(\beta+\delta)} + B(\hat{\lambda})\lambda^{-\alpha_\delta} + \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda}\mu^{\alpha_\delta-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu$$
$$+(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\infty}\mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big], \tag{12}$$

for $\lambda > \hat{\lambda}$ and

$$\begin{aligned} &\frac{1}{\beta\hat{\lambda}} + \frac{I_1 - I_2}{r} \\ &= \frac{1}{\hat{\lambda}(\beta+\delta)} + B(\hat{\lambda})\hat{\lambda}^{-\alpha_\delta} + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[(\alpha_\delta^* - 1)\hat{\lambda}^{-\alpha_\delta^*}\int_{\hat{\lambda}}^{\infty}\mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}\Big]d\mu\Big], \end{aligned} \tag{13}$$

for

$$B(\hat{\lambda}) \equiv A(\hat{\lambda}) + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\hat{\lambda}^{\alpha_\delta - 1}\ln\frac{1}{\hat{\lambda}}.$$

Here,

$$\Big(G(\mu) - \frac{I_1}{r}\Big)^+ = \max\Big\{G(\mu) - \frac{I_1}{r}, 0\Big\}.$$

Finally, we would like to derive the following relationship:

$$\begin{aligned} &\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K \\ &= \frac{\delta}{\beta}\ln\beta\hat{\lambda} + \hat{\lambda}(I_1 - I_2)\Big(1 + \frac{\alpha_\delta\theta^2}{2r}\Big) - \frac{\delta(\alpha_\delta^* - 1)}{\beta}\hat{\lambda}^{-\alpha_\delta^*+1}\int_{\hat{\lambda}}^{\infty}\mu^{\alpha_\delta^*-2}\ln\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu. \end{aligned} \tag{14}$$

2.1. The Derivation of Equation (11)

Recall that our value function $\Phi(x)$ follows

$$\Phi(x) \equiv \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}U_1\big(c(t)\big)dt + e^{-\beta(\tau\wedge\tau_U)}\int_{\tau\wedge\tau_U}^{\infty} e^{-\beta(t-\tau\wedge\tau_U)}U_1\big(Kc(t)\big)dt\Big]. \tag{15}$$

If we use the conditional expectation of τ_U, we can show that value function $\Phi(x)$ in (15) is equivalent to $\Phi(x)$ in (11). Define

$$U_2\big(X(\tau\wedge\tau_U)\big) \equiv \max_{(c,\pi)} E\Big[\int_{\tau\wedge\tau_U}^{\infty} e^{-\beta(t-\tau\wedge\tau_U)}U_1\big(Kc(t)\big)dt\Big].$$

If we let $s = \tau\wedge\tau_U$, then

$$U_2\big(X(s)\big) = \max_{(c,\pi)} E\Big[\int_s^{\infty} e^{-\beta(t-s)}U_1\big(Kc(t)\big)dt\Big].$$

Utilizing the dynamic programming approach in Merton (1969) or the martingale approach in Karatzas and Shreve (1998), we get

$$U_2\big(X(s)\big) = \frac{1}{\beta}\Big[\ln\Big\{\beta\big(X(s) + \frac{I_2}{r}\big)\Big\} + \frac{1}{\beta}\Big(r + \frac{\theta^2}{2} - \beta(1 - \ln K)\Big)\Big].$$

By the principle of dynamic programming, the value function $\Phi(x)$ in (15) becomes

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}U_1\big(c(t)\big)dt + e^{-\beta(\tau\wedge\tau_U)}U_2\big(X(\tau\wedge\tau_U)\big)\Big].$$

The conditional expectation of τ makes us to obtain the following result:

$$
\begin{aligned}
&E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}U_1\big(c(t)\big)dt + e^{-\beta(\tau\wedge\tau_U)}U_2\big(X(\tau\wedge\tau_U)\big)\Big] \\
&= E\Big[E\Big[\int_0^{\tau\wedge\tau_U} e^{-\beta t}U_1\big(c(t)\big)dt + e^{-\delta(\tau\wedge\tau_U)}U_2\big(X(\tau\wedge\tau_U)\big)\Big|\tau_U\Big]\Big] \\
&= E\Big[\int_0^{\infty} \delta e^{-\delta s}\int_0^{\tau\wedge s} e^{-\beta t}U_1\big(c(t)\big)dtds + \int_0^{\infty}\delta e^{-\delta s}e^{-\beta(\tau\wedge s)}U_2\big(W(\tau\wedge s)\big)ds\Big] \\
&= E\Big[\int_0^{\tau}\delta e^{-\delta s}\int_0^{s} e^{-\beta t}U_1\big(c(t)\big)dtds + \int_{\tau}^{\infty}\delta e^{-\delta s}\int_0^{\tau} e^{-\beta t}U_1\big(c(t)\big)dtds \\
&\quad + \int_0^{\tau}\delta e^{-\delta s}e^{-\beta s}U_2\big(X(s)\big)ds + \int_{\tau}^{\infty}\delta e^{-\delta s}e^{-\beta\tau}U_2\big(X(\tau)\big)ds\Big] \\
&= E\Big[\int_0^{\tau} e^{-\beta t}U_1\big(c(t)\big)\int_t^{\tau}\delta e^{-\delta s}dsdt + \int_0^{\tau} e^{-\beta t}U_1\big(c(t)\big)\int_{\tau}^{\infty}\delta e^{-\delta s}dsdt \\
&\quad + \int_0^{\tau} e^{-(\beta+\delta)s}\delta U_2\big(X(s)\big)ds + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big] \\
&= E\Big[\int_0^{\tau} e^{-\beta t}U_1\big(c(t)\big)\int_t^{\infty}\delta e^{-\delta s}dsdt + \int_0^{\tau} e^{-(\beta+\delta)s}\delta U_2\big(X(s)\big)ds + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big] \\
&= E\Big[\int_0^{\tau} e^{-(\beta+\delta)t}\Big\{U_1\big(c(t)\big) + \delta U_2\big(X(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big].
\end{aligned}
$$

2.2. The Derivation of Equation (12)

Recall that the general solution G follows

$$
G(\lambda) = \frac{1}{\lambda(\beta+\delta)} + A(\lambda)\lambda^{-\alpha_\delta} + A^*(\lambda)\lambda^{-\alpha_\delta^*}, \tag{16}
$$

which is subject to

$$
A'(\lambda)\lambda^{-\alpha_\delta} + (A^*(\lambda))'\lambda^{-\alpha_\delta^*} = 0.
$$

