E-Companion – Exact Analysis of Divergent Inventory Systems with Time-Based Shipment Consolidation and Compound Poisson Demand

Olof Stenius
Ayşe Gönül Karaarslan
Johan Marklund
A. G. de Kok

Appendix A: Algorithm for Determining TC_k^l

To find the lower bound for the total costs directly associated with retailer group k, TC_k, in Lemma 5,

$$TC_k^l = \min_{T_k, S_i} (TC_k | B_i(t_0) = 0, \forall\ i), \tag{A1}$$

we go through the following steps:

1. Determine a near optimal shipment interval for retailer group k, $\overline{T}_k$, using (29)

2. Given $\overline{T}_k$, determine near optimal reorder points for each retailer $\overline{S}_i$. This is done by optimizing each retailer separately as a single-echelon system. Note that the convexity property in Proposition 3 holds also for systems where $B_i = 0$ ($R_0 \to \infty$).

3. Calculate $\overline{TC}_k = TC_k(\overline{T}_k, \overline{S_i})$.

4. Obtain upper and lower bounds for T_k for the optimization. These bounds are obtained analogously to the bounds in Proposition 5 as

$$T_k^l = \frac{\overline{TC}_k - \sqrt{\overline{TC}_k^2 - 2\omega_k \sum_{i \in N_k} h_0 \lambda_i \mu_i}}{\sum_{i \in N_k} h_0 \lambda_i \mu_i} \tag{A2}$$

$$T_k^u = \frac{\overline{TC}_k + \sqrt{\overline{TC}_k^2 - 2\omega_k \sum_{i \in N_k} h_0 \lambda_i \mu_i}}{\sum_{i \in N_k} h_0 \lambda_i \mu_i}.. \tag{A3}$$

5. Search through all values of T_k in this interval and optimize S_i for each T_k to find the lower bound, TC_k^l.

Appendix B: Proofs

Proof of Proposition 1

Proof. The reserved stock on hand at the warehouse increase whenever an unreserved unit on hand is reserved or a backordered unit arrives to the central warehouse. Thus the process by which the reserved stock on hand accumulates depends on the customer demand process, and the warehouse replenishment process. For fixed policies these two processes are in the current system independent of the shipment process, i.e., the length of the shipment interval, and when shipments leave the central warehouse. Hence, the same holds for the accumulation process of the reserved stock on hand. Moreover, as all unsatisfied demand is backordered, and all units will be reserved stock on hand at the warehouse at some point in time, the average rate by which the reserved stock on hand for retailer i accumulate is equal to the demand rate at retailer i, $\lambda_i \mu_i$. Consequently, if the previous shipment to retailer i left at time t, the expected number of units on hand at the warehouse reserved by retailer i, at $t + \tau$, for any $\tau \in (0, T_i]$, is $\lambda_i \mu_i \tau$. This means that the expected amount of reserved stock on hand for retailer i will increase linearly between two consecutive shipments from 0 to $\lambda_i \mu_i T_i$. Taking the average over time renders (4). □

Proof of Lemma 1:

Proof. Conditioning on at least n retailer orders during $(t_0 - L_0, t_0]$, $\Phi_0 \geq n$, and that $\Psi^{n-1} = x$, for $x > 0$, there will be $u > 0$ partial backorders for retailer i caused by the n^{th} retailer order if two conditions are fulfilled: (i) The n^{th} retailer order originates from retailer i. The probability for this is $\frac{\lambda_i}{\lambda_0}$. (ii) The quantity of this order is $x + u$. The probability for this is $P\{Y_i = x + u\}$. This renders for $u > 0$,

$$P\{\hat{B}_i^n(t_0) = u | \Phi_0 \geq n \text{ and } \Psi^{n-1} = x\} = \frac{\lambda_i}{\lambda_0} P\{Y_i = x + u\}, \forall\ x > 0. \tag{B1}$$

(14) follows from unconditioning with respect to Φ_0 and Ψ^{n-1}. The latter by considering all possible positive values $x \in [1, S_0 - n + 1]$.