The first and second derivatives of G are given by

$$
G'(\lambda) = -\frac{1}{\lambda^2}\frac{1}{\beta+\delta} - \alpha_\delta A(\lambda)\lambda^{-\alpha_\delta-1} - \alpha_\delta^* A^*(\lambda)\lambda^{-\alpha_\delta^*-1}
$$

and

$$
\begin{aligned}
G''(\lambda) =& \frac{2}{\lambda^3}\frac{1}{\beta+\delta} + \alpha_\delta(\alpha_\delta+1)\lambda^{-\alpha_\delta-2}A(\lambda) + \alpha_\delta^*(\alpha_\delta^*+1)\lambda^{-\alpha_\delta^*-2}A^*(\lambda) \\
& - \alpha_\delta\lambda^{-\alpha_\delta-1}A'(\lambda) - \alpha_\delta^*\lambda^{-\alpha_\delta^*-1}(A^*(\lambda))',
\end{aligned}
$$

respectively. By using the general solution G and its first and second derivatives, we get the following relationship:

$$
\begin{aligned}
&-\frac{1}{2}\theta^2\lambda^2G''(\lambda) - \lambda G'(\lambda)(\theta^2+\beta+\delta-r) + rG(\lambda) \\
&= \frac{1}{\lambda} + \frac{\theta^2}{2}\Big(\alpha_\delta - \alpha_\delta^*\Big)\lambda^{1-\alpha_\delta}A'(\lambda).
\end{aligned}
$$

The first relationship in (1) shows that

$$\frac{\theta^2}{2}(\alpha_\delta-\alpha_\delta^*)\lambda^{1-\alpha_\delta}A'(\lambda)=-\frac{\delta}{\beta}\frac{G'(\lambda)}{G(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\lambda)>\frac{I_1}{r}\}$$

and

$$\frac{\theta^2}{2}(\alpha_\delta-\alpha_\delta^*)\lambda^{1-\alpha_\delta^*}(A^*(\lambda))'=\frac{\delta}{\beta}\frac{G'(\lambda)}{G(\lambda)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\lambda)>\frac{I_1}{r}\}.$$

Then we obtain that

$$A(\lambda)=A(\hat{\lambda})-\frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\int_{\hat{\lambda}}^{\lambda}\frac{\mu^{\alpha_\delta-1}G'(\mu)}{G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\lambda)>\frac{I_1}{r}\}d\mu$$

and

$$A^*(\lambda)=-\frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\int_{\lambda}^{\infty}\frac{\mu^{\alpha_\delta^*-1}G'(\mu)}{G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\lambda)>\frac{I_1}{r}\}d\mu.$$

As a result, the general solution G given by (16) is rewritten by

$$\begin{aligned}G(\lambda)=&\frac{1}{\lambda(\beta+\delta)}+A(\hat{\lambda})\lambda^{-\alpha_\delta}-\frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[\lambda^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda}\frac{\mu^{\alpha_\delta-1}G'(\mu)}{G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\mu)>\frac{I_1}{r}\}d\mu\\&+\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\infty}\frac{\mu^{\alpha_\delta^*-1}G'(\mu)}{G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\mu)>\frac{I_1}{r}\}d\mu\Big].\end{aligned}$$

Using the fact that

$$\frac{G'(\mu)}{G(\mu)-\frac{I_1}{r}+\frac{I_2}{r}}\mathbf{1}\{G(\mu)>\frac{I_1}{r}\}=\frac{d}{d\mu}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\Big\},$$

we restate the general solution G as the following:

$$\begin{aligned}G(\lambda)=&\frac{1}{\lambda(\beta+\delta)}+A(\hat{\lambda})\lambda^{-\alpha_\delta}\\&-\frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[-\lambda^{-\alpha_\delta}\hat{\lambda}^{\alpha_\delta-1}\ln\frac{1}{\hat{\lambda}}-\lambda^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda}(\alpha_\delta-1)\mu^{\alpha_\delta-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\Big\}d\mu\\&-\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\infty}(\alpha_\delta^*-1)\mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\Big\}d\mu\Big],\end{aligned}$$

where

$$\Big(G(\mu)-\frac{I_1}{r}\Big)^+=\max\Big\{G(\mu)-\frac{I_1}{r},0\Big\}.$$

If we define

$$B(\hat{\lambda})\equiv A(\hat{\lambda})+\frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\hat{\lambda}^{\alpha_\delta-1}\ln\frac{1}{\hat{\lambda}},$$

then we obtain the general solution G given by (12).

2.3. The Derivation of Equation (14)

We will show equation (14) can be obtained from the value-matching condition at $\lambda=\hat{\lambda}$, or equivalently, at $x=\hat{x}$. Recall that our free boundary problem can be specified by the following:

$$\begin{cases} (\beta+\delta)\phi(x)-(rx+I_1)\phi'(x)+\dfrac{\theta^2}{2}\dfrac{\phi'(x)^2}{\phi''(x)}+1+\ln\phi'(x)=\delta U_2(x), \quad -\dfrac{I_1}{r}<x<\hat{x}, \\ \phi(x)=U_2(x), \quad x\geq\hat{x}, \\ \phi(\hat{x})=U_2(\hat{x}), \\ \phi'(\hat{x})=\dfrac{1}{\beta}\dfrac{1}{\hat{x}+\frac{I_2}{r}}. \end{cases} \tag{17}$$

First, rearranging equation (17) we get an equality concerning $\phi(x)$:

$$(\beta+\delta)\phi(x)=(rx+I_1)\lambda(x)-\frac{\theta^2}{2}\lambda^2(x)G'\big(\lambda(x)\big)-\big(1+\ln\lambda(x)\big)+\delta U_2(x). \tag{18}$$

If we let

$$\begin{aligned} H(\lambda)\equiv\frac{1}{(\beta+\delta)}\Big(&rG(\lambda)\lambda-\frac{\theta^2}{2}\lambda^2G'(\lambda)-(1+\ln\lambda) \\ &+\frac{\delta}{\beta}\Big(\ln\Big\{\Big(G(\lambda)-\frac{I_1}{r}\Big)^+ +\frac{I_2}{r}\Big\}+\frac{1}{\beta}\Big(r+\frac{\theta^2}{2}-\beta(1-\ln K)\Big)\Big)\Big), \end{aligned} \tag{19}$$

then

$$\phi(x)=H\big(\lambda(x)\big). \tag{20}$$

Due to equations (1) and (19), we obtain

$$H'(\lambda)=\lambda G'(\lambda),$$

so

$$\phi'(x)=H'\big(\lambda(x)\big)\lambda'(x)=\frac{H'(\lambda(x))}{G'(\lambda(x))}=\lambda(x).$$

Therefore, we can say that $\phi(x)$ is a solution of equation (17) subject to a boundary condition of

$$\phi'(\hat{x})=\lambda(\hat{x})=\hat{\lambda}=\frac{1}{\beta}\frac{1}{\hat{x}+\frac{I_2}{r}}.$$

Using the condition of

$$\phi(\hat{x})=U_2(\hat{x})$$

in (17), we obtain the value of H at $\lambda=\hat{\lambda}$:

$$H(\hat{\lambda})=\frac{1}{\beta}\Big[\ln\frac{1}{\hat{\lambda}}+\frac{1}{\beta}\Big(r+\frac{\theta^2}{2}-\beta(1-\ln K)\Big)\Big]. \tag{21}$$

We can get

$$\begin{aligned} (\beta+\delta)\frac{1}{\beta}\Big[\ln\frac{1}{\hat{\lambda}}+\frac{1}{\beta}\Big(r+\frac{\theta^2}{2}-\beta(1-\ln K)\Big)\Big]&=r\hat{\lambda}\Big(\frac{1}{\beta\hat{\lambda}}+\frac{I_1-I_2}{r}\Big)-\frac{\theta^2}{2}\hat{\lambda}^2G'(\hat{\lambda}) \\ &-(1+\ln\hat{\lambda})+\frac{\delta}{\beta}\Big[\ln\frac{1}{\hat{\lambda}}+\frac{1}{\beta}\Big(r+\frac{\theta^2}{2}-\beta(1-\ln K)\Big)\Big] \end{aligned} \tag{22}$$

if we rearrange the relationship of (19) and rewrite it at the boundary $\lambda = \hat{\lambda}$. Then, (12) and (13) allow us to obtain

$$\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta)+\ln K = \frac{\delta}{\beta}\ln\beta\hat{\lambda}+\hat{\lambda}(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r}) - \frac{\delta(\alpha_\delta^*-1)}{\beta}\hat{\lambda}^{-\alpha_\delta^*+1}\int_{\hat{\lambda}}^\infty \mu^{\alpha_\delta^*-2}\ln\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu.$$

3. Theorems and Proofs

Theorem 2. *(Uniqueness) If* $\frac{2\delta}{\theta^2\beta\hat{\lambda}} < 1$*, the solution of (12) is unique.*

Proof. Let G_1 and G_2 be the two solutions of (12), then we get

$$\begin{aligned} G_1(\lambda)-G_2(\lambda) = \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[&(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda}\mu^{\alpha_\delta-2}\Big(\ln\beta\Big\{\Big(G_1(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\} \\ &-\ln\beta\Big\{\Big(G_2(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}\Big)d\mu \\ &+(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\infty}\mu^{\alpha_\delta^*-2}\Big(\ln\beta\Big\{\Big(G_1(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\} \\ &-\ln\beta\Big\{\Big(G_2(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}\Big)d\mu\Big]. \end{aligned}$$

However, we know that

$$\Big|\ln\beta\Big\{\Big(G_1(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\} - \ln\beta\Big\{\Big(G_1(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}\Big| \le |G_1(\mu)-G_2(\mu)|,$$

and this implies

$$|G_1(\lambda)-G_2(\lambda)| \le \frac{2\delta}{\theta^2\beta\hat{\lambda}}\sup_\mu |G_1(\mu)-G_2(\mu)|.$$

The proof is complete. **Q.E.D.**

The following theorem permits us to take a monotonically-decreasing $G(\lambda)$ under suitable parameter conditions.

Theorem 3. *(Monotonicity) Suppose that*

$$I_2 \ge r, \quad \frac{1}{\beta+\delta}+\frac{2}{\theta^2(\alpha_\delta-\alpha_\delta^*)}\ln(\beta\frac{I_2}{r}) > 0,$$

and

$$\frac{2\delta\alpha_\delta}{\theta^2\beta}\Big(\ln\beta+\ln\max\Big\{\frac{1}{\beta\hat{\lambda}},\frac{I_2}{r}\Big\}\Big) < \frac{1}{\beta+\delta}. \tag{23}$$

Then, any solution of (1) satisfies $G'(\lambda) < 0$.

Proof. Any solution of (1) satisfies the integral equation (12). From (13) and the assumption of Theorem 3 we deduce

$$\begin{aligned} B(\hat\lambda)\hat\lambda^{-\alpha_\delta} &= \Big[\frac{\delta}{\beta(\beta+\delta)}\frac{1}{\hat\lambda} + \frac{I_1-I_2}{r} \\ &\quad - \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big\{(\alpha_\delta^*-1)\hat\lambda^{-\alpha_\delta^*}\int_{\hat\lambda}^{\infty}\mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big\}\Big] \\ &\geq \frac{\delta}{\beta\hat\lambda}\Big[\frac{1}{\beta+\delta} + \frac{2}{\theta^2(\alpha_\delta-\alpha_\delta^*)}\ln(\beta\frac{I_2}{r})\Big] + \frac{I_1-I_2}{r} > 0. \end{aligned}$$

Since $B(\hat\lambda) > 0$,

$$\begin{aligned} G'(\lambda) &= -\frac{1}{\lambda^2(\beta+\delta)} - \alpha_\delta B(\hat\lambda)\lambda^{-\alpha_\delta-1} \\ &\quad - \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)\beta}\Big[\alpha_\delta(\alpha_\delta-1)\lambda^{-\alpha_\delta-1}\int_{\hat\lambda}^{\lambda}\mu^{\alpha_\delta-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu \\ &\quad + \alpha_\delta^*(\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*-1}\int_{\lambda}^{\infty}\mu^{\alpha_\delta^*-2}\ln\beta\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big] \\ &\quad + \frac{2\delta}{\theta^2\beta\lambda^2}\ln\beta\Big\{\Big(G(\lambda)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}, \\ &\leq -\frac{1}{\lambda^2}\Big[\frac{1}{\beta+\delta} - \frac{2\delta\alpha_\delta}{\theta^2\beta}\Big(\ln\beta + \ln\max\Big\{\frac{1}{\beta\hat\lambda}, \frac{I_2}{r}\Big\}\Big)\Big] - \alpha_\delta B(\hat\lambda)\lambda^{-\alpha_\delta-1}. \end{aligned} \tag{24}$$