In order to have 0 partial backorders for retailer i, when the n^{th} retailer order after $t_0 - L_0$ brings the nominal inventory position to a non-positive value, the proof is analogous. In this case, however, there are two different scenarios; either the n^{th} customer arrives from retailer i and demands exactly x units to move the nominal inventory position to 0, or the n^{th} customer arrives from another retailer and demands more than or equal to x units. □

Proof of Lemma 2:

Proof. The distribution of complete backorders depends on n, the retailer order that brings the nominal inventory position to a non-positive value. However, because of the memoryless property of the compound Poisson demand, the complete backorders are independent of which retailers these n first orders originated from, and the sizes of these orders.

Given that there are a orders to retailer i after the n^{th} retailer order, i.e., $\check{\Phi}_{i,n} = a$, it is clear that $P\{\check{B}_{i,n}(t_0) = v | \check{\Phi}_{i,n} = a\} = P\{Y_i^a = v\}$. (17) follows by taking the expectation over all possible outcomes of $\check{\Phi}_{i,n}$.

To arrive at (18) we note that given m retailer orders that are completely backordered, the probability that a of these originates with retailer i is $P\{Z_i^m = a\}$. Moreover, the probability of $n + m$ retailer orders in $(t_0 - L_0, t_0]$ given at least n orders in $(t_0 - L_0, t_0]$ is $P\{\Phi_0 = n + m\}/P\{\Phi_0 \geq n\}$. (18) follows as an expectation over all possible values of m. □

Proof of Proposition 2:

Proof. By definition $P\{\hat{B}_i^n(t_0) = u\}$ is the probability that the n^{th} retailer order brings Ψ^n to a non-positive value, causing u partial backorders for retailer i. Also, by definition $P\{\check{B}_{i,n}(t_0) = r - u\}$ is the probability that there are $r - u$ complete backorders for retailer i conditioned on that the n^{th} retailer order brings Ψ^n to a non-positive value. Taking the expectation over all possible values of n and u (noting that $u \leq r$) renders (20).

For $r = 0$ we also need to consider the probability that the inventory position never reaches zero during the replenishment lead time, p_0, rendering (21). □

Proof of Lemma 3:

Proof. In order for $\tilde{B}_i^{-m}(t_0) = u$ there are two possible scenarios; (a) The m^{th} order before $(t_0 - L_0)$ arrives from retailer $j \neq i$ and the size of this order is at least $-\Psi^{-(m-1)} = -x$ units $(x < 0)$, and (b) the m^{th} order originates with retailer i and is for at least $-\Psi^{-m} = -x$ units. Starting with (a) the probability that the m^{th} customer arrives from retailer $j \neq i$ and demands more than $-x$ units is $(\lambda_{i^c}/\lambda_0)P\{Y_{i^c} \geq -x\}$. In order for $\tilde{B}_i^{-m}(t_0) = u$ in this scenario, the $m-1$ next customer orders need to contain u units to retailer i and need to assure that the nominal inventory position is x, which can be expressed as the probability $P\{V_i^{m-1} = u \text{ and } \Psi^{-(m-1)} = x\}$. A summation over all possible values of x ($x \in [S_0 + m - 1, -1]$) generates the first part of (25).

The probability for scenario (b) is $(\lambda_i/\lambda_0)P\{Y_i \geq -x\}$. In this scenario, $-x$ units of the m^{th} order will be backordered at t_0. Thus, in order for $\tilde{B}_i^{-m}(t_0) = u$, the next $m-1$ customers need to order $u - (-x) = u + x$ units to retailer i and ensure that $\Psi^{-(m-1)} = x$, which can be expressed as the probability $P\{V_i^{m-1} = u + x \text{ and } \Psi^{-(m-1)} = x\}$. A summation over all possible values of x renders the second part of (25).

In order for $V_i^{m-1} = u$ and $\Psi^{-(m-1)} = x$, the last $m-1$ customer orders before $t_0 - L_0$ need to include u units to retailer i and $x - S_0 - u$ units to all other retailers (recall $S_0 < 0$ and $x < 0$). With Z_i^{m-1} defined as in Section 3.3.1 and determined by (16) we get (26). □

Proof of Proposition 3:

Proof. Neither the shipment costs, $\sum_{k=1}^{K} \frac{\omega_k}{T_k}$, nor the holding costs at the central warehouse are affected by S_i. Furthermore, the holding cost and backorder cost at retailer i are unaffected by the order-up-to levels at other retailers. Thus, for fixed R_0 and $\mathbf{T}$, $TC(R_0, \mathbf{S}, \mathbf{T})$ is separable in the retailer order-up-to levels. To assert the convexity in S_i we define the holding and backorder costs at retailer i as

$$RC_i(R_0, S_i, T_i) \quad = \quad h_i E[IL_i^+(s)] + \beta_i E[IL_i^-(s)]. \tag{B2}$$

It is sufficient to show convexity for $RC_i(R_0, S_i, T_i)$ with respect to S_i for each retailer i. We define the difference function $\Delta G(s)$ as follows:

$$\Delta G(s) \quad = \quad RC_i(R_0, s+1, T_i) - RC_i(R_0, s, T_i). \tag{B3}$$

To prove convexity, we need to show that $\Delta G(s) - \Delta G(s-1) \geq 0$. First, by using (7), (8) and (9), we rewrite $RC_i(R_0, s, T_i)$ as:

$$\begin{aligned} RC_i(R_0, s, T_i) &= (h_i + \beta_i)E[IL_i^+(s)] - \beta_i E[IL_i(s)], \\ &= (h_i + \beta_i)\frac{1}{T_i}\sum_{j=1}^{s}\sum_{r=0}^{s-j} jP\{B_i(t_0) = r\}\int_0^{T_i} P\{D_i(L_i + x) = s - j - r\}dx, \\ &\quad -\beta_i(s - E[B_i(t_0)] - \lambda_i\mu_i L_i). \end{aligned} \tag{B4}$$

Note that the probability mass function and expectation of $B_i(t_0)$ does not depend on the order-up-to levels of the retailers. Next, we derive $RC_i(R_0, s+1, T_i)$ in terms of $RC_i(R_0, s, T_i)$:

$$
\begin{aligned}
& RC_i(R_0, s+1, T_i) = \\
& \quad (h_i+\beta_i)\frac{1}{T_i}\sum_{j=1}^{s+1}\sum_{r=0}^{s+1-j} jP\{B_i(t_0)=r\}\int_0^{T_i} P\{D_i(L_i+x)=s+1-j-r\}dx \\
& \quad -\beta_i(s+1-E[B_i(t_0)]-\lambda_i\mu_i L_i), \\
= & \quad (h_i+\beta_i)\frac{1}{T_i}\sum_{z=0}^{s}\sum_{r=0}^{s-z}(z+1)P\{B_i(t_0)=r\}\int_0^{T_i} P\{D_i(L_i+x)=s-z-r\}dx \\
& \quad -\beta_i(s+1-E[B_i(t_0)]-\lambda_i\mu_i L_i), \\
= & \quad (h_i+\beta_i)\frac{1}{T_i}\sum_{z=0}^{s}\sum_{r=0}^{s-z} zP\{B_i(t_0)=r\}\int_0^{T_i} P\{D_i(L_i+x)=s-z-r\}dx \\
& \quad +(h_i+\beta_i)\frac{1}{T_i}\sum_{z=0}^{s}\sum_{r=0}^{s-z} P\{B_i(t_0)=r\}\int_0^{T_i} P\{D_i(L_i+x)=s-z-r\}dx \\
& \quad -\beta_i(s-E[B_i(t_0)]-\lambda_i\mu_i L_i)-\beta_i, \\
= & \quad (h_i+\beta_i)\frac{1}{T_i}\sum_{z=0}^{s}\sum_{r=0}^{s-z} P\{B_i(t_0)=r\}\int_0^{T_i} P\{D_i(L_i+x)=s-z-r\}dx \\
& \quad -\beta_i + RC_i(R_0, s, T_i). \qquad \text{(B5)}
\end{aligned}
$$

As a result the first order difference is equal to:

$$
\Delta G(s) = (h_i+\beta_i)\frac{1}{T_i}\sum_{j=0}^{s}\sum_{r=0}^{s-j} P\{B_i(t_0)=r\}\int_0^{T_i} P\{D_i(L_i+x)=s-j-r\}dx - \beta_i. \qquad \text{(B6)}
$$

By following the similar line of thought, we get the following for the second order difference:

$$
\Delta G(s) - \Delta G(s-1) = (h_i+\beta_i)\frac{1}{T_i}\sum_{r=0}^{s} P\{B_i(t_0)=r\}\int_0^{T_i} P\{D_i(L_i+x)=s-r\}dx \geq 0. \qquad \text{(B7)}
$$