From the assumption of Theorem (3), we find the fact of $G'(\lambda) < 0$. **Q.E.D.**

THEOREM 4. *(Uniqueness of free boundary $\hat\lambda$) Suppose*

$$\begin{aligned} & I_1 > \frac{\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K - \frac{\delta}{\beta}\ln\frac{r}{I_2}}{\frac{r}{\beta I_2}(1+\frac{\alpha_\delta\theta^2}{2r})} + I_2, \\ & \frac{\delta}{\beta}\ln\frac{2\delta}{\theta^2} + \frac{2\delta}{\theta^2\beta}(I_1-I_2)\Big(1+\frac{\alpha_\delta\theta^2}{2r}\Big) + \frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{\theta^2}{2\delta}, \frac{I_2}{r}\Big\}\Big) < \frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K, \quad \textit{and} \\ & \frac{\delta}{\beta}\ln\beta - \frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)} + e^{-\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}}(I_1-I_2)\Big(1+\frac{\alpha_\delta\theta^2}{2r}\Big) \\ & \qquad + \frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{1}{\beta}e^{\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}}, \frac{I_2}{r}\Big\}\Big) < \frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K. \end{aligned}$$

Then there exists a unique solution $\hat\lambda$ of (14) satisfying $0 < \hat\lambda < \dfrac{r}{\beta I_2}$.

Proof. Recall the relationships of

$$\begin{aligned} \psi_\delta(\hat\lambda) &= \frac{\delta}{\beta}\ln\beta\hat\lambda + \hat\lambda(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r}) \\ &\quad - \frac{\delta(\alpha_\delta^*-1)}{\beta}\hat\lambda^{-\alpha_\delta^*+1}\int_{\hat\lambda}^{\infty}\mu^{\alpha_\delta^*-2}\ln\Big\{\Big(G(\mu)-\frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu, \\ L_\delta &= \frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K, \\ \underline{\phi}_\delta(\hat\lambda) &= \frac{\delta}{\beta}\ln\beta\hat\lambda + \hat\lambda(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r}), \\ \overline{\phi}_\delta(\hat\lambda) &= \frac{\delta}{\beta}\ln\beta\hat\lambda + \hat\lambda(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r}) + \frac{\delta}{\beta}\ln\max\Big\{\frac{1}{\beta\hat\lambda}, \frac{I_2}{r}\Big\}. \end{aligned}$$

It is obvious that

$$\psi_\delta(\hat{\lambda}) \geq \underline{\phi}_\delta(\hat{\lambda}). \tag{25}$$

Since $\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r} \leq \max\Big\{\frac{1}{\beta\hat{\lambda}}, \frac{I_2}{r}\Big\}$, under the assumption of $I_2 \geq r$,

$$\psi_\delta(\hat{\lambda}) \leq \frac{\delta}{\beta}\ln\beta\hat{\lambda} + \hat{\lambda}(I_1 - I_2)(1 + \frac{\alpha_\delta\theta^2}{2r}) + \frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{1}{\beta\hat{\lambda}}, \frac{I_2}{r}\Big\}\Big) = \overline{\phi}_\delta(\hat{\lambda}). \tag{26}$$

Since both $\overline{\phi}_\delta(\hat{\lambda})$ and $\underline{\phi}_\delta(\hat{\lambda})$ are monotonically-increasing and continuous functions with

$$\begin{aligned} &\overline{\phi}_\delta(0) = \underline{\phi}_\delta(0) = -\infty, \text{ and} \\ &\overline{\phi}_\delta(+\infty) = \underline{\phi}_\delta(+\infty) = +\infty, \end{aligned} \tag{27}$$

λ_δ^0 and λ_δ^1 are lower and upper bounds for $\hat{\lambda}$. From (25), (26) and (27) and the continuity of the function $\psi_\delta(\hat{\lambda})$ we conclude that there exists at least one solution of

$$\psi_\delta(\hat{\lambda}) = L_\delta. \tag{28}$$

Notice that the following two assumptions of

$$\frac{2\delta}{\theta^2\beta\lambda_\delta^0} < 1 \tag{29}$$

and

$$\frac{2\delta\alpha_\delta}{\theta^2\beta}\Big(\ln\beta + \ln\max\Big\{\frac{1}{\beta\lambda_\delta^0}, \frac{I_2}{r}\Big\}\Big) < \frac{1}{\beta+\delta} \tag{30}$$

guarantee the fact that a solution of (28) satisfies (23), and imply the assumption in Theorem 2. Also notice that (29), (30) are equivalent to

$$\lambda_\delta^0 > \frac{2\delta}{\theta^2\beta}, \quad \lambda_\delta^0 > e^{-\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}}, \tag{31}$$

respectively. Using the definition of λ_δ^0 and the two inequalities in (31) we can show that

$$\overline{\phi}_\delta\Big(\frac{2\delta}{\theta^2\beta}\Big) < L_\delta, \quad \overline{\phi}_\delta\Big(e^{-\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}}\Big) < L_\delta. \tag{32}$$

The first inequality in (32) can be rewritten as

$$\begin{aligned} \frac{\delta}{\beta}\ln\frac{2\delta}{\theta^2} + \frac{2\delta}{\theta^2\beta}(I_1 - I_2)\Big(1 + \frac{\alpha_\delta\theta^2}{2r}\Big) + \frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{\theta^2}{2\delta}, \frac{I_2}{r}\Big\}\Big) \\ < \frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1 - \alpha_\delta) + \ln K, \end{aligned} \tag{33}$$

and the second one also can be restated as

$$\begin{aligned} \frac{\delta}{\beta}\ln\beta - \frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)} + e^{-\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}}(I_1 - I_2)\Big(1 + \frac{\alpha_\delta\theta^2}{2r}\Big) + \frac{\delta}{\beta}\ln\Big(\max\Big\{\frac{1}{\beta}e^{\frac{\theta^2\beta}{2\delta\alpha_\delta(\beta+\delta)}}, \frac{I_2}{r}\Big\}\Big) \\ < \frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1 - \alpha_\delta) + \ln K. \end{aligned} \tag{34}$$