□

Proof of Proposition 4:

Proof. It is clear from (3) that $h_0E[IL_0^+]$ is increasing in R_0 for all values of $R_0 \geq -Q_0$ and that the total cost $TC(R_0, \mathbf{S}, \mathbf{T}) = h_0E[IL_0^+] + \sum_{k\in K} TC_k(R_0, \mathbf{S}, T_k) \geq h_0E[IL_0^+] + \sum_{k\in K} TC_k^l$. With $R_0^u = min\{R_0 : h_0E[IL_0^+] + \sum_{k\in K} TC_k^l \geq \overline{TC}\}$ it follows that for all $R_0 > R_0^u$, $TC(R_0, \mathbf{S}, \mathbf{T}) \geq \overline{TC}$ and searching this region cannot reduce the costs. □

Proof of Proposition 5:

Proof. Let

$$\Theta_k(T_k) = h_0 \sum_{i \in N_k} E[W_i] + \frac{\omega_k}{T_k} = \frac{1}{2} \sum_{i \in N_k} h_0 \lambda_i \mu_i T_k + \frac{\omega_k}{T_k}, \tag{B8}$$

be the costs directly related to shipment interval T_k. Note that $\Theta_k(T_k)$ is strictly convex in T_k as (for $T_k > 0$)

$$\frac{\partial^2 \Theta_k}{\partial T_k^2} = \frac{\partial^2 (\frac{1}{2} \sum_{i \in N_k} h_0 \lambda_i \mu_i T_k + \frac{\omega_k}{T_k})}{\partial T_k^2} = 0 + 2 \frac{\omega_k}{T_k^3} > 0. \tag{B9}$$

Note also that

$$\begin{aligned} TC(R_0, \mathbf{S}, \mathbf{T}) &= h_0 E[IL_0^+] + \sum_{\kappa \in K} TC_\kappa(R_0, \mathbf{S}, T_\kappa) \\ &\geq TC_k(R_0, \mathbf{S}, T_k) + \sum_{\kappa \in K \setminus \{k\}} TC_\kappa(R_0, \mathbf{S}, T_\kappa) \geq \Theta_k(T_k) + \sum_{\kappa \in K \setminus \{k\}} TC_\kappa^l. \end{aligned} \tag{B10}$$

It follows that no T_k satisfying

$$\Theta_k(T_k) \geq \overline{TC} - \sum_{\kappa \in K \setminus \{k\}} TC_\kappa^l \tag{B11}$$

can render a lower total expected cost than $\overline{TC}$. From (B8), (B11) and the convexity of $\Theta_k(T_k)$ we arrive at (31) and (32) by solving

$$\frac{1}{2} \sum_{i \in N_k} h_0 \lambda_i \mu_i T_k + \frac{\omega_k}{T_k} = \overline{TC} - \sum_{\kappa \in K \setminus \{k\}} TC_\kappa^l \tag{B12}$$

with respect to T_k. When T_k is smaller than the smallest root, T_k^l, or larger than the largest root, T_k^u, (B11) is always satisfied and these regions do not need to be searched. □

Proof of Lemma 4:

Proof. From Proposition 5 we know that the costs directly related to the shipment interval T_k, $\Theta_k(T_k) = \sum_{i \in N_k} E[W_i] + \frac{\omega_k}{T_k}$ is convex in T_k. Hence, the shipment interval $T_{k,l}$ that minimizes $\Theta_k(T_k)$ is obtained from the first order optimality condition

$$\frac{\delta \Theta_k}{\delta T_k} = \frac{\delta}{\delta T_k} \left(\frac{1}{2} \sum_{i \in N_k} h_0 \lambda_i \mu_i T_k + \frac{\omega_k}{T_k} \right) = 0, \tag{B13}$$

which renders

$$T_{k,l} = \sqrt{\frac{2\omega_k}{\sum_{i \in N_k} h_0 \lambda_i \mu_i}}. \tag{B14}$$

(33) follows from

$$TC_k(T_k) \geq \Theta_k(T_k) \geq \Theta_k(T_{k,l}) = \sqrt{2 h_0 \omega_k \sum_{i \in N_k} \lambda_i \mu_i}. \tag{B15}$$

□

Proof of Lemma 5:

Proof. From (27) $TC_k(R_0, \mathbf{S}, T_k) = \frac{\omega_k}{T_k} + \sum_{i \in N_k} h_0 E[W_i] + h_i E[IL_i^+] + \beta_i E[IL_i^-]$. Note that neither W_i nor $\frac{\omega_k}{T_k}$ depend on R_0. Moreover, the retailer costs, $RC_i(R_0, S_i, T_k) = h_i E[IL_i^+(s)] + \beta_i E[IL_i^-(s)]$, depend on R_0 solely through the backorder distribution, $B_i(t_0)$. From (6) we can show that the probability of an inventory level j at time $t_0 + L_i + t$ at retailer i when $B_i(t_0) = b_i$ is

$$\begin{aligned} P\{IL_i(t_0 + L_i + t) = j\} &= P\{b_i + D_i(t_0, t_0 + L_i + t) = S_i - j\} \\ &= P\{(b_i + 1) + D_i(t_0, t_0 + L_i + t) = (S_i + 1) - j\}, \end{aligned} \quad \text{(B16)}$$

which implies that

$$RC_i(S_i, T_k | B_i(t_0) = b_i) = RC_i(S_i + 1, T_k | B_i(t_0) = b_i + 1). \quad \text{(B17)}$$

For reasons of exposition and without loss of generality we renumber the retailers in retailer group k, $\{1, 2, ..., N_k\}$. Now, let $\mathbf{S}_{N_k}$ denote the vector of all order-up-to levels within retailer group k, $\{S_1, ..., S_{N_k}\}$, and $\mathbf{B}_{N_k}(t_0)$ denote the vector of backordered units to each retailer in retailer group k at t_0, $\{B_1(t_0), ..., B_{N_k}(t_0)\}$. The total costs for retailer group k, $TC_k(R_0, \mathbf{S}_{N_k}, T_k)$, for any values of R_0, S_{N_k} and T_k can then be seen as a sum over weighted averages of costs for all possible backorder combinations:

$$TC_k(R_0, \mathbf{S}_{N_k}, T_k) = \sum_{b_1=0}^{\infty} ... \sum_{b_k=0}^{\infty} P\{\mathbf{B}_{N_k}(t_0) = \{b_1, ..., b_k\}\} TC_k(\mathbf{S}_{N_k}, T_k | \mathbf{B}_{N_k}(t_0) = \{b_1, ..., b_k\}). \quad \text{(B18)}$$

This gives us for any value of R_0

$$\begin{aligned} TC_k(R_0, \mathbf{S}, T_k) &\geq \min_{T_k, \mathbf{S}_{N_k}} [TC_k(R_0, \mathbf{S}, T_k)] \\ &= \min_{T_k, \mathbf{S}_{N_k}} \left[\sum_{b_1=0}^{\infty} ... \sum_{b_k=0}^{\infty} P\{\mathbf{B}_{N_k}(t_0) = \{b_1, ..., b_k\}\} TC_k(\mathbf{S}_{N_k}, T_k | \mathbf{B}_{N_k}(t_0) = \{b_1, ..., b_k\}) \right] \\ &\geq \sum_{b_1=0}^{\infty} ... \sum_{b_k=0}^{\infty} P\{\mathbf{B}_{N_k}(t_0) = \{b_1, ..., b_k\}\} \min_{T_k, \mathbf{S}_{N_k}} [TC_k(\mathbf{S}_{N_k}, T_k | \mathbf{B}_{N_k}(t_0) = \{b_1, ..., b_k\})] \\ &= \sum_{b_1=0}^{\infty} ... \sum_{b_k=0}^{\infty} P\{\mathbf{B}_{N_k}(t_0) = \{b_1, ..., b_k\}\} \min_{T_k, \mathbf{S}_{N_k}} [TC_k(\mathbf{S}_{N_k}, T_k | \mathbf{B}_{N_k}(t_0) = \{0, ..., 0\})] \\ &= \min_{T_k, S_i \forall\ i \in N_k} (TC_k | B_i(t_0) = 0, \forall\ i \in N_k). \end{aligned} \quad \text{(B19)}$$