Therefore, we can say that, if (33) and (34) hold (these conditions require a sufficiently small δ), there exists a unique solution of (28). Moreover, the inequality $\lambda_\delta^1 < \dfrac{r}{\beta I_2}$ is a sufficient condition of the inequality of $\hat{\lambda} < \dfrac{r}{\beta I_2}$. It can be rewritten as

$$\underline{\phi}_\delta\Big(\frac{r}{\beta I_2}\Big) > L_\delta,$$

and it is equivalent to

$$\frac{\delta}{\beta}\ln\frac{r}{I_2} + \frac{r}{\beta I_2}(I_1 - I_2)\Big(1 + \frac{\alpha_\delta \theta^2}{2r}\Big) > \frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K - \frac{\delta}{\beta}\ln\frac{r}{I_2}.$$

Therefore, we get the first assumption of the theorem

$$I_1 > \frac{\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \ln K - \frac{\delta}{\beta}\ln\frac{r}{I_2}}{\frac{r}{\beta I_2}\Big(1 + \frac{\alpha_\delta\theta^2}{2r}\Big)} + I_2.$$

Q.E.D.

For the unique solution $\hat{\lambda}$ of (28), the corresponding $G(\lambda)$ is also the unique solution of (1).

4. Verification for Optimal Stopping Problem

Recall that our optimal stopping problem is given by

$$\Phi(x) = \max_\tau J_\tau(x), \tag{35}$$

where

$$J_\tau(x) \equiv \max_{(c,\pi)} E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{U_1\big(c(t)\big) + \delta U_2\big(X(t)\big)\Big\}dt + e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big],$$

for a fixed stopping time τ. We also recall that the variational inequality associated with the optimal stopping problem follows

$$\begin{aligned}(\beta+\delta)\phi(x) - (rx+I_1)\phi'(x) + \frac{\theta^2}{2}\frac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) \geq \delta U_2(x),\\ \phi(x) \geq U_2(x),\\ \Big[(\beta+\delta)\phi(x) - (rx+I_1)\phi'(x) + \frac{\theta^2}{2}\frac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) - \delta U_2(x)\Big]\Big(\phi(x) - U_2(x)\Big) = 0,\end{aligned} \tag{36}$$

for any $x > -I_1/r$. The verification for our optimal stopping problem (35) is executed in the following two steps: we first verify that the solution $\phi(x)$ to the variational inequality (36) is the solution to the optimal stopping problem (35). Next, we verify that the solution to the free boundary problem (17) satisfies the variational inequality (36). We reuse some notations and definitions in Øksendal (2007).

First, we fix a domain G in $\mathbf{R}^k$ and let

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad Y_0 = y,$$

be an Itô diffusion in $\mathbf{R}^k$. Define

$$\tau_G = \tau_G(y,\omega) = \inf\{t > 0; Y_t(\omega) \notin G\}.$$

Let $f : \mathbf{R}^k \to \mathbf{R}$ and $g : \mathbf{R}^k \to \mathbf{R}$ be continuous functions satisfying

(a) $E^y\Big[\int_0^{\tau_G} f^-(Y_t)dt\Big] < \infty$ for all $y \in \mathbf{R}^k$, and

(b) the family $\{g^-(Y_\tau); \tau$ stopping time, $\tau \le \tau_G\}$ is uniformly integrable with respect to the probability law of Y_t for all $y \in \mathbf{R}^k$.

Let $\mathcal{T}$ denote the set of all stopping times $\tau \le \tau_G$. Consider the following problem: Find $\Phi(y)$ and $\tau^* \in \mathcal{T}$ such that

$$\Phi(y) = \sup_{\tau \in \mathcal{T}} J^\tau(y) = J^{\tau^*}(y),$$

where

$$J^\tau(y) = E^y\Big[\int_0^\tau f(Y_t)dt + g(Y_\tau)\Big] \quad \text{for} \quad \tau \in \mathcal{T}.$$

Note that since $J^0(y) = g(y)$ we have

$$\Phi(y) \ge g(y) \quad \text{for all} \quad y \in G.$$

We can now formulate the variational inequalities. As usual we let

$$L \equiv L_Y = \sum_{i=1}^k b_i(y)\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^k (\sigma\sigma^T)_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j}$$

be the partial differential operator.

Now, we can state Theorem 10.4.1 in Øksendal (2007).

Theorem 5. *(Variational inequalities for optimal stopping)*

a) Suppose we can find a function $\phi : \overline{G} \to \boldsymbol{R}$ *such that*

(i) $\phi \in C^1(G) \cap C(\overline{G})$

(ii) $\phi \ge g$ *on* G $\lim_{t \to \tau_G^-} \phi(Y_t) = g(Y_{\tau_G})\chi_{\{\tau_G < \infty\}}$ *a.s.*

Define

$$D = \{x \in G; \phi(x) > g(x)\}.$$

Suppose Y_t *spends* 0 *time on* ∂D *a.s., i.e.*

(iii) $E^y\Big[\int_0^{\tau_G} \chi_{\partial D}(Y_t)dt\Big] = 0$ *for all* $y \in G$

and suppose that

(iv) ∂D is a Lipschitz surface, i.e. ∂D is locally the graph of a function $h:\boldsymbol{R}^{k-1}\to\boldsymbol{R}$ such that there exists $K<\infty$ with

$$|h(x)-h(y)|\leq K|x-y| \quad \textit{for all} \quad x,y.$$

Moreover, suppose the following:

(v) $\phi\in C^2(G\backslash\partial D)$ and the second order derivatives of ϕ are locally bounded near ∂D

(vi) $L\phi+f\leq 0$ on $G\backslash D$.

Then

$$\phi(y)\geq\Phi(y) \quad \textit{for all} \quad y\in G.$$

b) Suppose, in addition to the above, that

(vii) $L\phi+f=0$ on D

(viii) $\tau_D\equiv\inf\{t>0;Y_t\notin D\}<\infty$ a.s. the probability law of Y_t for all $y\in G$

and

(ix) the family $\{\phi(Y_\tau);\tau\leq\tau_D,\tau\in\mathcal{T}\}$ is uniformly integrable with respect to the probability law of Y_t for all $y\in G$.

Then

$$\phi(y)=\Phi(y)=\sup_{\tau\in\mathcal{T}}E^y\Big[\int_0^\tau f(Y_t)dt+g(Y_\tau)\Big]; \quad y\in G$$

and

$$\tau^*=\tau_D$$

is an optimal stopping time for this problem.

Recall the optimal stopping problem. For a fixed stopping time τ,

$$\Phi(x)=\max_\tau J_\tau(x),$$

where

$$J_\tau(x)=\max_{(c,\pi)}E\Big[\int_0^\tau e^{-(\beta+\delta)t}\Big\{U_1\big(c(t)\big)+\delta U_2\big(X(t)\big)\Big\}dt+e^{-(\beta+\delta)\tau}U_2\big(X(\tau)\big)\Big].$$

We let $c^*(t)$ and $\pi^*(t)$ denote optimal consumption and optimal risky portfolio, respectively. The partial differential operator

$$L=\frac{\partial}{\partial t}+\Big(rx-c^*(t)+I_1+\pi^*(t)\sigma\theta\Big)\frac{\partial}{\partial x}+\frac{1}{2}\pi^*(t)^2\sigma^2\frac{\partial^2}{\partial x^2}.$$

In our model, the domain G is given by

$$G=\{(x,t)\in\mathbf{R}\times\mathbf{R};x>-I_1/r,t\geq 0\}.$$

Also, the domain D follows

$$D = \{(x,t) \in G; \tilde{\phi}(x,t) > e^{-(\beta+\delta)t}U_2(x)\}$$

for a function $\tilde{\phi} : \overline{G} \to \mathbf{R}$. Then

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} = &\frac{\partial \tilde{\phi}}{\partial t} + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\frac{\partial \tilde{\phi}}{\partial x} \\ &+ \frac{1}{2}\pi^*(t)^2\sigma^2\frac{\partial^2 \tilde{\phi}}{\partial x^2} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\}. \end{aligned}$$

Now, we derive variational inequalities for our optimal stopping problem. We suppose that the function $\tilde{\phi}$ satisfies the following variational inequalities:

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} = 0 \quad \text{on} \quad D, \\ L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} \le 0 \quad \text{on} \quad G\backslash D. \end{aligned}$$

Accordingly, the above variational inequalities are equivalent to the following:

$$\begin{aligned} L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\} \le 0, \\ \tilde{\phi}(x,t) \ge e^{-(\beta+\delta)t}U_2(x), \\ \Big[L\tilde{\phi} + e^{-(\beta+\delta)t}\Big\{U_1\big(c^*(t)\big) + \delta U_2(x)\Big\}\Big]\Big(\tilde{\phi}(x,t) - e^{-(\beta+\delta)t}U_2(x)\Big) = 0. \end{aligned} \tag{37}$$

We conjecture the form of $\tilde{\phi}$ as

$$\tilde{\phi}(x,t) = e^{-(\beta+\delta)t}\phi(x).$$

Then substituting the conjectured $\tilde{\phi}$ into the inequality (37) we can show that

$$\begin{aligned} &\Big[-(\beta+\delta)\phi(x) + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\phi'(x) \\ &\qquad + \frac{1}{2}\pi^*(t)^2\sigma^2\phi''(x) + U_1\big(c^*(t)\big) + \delta U_2(x)\Big] \le 0, \\ &\qquad \phi(x) \ge U_2(x), \\ &\Big[-(\beta+\delta)\phi(x) + \Big(rx - c^*(t) + I_1 + \pi^*(t)\sigma\theta\Big)\phi'(x) \\ &\qquad + \frac{1}{2}\pi^*(t)^2\sigma^2\phi''(x) + U_1\big(c^*(t)\big) + \delta U_2(x)\Big]\Big(\phi(x) - U_2(x)\Big) = 0. \end{aligned}$$

Optimality conditions for optimal consumption and risky portfolio are given by

$$c^*(t) = \frac{1}{\phi'(x)} \quad \text{and} \quad \pi^*(t) = -\frac{\theta}{\sigma}\frac{\phi'(x)}{\phi''(x)}.$$

Therefore, we obtain the variational inequality (36). If we apply Theorem 5, then the solution $\phi(x)$ to the variational inequality (36) is the solution $\Phi(x)$ of our optimal stopping problem (35).

Now we provide a theorem verifying that the solution $\phi(x)$ to the free boundary problem (17) satisfies the variational inequality (36).

Theorem 6. *If*

$$\frac{1}{\beta}\frac{r(I_1-I_2)}{I_2}(1+\frac{\alpha_\delta\theta^2}{2r}) < \frac{\theta^2}{2}\frac{\delta(1-\alpha_\delta)}{\beta(\beta+\delta)} + \ln K \quad and$$
$$\frac{1}{\beta}e^{\frac{\theta^2}{2}\frac{(1-\alpha_\delta)}{(\beta+\delta)}}(I_1-I_2)(1+\frac{\alpha_\delta\theta^2}{2r}) + \frac{\delta I_2}{\beta r} < \ln K$$

are true, the solution $\phi(x)$ to the free boundary problem (17) satisfies the variational inequality (36).

Proof. Since

$$\phi(\hat{x}) = U_2(\hat{x}),$$

for

$$\psi(x) \equiv \phi(x) - U_2(x)$$

we know $\psi(\hat{x}) = 0$. If we show that $\psi'(x) \le 0$ for $-I_1/r < x \le \hat{x}$, the second inequality of (36) will follow. It is enough to show that

$$G(\lambda) < \frac{1}{\beta\lambda} + \frac{I_1 - I_2}{r}, \tag{38}$$

for $\lambda > \hat{\lambda}$. Define

$$\Gamma(\lambda) \equiv G(\lambda) - \frac{1}{\beta\lambda} - \frac{I_1 - I_2}{r},$$

then, $\Gamma(\hat{\lambda}) = 0$ and $\Gamma(\infty) = -\frac{I_1 - I_2}{r} < 0$. From the first equality of (1) we can get

$$\begin{aligned} -\frac{1}{2}\theta^2\lambda^2\Gamma''(\lambda) - \lambda(\theta^2+\beta+\delta-r)\Gamma'(\lambda) + r\Gamma(\lambda) + \frac{\delta}{\beta}\frac{\Gamma'(\lambda)}{\Gamma(\lambda)+\frac{1}{\beta\lambda}} \\ = -\frac{\delta}{\beta\lambda} - (I_1 - I_2) + \frac{\delta}{\beta^2}\frac{1}{\Gamma(\lambda)+\frac{1}{\beta\lambda}}\frac{1}{\lambda^2}. \end{aligned} \tag{39}$$

Since $\lambda > \hat{\lambda} > \lambda_\delta^0$, the right hand side of (39) have an upper bound of $-\frac{\delta}{\beta\lambda}\left(1 - \frac{r}{\beta I_2 \lambda_\delta^0}\right) < 0$ under the condition of

$$\lambda_\delta^0 > \frac{r}{\beta I_2}. \tag{40}$$

Applying the comparison principle (Friedman, 1982) to (39), we obtain $\Gamma(\lambda) < 0$, which is equivalent to (38).