The first equality follows from (B18). The second inequality is a consequence of relaxing the constraint forcing the same values of T_k and $\mathbf{S}_{N_k}$ for all values of $\mathbf{B}_{N_k}(t_0)$. The second equality follows from (B17) and the fact that only the retailer costs are affected by the backorder distribution. The last equality follows directly as probabilities must sum to 1, completing the proof of (34) □

Appendix C: A Small Numerical Example

To illustrate the analysis we consider a system consisting of 3 retailers belonging to 2 retailer groups; retailers $\{1, 2\}$ constitute the first group (and therefore have equal shipment intervals) and retailer 3 the second. Each retailer face compound Poisson demand with a logarithmic compounding distribution, i.e. with a variance to mean ratio of the demand at retailer i of $\rho_i = Var[D_i]/E[D_i]$ we have: $P\{Y_i = y\} = -\alpha_i^y/(ln(1 - \alpha_i)y)$ and $\lambda_i = -E[D_i](1 - \alpha_i)ln(1 - \alpha_i)/\alpha_i$, where $\alpha_i = 1 - \rho_i^{-1}$. The considered problem parameters are presented in Table 2.

Table 2: Values for parameters and decision variables in the illustrative example

$E[D_i] = \{1, 1, 1\}$	$R_0 = -2$	$T_i = \{0.5, 0.5, 1\}$	$L_i = \{0.5, 0.5, 1, 0.5\}$	$\beta_i = \{10, 10, 10\}$
$\rho_i = \{4, 2, 1.5\}$	$Q_0 = 5$	$S_i = \{4, 4, 4\}$	$h_i = \{1, 1, 1, 1\}$	$\omega_k = \{2, 2\}$

As seen in Section 3.2, to analyze retailers i ($i = 1, 2, 3$) we need to determine the probability of r warehouse backorders designated to retailer i, $P\{B_i(t_0) = r\}$, for all $r = [0, S_i - 1]$, and the expected amount of backorders to retailer i, $E[B_i(t_0)]$. The computations are based on the analysis in Section 3.3. Two examples of the backorder distribution to retailer 1 conditioned on the inventory positions are $\{P\{B_1(t_0) = r|IP_0(t_0 - L_0) = -1\}, r = 0, 1, 2, 3\} = \{0.607, 0.262, 0.057, 0.028\}$ and $\{P\{B_1(t_0) = r|IP_0(t_0 - L_0) = 3\}, r = 0, 1, 2, 3\} = \{0.942, 0.023, 0.012, 0.008\}$. Taking the average over all possible inventory positions ($IP_0(t_0 - L_0) = [-1, 3]$) we get the steady state distributions of the backorders designated to each retailer, presented in Table 3. Exemplifying the Remark on page 20, Table 3 also presents the expected backorders designated to each retailer and illustrates their disproportions to the demand per time unit. One result of this is that the delay in the expected replenishment lead time to retailer 1, caused by backorders at the central warehouse, will be higher than the delay experienced by retailers 2 and 3.

Table 3: Distribution of warehouse backorders designated to each retailer

Retailer (i)	$\{P\{B_i(t_0) = r\}, r = 0, 1, 2, 3\}$	$E[B_i(t_0)]$
1	$\{0.824, 0.096, 0.032, 0.017\}$	0.399
2	$\{0.773, 0.144, 0.048, 0.020\}$	0.373
3	$\{0.754, 0.165, 0.054, 0.018\}$	0.367

Knowing the backorder distributions, the expected stock on hand, $E[IL_i^+]$, the expected backorders, $E[IL_i^-]$, and the fill rates, γ_i, at the retailers can be determined from (7), (9) and (10) respectively. Calculating the expected stock on hand at the central warehouse using (3) and (4), the total cost of the system can be determined from (2). The results, determined analytically by the suggested approach, and simulated in a discrete event simulation program (Extend), are presented in Table 4.

Table 4: Results from exact analysis and simulation (Sim)

	$E[IL_0^+]$	$E[IL_i^+]$	$E[IL_i^-]$	TC	$\gamma_i(\%)$
Exact	1.639	{3.087,2.541,2.704}	{0.236,0.165,0.071}	20.691	{72.6,79.5,88.1}
Sim[1]	1.639	{3.087,2.541,2.704}	{0.236,0.165,0.071}	20.691	{72.6,79.5,88.1}

[1]The Standard deviations of the simulated results were < 0.001.