Now, we will verify the first inequality in (36). For $x \ge \hat{x}$, the following equality holds:

$$\begin{aligned} (\beta+\delta)\phi(x) - (rx+I_1)\phi'(x) + \frac{\theta^2}{2}\frac{\phi'(x)^2}{\phi''(x)} + 1 + \ln\phi'(x) - \delta U_2(x) \\ = -\frac{I_1 - I_2}{\beta(x+\frac{I_2}{r})} + \ln K, \end{aligned} \tag{41}$$

since $\phi(x) = U_2(x)$. The function $-\frac{I_1 - I_2}{\beta(x + I_2/r)} + \ln K$ is monotonically increasing, so it is sufficient to show

$$-\frac{I_1 - I_2}{\beta(\hat{x} + \frac{I_2}{r})} + \ln K \ge 0,$$

or equivalently,

$$\ln K - \hat{\lambda}(I_1 - I_2) \geq 0.$$

Using (14), we get

$$\begin{aligned}\ln K - \hat{\lambda}(I_1 - I_2) &\geq -\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \frac{\delta}{\beta}\ln(\beta\hat{\lambda}) + \hat{\lambda}(I_1 - I_2)\frac{\alpha_\delta\theta^2}{2r} \\ &\geq -\frac{\theta^2}{2}\frac{\delta}{\beta(\beta+\delta)}(1-\alpha_\delta) + \frac{\delta}{\beta}\ln(\beta\lambda_\delta^0) + \lambda_\delta^0(I_1 - I_2)\frac{\alpha_\delta\theta^2}{2r} > 0\end{aligned}$$

where

$$\lambda_\delta^0 > \frac{1}{\beta}e^{\frac{\theta^2}{2}\frac{(1-\alpha_\delta)}{(\beta+\delta)}}. \tag{42}$$

We can rewrite the restrictions (40) and (42) as the following inequalities by using the function $\overline{\phi}_\delta(\cdot)$ defined in (26):

$$\overline{\phi}_\delta\Big(\frac{r}{\beta I_2}\Big) < L_\delta, \quad \overline{\phi}_\delta\Big(\frac{1}{\beta}e^{\frac{\theta^2}{2}\frac{(1-\alpha_\delta)}{(\beta+\delta)}}\Big) < L_\delta. \tag{43}$$

Therefore, the two inequalities in (43) become our assumptions:

$$\begin{aligned}&\frac{1}{\beta}\frac{r(I_1 - I_2)}{I_2}(1 + \frac{\alpha_\delta\theta^2}{2r}) < \frac{\theta^2}{2}\frac{\delta(1-\alpha_\delta)}{\beta(\beta+\delta)} + \ln K, \quad \text{and} \\ &\frac{1}{\beta}e^{\frac{\theta^2}{2}\frac{(1-\alpha_\delta)}{(\beta+\delta)}}(I_1 - I_2)(1 + \frac{\alpha_\delta\theta^2}{2r}) + \frac{\delta I_2}{\beta r} < \ln K.\end{aligned}$$

Q.E.D.

5. Convergence of the Iterative Procedure

We show that the approximation function $G(\cdot)$ obtained from the iterative procedure converges to the to the implicit equation (12) by using the Banach fixed-point theorem.

Consider a set $X = [\hat{\lambda}, \infty)$ which is the domain of $\lambda(\cdot)$. Since the set $\mathbf{R}$ of real numbers is complete, the set $\mathcal{B}(X, \mathbf{R})$ of all bounded functions $f : X \to \mathbf{R}$ is a complete metric space with the supremum norm

$$d(f, g) \equiv \sup\{|f(x) - g(x)| : x \in X\}.$$

Note that the set $C_b(X, \mathbf{R})$ consisting of all continuous bounded functions $f : X \to \mathbf{R}$ is a closed subspace of $\mathcal{B}(X, \mathbf{R})$, so that, $C_b(X, \mathbf{R})$ is also a complete metric space. Hence, the continuous and decreasing function $G(\lambda)$, which is a solution to the differential equation (1) satisfying

$$0 \leq G(\lambda) \leq G(\hat{\lambda}),$$

should be in $C_b(X, \mathbf{R})$.

Define for any $G(\lambda) \in C_b(X, \mathbf{R})$

$$\begin{aligned}T\big(G(\lambda)\big) \equiv \frac{1}{\lambda(\beta+\delta)} + B(\hat{\lambda})\lambda^{-\alpha_\delta} + \frac{2\delta}{\theta^2(\alpha_\delta - \alpha_\delta^*)\beta}\Big[&(\alpha_\delta - 1)\lambda^{-\alpha_\delta}\int_{\hat{\lambda}}^{\lambda}\mu^{\alpha_\delta - 2}\ln\beta\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu \\ &+(\alpha_\delta^* - 1)\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\infty}\mu^{\alpha_\delta^* - 2}\ln\beta\Big\{\Big(G(\mu) - \frac{I_1}{r}\Big)^+ + \frac{I_2}{r}\Big\}d\mu\Big],\end{aligned}$$

then T is continuous and $T\big(G(\lambda)\big)$ is in $C_b(X,\mathbf{R})$ by the relationship of

$$|T\big(G(\lambda)\big)| \le \frac{2\delta}{\theta^2\beta\hat{\lambda}} \sup_{\lambda} |G(\lambda)|.$$

Moreover, if we take an assumption of

$$\frac{2\delta}{\theta^2\beta\hat{\lambda}} < 1,$$

then the map $T: C_b(X,\mathbf{R}) \to C_b(X,\mathbf{R})$ is a contraction mapping. This is because, for any $G_1(\lambda),\ G_2(\lambda) \in C_b(X,\mathbf{R})$, T satisfies the following:

$$\sup_{\lambda} |T\big(G_1(\lambda)\big) - T\big(G_2(\lambda)\big)| = \frac{2\delta}{\theta^2\beta\hat{\lambda}} \sup_{\lambda} |G_1(\lambda) - G_2(\lambda)|.$$

Let $G^i(\lambda)$ be a function and $B^i(\hat{\lambda}^i),\ \hat{\lambda}^i$ be the two constants obtained from the i-th iteration. By the Banach fixed-point theorem, we have that $G^i(\lambda)$ converges uniformly to $G(\lambda)$ on the region of $[\hat{\lambda},\infty)$. Obviously, we know that $B^i(\hat{\lambda}^i) \to B(\hat{\lambda})$ and $\hat{\lambda}^i \to \hat{\lambda}$ as $i \to \infty$.

We display an example of the numerical experiments in the next subsection.

5.1. An example

Let $G^i(\lambda)$ be the numerical solution derived by the i-th iteration. Then the corresponding value function $\Phi^i(x)$, $i = 0,1,2,...$, can be calculated by the first relationship in (17): for any $-I_1/r < x < \hat{x}$,

$$\Phi^i(x) = \begin{cases} \dfrac{1}{\beta}\Big[rG^0\big(\lambda^*(x)\big)\lambda^*(x) - \dfrac{\theta^2}{2}\Big(G^0\big(\lambda^*(x)\big)\Big)'\big(\lambda^*(x)\big)^2 - 1 - \ln\lambda^*(x)\Big], & \text{if} \quad i = 0, \\ \dfrac{1}{\beta+\delta}\Big[rG^i\big(\lambda^*(x)\big)\lambda^*(x) - \dfrac{\theta^2}{2}\Big(G^i\big(\lambda^*(x)\big)\Big)'\big(\lambda^*(x)\big)^2 - 1 - \ln\lambda^*(x) \\ \qquad + \delta U_2\Big(G^i\big(\lambda^*(x)\big) - I_1/r\Big)\Big], & \text{if} \quad i = 1,2,..., \end{cases}$$

where $G^i(\cdot)$, $B^i(\hat{\lambda}^i)$, $\hat{\lambda}^i$ are obtained from the iterative procedure. Note that $\Phi^0(x)$ represents the value function without forced unemployment risks (i.e., $\delta = 0$). Then $\Phi^i(x)$ $(i = 1,2,...)$ must satisfy the following inequality: for any $-I_1/r < x < \hat{x}$,

$$U_2(x) < \Phi^i(x) < \Phi^0(x). \tag{44}$$

The left inequality comes from the variational inequality of (36) and the right one must hold due to the fact that the value function without forced unemployment risks is clearly larger than that with the forced unemployment risks.

PROPOSITION 5.1. If $I_2 \ge r$,

$$\lambda_\delta^0 \le \hat{\lambda} \le \lambda_\delta^1, \tag{45}$$

where λ_δ^0 and λ_δ^1 are found by

$$\overline{\phi}_\delta(\lambda_\delta^0) = L_\delta \quad \text{and} \quad \underline{\phi}_\delta(\lambda_\delta^1) = L_\delta.$$

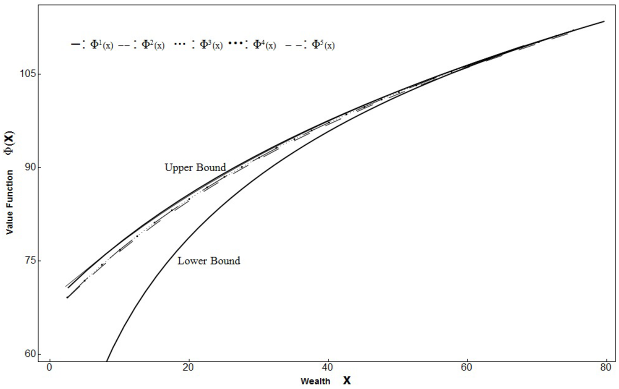

Figure 1 Numerical solutions as a function of initial wealth: $\delta = 0.01$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$ are used for parameter values.

Figure 1 shows the convergence of the numerical solution for a reasonable parameter set. The upper bound (lower bound) in the Figure 1 represents $\Phi^0(x)$ ($U_2(x)$, respectively). The figure shows that $\Phi^i(x)$ converges as the number of iteration increases and the numerical solutions satisfy the relationships in (44).

Table 1 shows the numerical solutions of $\Phi^i(x)$ $(i = 0, 1, ..., 5)$ for various initial wealth levels. It shows that the numerical results apparently seem to be convergent and are bounded by $U_2(x)$ and $\Phi^0(x)$. On the other hand, if we let $\hat{\lambda}^i$ $(i = 1, 2, ...)$ be the free boundary obtained from the i-th iteration, then we can observe in Table 2 that the numerical results for the free boundary seem to be convergent and stay between the lower bound of λ_δ^0 and the upper bound of λ_δ^1 (see the inequality (45) in Proposition 5.1) for various unemployment intensity δ's.

	lower bound	numerical solution					upper bound
x	$U_2(x)$	$\Phi^1(x)$	$\Phi^2(x)$	$\Phi^3(x)$	$\Phi^4(x)$	$\Phi^5(x)$	$\Phi^0(x)$
10	63.1217	77.8138	76.4976	76.7837	76.7839	76.7825	77.8959
20	78.7800	85.5165	84.6669	84.9565	84.9435	84.9428	85.732
30	88.6202	91.9625	91.3259	91.6085	91.5910	91.5908	92.1351
40	95.8131	97.4722	96.9468	97.2192	97.1998	97.2004	97.5626
50	101.4850	102.2700	101.8220	102.0880	102.0660	102.0660	102.2820
60	106.1690	106.5140	106.0870	106.3540	106.3310	106.3320	106.4620

Table 1 Numerical solutions for various initial wealth levels: $\delta = 0.01$, $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$ are used for parameter values.

	lower bound	numerical result					upper bound
δ	λ_δ^0	$\hat{\lambda}^1$	$\hat{\lambda}^2$	$\hat{\lambda}^3$	$\hat{\lambda}^4$	$\hat{\lambda}^5$	λ_δ^1
0.01	0.252839	0.380467	0.342191	0.345989	0.345784	0.345790	0.548594
0.02	0.183758	0.498807	0.335688	0.363804	0.361220	0.361278	0.707226
0.03	0.119902	0.641149	0.304488	0.387526	0.376163	0.376625	0.831393

Table 2 Numerical results of the free boundary $\hat{\lambda}$ for various unemployment intensity δ's: $\beta = 0.0371$, $r = 0.0371$, $\mu = 0.1123$, $\sigma = 0.1954$, $K = 3$, $I_1 = 1$, and $I_2 = 0.10$ are used for parameter values.