Appendix D: Numerical Study

For all 128 problem settings defined in Section 5, the shipment intervals, $T_k \ \forall\ k \in K$, the reorder points at the central warehouse, R_0, and the order up to levels at the retailers, $S_i \ \forall\ i \in N$, are optimized using the method described in Section 4. For optimizing the shipment intervals we have used a step size of of 0.01 time units. The complete results for all settings are available from the authors upon request. Table 5 summarizes the results in terms of average effects on: the optimal total cost, TC^*, the optimal reorder point at the central warehouse, R_0^*, the average of the optimal order-up-to level at the retailers, S_i^*, and the average of the optimal shipment intervals, T_k^*. It also includes the relative difference between the heuristic shipment intervals and the optimal, ΔT, the relative increase in the associated costs, ΔC, the difference between the upper bound on R_0, R_0^u, and the optimal value, R_0^*, ΔR_0^u, the difference between the optimal value on T_k, T_k^*, and the lower bound, T_k^l, ΔT_k^l, and the difference between the upper bound on T_k, T_k^u, and the optimal value, T_k^*, ΔT_k. The results associated with $\rho_i = 1$ are averages across all 64 problems where $\rho_i = 1$, and analogously for all other parameters.

Table 5: Average results for the test series, for low and high values of N, ρ_i, β_i, ω_k, L_0, L_i and Q_0 as well as averages over all problems and minimum and maximum values

	$E[TC^*]$	$E[R_0^*]$	$E[S_i^*]$	$E[T_k^*]$	$E[\Delta C]$	$E[\Delta T]$	$E[\Delta R_0^u]$	$E[\Delta T_k^l]$	$E[\Delta T_k^u]$
ρ_i[1] $= 1$	82.02	19.11	15.08	3.24	0.12%	-0.60%	4.91	2.05	15.81
ρ_i[1] $= 5$	141.00	20.67	23.75	3.05	0.16%	6.57%	9.92	2.33	31.28
$\beta_i = 10$	93.66	18.19	15.88	3.30	0.08%	-2.27%	7.42	2.21	18.88
$\beta_i = 100$	129.37	21.59	22.94	2.99	0.21%	8.24%	7.41	2.17	28.21
$\omega_k = 10$	81.96	21.25	15.97	1.54	0.13%	1.90%	7.28	1.25	19.07
$\omega_k = 100$	141.07	18.53	22.85	4.75	0.15%	4.07%	7.55	3.13	28.02
$N = 3$	84.78	10.86	20.45	3.68	0.15%	3.18%	6.53	2.52	25.82
$N = 6$	138.24	28.92	18.38	2.61	0.14%	2.79%	8.30	1.86	21.27
$L_0 = 1$	108.36	1.27	18.86	3.11	0.17%	4.41%	5.06	2.11	22.05
$L_0 = 5$	114.66	38.52	19.96	3.18	0.11%	1.56%	9.77	2.27	25.04
$L_i = 1$ and 2	107.98	20.45	17.26	3.10	0.18%	4.94%	7.48	2.13	22.96
$L_i = 2$ and 4	115.04	19.33	21.57	3.19	0.11%	1.03%	7.34	2.26	24.14
$Q_0 = 2$	110.40	24.50	19.13	3.13	0.15%	3.49%	6.33	2.15	22.96
$Q_0 = 20$	112.63	15.28	19.69	3.16	0.14%	2.48%	8.50	2.23	24.13
Average	111.51	19.89	19.41	3.15	0.14%	2.99%	7.41	2.19	23.55
Minimum	31.61	-10	7.67	1.09	0.00%	-9.79%	0	0.91	7.05
Maximum	264.19	68	38.33	6.11	0.66%	18.60%	24	4.23	51.26

[1] $\rho_i = 1$ corresponds to Poisson demand at all retailers and $\rho_i = 5$ to compound Poisson demand with logarithmic compounding distributions so that the variance-to-mean ratio of the demand per time unit at retailer i is 5 $\forall\ i \in N$.

Focusing first on the computational aspects and the optimality bounds, it is relevant to know that the optimization times for studied problems were between 0.2 and 120 minutes on a Dell Latitude 6400 lap top. The parameters that seem to affect the computational times the most are λ_0, L_0, N and K. An important observation is that the fairly time consuming calculations of the backorder distributions at the central warehouse only needs to be computed once for each value of R_0 (in a given problem). Especially for the computationally more demanding problems, most of the computational time was spent on calculating these distributions. An explanation for this is that the complexity of determining these distributions increase with $R_0 + Q_0$. As a result, the upper bound on the reorder level at the central warehouse, R_0^u is the most important bound. Table 5 shows that this bound is rather tight, exceeding the optimal value R_0^* by on average with only 7.41 units. It is especially tight in the cases where $\rho_i = 1$ (on average only 4.91 units above optimum) and when $L_0 = 1$ (on average only 5.06 units above optimum). In fact, in the 32 problem settings investigated, where $\rho_i = 1$ and $L_0 = 1$,

$E[\Delta R_0^u] = 3.53$. The fact that $E[\Delta R_0^u]$ is lower in systems where L_0 is lower is intuitive, as the optimal reorder level, R_0^*, is much lower for these systems. The effect of ρ_i on the performance of the bound is perhaps less obvious. In order to explain this, recall that this bound is based on an estimation of the minimum costs at the retailers, TC_k^l, that assumes no backorders at the central warehouse (see Proposition 4). When the demand has a higher variance-to-mean ratio, a higher R_0-value is required for reaching this situation and the bound will therefore become looser.

Turning to the bounds on T_k, Table 5 shows that the lower bound is on average 2.19 time units below the optimal value (which is on average 3.15). The upper bound is looser as it is on average 23.55 time units above the optimal value. These bounds play a less important role in reducing the computational time than the upper bound on R_0^*.

Shifting our attention to the EOI heuristic, Marklund (2011) shows that it performs very well for Poisson demand. Based on Table 5 this seems to hold also for compound Poisson demand. The expected relative cost increase for all problem settings is only 0.14%, although the heuristic tends to overestimate the optimal shipment intervals with on average 2.99%. The relative cost increase is also only slightly higher in the systems where the variability in the demand is high 0.16% compared to 0.12% for the Poisson systems. There is a stronger tendency to overestimate the shipment intervals for the systems where $\rho_i = 5$ ($E[\Delta T] = 6.57\%$), but because the total costs are much higher in these systems, the relative increase is still small. The parameter that seems to have the biggest influence on the performance of the heuristic is, in fact, the backorder costs. In systems where the backorder costs are high $E[\Delta C] = 0.21\%$. An explanation may be the desire to increase the flexibility in these systems by reducing the shipment intervals (the shipment intervals are overestimated by 8.24% in systems where $\beta_i = 100$ when using the heuristic).

Considering the behavior of the optimal solutions, Table 5 illustrates that when the variability increases (i.e. comparing $\rho_i = 1$ to $\rho_i = 5$), the biggest difference in the control parameters is seen in the order-up-to levels of the retailers, which increase from on average 15.08 to 23.75. However there are also effects on the shipment intervals and reorder points at the central warehouse. The optimal warehouse reorder point increases from on average 19.11 to 20.67, thus raising the amount of available units at the central warehouse to handle the variability. The shipment intervals decrease from on average 3.24 to 3.05 with the effect that there is more flexibility in the system (the system can react faster if there is a big order at a specific retailer). Similar effects can be seen when increasing the backorder cost (β_i). The largest effect on the the optimal control parameters is an increase in the average order-up-to levels at the retailers from 15.88 to 22.94. However, we also see an increase in the average reorder point at the central warehouse from 18.19 to 21.59 and a decrease in the average shipment intervals (from 3.30 to 2.99), increasing the flexibility.

With regards to the shipment costs, we can see from Table 5 that as $\omega_k \ \forall \ k$ increase, the system reacts by increasing the shipment intervals (from on average 1.54 to 4.75). Moreover, the order-up-to levels at the retailers need to be raised accordingly in order to ensure stock for a longer replenishment cycle. Maybe less intuitively, the average optimal reorder points at the central warehouse decreases for these systems. This can be explained by the fact that increased shipment intervals results in longer replenishment lead times to the retailers, which may reduce the relative impact of inventory pooling at the central warehouse. Another contributing factor may be that the consolidation stock at the central warehouse increases with the shipment intervals. Thus, in order to avoid too much stock at the warehouse, the reorder point is reduced. For the other parameters, N, L_0, L_i and Q_0, the behavior of the optimal solutions did not offer any insights beyond the obvious.